# An Alternative Derivation of the Black-Scholes Formula

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Abstract

We present a new way of solving the Black-Scholes differential equation.

## 1   Introduction

The approach of Black and Scholes in [1] marks a major milestone in the research on the pricing of options. Over the last decades numerous variants of the model have been developed modifying the underlying model for the stock price or describing exotic variants of options.

In the present short note we step back, reconsider the original Black-Scholes differential equation and present a new and - as we think - elegant way of solving it.

## 2   The Calculation

It is well known that by using the Feynman-Kac formula the solution of the Black-Scholes differential equation may be cast in the simple form

$$C(S,t) = e^{-r(T-t)}\mathbb{E}(\max(0, S(T) - K)|S(t) = S), \tag{1}$$

where $C(S,t)$ represents the price of the option at time $t$ with expiration time $T$, strike price $K$ and interest rate $r$. The underlying stock price is modelled by a stochastic process $S(t)$. We rephrase the argument of the expectation operator in a better suited way for analytical manipulations by

$$\max(0, S(T) - K) = (S(T) - K)\,\theta(S(T) - K), \tag{2}$$

where $\theta$ denotes the Heaviside stepfunction.[3] Using the well known solution of the underlying stochastic differential equation (geometrical Brownian motion) which determines the

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[3]The Heaviside stepfunction is defined by

$$\theta(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0. \end{cases}$$

evolution of the stock price to be

$$S(T) = S(t) \exp\left[(r - \frac{1}{2}\sigma^2)(T - t) + \sigma(W(T) - W(t))\right]$$

(with diffusion parameter $\sigma$ and Brownian motion $W(t)$) for $S(T)$ in expression (2) one finds[4]

$$\begin{aligned}
\theta(S(T) - K) &= \theta[S\exp(\nu + \sigma W(T)) - K] \\
&= \theta(W(T) + \sqrt{T}d_2),
\end{aligned} \tag{3}$$

where the definition $d_2 = [\ln(S/K) + \nu]/\gamma$, with $\nu = (r - \sigma^2/2)T$ and $\gamma^2 = \sigma^2 T$ is standard. Note that the second line follows from the property $\theta(a^2 e^{b+ct} - d^2) = \theta(t - \frac{1}{c}[\ln(a^2/d^2) + b])$ which is a special case of a more general statement for strictly monotone and positive functions which we do not state here.

Using eq. (2) with $\theta$ given in eq. (3) in formula (1) we find for the option price

$$C(S,t) = -Ke^{-rT}\mathcal{A} + Se^{-rT+\nu}\mathcal{B} \tag{4}$$

with

$$\mathcal{A} = \mathbb{E}\left[\theta(W(T) + \sqrt{T}d_2)\right] \tag{5}$$

and

$$\mathcal{B} = \mathbb{E}\left[e^{\sigma W(T)}\theta(W(T) + \sqrt{T}d_2)\right]. \tag{6}$$

The next step is to compute $\mathcal{A}$ and $\mathcal{B}$ explicitly. We start with $\mathcal{A}$.

## 2.1   Calculating $\mathcal{A}$

The Fourier-transform of the step function is given by

$$\theta(x) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \frac{dk}{k - i\varepsilon} e^{ikx},$$

see e.g. p. 643 in [2]. Using this formula in eq. (5) and interchanging the integration with the expectation operation one finds

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \frac{dk}{k - i\varepsilon} e^{ik\sqrt{T}d_2} \mathbb{E}\left(e^{ikW(T)}\right) \\
&= \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \frac{dk}{k - i\varepsilon} e^{-\frac{1}{2}k^2 T + ik\sqrt{T}d_2},
\end{aligned} \tag{7}$$

where

$$\mathbb{E}\left(e^{\alpha W(T)}\right) = e^{\frac{1}{2}\alpha^2 T} \tag{8}$$

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[4]For simplicity we set $t = 0$ in what follows and denote $S \equiv S(t = 0)$.

for the characteristic function has been used. The integral in eq. (7) is computed as follows: The term $\sim 1/k$ can be rephrased by an integral over[5] $d_2$. Interchanging the order of integration, one finds

$$\mathcal{A} = \frac{\sqrt{T}}{2\pi} \int dd_2 \int_{-\infty}^{\infty} dk e^{-\frac{1}{2}k^2 T + ik\sqrt{T}d_2},$$

where we have dropped $\varepsilon$. Using elementary properties of the integrand and the definition of the complex exponential, $\mathcal{A}$ results in

$$\mathcal{A} = \frac{\sqrt{T}}{\pi} \int dd_2 \int_0^{\infty} dk e^{-\frac{1}{2}k^2 T} \cos k\sqrt{T}d_2.$$

The $k$-integral is a standard integral and may be found in [4], sec. 3.896. The relevant formula is

$$\int_0^{\infty} dt e^{-at^2} \cos(2xt) = \frac{1}{2}\sqrt{\frac{\pi}{a}} \, e^{-\frac{x^2}{a}}.$$

Using this result we can evalute the desired integral to be

$$\mathcal{A} = \frac{1}{\sqrt{2\pi}} \int dd_2 \, e^{-\frac{1}{2}d_2^2} = \Phi(d_2), \tag{9}$$

with $\Phi$ being the cumulative normal distribution function.

## 2.2  Calculating $\mathcal{B}$

The calculation of $\mathcal{B}$ proceeds along the lines described above. Using the Fourier decomposition of the step function, $\mathcal{B}$ is calculated to be

$$\begin{aligned}
\mathcal{B} &= \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \frac{dk}{k - i\varepsilon} e^{ik\sqrt{T}d_2} \mathbb{E}\left(e^{W(T)(\sigma + ik)}\right) \\
&= \frac{1}{2\pi i} e^{\sigma^2 T/2} \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \frac{dk}{k - i\varepsilon} e^{-\frac{1}{2}k^2 T + ik\sqrt{T}d_1},
\end{aligned}$$

with $d_1 = d_2 + \gamma$. In the last step we used again eq. (8). Comparing this integral with the one given in (7), we see that both integrals are identical up to the replacement $d_1 \to d_2$. Thus we can read of the solution directly from eq. (9) and find

$$\mathcal{B} = e^{\sigma^2 T/2} \Phi(d_1). \tag{10}$$

## 2.3  Completion of the Calculation

We are now in the position to use the results for $\mathcal{A}$ and $\mathcal{B}$ given in eqs. (9) and (10) in eq. (4) and find the final and well known result

$$C(S,t) = S\Phi(d_1) - Ke^{-rT}\Phi(d_2).$$

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[5]This is similar to Feynman parameter integrals in quantum field theory, see e.g. [3].

# References

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