An application of the put-call-parity to variance reduced Monte-Carlo option pricing

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Abstract: The standard error of Monte Carlo estimators for derivatives typically decreases at a rate $\propto 1/\sqrt{N}$ where N is the sample size. To reduce empirical variance for estimators of several in-the-money options an application of the put-call-parity is analyzed. Instead of directly simulating a call option, first the corresponding put option is simulated. By employing the put-call-parity the desired call price is calculated. Of course, the approach can also be applied vice versa. By employing this approach for in-the-money options, significant variance reductions are observed.

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1 Introduction

It is a well known-property of certain options that under the assumption of arbitrage free markets a put-call-parity holds, relating the price of a put option to a call option. The put-call-parity can be derived based on the assumption that two financial products with the same pay-off and the same risk-profile must have the same fair market value. Otherwise, arbitrage would be possible [1, 2].

As described by Reider (1994) [3], put-call-parities can be applied to reduce the variance of Monte-Carlo estimators for option prices. E.g., when pricing an in-the-money call, one can instead simulate the price of a put option with same parameters and independent variables. The put-call-parity then yields the price of the in-the-money call, while the variance of the estimator can be dramatically reduced.

This paper is organized as follows: Monte-Carlo simulations of the Feynman-Kac formula as an approach to option pricing are introduced in section 2. In section 3 put-call-parities for European, Arithmetic Asian, Digital and Basket options are derived. Results for variance reduced Monte-Carlo simulations of these options are presented in section 4 and subsequently discussed in section 5. Section 6 concludes this paper.

2 Monte-Carlo simulation of the Feynman-Kac formula

Black-Scholes PDE In 1973, Fischer Black and Myron Scholes published an important contribution to option pricing theory [4]. Their option pricing model had tremendous impact on the further development of the trading of financial derivatives. Their work was honored with a noble price in economics later [1]. To derive the central constituent of their model, Black and Scholes derived a partial differential equation resulting from a no-arbitrage argument [2].

The *hedge portfolio*

$$V = C + \tau S \tag{1}$$

consists of one call and τ units of the underlying with $C \equiv C\left(S\left(t\right),t\right)$. The

[2] The representation follows Singer (1999) [5].

differential $\mathrm{d}V$ can be calculated by applying Itō's formula[3]:

$$\begin{aligned} \mathrm{d}V &= \mathrm{d}C + \tau \mathrm{d}S \\ &= C_S \mathrm{d}S + C_t \mathrm{d}t + \frac{1}{2} C_{SS} \mathrm{d}S^2 + \frac{1}{2} C_{tt} \mathrm{d}t^2 + C_{St} \mathrm{d}S \mathrm{d}t + \tau \mathrm{d}S \\ &= (C_S + \tau)\, \mathrm{d}S + \left(C_t + \frac{1}{2} C_{SS} g^2 \right) \mathrm{d}t + \mathcal{O}\left(\mathrm{d}t^{3/2}\right) \end{aligned} \tag{2}$$

Here, the differential equation

$$\mathrm{d}S = f \mathrm{d}t + g \mathrm{d}W \tag{3}$$

and $\mathrm{d}W^2 \propto \mathrm{d}t$ were used. Terms of the order $\mathrm{d}t^{3/2}$ or higher were suppressed.

By choosing $\tau = -C_S$ the hedge portfolio V becomes riskless: The differential $\mathrm{d}V$ then depends only on deterministic terms as the first bracket term in the last row of equation (2) vanishes. In consequence, V must earn the risk-free rate r:

$$\begin{aligned} \mathrm{d}V &= \left(C_t + \frac{1}{2} C_{SS} g^2 \right) \mathrm{d}t \\ &\stackrel{!}{=} rV \mathrm{d}t \\ &= (rC - rC_S S)\, \mathrm{d}t \end{aligned} \tag{4}$$

In the first row the term (2) was repeated and in the third row definition (1) was inserted. Rearranging elements and choosing a Geometric Brownian Motion ($f = rS$ and $g = \sigma S$) yields the well known Black-Scholes equation

$$C_t + rSC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} - rC = 0. \tag{5}$$

Black and Scholes (1973) [4] presented a closed form solution for an European call with boundary condition $C_T = (S_T - K)^+$:

$$C(S(0), 0) = S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2) \tag{6}$$

with

$$d_{1,2} = \frac{\ln \frac{S(0)}{K} + \left(r \pm \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}.$$

Feynman-Kac formula An alternative approach to option pricing was introduced by Cox and Ross in 1976 [6]. The authors state that in a risk-neutral world

[3] In this paper, the abbreviation $\frac{\partial C}{\partial S} \equiv C_S$ is frequently used.

the expected rates on the underlying and on the option must equal the risk-free rate:

$$\begin{aligned}\mathbb{E}\left[S(T)/S(0)|S(0)=S_0\right] &= e^{rT}\\ \mathbb{E}\left[C(S(T),T)/C(S(0),0)|S(0)=S_0\right] &= C(S(0),0)^{-1}\mathbb{E}\left[C(S(T),T)|S(0)=S_0\right]\\ &= e^{rT}\end{aligned} \tag{7}$$

Setting $C(S(T),T) = h\left(S(T)\right)$ and rearranging yields the Feynman-Kac formula for option pricing[4]:

$$\begin{aligned}C(S(0),0) &= e^{-rT}\mathbb{E}\left[h\left(S(T)\right)|S(0)=S_0\right]\\ &= e^{-rT}\int h\left(S'(T)\right)p\left(S'(T),T|S(0),0\right)\mathrm{d}S'(T)\end{aligned} \tag{8}$$

The advantage of this representation is that for more difficult boundary conditions $h\left(S(T)\right)$, where no analytical solution to the Black-Scholes PDE (5) exists, option prices can be approximated by Monte-Carlo simulations. To obtain a Monte-Carlo estimator, N i.i.d. random numbers are drawn from the transition density $p\left(S'(T),T|S(0),0\right)$ yielding

$$\hat{C} \equiv \hat{C}(S(0),0) = \frac{1}{N}\sum_{i=1}^{N}\hat{C}_i \tag{9}$$

with standard error

$$\delta\hat{C} = \sqrt{\frac{1}{N\left(N-1\right)}\left(\sum_{i=1}^{N}\hat{C}_i^2 - N\hat{C}^2\right)}. \tag{10}$$

Path-dependent options A similar approach can be applied to path-dependent options [8]. Here, we consider an Arithmetic Asian option depending on the entire path of the underlying:

$$h\left(S(T),Y(T),T\right) \equiv C\left(S(T),Y(T),T\right) = \left(\frac{Y(T)}{T} - K\right)^{+} \tag{11}$$

with

$$Y(t) := \int_0^t S(u)\mathrm{d}u, \text{ for } t \in [0,T]. \tag{12}$$

[4]This representation follows Singer (1999) [5]. It can be shown that this intuitive representation of the option price as discounted expected value of the pay-off function solves the partial differential equation 5. For further details on the Feynman-Kac formula see [2, 7].

Differentiating equation (12) yields

$$\mathrm{d}Y = S(t)\mathrm{d}t. \tag{13}$$

A partial differential equation for the option price can be derived as above applying Itō's formula:

$$\begin{aligned}
\mathrm{d}V &= \mathrm{d}C + \tau\mathrm{d}S \\
&= C_S\mathrm{d}S + C_Y\mathrm{d}Y + C_t\mathrm{d}t + \frac{1}{2}C_{SS}\mathrm{d}S^2 + \frac{1}{2}C_{YY}\mathrm{d}Y^2 + \frac{1}{2}C_{tt}\mathrm{d}t^2 \\
&\quad + C_{SY}\mathrm{d}S\mathrm{d}Y + C_{St}\mathrm{d}S\mathrm{d}t + C_{Yt}\mathrm{d}Y\mathrm{d}t + \tau\mathrm{d}S \\
&= \left(C_S + \tau\right)\mathrm{d}S + \left(C_t + C_Y S + \frac{1}{2}C_{SS}g^2\right)\mathrm{d}t + \mathcal{O}\left(\mathrm{d}t^{3/2}\right)
\end{aligned} \tag{14}$$

Choosing again $\tau = -C_S$ yields an equation similar to (5) including the additional term $C_Y S$:

$$0 = C_t + rSC_S + C_Y S + \frac{1}{2}\sigma^2 S^2 C_{SS} - rC \tag{15}$$

For this type of option no explicit analytical solution is available [9]. However, the option can be replicated by a portfolio (the *replicating portfolio*) $X(t)$ consisting of $-\tau = C_S$ units of the underlying and a risk-free position [8, 10]. It can be shown that the discounted replicating portfolio $e^{-rt}X(t)$ is a martingale [8] and that therefore

$$X(t) = \mathbb{E}\left[e^{-r(T-t)}X(T)\middle|S(t) = S_t, Y(t) = Y_t\right] \tag{16}$$

holds. This implies

$$\begin{aligned}
C(S(t),Y(t),t) &= \mathbb{E}\left[e^{-r(T-t)}h\left(S(T),Y(T),T\right)\middle|S(t) = S_t, Y(t) = Y_t\right] \\
&= e^{-r(T-t)}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S(u)du - K\right)^+\middle|S(t) = S_t, Y(t) = Y_t\right]
\end{aligned} \tag{17}$$

and

$$C(S(0),Y(0),0) = e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S(u)du - K\right)^+\middle|S(0) = S_0, Y(0) = Y_0\right]. \tag{18}$$

As a consequence, also prices of path-dependent Arithmetic Asian options can be estimated by Monte-Carlo simulations.

3 Put-call-parities for several options

European options In the case of a European option with pay-off function

$$C_T = (S_T - K)^+ \tag{19}$$

the put call parity can be derived as follows[5]:

Transaction	Current Value	Value at expiration $S_T \leq K$	$S_T > K$
European call			
Buy call	C_0	0	$S_T - K$
Synthetic European call			
Buy put	P_0	$K - S_T$	0
Buy underlying asset	S_0	S_T	S_T
Issue bond	$-Ke^{-rT}$	$-K$	$-K$
Total	$P_0 + S_0 - Ke^{-rT}$	0	$S_T - K$

Table 1: European call and synthetic European call

The pay-off function of a European long call position with strike K is the same as the pay-off function of a portfolio containing a European long put position with the same strike K, a long position of the underlying asset S_0 and a short position of a bond with nominal value K. In an arbirage-free market, this portfolio (a so called "synthetic call") must therefore have the same price yielding the put-call-parity for European options:

$$C_0 = P_0 + S_0 - Ke^{-rT} \tag{20}$$

Arithmetic Asian options For Arithmetic Asian options similar relations hold both in the fixed- and in the floating-strike case [12, 13]. Here, in the same manner as above, a put-call-parity is derived for a Asian call option with the following pay-off function:

$$C_T = \left(\bar{S}_T - K\right)^+ \text{ with } \bar{S}_T = \frac{1}{m}\sum_{i=1}^{m} S_i. \tag{21}$$

It is assumed that $S_m = S_T$ and that all S_i are equidistant. Otherwise the following parameter μ would have to be adjusted accordingly.

[5]The derivation follows Chance (2015) [11].

In the following table, the pay-offs for several transactions are given setting

$$\mu = \left(e^{-\frac{rT(m-1)}{m}} + \ldots + e^{-\frac{rT}{m}} + 1\right): \tag{22}$$

Transaction	Current Value	Value at expiration $\bar{S}_T \leq K$	 $\bar{S}_T > K$
Asian call			
Buy m calls	mC_0	0	$\sum_{i=1}^{m} S_i - mK$
Synthetic Asian call			
Buy m puts	mP_0	$mK - \sum_{i=1}^{n} S_i$	0
Buy μ underlying assets and sell assets for $S_i e^{-\frac{rT(m-i)}{m}}$ at each $i = 1, \ldots, m$	μS_0	$\sum_{i=1}^{m} S_i$	$\sum_{i=1}^{m} S_i$
Issue m bonds	$-mKe^{-rT}$	$-mK$	$-mK$
Total	$mP_0 + \mu S_0 - mKe^{-rT}$	0	$\sum_{i=1}^{m} S_i - mK$

Table 2: Asian call and synthetic Asian call

As the pay-off of the synthetic Asian call is the same as the pay-off of the Asian call, the current prices must equal. Dividing by m yields a put-call-parity slightly different from the European case (20):

$$C_0 = P_0 + \frac{\mu}{m} S_0 - Ke^{-rT} \tag{23}$$

Alziary et al. (1997) [13] discuss the case of Arithmetic Asian options with integral pay-off function of the type of equation (11), i.e. the case where $m \to \infty$.

Basket options The approach of constructing a synthetic portfolio that exactly replicates the option to be priced can be extended to Basket options depending on more than one underlying asset. In the case of a linear dependence of the option value on the underlying assets, in principle put-call-parities for Basket options with an infinite amount of underlyings can be derived. For simplicity, in this paper only the two-asset-case is considered.

We examine a Basket option depending on the terminal value at T of two underlying assets S_1 and S_2 with pay-off function

$$C_T = (S_{1,T} + S_{2,T} - K)^+ . \tag{24}$$

Again, we construct a synthetic portfolio that perfectly replicates the Basket call.

Transaction	Current Value	Value at expiration $S_{1,T}+S_{2,T}\leq K$	 $S_{1,T}+S_{2,T}>K$
Basket call			
Buy call	C_0	0	$S_{1,T}+S_{2,T}-K$
Synthetic Basket call			
Buy put	P_0	$K-S_{1,T}-S_{2,T}$	0
Buy underlying assets	$S_{1,0}+S_{2,0}$	$S_{1,T}+S_{2,T}$	$S_{1,T}+S_{2,T}$
Issue bond	$-Ke^{-rT}$	$-K$	$-K$
Total	$P_0+S_{1,0}+S_{2,0}$ $-Ke^{-rT}$	0	$S_{1,T}+S_{2,T}-K$

Table 3: Basket call and synthetic Basket call

This yields a put-call-parity of the following form similar to the European case (20):

$$C_0 = P_0 + S_{1,0} + S_{2,0} - Ke^{-rT} \tag{25}$$

Note that no assumption on the correlation of the two assets $S_{1,t}$ and $S_{2,t}$ has been made.

Digital options Let's now consider a Digital call option with pay-off function

$$C_T = \theta(S-K) \tag{26}$$

where $\theta(x)$ is the Heaviside step function.

Here the approach of constructing a synthetic replicating portfolio for a call is slightly different. Instead of buying the put, buying the underlying and selling a bond in this case it is necessary to sell a put and to buy a bond with nominal value 1 (not K!):

Thus, a put-call-parity relation of the form

$$C_0 = -P_0 + e^{-rT} \tag{27}$$

results.

In the next chapters the put-call-parities (20), (23), (25) and (27) will be applied to analyze variance reduced option price estimators. The deterministic relation between put and call prices will be exploited to price in-the-money calls.

Transaction	Current Value	Value at expiration $S_T \leq K$	$S_T > K$
Digital call			
Buy call	C_0	0	1
Synthetic Digital call			
Sell put	$-P_0$	-1	0
Buy bond	e^{-rT}	1	1
Total	$-P_0 + e^{-rT}$	0	1

Table 4: Digital call and synthetic Digital call

4 Numerical Results

Introductory remarks Several Monte-Carlo simulation of option prices were conducted to investigate the variance reduction effect of applying the put-call-parities (20), (23), (25) and (27) derived in section 3.

For all options investigated the Black-Scholes stock price model [4] involving a Geometric Brownian Motion as introduced in section 2 was employed for reasons of simplicity. However, as the put-call-parities presented in section 3 do not depend on a particular stock price model, other models like the CEV model as discussed by Cox in 1975 [14] and by Cox and Ross in 1976 [6], the CIR model introduced by Cox, Ingersoll and Ross in 1985 [15] or Scott's stochastic volatility model introduced in 1987 [16] could have been used as well.

To simulate sample paths, the differential equation (3) for the special case of a Geometric Brownian Motion with $f = rS$ and $g = \sigma S$ was discretized using the Euler-Maruyama approximation [17]:

$$S_{i+1} = S_i + rS_i\Delta t + \sigma S_i \epsilon_i \sqrt{\Delta t} \tag{28}$$

where ϵ_i is a standard normal random number. For all options, $T = 1$ and $n = 1.000$ was selected, involving $\Delta t = 0.001$. In all four examples presented subsequently, option prices were estimated based on a sample size of $N = 100$ both for the simulation of call options and for the simulation of put options. This rather small sample size was used mainly to make standard errors visible in figures 1, 2 and 4.

In all cases, first a straight forward Monte-Carlo simulation of the call option (named "Direct MC" subsequently) was conducted. In a second step, a Monte-Carlo simulation of the corresponding put option with same parameters was carried out (named "Put-Call-Parity MC"). Call values were then calculated by applying the put-call-parities (20), (23), (25) and (27). In order to compare the performance of

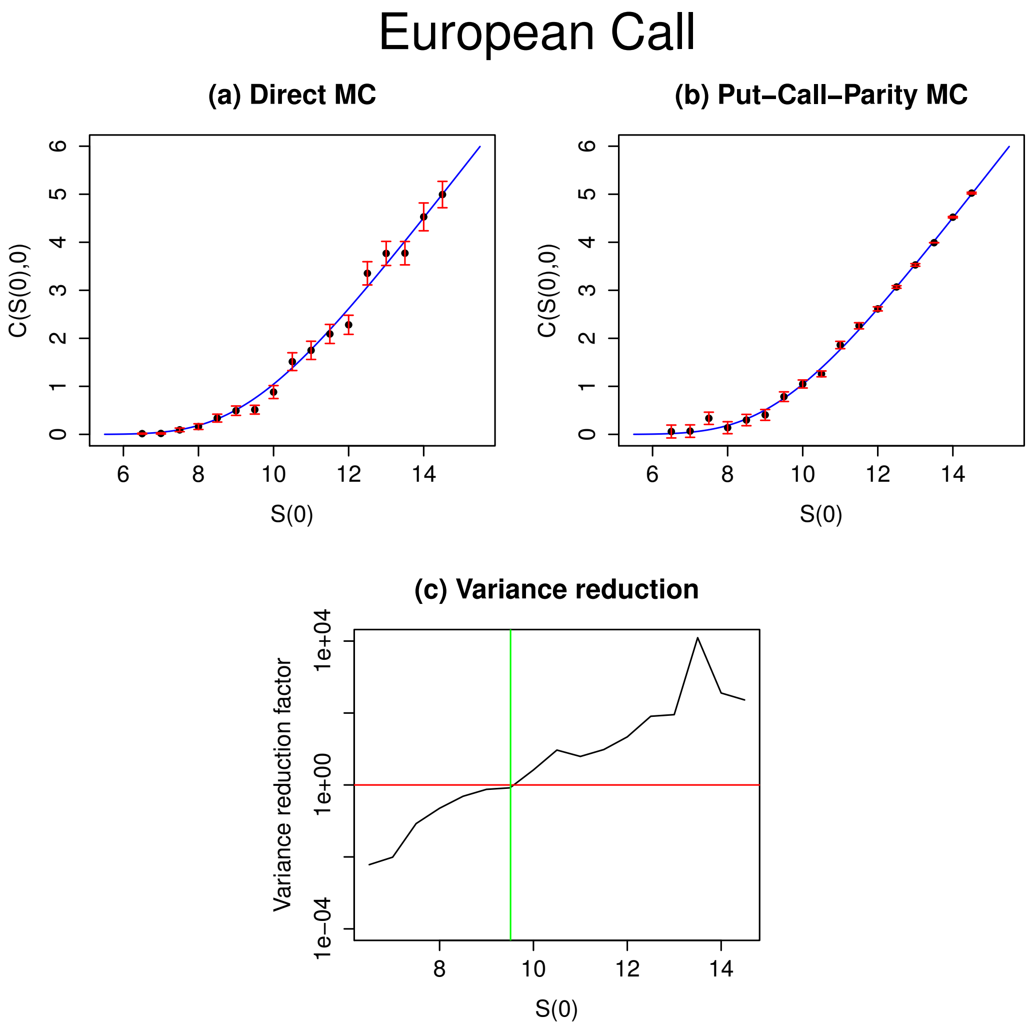

Figure 1: **(a) Direct MC**: Black dots: direct Monte-Carlo simulation of European call with pay-off function (19) with $r = 0.05$, $\sigma = 0.2$, $T = 1$, $K = 10$, $n = 1.000$ discretization steps and $N = 100$ trajectories simulated. Red bars: Standard error of simulated option value. Blue line: Analytic option price calculated with Black-Scholes formula (6). **(b) Put-Call-Parity MC**: Black dots: Call prices calculated employing equation (20). Required put prices were estimated by Monte-Carlo simulation with same parameters as in (a). Red bars: as in (a). Blue line: as in (a). **(c) Variance reduction**: Black line: Variance reduction achieved by employing put-call-parity (20). The variance reduction factor is calculated as ratio between the empirical variances of the two Monte-Carlo estimators presented in (a) and (b). Red line: Line where the variance reduction factor equals one, i.e. empirical variance of estimators is not influenced by employing the put-call-parity. Green line: Line where $S(0) = Ke^{-rT}$.

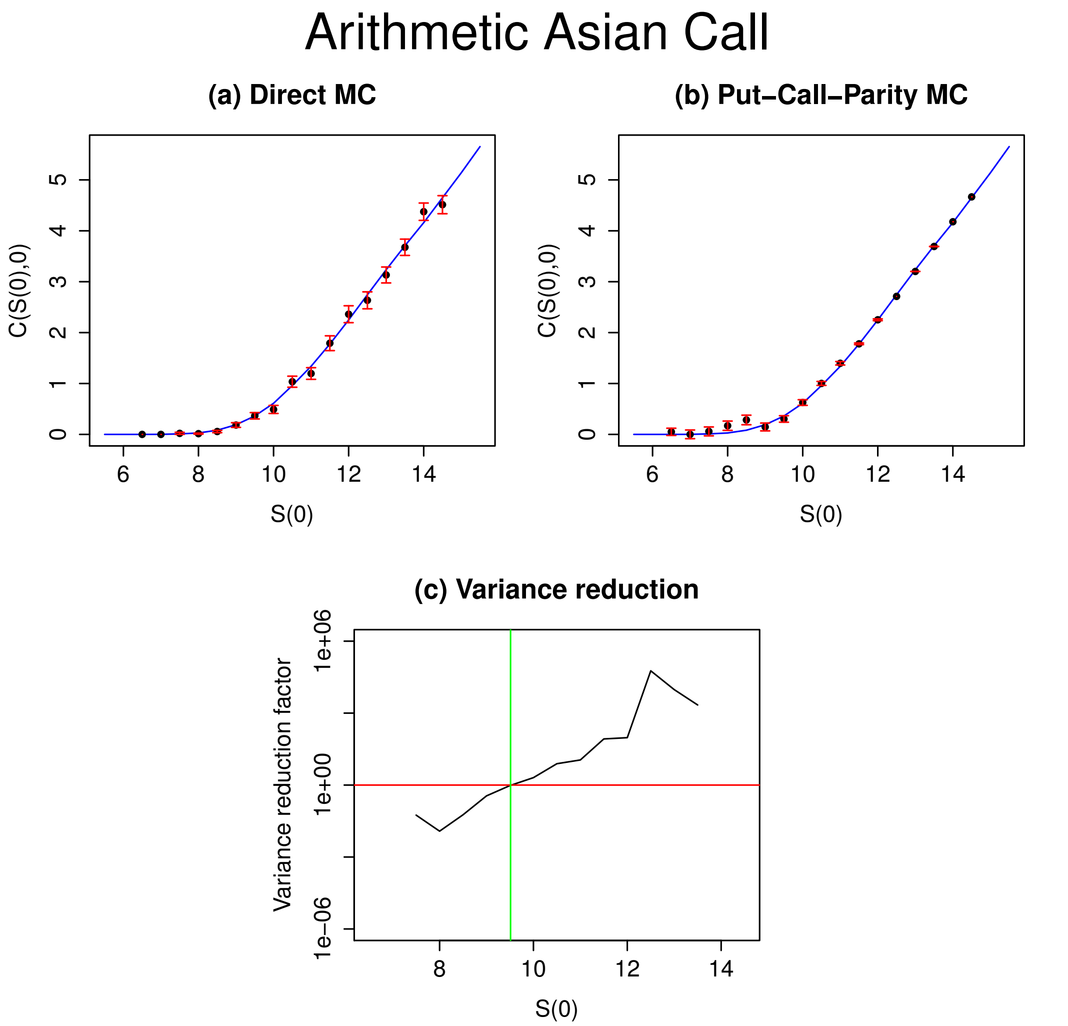

Figure 2: **(a) Direct MC**: Black dots: direct Monte-Carlo simulation of Arithmetic Asian call with pay-off function (21) with $r = 0.05$, $\sigma = 0.2$, $T = 1$, $K = 10$, $n = 1.000$ discretization steps, $N = 100$ trajectories simulated and $m = 10$, i.e. the course trajectory was divided in m equal parts and the final S value of each interval was taken to calculate $\bar{S}$. Red bars: Standard error of simulated option value. Blue line: Option price simulated with same parameters as before, but with increased number of trajectories $N_{\text{reference}} = 10,000$. **(b) Put-Call-Parity MC**: Black dots: Call prices calculated employing equation (23). Required put prices were estimated by Monte-Carlo simulation with same parameters as in (a). Red bars: as in (a). Blue line: as in (a). **(c) Variance reduction**: Black line: Variance reduction achieved by employing put-call-parity (23). The variance reduction factor is calculated as ratio between the empirical variances of the two Monte-Carlo estimators presented in (a) and (b). Due to low sample size ($n = 100$) for low (Direct MC) and high (Put-Call-Parity MC) S_0 values variances could not be calculated as all trajectories yielded a terminal value of 0. Red line: Line where the variance reduction factor equals one, i.e. empirical variance of estimators is not influenced by employing the put-call-parity. Green line: Line where $S(0) = Ke^{-rT}$.

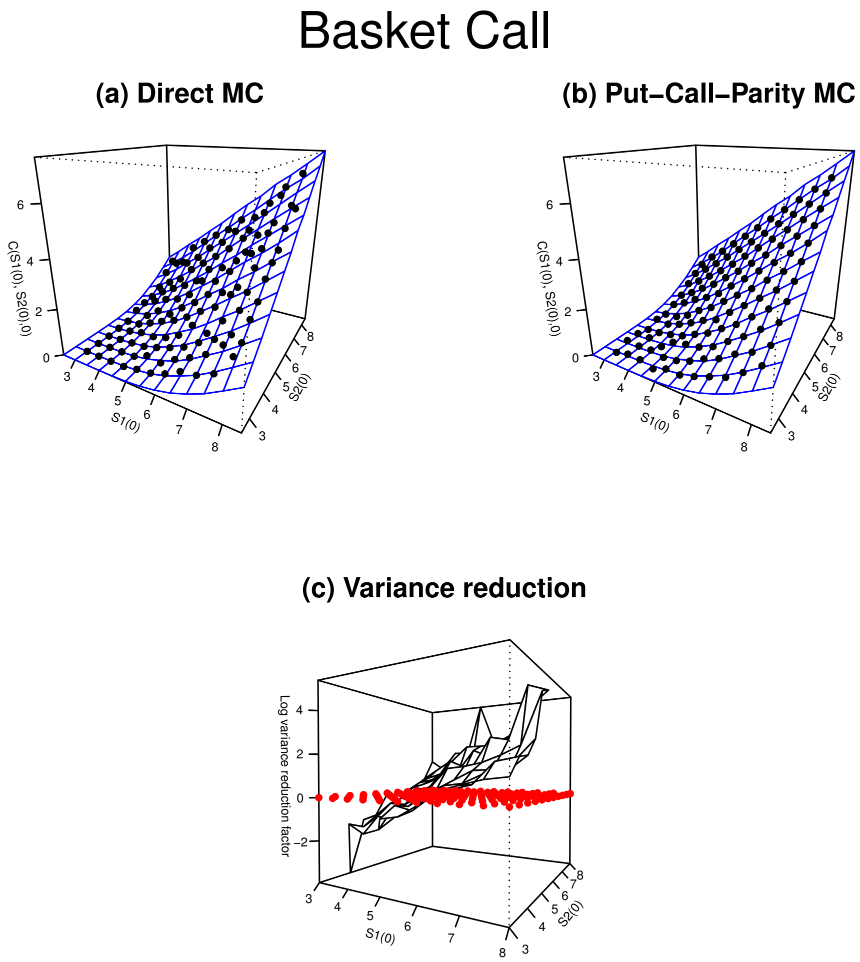

Figure 3: **(a) Direct MC**: Black dots: direct Monte-Carlo simulation of Basket call with pay-off function (24) with $r = 0.05$, $\sigma = 0.2$, $T = 1$, $K = 10$, $n = 1.000$ discretization steps and $N = 100$ trajectories simulated. Blue surface: Option price simulated with same parameters as before, but with increased number of trajectories $N_{\text{reference}} = 10,000$. For reasons of clarity error bars where omitted. However, also a visual inspection yields that the black dots fit more smoothly to the blue surface for low underlying values. **(b) Put-Call-Parity MC**: Black dots: Call prices calculated employing equation (25). Required put prices were estimated by Monte-Carlo simulation with same parameters as in (a). Blue surface: as in (a). Here, black dots fit more smoothly to the blue surface for high underlying values. **(c) Variance reduction**: Black surface: Variance reduction achieved by employing put-call-parity (25). The log variance reduction factor is calculated as logarithm of the ratio between the empirical variances of the two Monte-Carlo estimators presented in (a) and (b). Red dots: Plane where the log variance reduction factor equals zero, i.e. empirical variance of estimators is not influenced by employing the put-call-parity.

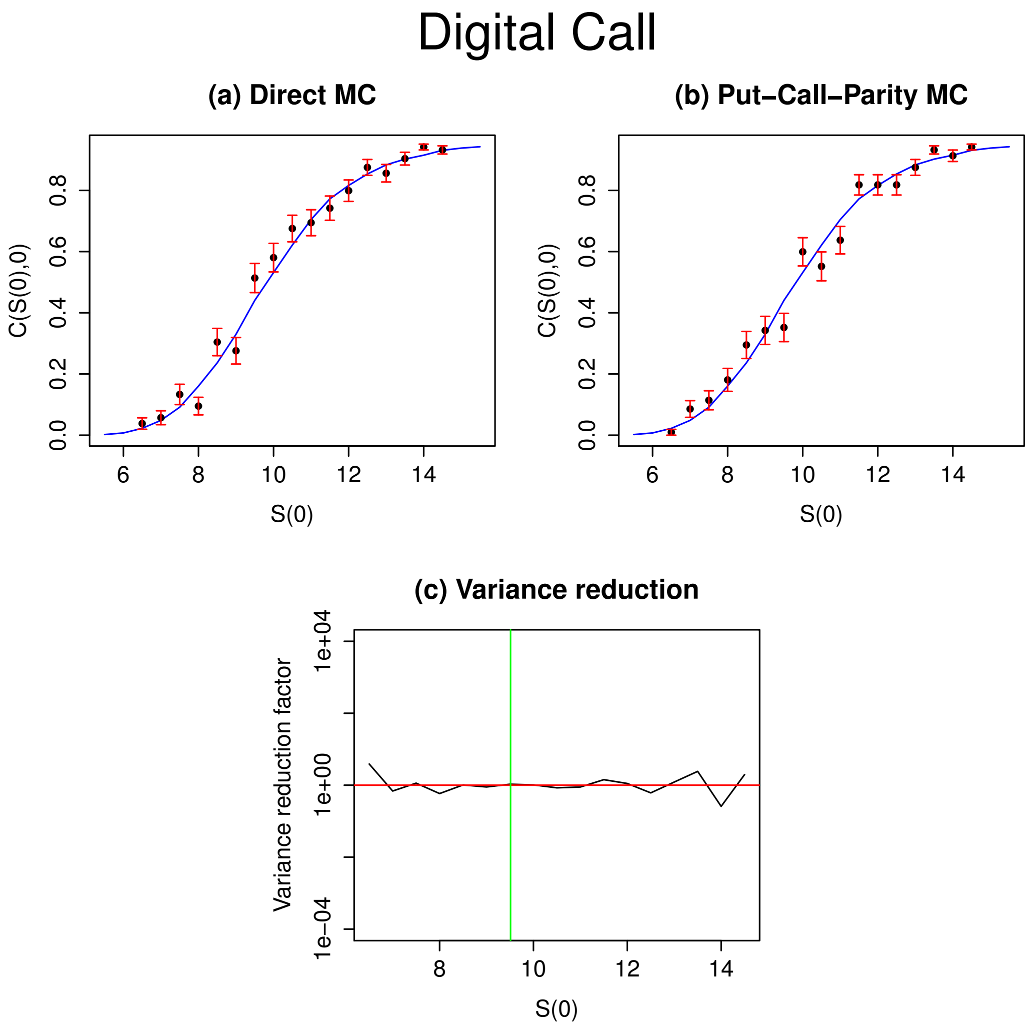

Figure 4: **(a) Direct MC**: Black dots: direct Monte-Carlo simulation of Digital call with pay-off function (26) with $r = 0.05$, $\sigma = 0.2$, $T = 1$, $K = 10$, $n = 1.000$ discretization steps and $N = 100$ trajectories simulated. Red bars: Standard error of simulated option value. Blue line: Option price simulated with same parameters as before, but with increased number of trajectories $N_{\text{reference}} = 10,000$. **(b) Put-Call-Parity MC**: Black dots: Call prices calculated employing equation (27). Required put prices were estimated by Monte-Carlo simulation with same parameters as in (a). Red bars: as in (a). Blue line: as in (a). **(c) Variance reduction**: Black line: Variance reduction achieved by employing put-call-parity (27). The variance reduction factor is calculated as ratio between the empirical variances of the two Monte-Carlo estimators presented in (a) and (b). Red line: Line where the variance reduction factor equals one, i.e. empirical variance of estimators is not influenced by employing the put-call-parity. Green line: Line where $S(0) = Ke^{-rT}$.

both approaches, variance reduction factors were calculated dividing the empirical variance of the Direct MC estimator by the empirical variance of the Put-Call-Parity MC estimator.

European call For the case of a European call option with pay-off function (19) the results for the two Monte-Carlo simulation approaches are shown in figure 1. Monte-Carlo estimators including standard errors are shown in subfigures (a) and (b). To examine possible biases the analytically exact option price calculated using the Black-Scholes formula 6 is depicted as well. Most Monte-Carlo estimators range within one standard deviation or less from the analytical value. Positive variance reduction results have been achieved by applying the put-call-parity to Monte-Carlo simulations of in-the-money calls. Reduction factors up to 10^4 have been achieved.

Arithmetic Asian call Similar results have been achieved for an Arithmetic Asian call option with pay-off function (21) (see figure 2). The exact option price curve here could not be calculated analytically as no closed form solution to the PDE (15) exists [18] which makes Monte-Carlo simulations even more interesting. Therefore, the blue reference curve was simulated employing a bigger sample size $N_{\text{reference}} = 10,000$. Again, significant variance reduction was achieved for in-the-money calls.

Basket call Also for the analyzed Basket call with pay-off function (24) the approach works well yielding variance reduction for in-the-money calls (see figure 3). Again, the reference surface has been simulated employing a bigger sample size $N_{\text{reference}} = 10,000$.

Digital call For the analyzed Digital call with pay-off function (26) results are presented in figure 4. The main result differs from the other options: no variance reduction could be achieved by applying the put-call-parity to in-the-money calls. Also here the blue reference curve has been simulated employing a bigger sample size $N_{\text{reference}} = 10,000$.

5 Discussion

Variance reduction by applying put-call-parities The results presented in section 4 indicate that considerable variance reduction can be achieved by applying put-call-parities to the valuation of several in-the-money options by Monte-Carlo simulation. It appears useful to apply put-call-parities whenever they allow to represent a stochastic quantity by deterministic components and a residual stochastic quantity reduced in absolute size. The standard error of a deterministic component equals zero per definition. The absolute size of the standard error of a stochastic quantity tends to decrease with decreased quantity size. Consequently, Gaussian propagation of uncertainty leads to variance reduced Monte-Carlo estimators. This is the case for the analyzed European, Asian and Basket calls. However, the profile of the Digital call's pay-off function involves that uncertainty does not decrease by applying the put-call-parity. This is not surprising as the relative size of the stochastic component compared with the deterministic component remains approximately the same. In contrast to other put-call-parities, the deterministic amount in equation (27) for all underlying values S_0 is low. As a consequence no variance reduction can be achieved.

In the three cases where variance reduction was achieved, the put-call-parity came in handy for in-the-money options. For practical purposes, as a rule of thumb the put-call-parity can be applied as soon as the current value of the underlying S_0 exceeds the discounted strike price Ke^{-rT}.

Setting the stage for more efficient importance sampling Another interesting consequence from the presented results is that by applying the put-call-parity the valuation of an in-the-money (call/put) option can be transformed to the valuation of an out-of-the-money (put/call) option. This sets the stage for the application of efficient importance sampling techniques.

Importance sampling implies the consideration of an additional drift terms in the differential equation of the underlying stock. As shown in several publications [5, 18, 19, 20] this approach can lead to considerable variance reduction and may be more powerful in the case of out-of-the-money options. The high suitability of this approach for the pricing of out-of-the-money options follows from the fact that without importance sampling many trajectories end below the strike price (in the case of call options) or above the strike price (in the case of put options). The additional drift term then has the effect of pushing an increased amount of trajectories above strike price for call options or below the strike price for put options. As a result more trajectories contribute to the Monte-Carlo simulation leading to an unbiased estimator with reduced variance[6]. For in-the-money options the effect generally is smaller as already without importance sampling most trajectories

[6]For details see Singer (2014) [20].

contribute to the Monte-Carlo estimator.

As a consequence, applying the put-call-parity Monte-Carlo approach when pricing an in-the-money option not only reduces the estimator variance *per se*. It also transforms the Monte-Carlo simulation into a regime where importance sampling works especially well. The joint impact of applying the put-call-parity and importance sampling techniques will be studied in a subsequent paper.

6 Conclusion

Put-call-parities for different types of options have been derived. For in-the-money call options it has been shown that the simulation of the corresponding put options and subsequent calculation of the call price by applying put-call-parities can lead to considerable variance reduction. So, the application of put-call-parities can contribute to the acceleration of derivative pricing and portfolio valuation by Monte-Carlo simulations.

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References

[1] Hull J. Optionen, Futures und andere Derivate. 6th ed. wi - wirtschaft. München: Pearson Studium; 2006.

[2] Duffie D. Dynamic asset pricing theory. Princeton University Press; 2010.

[3] Reider RL. Two applications of Monte Carlo techniques to finance. University of Pennsylvania. 1994;.

[4] Black F, Scholes M. The Pricing of Options and Corporate Liabilities. The Journal of Political Economy. 1973 May-June;81(3):637–654.

[5] Singer H. Finanzmarktökonometrie. Wirtschaftswissenschaftliche Beiträge Nr. 171. Physica-Verl.; 1999.

[6] Cox JC, Ross SA. The valuation of options for alternative stochastic processes. Journal of Financial Economics. 1976;3(1–2):145 – 166.

[7] Karatzas I, Shreve S. Brownian motion and stochastic calculus. vol. 113. Springer Science & Business Media; 1988.

[8] Dang VX, Potter H, Glasgow S, Taylor S. Pricing the Asian Call Option. Electronic Proceedings of Undergraduate Mathematics Day. 2008;.

[9] Kemna AGZ, Vorst A. A pricing method for options based on average asset values. Journal of Banking & Finance. 1990;14(1):113–129.

[10] Jacques M. On the hedging portfolio of Asian options. Astin Bulletin. 1996;26(02):165–183.

[11] Chance D. Derivatives and Portfolio Management - Reading 49: Option markets and Contracts. vol. 6. CFA Institute - CFA Level II curriculum; 2015.

[12] Bouaziz L, Briys E, Crouhy M. The pricing of forward-starting Asian options. Journal of Banking & Finance. 1994;18(5):823–839.

[13] Alziary B, Décamps JP, Koehl PF. A PDE approach to Asian options: analytical and numerical evidence. Journal of Banking & Finance. 1997;21(5):613–640.

[14] Cox J. Notes on option pricing I: Constant elasticity of variance diffusions. Unpublished note, Stanford University, Graduate School of Business. 1975;.

[15] Cox JC, Ingersoll Jr JE, Ross SA. A theory of the term structure of interest rates. Econometrica: Journal of the Econometric Society. 1985;p. 385–407.

[16] Scott LO. Option Pricing when the Variance Changes Randomly: Theory, Estimation, and an Application. The Journal of Financial and Quantitative Analysis. 1987;22(4):pp. 419–438.

[17] Kloeden PE, Platen E. Numerical solution of stochastic differential equations. Berlin ; New York: Springer-Verlag; 1992.

[18] Glasserman P, Heidelberger P, Shahabuddin P. Asymptotically Optimal Importance Sampling and Stratification for Pricing Path-Dependent Options. Mathematical Finance. 1999;9(2):117–152.

[19] Zhao Q, Liu G, Gu G. Variance Reduction Techniques of Importance Sampling Monte Carlo Methods for Pricing Options. Journal of Mathematical Finance. 2013;3(4):431–436.

[20] Singer H. Importance sampling for Kolmogorov backward equations. AStA Advances in Statistical Analysis. 2014;98(4):345–369.

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