„Beyond Balanced Growth“: Some Further Results

by

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Abstract

Kongsamut et al. (2001) have demonstrated that contrary to earlier opinion balanced growth of aggregated variables and structural change can be simultaneously generated in a model of exogenous technological progress. That is, they have presented a model that is simultaneously consistent with Kaldor's stylized facts and stylized facts of labor reallocation between sectors. However, they used in their model sectoral production functions that differ only by a multiplicative constant. Thus their model is not consistent with the empirical fact of different labor shares of income across sectors. We generalize their model and show that a model of exogenous growth can simultaneously be consistent with Kaldor's stylized facts, stylized facts of labor reallocation and different labor shares of income across sectors (i.e. completely different sectoral production functions).

Keywords: balanced growth, structural change, Kaldor facts, labor reallocation, labor shares of income.

JEL Codes: O14, O 41

1. Introduction

Kongsamut et al. (2001) have demonstrated that contrary to earlier opinion balanced growth of aggregated variables and structural change[1] can be simultaneously generated in a model of exogenous technological progress. Therefore, their model is simultaneously consistent with Kaldor's stylized facts[2] and with facts about structural change. Unfortunately, the authors assumed sectoral production functions, which differ only by a constant productivity parameter. This is an unrealistic assumption, because there is empirical evidence that the labor shares of income differ across sectors.[3] Therefore, Kongsamut et al. (1997) had introduced sectoral production functions that differ completely, i.e. in every parameter. There they proved that in this case structural change is only consistent with a constant real rate of return. Simultaneous balanced growth is no longer feasible. Thus not all Kaldor facts are satisfied. However, their model featured only three sectors (manufacturing, agriculture, services).[4]

The overall conclusion that can be drawn from this research is that balanced growth of aggregated variables and structural change cannot be simultaneously unified in this model, as long as sectoral production functions are completely different. In other words, the research by Kongsamut et al. (1997) and (2001) conveys the impression that Kaldor's stylized facts, structural change facts and different labor shares of income across sectors cannot be unified in a "neoclassical" model.

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[1] Structural change stands here for labor reallocation between sectors (such as manufacturing, agriculture and services).

[2] Kaldor's stylized facts state that growth rate of per capita output, capital-to-output ratio, shares of labor and capital in national income and rate of return to capital are nearly constant in the long run; and capital per worker grows over time.

[3] See Kongsamut et al. (1997), p. 20.

[4] Kongsamut et al. (2001) and (1997) used a three sector framework, because their aim was to fit the model to the empirical facts of US-development during the last century.

Our aim is to demonstrate that this is not true. The reason why Kongsamut et al. (2001) and (1997) were not "successful" in this respect is that they did not use "enough" sectors to prove this result. We use the model presented by Kongsamut et al. (2001) for our proof. But, unlike Kongsamut et al. (2001), we assume that there is an arbitrary number of sectors (instead of only three sectors) in the economy and the sectoral production functions are completely different. We use the concept of a "balanced growth path of aggregated variables". Such a growth path features constant growth rates of aggregated variables, a constant real rate of return to capital and constant relative prices. But, along this growth path, the growth rates of disaggregated variables (such as sectoral output etc.) need not to be constant. We show that a necessary condition for simultaneous balanced growth of aggregated variables and structural change is the existence of at least four sectors in this framework.

Overall, the model that is presented here is a more general version of the model presented in Kongsamut et al. (2001), because the (unrealistic) assumption of nearly identical sectoral production functions is not necessary. Thus, this model is consistent at the same time with three empirical findings: Kaldor's stylized facts, structural change facts, and different labor shares of income across sectors, whereas the version of Kongsamut et al. (2001) is only consistent with the first two empirical findings.

In the next section of this paper we specify the production sector and its efficiency conditions. Following this, we describe the household sector and solve its dynamic optimization problem by using results from the production sector. In the fourth part we look at the balanced growth path of this model and derive the necessary conditions for its existence. The fifth part is about structural change along this "balanced" growth path. Finally, we summarize our results.

2. Production Sector

Like Kongsamut et al. (2001), we assume that there are two production factors: capital (K_t) and labor. The total amount of labor available in the economy is exogenously given and normalized to one at every point of time. There is no population growth.[5] The growth rate (g) of labor-augmenting technical progress (X_t) is constant, exogenously given and equal across sectors. The output of the manufacturing sector (i = M) can be used as capital and consumed, as well. The output of the other sectors can only be consumed. That is, only the manufacturing sector produces capital.[6] Therefore, the output of the manufacturing sector has to be numéraire. All capital and labor available has to be used in production. Unlike Kongsamut et al. (2001), we assume that each sector i produces its output (Y_t^i) by a sector specific Cobb-Douglas production function and the number of sectors (n) is arbitrary. Thus the equations describing the production side of the economy are:

$$Y_t^i = B_i\left(\phi_t^i K_t\right)^{\alpha_i}\left(N_t^i X_t\right)^{1-\alpha_i} = B_i N_t^i X_t\left(\frac{\phi_t^i K_t}{N_t^i X_t}\right)^{\alpha_i} \qquad \forall i = 1,\ldots n \tag{1}$$

$$\sum_i \phi_t^i = 1 \;\; \forall t \tag{2}$$

$$\sum_i N_t^i = 1 \;\; \forall t \tag{3}$$

$$\dot{X}_t = gX_t \tag{4}$$

$$Y_t^M = \dot{K}_t + \delta K_t + C_t^M \tag{5}$$

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[5] Similar results can be derived with population growth.
[6] This assumption is empirically reasonable at least for the USA. See Kongsamut et al. (2001).

$$Y_t^i = C_t^i, \ \forall i \neq M \tag{6}$$

$$p_M = 1 \tag{7}$$

The parameters α_i are different across sectors, i.e. $\alpha_i \neq \alpha_j$, $\forall i \neq j$. ϕ_t^i (N_t^i) represents the fraction of capital (labor) devoted to sector i; C_t^i denotes consumption of good i; p_i is the relative price of good i expressed in manufacturing terms; δ denotes the economy-wide depreciation rate and t is the time index.

In order to facilitate the calculations, we follow Kongsamut et al. (2001) and (1997) and elaborate the efficiency conditions in production:[7] Efficient allocation across sectors requires that the marginal rates of technical substitution are equal across sectors, i.e.:[8]

$$\frac{\partial Y_t^i / \partial(\phi_t^i K_t)}{\partial Y_t^i / \partial(N_t^i X_t)} = \frac{\partial Y_t^M / \partial(\phi_t^M K_t)}{\partial Y_t^M / \partial(N_t^M X_t)} \quad \forall \mathrm{i}$$

$$\Leftrightarrow \frac{\alpha_i}{1-\alpha_i}\frac{N_t^i}{\phi_t^i} = \frac{\alpha_M}{1-\alpha_M}\frac{N_t^M}{\phi_t^M} \qquad \forall \mathrm{i} \tag{8}$$

Moreover, it also requires that the marginal productivity of labor is equal across sectors, which implies:

$$p_i = \frac{\partial Y_t^M / \partial(N_t^M X_t)}{\partial Y_t^i / \partial(N_t^i X_t)} = \frac{B_M(1-\alpha_M)\left(\dfrac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M}}{B_i(1-\alpha_i)\left(\dfrac{\phi_t^i K_t}{N_t^i X_t}\right)^{\alpha_i}} \quad \forall \mathrm{i} \tag{9}$$

Profit-maximizing producers employ capital and labor in such a manner that the sum of the real rate of return on capital and the depreciation rate is equal to the marginal productivity of capital, and the real wage rate is equal to the marginal productivity of labor, i.e.:

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[7] Because the optimization with the Hamiltonian yields these efficiency conditions anyway, it facilitates the calculations if we elaborate these conditions now and use them in dynamic optimization.

[8] See Kongsamut et al. (1997) as well.

$$(r+\delta) = p_i \frac{\partial Y_t^i}{\partial(\phi_t^i K_t)} \quad \forall i$$

$$w_t = p_i \frac{\partial Y_t^i}{\partial N_t^i} \quad \forall i$$

For i = M this results in (because of eq. (1) and (7)):[9]

$$r+\delta = B_M \alpha_M \left(\frac{\phi_t^M K_t}{N_t^M X_t} \right)^{\alpha_M - 1} \tag{10}$$

$$w_t = B_M (1-\alpha_M) \left(\frac{\phi_t^M K_t}{N_t^M X_t} \right)^{\alpha_M} X_t \tag{11}$$

Aggregated output in manufacturing terms is given by (because of eq. (1) and (9)):

$$Y_t \equiv \sum_i p_i Y_t^i = B_M (1-\alpha_M) \left(\frac{\phi_t^M K_t}{N_t^M X_t} \right)^{\alpha_M} X_t \left(\frac{\alpha_M}{1-\alpha_M} \frac{N_t^M}{\phi_t^M} + 1 \right) \tag{12}$$

A proof of equation (12) is given in Appendix A.

3. Preferences

The preference structure is the same as in Kongsamut et al. (2001). The only difference is that we generalize the utility function: the number of goods (n) is arbitrary (instead of being three):

$$U = \int_0^\infty \frac{\left[\prod_i \left(C_t^i - \overline{C}^i \right)^{\beta_i} \right]^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt, \qquad i = 1,\ldots n$$

where

[9] The same results are obtained in Kongsamut et al. (1997), p. 22.

$$\sigma, \rho, \beta_i > 0 \qquad \forall \mathrm{i} \tag{13}$$

$$\sum_i \beta_i = 1 \tag{14}$$

The constants $\overline{C}^i$ can be interpreted as subsistence levels (if $\overline{C}^i$ is positive) or as the home production (if $\overline{C}^i$ is negative) of good i (because the marginal utility approaches infinity when C_t^i approaches $\overline{C}^i$). β_i defines by how much the consumption of good i (C_t^i) contributes to the utility of the household. These preferences are non-homothetic, i.e. the income elasticity of demand differs across goods, as long as not all $\overline{C}^i = 0$. This can cause some structural change.

The representative household maximizes its lifetime utility subject to its dynamic budget-restriction, which is given by: [10]

$$\dot{K}_t = Y_t - \delta K_t - E_t, \tag{15}$$

where the aggregated consumption expenditures are given by:

$$E_t \equiv \sum_i p_i C_t^i \tag{16}$$

According to Kongsamut et al. (2001) we assume[11]

$$\overline{C}^M = 0 \tag{17}$$

in order to ensure that the labor share of the manufacturing sector stays constant, which is consistent with empirical facts.

This optimal control problem can be solved by using a Hamiltonian. The transversality condition is given by $\lim_{t \to \infty} \{\psi_t K_t\} = 0$, where ψ_t is the operator of the Hamiltonian

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[10] This restriction can be obtained by adding $\sum_{i \neq M} p_i C_t^i$ to both sides of eq. (5). Because of eq. (6) and (7), this results in the dynamic restriction above.

[11] This is consistent with the empirical facts, at least for the developed economies during the last 100 years; see e.g. Kongsamut et al. (2001).

(shadow price of capital). The optimal solution is the same as in Kongsamut et al. (2001):[12]

$$\frac{p_i\left(C_t^i - \overline{C}^i\right)}{\beta_i} = \frac{C_t^M}{\beta_M}, \qquad \forall i \tag{18}$$

$$\frac{\dot{C}_t^M}{C_t^M} = \frac{r_t - \rho}{\sigma} \tag{19}$$

These results are proved in Appendix B.

We can now derive the consumption expenditures E_t (because of eq. (9), (14), (16) and (18)):

$$E_t \equiv \sum_i p_i C_t^i = \frac{C_t^M}{\beta_M} + B_M(1-\alpha_M)\left(\frac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M} \sum_i \frac{1}{1-\alpha_M} \frac{\overline{C}^i}{B_i\left(\frac{\phi_t^i K_t}{N_t^i X_t}\right)^{\alpha_i}} \tag{21}$$

4. Balanced Growth of Aggregated Variables

A balanced growth path of aggregated variables[13], which is consistent with Kaldor's stylized facts of economic growth, requires r_t to be constant and $\frac{\dot{Y}_t}{Y_t} = \frac{\dot{K}_t}{K_t} = \frac{\dot{E}_t}{E_t} = \frac{\dot{w}_t}{w_t} = \frac{\dot{X}_t}{X_t} \equiv g$.

[12] This solution is only true when relative prices are constant. We will see that relative prices are constant along the balanced growth path that we focus on. Kongsamut et al. (1997) achieve the same results, as well; see Kongsamut et al. (1997), p. 21.

[13] A "balanced growth path" in the traditional sense refers to a growth path that features constant growth rates of all variables. Thus, no structural change takes place along this type of growth path. When we refer to a "balanced growth path of aggregated variables" in this paper, we mean a growth path that features constant growth rates of aggregated variables, constant relative prices and a constant real rate of return to capital, but not necessarily constant growth rates of disaggregated variables, such as sectoral output etc.

Let us now assume that N_t^M is constant[14] and $\frac{\phi_t^M}{N_t^M}$ is constant as well. It can be seen at first sight that in this case a balanced growth path of aggregated variables (which is consistent with Kaldor's stylized facts) exists when K_t grows at rate g: r is constant (eq. (10)); Y_t (eq. (12)), w_t (eq. (11)) and E_t (eq. (15)[15]) each grow at rate g. Thus, the Kaldor facts are satisfied. Additionally, relative prices are constant (eq. (9))[16] and C_t^M grows at rate g (eq. (5) and (1))[17].

The remaining task is to elaborate the necessary conditions that ensure that N_t^M and $\frac{\phi_t^M}{N_t^M}$ are constant: For N_t^M to be constant it is necessary that $\sum_{i \neq M} N_t^i$ is constant (eq. (3)). It can be derived from equations (2), (3) and (8), that $\frac{\phi_t^M}{N_t^M}$ is given by

$$\frac{\phi_t^M}{N_t^M} = \frac{1}{\frac{1-\alpha_M}{\alpha_M}\left[\frac{\alpha_M}{1-\alpha_M} + \sum_{i \neq M} \frac{\alpha_i}{1-\alpha_i} N_t^i - \frac{\alpha_M}{1-\alpha_M} \sum_{i \neq M} N_t^i\right]}$$

The explicit proof is in Appendix C.

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[14] This assumption is necessary in order to facilitate the solution of differential equations, which we will need later on. The assumption is consistent with empirical findings (see Kongsamut et al. (2001)).

[15] Y_t, δK_t and $\dot{K}_t$ each grow at rate g. Thus, equation (15) can only be fulfilled in every point of time if E_t grows at rate g as well.

[16] When $\frac{\phi_t^M}{N_t^M}$ is constant, $\frac{\phi_t^i}{N_t^i}$ is constant in all sectors as well, because of eq. (8).

[17] It can be seen from eq. (1) that Y_t^M grows at rate g along the balanced growth path of aggregated variables, because we assumed that N_t^M is constant. Thus, because Y_t^M, δK_t and $\dot{K}_t$ each grow at rate g along the balanced growth path of aggregated , eq. (5) can only be satisfied at any point of time if C_t^M grows at rate g as well.

Thus, $\frac{\phi_t^M}{N_t^M}$ is constant, when $\sum_{i \neq M} \frac{\alpha_i}{1-\alpha_i} N_t^i$ is constant. (We have already assumed that $\sum_{i \neq M} N_t^i$ is constant). Overall, the two requirements that are necessary for N_t^M and $\frac{\phi_t^M}{N_t^M}$ to be constant, are:

$$\sum_{i \neq M} N_t^i = \text{constant} \tag{23}$$

$$\sum_{i \neq M} \frac{\alpha_i}{1-\alpha_i} N_t^i = \text{constant} \tag{24}$$

It can be proved by using equations (1), (6), (8), (18) and (22) that along our balanced growth path of aggregated variables sectoral labor shares are given by: [18]

$$N_t^i = N_0^i - \Gamma_i + \Gamma_i \exp(-gt), \quad \forall \mathrm{i} \neq \mathrm{M}, \tag{25}$$

where

$$\Gamma_i = \frac{\overline{C}^i}{X_0 B_i \left[\left(\frac{\sigma g + \rho + \delta}{\alpha_M B_M} \right)^{1/(\alpha_M - 1)} \frac{1-\alpha_M}{\alpha_M} \frac{\alpha_i}{1-\alpha_i} \right]^{\alpha_i}} \tag{26}$$

The explicit proof is in Appendix D.

With this result, we can express the equations (23) and (24) as functions of model-parameters:

$$\sum_{i \neq M} \Gamma_i = 0 \tag{27}$$

$$\sum_{i \neq M} \frac{\alpha_i}{1-\alpha_i} \Gamma_i = 0 \tag{28}$$

[18] See also Kongsamut et al. (1997), p. 23.

The question that now has to be answered is whether these two requirements can be satisfied simultaneously. The answer is yes, but there have to be at least three goods/sectors with a $\overline{C}^i \neq 0$.

If there are only two sectors with a $\overline{C}^i \neq 0$ (as in the paper presented by Kongsamut et al. (1997) and (2001)), the equations (27) and (28) form a homogenous linear-equation system, that is fully determined (i.e. there are two unknowns[19] and two linearly independent equations). Thus, there is only a trivial solution to this system, i.e. $\Gamma_i = 0, \forall i$. It can be seen from equations (25) and (26) that in this case $\overline{C}^i$ has to be equal to zero in all sectors, and no structural change takes place, i.e. labor shares stay constant. Of course, if sectoral production functions are identical up to a constant (i.e. $\alpha_i = \alpha_j, \forall i, j$), the equations (27) and (28) are linearly dependent. Thus, there exist an infinite number of solutions of the system. This is the case in Kongsamut et al. (2001). Overall, when only two sectors have a $\overline{C}^i \neq 0$, the equations (27) and (28) can only be satisfied simultaneously, either when there is no structural change or when sectoral production functions are identical up to a constant.

If there are at least three sectors with a $\overline{C}^i \neq 0$, the equations (27) and (28) are a homogenous linear-equation system that is not fully determined, i.e. the number of equations is smaller than the number of unknowns (two equations and at least three unknowns). Thus, there exist an infinite number of solutions of the system, even when

[19] It can be seen from equation (26) that Γ_i is equal to zero when the corresponding $\overline{C}_i = 0$. Thus, when there are only two sectors with $\overline{C}^i \neq 0$ in the economy there are also only two (corresponding) unknowns Γ_i in the equations (27) and (28) (the other Γ_i's are equal to zero).

the production function parameters α_i differ in all sectors. Therefore, the equations (27) and (28) can simultaneously be satisfied.[20]

The remaining question is: how can the two conditions (27) and (28) be interpreted? Equation (27) is necessary, because we want our model to have a constant N_t^M, which is an empirical fact. As explained above, N_t^M is constant when equation (27) is satisfied.

By using equations (9), (17), (26) and (27) it can be proved that if condition (28) is satisfied, $\sum_i p_i \overline{C}^i = 0$ is satisfied as well (see Appendix E). An interpretation of this requirement has already been suggested by Kongsamut et al. (2001): If we interpret the utility parameters $\overline{C}^i$ as the household's initial endowments of goods i, this condition states that the market value of initial endowments has to be equal to zero.

5. Structural Change

What about structural change? It can be seen from equations (25) and (26) that sectors with $\overline{C}^i < 0$ (>0) have increasing (decreasing) labor shares along the balanced growth path of aggregated variables.[21] (The labor share of the manufacturing sector is constant, as mentioned above.) Thus, our balanced growth path of aggregated variables also features structural change. The reason for structural change is different income elasticity of demand across goods as in Kongsamut et al. (1997) and (2001). The direction of structural change depends on the relation between demand growth and productivity growth along the balanced growth path of aggregated variables. As long as demand

[20] See Meckl (2002) for similar conditions in another model of balanced growth and structural change.
[21] See footnote 13.

grows at a rate higher than g, the increase in productivity due to technological progress is not sufficient to make supply keep pace with demand growth. Thus, labor input has to be increased in the corresponding sector, and vice versa. In the manufacturing sector, where demand grows at rate g there is no need to change the factor inputs.

How can the sectors of this economy be interpreted?

- As already mentioned, the sector i = M with $\overline{C}^{M} = 0$ is the manufacturing sector, which features no labor reallocation.
- One sector with $\overline{C}^{i} > 0$ can be interpreted as the agriculture sector. As explained above, sectors with a positive $\overline{C}_{i}$ have decreasing labor shares in our model. Thus, the labor share of the agriculture sector is decreasing in our model. This is consistent with the empirical facts of structural change.[22]
- There are two alternatives for the interpretation of the remaining sectors with a $\overline{C}^{i} \neq 0$: **Either**, they can be interpreted as subsectors of the service sector. They should have increasing labor shares.[23] As stated above, this requires $\overline{C}^{i} < 0$ in these sectors. **Or**, if there are only two remaining sectors: One sector with $\overline{C}^{i} < 0$ can be interpreted as the service sector (increasing labor share). The other sector can be interpreted as the public sector that provides public services that correspond to government spending. Wagner's law states that the ratio of government spending to total output increases in the long run (see for example Oxley (1994)). The ratio of sectoral output to aggregate output is increasing in

[22] For a review of empirical facts regarding structural change see, e.g. Kuznets (1976), Kongsamut et al. (2001) and (1997), or Ngai/Pissarides (2004). These empirical findings state that the labor share of the service sector (agricultural sector) is increasing (decreasing). With respect to the manufacturing sector, Kongsamut et al. (1997) and (2001) state that its labor share can be regarded as constant in the last century in the developed countries. Other authors (e.g. Ngai/Pissarides (2004)) state that the evolution of the labor share of the manufacturing sector might be rather described as "hump-shaped" (in the longer run).

[23] See footnote 22.

our model as long as the corresponding $\overline{C}^{i} < 0$.[24] Thus, we have to assume that $\overline{C}^{i}$ is negative in the sector that is interpreted as the public sector. When this assumption is made the ratio of public sector output to aggregate output increases with time in our model, corresponding to Wagner's law.

Overall, the same structural change dynamic as in Kongsamut et al. (2001) is feasible in this model.

6. Résumé

The work of Kongsamut et al. (2001) and (1997) conveys the impression that balanced growth of aggregated variables and structural change cannot be simultaneously generated in a "neoclassical" model of exogenous growth when the sectoral production functions differ in all parameters. It seemed therefore that Kaldor's stylized facts, structural change facts and different sectoral labor shares of income were not simultaneously feasible in such a model.

We have proved in this paper that this is not true. The framework for this proof is similar to the framework of Kongsamut et al. (2001). The necessary condition for simultaneous balanced growth of aggregated variables and structural change is the existence of at least three sectors with a $\overline{C}^{i} \neq 0$. In this case the parameter restrictions (27) and (28) can be satisfied simultaneously. These two restrictions involve 16

[24] The ratio of sectoral output to aggregated output is given by $p_i Y_t^i / Y_t$. We know that relative prices (p_i) are constant and aggregate output (Y_t) grows at rate g along the balanced growth path of aggregated variables. By using equations (18) and (6) it can be shown that the growth rate of sectoral output is given by $\dot{Y}_t^i / Y_t^i = g\left(1 - \overline{C}^{i} / Y_t^i\right), \ \forall i \neq M$ in our model. Thus, when $\overline{C}^{i} < 0$, the sectoral output (Y_t^i) is growing at a rate higher than g along the balanced growth path of aggregated variables. Therefore, the ratio of sectoral output to aggregate output $p_i Y_t^i / Y_t$ is increasing along the balanced growth path of aggregated variables if $\overline{C}^{i} < 0$.

parameters. Therefore, they will probably not restrict the universality of the model. When these two restrictions are satisfied the economy is on a balanced growth path, which features simultaneously structural change, every time when the aggregated capital grows at rate g.

Overall, we managed to get the same patterns of structural change as in Kongsamut et al. (2001) without using the unrealistic assumption of nearly identical sectoral production functions. Thus, our model is consistent with three kinds of empirical findings: Kaldor’s stylized facts, structural change facts and completely different sectoral production functions (i.e. different labor income shares across sectors).

APPENDIX A

Inserting equations (9) and (1) into $Y_t := \sum_i p_i Y_t^i$ results in:

$$Y_t \equiv \sum_i p_i Y_t^i = B_M (1-\alpha_M)\left(\frac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M} \sum_i \frac{B_i N_t^i X_t \left(\frac{\phi_t^i K_t}{N_t^i X_t}\right)^{\alpha_i}}{B_i (1-\alpha_i)\left(\frac{\phi_t^i K_t}{N_t^i X_t}\right)^{\alpha_i}}$$

$$= B_M (1-\alpha_M)\left(\frac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M} X_t \sum_i \frac{N_t^i}{(1-\alpha_i)}$$

$$= B_M (1-\alpha_M)\left(\frac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M} X_t \left[\sum_i \frac{\alpha_i}{(1-\alpha_i)} N_t^i + \sum_i N_t^i\right]$$

$$= B_M (1-\alpha_M)\left(\frac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M} X_t \left[\sum_i \frac{\alpha_i}{(1-\alpha_i)} N_t^i + 1\right]$$

Solving equation (8) for ϕ_t^i and inserting it into equation (2) yields:

$$\sum_i \frac{\alpha_i}{1-\alpha_i} N_t^i = \frac{\alpha_M}{1-\alpha_M} \frac{N_t^M}{\phi_t^M}$$

Inserting this result into the equation above results in:

$$Y_t \equiv \sum_i p_i Y_t^i = B_M (1-\alpha_M)\left(\frac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M} X_t \left(\frac{\alpha_M}{1-\alpha_M} \frac{N_t^M}{\phi_t^M} + 1\right) \qquad \text{q.e.d.}$$

APPENDIX B

The Hamiltonian for this optimization problem is given by:

$$H_t = u_t + \psi_t (Y_t - E_t - \delta K_t)$$

where

$$u_t = \frac{C_t^{1-\sigma} - 1}{1-\sigma}$$

$$C_t = \prod_i (C_t^i - \overline{C}_i)^{\beta_i} \tag{B.1}$$

$$E_t \equiv \sum_i p_i C_t^i$$

$$Y_t \equiv \sum_i p_i Y_t^i = B_M (1-\alpha_M) \left(\frac{\phi_t^M K_t}{N_t^M X_t} \right)^{\alpha_M} X_t \left(\frac{\alpha_M}{1-\alpha_M} \frac{N_t^M}{\phi_t^M} + 1 \right)$$

$$p_i = \frac{\partial Y_t^M / \partial (N_t^M X_t)}{\partial Y_t^i / \partial (N_t^i X_t)} = \frac{B_M (1-\alpha_M) \left(\frac{\phi_t^M K_t}{N_t^M X_t} \right)^{\alpha_M}}{B_i (1-\alpha_i) \left(\frac{\phi_t^i K_t}{N_t^i X_t} \right)^{\alpha_i}} \quad \forall \mathrm{i}$$

$C_t^i \ \forall i$ are control variables and K_t is the state variable. The transversality condition is given by $\lim_{t\to\infty} \{\psi_t K_t\} = 0$. The well known optimality conditions state that:

$$\frac{\partial H_t}{\partial C_t^j} \overset{!}{=} 0, \quad j = 1,...n \tag{B.2}$$

$$-\frac{\partial H_t}{\partial K_t} \overset{!}{=} \dot{\psi}_t - \rho \psi_t \tag{B.3}$$

From the first optimality condition (B.2) it follows that:

$$\frac{\partial u_t}{\partial C_t^j} - \psi_t \frac{\partial E_t}{\partial C_t^j} = C_t^{-\sigma} \prod_i (C_t^i - \overline{C}^i)^{\beta_i} \beta_j \frac{1}{C_t^j - \overline{C}^j} - \psi_t p_i = C_t^{1-\sigma} \frac{\beta_j}{C_t^j - \overline{C}^j} - \psi_t p_j \overset{!}{=} 0, \quad j = 1,...n$$

$$\Leftrightarrow C_t^{1-\sigma}\frac{\beta_i}{p_i(C_t^i-\overline{C}^i)}=\psi_t,\ \forall i \tag{B.4}$$

Inserting equation (7) and (17) into equation (B.4) yields for i = M:

$$C_t^{1-\sigma}\frac{\beta_M}{C_t^M}=\psi_t \tag{B.5}$$

Thus, it follows from equation (B.5):

$$(1-\theta)\frac{\dot{C}_t}{C_t}-\frac{\dot{C}_t^M}{C_t^M}=\frac{\dot{\psi}_t}{\psi_t} \tag{B.6}$$

Setting (B.4) = (B.5) yields:

$$\frac{p_i\left(C_t^i-\overline{C}^i\right)}{\beta_i}=\frac{C_t^M}{\beta_M},\quad \forall i \qquad\qquad \text{q.e.d.} \tag{B.7}$$

Solving (B.7) for $(C_t^i-\overline{C}^i)$ and inserting it into equation (B.1) results in:

$$C_t=\prod_i\left(\frac{1}{p_i}\frac{\beta_i}{\beta_M}C_t^M\right)^{\beta_i}=\frac{C_t^M}{\beta_M}\prod_i\left(\frac{\beta_i}{p_i}\right)^{\beta_i}$$

Thus, as long as prices are constant we have:

$$\frac{\dot{C}_t}{C_t}=\frac{\dot{C}_t^M}{C_t^M} \tag{B.8}$$

Inserting (B.8) in (B.6) yields:

$$-\theta\frac{\dot{C}_t^M}{C_t^M}=\frac{\dot{\psi}_t}{\psi_t} \tag{B.9}$$

From the second optimality condition (B.3) it follows that:

$$-\psi_t\left(\frac{\partial(Y_t-E_t)}{\partial K_t}-\delta\right)\overset{!}{=}\dot{\psi}_t-\rho\psi_t \tag{B.10}$$

Because of equations (6) and (7) we know that

$$Y_t - E_t = \sum_i p_i Y_t^i - \sum_i p_i C_t^i = Y_t^M - C_t^M + \sum_{i \neq M} p_i Y_t^i - \sum_{i \neq M} p_i C_t^i$$
$$= Y_t^M - C_t^M + \sum_{i \neq M} p_i Y_t^i - \sum_{i \neq M} p_i Y_t^i = Y_t^M - C_t^M \qquad \text{(B.11)}$$

Thus, because of equation (10) and (B.11): [25]

$$\frac{\partial (Y_t - E_t)}{\partial K_t} = \frac{\partial (Y_t^M - C_t^M)}{\partial K_t} = \frac{\partial Y_t^M}{\partial K_t} = r_t + \delta \qquad \text{(B.12)}$$

Inserting (B.12) into (B.10) results in:

$$-r_t + \rho = \frac{\dot{\psi}_t}{\psi_t} \qquad \text{(B.13)}$$

Setting (B.9) = (B.13) yields:

$$\frac{\dot{C}_t^M}{C_t^M} = \frac{r_t - \rho}{\sigma} \qquad \text{q.e.d.}$$

[25] Remember: C_t^M is a control variable.

APPENDIX C

By solving equation (8) for ϕ_t^i, inserting it into equation (2) and solving for $\frac{\phi_t^M}{N_t^M}$, we get:

$$\frac{\phi_t^M}{N_t^M} = \frac{1}{\frac{1-\alpha_M}{\alpha_M}\sum_i \frac{\alpha_i}{1-\alpha_i} N_t^i} = \frac{1}{\frac{1-\alpha_M}{\alpha_M}\left[\frac{\alpha_M}{1-\alpha_M} N_t^M + \sum_{i \neq M} \frac{\alpha_i}{1-\alpha_i} N_t^i\right]}$$

Substituting N_t^M in this equation by using equation (3) results in

$$\frac{\phi_t^M}{N_t^M} = \frac{1}{\frac{1-\alpha_M}{\alpha_M}\left[\frac{\alpha_M}{1-\alpha_M} + \sum_{i \neq M} \frac{\alpha_i}{1-\alpha_i} N_t^i - \frac{\alpha_M}{1-\alpha_M}\sum_{i \neq M} N_t^i\right]}$$ q.e.d.

APPENDIX D

Solving equation (1) for N_t^M and substituting C_t^i by using equation (6), gives

$$N_t^i = \frac{C_t^i}{B_i X_t \left(\frac{\phi_t^i K_t}{N_t^i X_t} \right)^{\alpha_i}}, \quad \forall i \neq M$$

Differentiating this equation with respect to time results in:[26]

$$\frac{\dot{N}_t^i}{N_t^i} = \frac{\dot{C}_t^i}{C_t^i} - g, \quad \forall i \neq M$$

The solution to this differential equation is

$$N_t^i = \frac{N_0^i}{C_0^i} \exp\left(\int \frac{\dot{C}_t^i}{C_t^i} - g dt\right) = \frac{N_0^i}{C_0^i} C_t^i \exp(-gt), \quad \forall \text{i} \neq \text{M} \tag{D.1}$$

From equation (18) it follows, that on the balanced growth path (i.e. when prices are constant and when C_t^M grows at rate g):

$$\frac{\dot{C}_t^i}{C_t^i - \overline{C}^i} = g \Leftrightarrow \dot{C}_t^i - gC_t^i = -g\overline{C}^i, \quad \forall i$$

The solution of this differential equation is given by

$$C_t^i = (C_0^i - \overline{C}^i) \exp(gt) + \overline{C}^i, \quad \forall i \tag{D.2}$$

Because of equation (6) it follows from equation (1), that

$$C_0^i = B_i N_0^i X_0 \left(\frac{\phi_0^i K_0}{N_0^i X_0} \right)^{\alpha_i}, \quad \forall i \neq M \tag{D.3}$$

Now we substitute in equation (10) as follows: $\frac{\phi_t^M}{N_t^M}$ by using equation (8) and r_t by using equation (19)[27]. Thus we get

[26] This equation is only true along the balanced growth path.

[27] Remember that C_t^M grows at rate g along the balanced growth path.

$$\frac{\phi_0^i K_0}{N_0^i X_0} = \left[\left(\frac{\sigma g + \rho + \delta}{\alpha_M B_M} \right)^{1/(\alpha_M - 1)} \frac{1-\alpha_M}{\alpha_M} \frac{\alpha_i}{1-\alpha_i} \right], \quad \forall i \tag{D.4}$$

Now we substitute in equation (D.1) as follows: first C_t^i by using equation (D.2), then C_0^i by using equation (D.3), and finally $\frac{\phi_0^i K_0}{N_0^i X_0}$ by using equation (D.4). Thus we get:

$$N_t^i = N_0^i - \Gamma_i + \Gamma_i \exp(-gt), \quad \forall \mathrm{i} \neq \mathrm{M},$$

where

$$\Gamma_i = \frac{\overline{C}^i}{X_0 B_i \left[\left(\frac{\sigma g + \rho + \delta}{\alpha_M B_M} \right)^{1/(\alpha_M - 1)} \frac{1-\alpha_M}{\alpha_M} \frac{\alpha_i}{1-\alpha_i} \right]^{\alpha_i}} \qquad \text{q.e.d.}$$

APPENDIX E

Because of equations (9) and (17), it follows that[28]

$$\sum_i p_i \overline{C}^i = B_M(1-\alpha_M)\left(\frac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M}\sum_i \frac{1}{1-\alpha_i}\frac{\overline{C}^i}{B_i\left(\frac{\phi_t^i K_t}{N_t^i X_t}\right)^{\alpha_i}}$$

$$= B_M(1-\alpha_M)\left(\frac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M}\sum_{i\neq M} \frac{1}{1-\alpha_i}\frac{\overline{C}^i}{B_i\left(\frac{\phi_t^i K_t}{N_t^i X_t}\right)^{\alpha_i}}$$

$$= B_M(1-\alpha_M)\left(\frac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M}\sum_{i\neq M} \frac{1}{1-\alpha_i}\frac{\overline{C}^i}{B_i\left(\frac{\phi_0^i K_0}{N_0^i X_0}\right)^{\alpha_i}}$$

By inserting equation (D.4) from Appendix D into this equation we get (because of equation (26) and (27)):

$$\sum_i p_i \overline{C}^i = B_M(1-\alpha_M)\left(\frac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M} X_0 \sum_{i\neq M} \frac{1}{1-\alpha_i}\frac{\overline{C}^i}{B_i X_0\left[\left(\frac{\sigma g+\rho+\delta}{\alpha_M B_M}\right)^{1/(\alpha_M-1)}\frac{1-\alpha_M}{\alpha_M}\frac{\alpha_i}{1-\alpha_i}\right]^{\alpha_i}}$$

$$= B_M(1-\alpha_M)\left(\frac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M} X_0 \sum_{i\neq M} \frac{1}{1-\alpha_i}\Gamma_i$$

$$= B_M(1-\alpha_M)\left(\frac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M} X_0 \left[\sum_{i\neq M} \frac{\alpha_i}{1-\alpha_i}\Gamma_i + \sum_{i\neq M}\Gamma_i\right]$$

$$= B_M(1-\alpha_M)\left(\frac{\phi_t^M K_t}{N_t^M X_t}\right)^{\alpha_M} X_0 \left[\sum_{i\neq M} \frac{\alpha_i}{1-\alpha_i}\Gamma_i\right]$$

We can see now that $\sum_i p_i \overline{C}^i = 0$, when $\sum_{i\neq M}\frac{\alpha_i}{1-\alpha_i}\Gamma_i = 0$.

[28] Remember: $\frac{\phi_t^i K_t}{N_t^i X_t}$ is constant along the balanced growth path.

References

KONGSAMUT, P., REBELO, A. and XIE, D. (1997), “Beyond Balanced Growth“, *NBER Working Paper No. 6159.*

KONGSAMUT, P., REBELO, A. and XIE, D. (2001), “Beyond Balanced Growth“, *Review of Economic Studies*, **68**, 869-882.

KUZNETS, S. (1976), “Modern Economic Growth: Rate, Structure and Spread” (Yale University Press, 7th edition).

MECKL, J. (2002), “Structural Change and Generalized Balanced Growth”, *Journal of Economics*, **77**, 3, 241-266.

NGAI, R. L. and PISSARIDES C. A. (2004), “Balanced Growth with Structural Change” (London School of Economics).

OXLEY, L. (1994), “Cointegration, Causality and Wagner’s Law: A Test for Britain 1870-1913”, Scottish Journal of Political Economy, **41**, 3, 286-298.