Continuous Time Models for Panel Data

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Abstract

Continuous time stochastic processes are used as dynamical models for discrete time measurements (time series and panel data). Thus causal effects are formulated on a fundamental infinitesimal time scale. Interaction effects over the arbitrary sampling interval can be expressed in terms of the fundamental structural parameters. It is demonstrated that the choice of the sampling interval can lead to different causal interpretations although the system is time invariant. Maximum likelihood estimates of the structural parameters are obtained by using Kalman filtering (KF) or nonlinear structural equations models (SEM).

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Sampling;
Exact discrete model;
Continuous-discrete state space models;
Kalman filtering;
SEM modeling

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1 Overview

1.1 Differential equations

growth model

$$dY(t)/dt = aY(t) \tag{1}$$

$$Y(t) = \exp[a(t - t_0)]Y(t_0) \tag{2}$$

stochastic differential equation (SDE)

$$dY(t)/dt = aY(t) + g\zeta(t) \tag{3}$$

$\zeta(t)$ **= Gaussian white noise process**

$\gamma(t - s) = E[\zeta(t)\zeta(s)] = \delta(t - s)$

solution:

$$Y(t) = \exp[a(t - t_0)]Y(t_0) + \int_{t_0}^{t} \exp(a(t - s))g\zeta(s)ds. \tag{4}$$

symbolic notation (Itô calculus):

$$\begin{aligned} dY(t) &= aY(t)dt + gdW(t) && (5) \\ Y(t) &= \exp[a(t - t_0)]Y(t_0) + \int_{t_0}^{t} \exp[a(t - s)]gdW(s). && (6) \end{aligned}$$

exact discrete model (EDM)

Bergstrom (1976, 1988)

$$Y_{i+1} = \exp[a(t_{i+1} - t_i)]Y_i + \int_{t_i}^{t_{i+1}} \exp[a(t_{i+1} - s)]g dW(s), \quad (7)$$

$$Y_{i+1} = \Phi(t_{i+1}, t_i)Y_i + u_i, \quad (8)$$

- Φ = fundamental matrix; $Y_i := Y(t_i)$
- nonlinear restrictions

$$\text{Var}(u_i) = \int \Phi(t_{i+1}, s)^2 g^2 ds \quad (9)$$

- Software: implement nonlinear restrictions
- multivariate case: (time ordered) matrix exponentials (Phillips, 1976, Jones, 1984, Hamerle et al., 1991, 1993, Singer, 1998).

Models with time-varying matrices:

- development psychology: children get older in a longitudinal study, causal effects are time dependent.
- Factor structure of a depression questionaire: time dependent due to the psychological state of the subjects.

1.2 Advantages of differential equations

(cf. Möbus and Nagl, 1983):

1. System dynamics: independent of the measurement scheme
   Process level of the phenomenon
   micro causality: infinitesimal time interval dt
2. Design of the study: **measurement model**
   Independent of the systems dynamics.
3. **Changes** of the variables: at any time at and between measurements.
   State defined for any time point, even if not measured.
4. **Extrapolation** and **interpolation** of data points: arbitrary times.
   not constrained to panel waves.
5. Studies with different or **irregular sampling** intervals: can be compared
   parameters do not depend on the measurement intervals.
6. Data sets with different sampling intervals: analyzed together as one vector series.
7. **Irregular sampling, missing data**: unified framework.
   Parametrization is **parsimonious**:
   only estimate the fundamental continuous time parameters
8. Cumulated or integrated data (flow data): represented explicitly.
9. Nonlinear transformations of data and variables: differential calculus (Itô calculus).

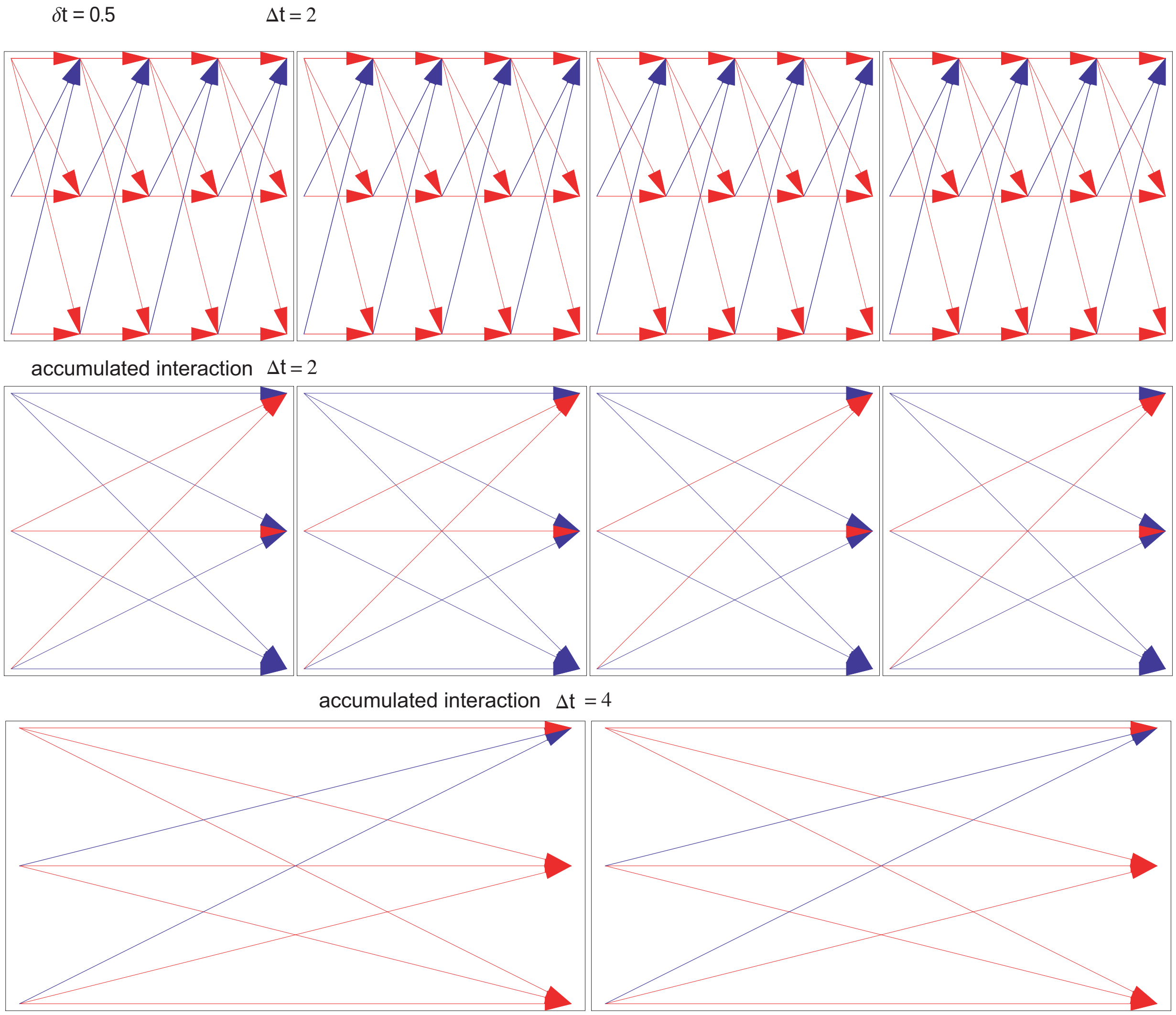

Figure 1: 3 variable model: Product representation of interactions within the measurement interval $\Delta t = 2$. Discretization interval $\delta t = 2/4 = 0.5$. Positive causal actions = red; Negative causal actions = blue.

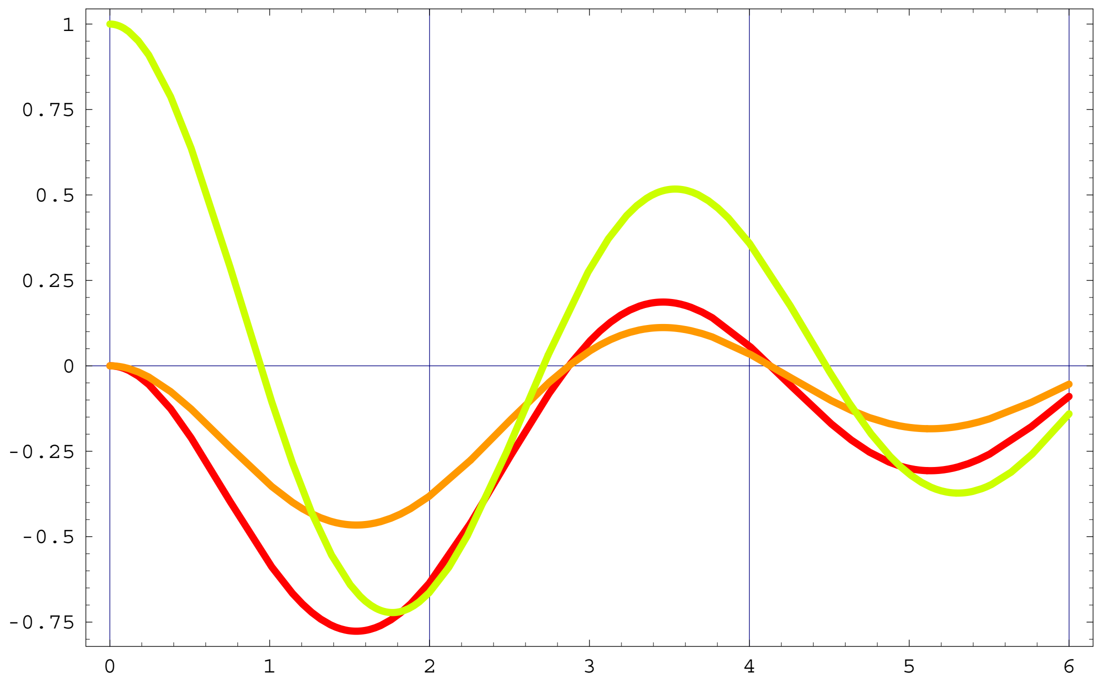

Figure 2: 3 variable model: Exact discrete matrix $A^* = \exp(A\Delta t)$ as a function of measurement interval Δt. Matrix elements A^*_{12}, A^*_{21}, A^*_{33}. Note that the discrete time coefficients change their strength and even sign.

1.3 Example 1: fine grid with interval $\delta t = \Delta t/2$

$$\exp(A\Delta t) \approx (I + A\Delta t/2)^2 = I + A\Delta t + A^2\Delta t^2/4 \tag{10}$$

no direct interaction between Y_1 and Y_2, i.e. $A_{12} = 0 = A_{21}$

Second order terms:

$$[\exp(A\Delta t)]_{12} \approx A_{13}A_{32}\Delta t^2/4 \tag{11}$$

- Indirect interactions mediated through third variable: appear at finite sampling interval.
- different signs: positive and negative contributions
- overall sign is dependent on the sampling interval.

$$A = \begin{bmatrix} -0.3 & \mathbf{0} & 1 \\ \mathbf{0} & -0.5 & 0.6 \\ -2 & -2 & 0 \end{bmatrix} \tag{12}$$

$$\lambda(A) = \{-0.18688 + 1.77645i, -0.18688 - 1.77645i, -0.42624\} \tag{13}$$

$$\exp[A(\Delta t = 2)] = \begin{bmatrix} -0.242254 & -0.634933 & -0.131455 \\ -0.38096 & 0.0697566 & -0.116969 \\ 0.262911 & 0.389897 & -0.66265 \end{bmatrix} \tag{14}$$

2 Model specification and interpretation

2.1 Linear continuous/discrete state space model

(Jazwinski, 1970)

$$dY(t) = [A(t,\psi)Y(t) + b(t,\psi)]dt + G(t,\psi)dW(t) \quad (15)$$

$$Z_i = H(t_i,\psi)Y(t_i) + d(t_i,\psi) + \epsilon_i \quad (16)$$

measurement times $t_i, i = 0, ..., T$

2.2 Exact discrete model (EDM)

$$Y(t_{i+1}) = \Phi(t_{i+1},t_i)Y(t_i) + + \int_{t_i}^{t_{i+1}} \Phi(t_{i+1},s)b(s)ds + \int_{t_i}^{t_{i+1}} \Phi(t_{i+1},s)G(s)dW(s) \quad (17)$$

2.3 Parameter functionals

(Arnold, 1974)

$$A_i^* := \Phi(t_{i+1},t_i) \quad (18)$$

$$b_i^* := \int_{t_i}^{t_{i+1}} \Phi(t_{i+1},s)b(s)ds \quad (19)$$

$$\Omega_i^* := \mathrm{Var}(u_i) = \int_{t_i}^{t_{i+1}} \Phi(t_{i+1},s)G(s)G'(s)\Phi'(t_{i+1},s)ds. \quad (20)$$

2.4 State transition matrix

$$\frac{d}{dt}\Phi(t,t_i) = A(t)\Phi(t,t_i) \quad (21)$$

$$\Phi(t_i,t_i) = I. \quad (22)$$

2.5 Time invariant and uniform sampling case

$$\Phi(t_{i+1},t_i) = A^* = \exp(A\Delta t). \quad (23)$$

2.6 Matrix exponential function

Taylor series of fundamental interaction matrix A

$$\exp(A\Delta t) = \sum_{j=0}^{\infty}(A\Delta t)^j/j!, \quad (24)$$

2.7 second order contribution: Y_k and Y_m

$$[(A\Delta t)^2]_{km} = \sum_l A_{kl}A_{lm}\Delta t^2, \quad (25)$$

2.8 Product representation

$$\exp(A\Delta t) = \lim_{J\to\infty}\prod_{j=0}^{J}(I + A\Delta t/J). \quad (26)$$

2.9 Example 2: Linear oscillator; CAR(2)

synonyms:
pendulum, swing

γ = friction = 4,
$\omega_0 = 2\pi/T_0 = 4$ = angular frequency,
T_0 = period of undamped oscillation

applications: systems with **periodic behaviour**

$$\ddot{y} + \gamma\dot{y} + \omega_0^2 y = bx(t) + g\zeta(t) \tag{27}$$

$$d\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} := \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\gamma \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}dt + \begin{bmatrix} 0 \\ b \end{bmatrix}dt + \begin{bmatrix} 0 & 0 \\ 0 & g \end{bmatrix}d\begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix} \tag{28}$$

$$z_i := \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} y_1(t_i) \\ y_2(t_i) \end{bmatrix} + \epsilon_i \tag{29}$$

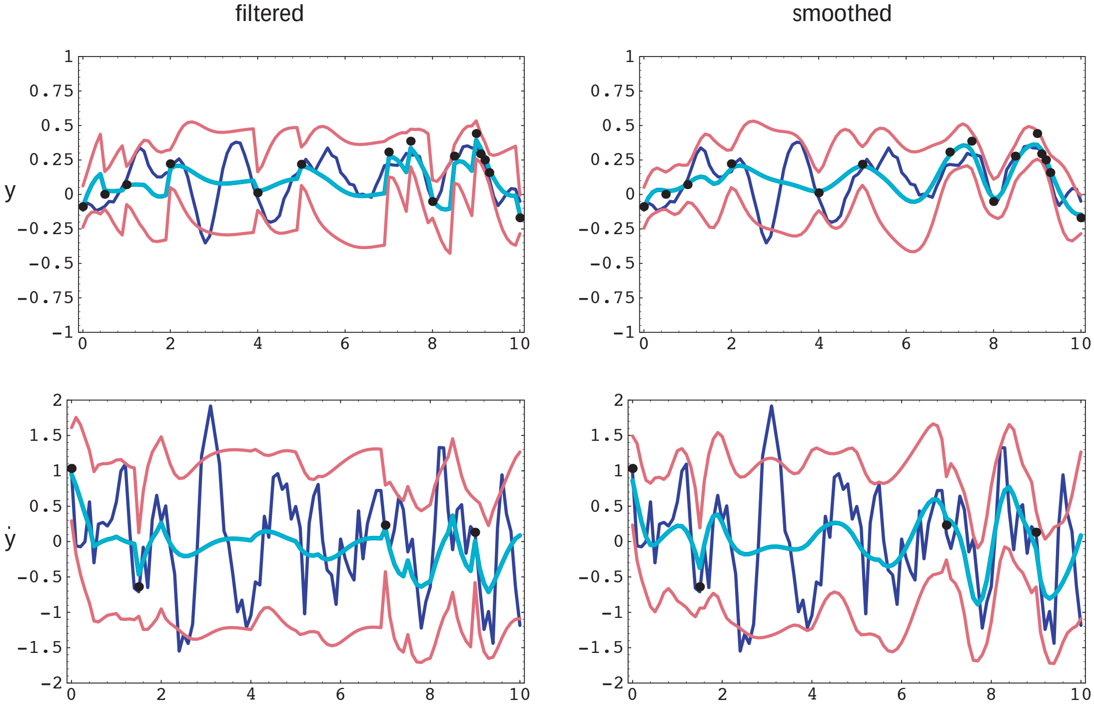

Figure 3: Linear oscillator with irregularly measured states (dots): Filtered state (left), smoothed state (right) with 95%-HPD confidence intervals. Measurements at $\tau_1 = \{0, .5, 1, 2, 4, 5, 7, 7.5, 8, 8.5, 9, 9.1, 9.2, 9.3, 10\}$ (first component; 1 st line), $\tau_2 = \{0, 1.5, 7, 9\}$ (2 nd component, 2 nd line). Discretization interval $\delta t = 0.1$. The controls $x(t)$ were measured at $\tau_3 = \{0, 1.5, 5.5, 9, 10\}$.

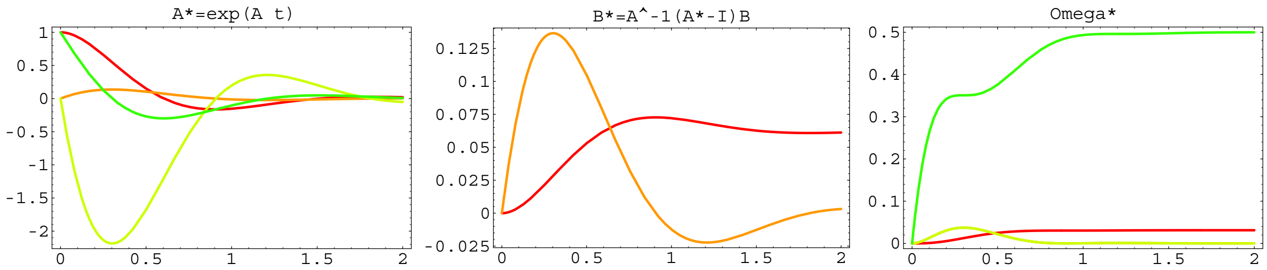

Figure 4: Linear oscillator: Exact discrete matrices $A^* = \exp(A\Delta t)$, $B^* = A^{-1}(A^* - I)B$, $\Omega^* = \int_0^{\Delta t} \exp(As)\Omega\exp(A's)ds$ as a function of measurement interval. Note that the discrete time coefficients change their strength and even sign.

2.10 Conclusion 1

- Researchers using different sampling intervals:
  dispute over strength and even **sign** of causal relation.
  only if using a discrete time model
  without deeper structure of the continuous time approach.
- **continuous time approach: estimate parameters related to interval** dt
  irrespective of measurement intervals $\Delta t_1, \Delta t_2, ...$ of different studies
  or irregular intervals in one study.
- sampling: can be **completely irregular** for each panel unit and within the variables.
- always point to the **same** fundamental level of the theory.

3 Estimation

3.1 General and historical remarks

- Exact Discrete Model: vector autoregression with special restrictions to be incorporated into the estimation procedure.
- Otherwise: serious **embeddability and identification problems** (cf. Phillips, 1976a, Hamerle et al. 1991, 1993, Singer, 1992).
- Small sampling interval: EDM may be linearized:
  time series or SEM software can be used.
  Proposed by Bergstrom in the 19sixties (rectangle or trapezium approximation).
- Later: estimate a reparametrized version of the EDM;
  infer the continuous time parameters **indirectly**.
- **Serious problems:**
  restrictions of $A, B, ..$ (see above) cannot be implemented
  no restrictions: embeddability and identification problems arise.
- express likelihood function $p(Z_T, ...Z_0; \psi)$ in terms of nonlinear EDM-matrices

$$A_i^* = \exp(A\Delta t_i) \tag{30}$$

$$B_i^* = A^{-1}(A_i^* - I)B \tag{31}$$

$$\Omega_i^* = \int_0^{\Delta t_i} \exp(As)\Omega\exp(A's)ds \tag{32}$$

  and $A = A(\psi), B = B(\psi), ...$

3.2 Exact estimation methods

1. Recursively by using the **Kalman filter**.
2. Non-recursively by using **nonlinear** simultaneous equations.

Exact Discrete Model

(panel index $n = 1, ...N; i = 0, ...T$)

$$Y_{i+1,n} = A^*_{in} Y_{in} + b^*_{in} + u_{in} \tag{33}$$

$$Z_{in} = H_{in} Y_{in} + d_{in} + \epsilon_{in} \tag{34}$$

$$A^*_{in} := \Phi_n(t_{i+1}, t_i) = \overleftarrow{T} \exp[\int_{t_i}^{t_{i+1}} A(s, x_n(s))ds] \tag{35}$$

$$b^*_{in} := \int_{t_i}^{t_{i+1}} \Phi_n(t_{i+1}, s) b(s, x_n(s))ds \tag{36}$$

$$\mathrm{Var}(u_{in}) := \Omega^*_{in} = \int_{t_i}^{t_{i+1}} \Phi_n(t_{i+1}, s) G(s, x_n(s))) G'(s, x_n(s))) \Phi'_n(t_{i+1}, s)ds. \tag{37}$$

- Matrices are **noncommutative**, i.e $A(t)A(s) \neq A(s)A(t)$
- $\overleftarrow{T}\, A(t)A(s) = A(s)A(t); t < s$
  Wick time ordering operator (cf. Abrikosov et al., 1963)

3.3 Kalman filter approach

likelihood (prediction error decomposition)

(panel index n is dropped)

$$l(\psi; Z) = \log p(Z_T, \ldots, Z_0; \psi) = \sum_{i=0}^{T-1} \log p(Z_{i+1}|Z^i; \psi) p(Z_0), \tag{38}$$

- $p(Z_{i+1}|Z^i; \psi) = \phi(\nu(t_{i+1}|t_i); 0, \Gamma(t_{i+1}|t_i))$
  transition densities (Gauss distributions)
- $\nu(t_{i+1}|t_i)$: **prediction error**
  (measurement minus prediction using information
  $Z^i := \{Z_i, \ldots, Z_0\}$ up to time t_i)
  Γ: prediction error covariance matrix.
- sequence of prediction and correction steps (time and measurement update).
- first discovered: engineering context by Kalman (1960).
- An implementation for panel data is LSDE (Singer, 1991, 1993, 1995).

Kalman filter algorithm

(Liptser and Shiryaev, 1977, 2001, Harvey and Stock, 1985, Singer, 1998).

conditional moments

- $\mu(t|t_i) = E[Y(t)|Z^i]$
- $\Sigma(t|t_i) = \text{Var}[Y(t)|Z^i]$,
- $Z^i = \{Z_i, ..., Z_0\}$ are the measurements up to time t_i.

time update

$$(d/dt)\mu(t|t_i) = A(t,\psi)\mu(t|t_i) + b(t,\psi) \tag{39}$$

$$(d/dt)\Sigma(t|t_i) = A(t,\psi)\Sigma(t|t_i) + \Sigma(t|t_i)A'(t,\psi) + \Omega(t,\psi) \tag{40}$$

Kalman filter algorithm

measurement update

$$\begin{aligned}
\mu(t_{i+1}|t_{i+1}) &= \mu(t_{i+1}|t_i) + K(t_{i+1}|t_i)\nu(t_{i+1}|t_i) && (41)\\
\Sigma(t_{i+1}|t_{i+1}) &= [I - K(t_{i+1}|t_i)H(t_{i+1})]\Sigma(t_{i+1}|t_i) && (42)\\
\nu(t_{i+1}|t_i) &= Z_{i+1} - Z(t_{i+1}|t_i) && (43)\\
Z(t_{i+1}|t_i) &= H(t_{i+1})\mu(t_{i+1}|t_i) + d(t_{i+1}) && (44)\\
\Gamma(t_{i+1}|t_i) &= H(t_{i+1})\Sigma(t_{i+1}|t_i)H'(t_{i+1}) + R(t_{i+1}) && (45)\\
K(t_{i+1}|t_i) &:= \Sigma(t_{i+1}|t_i)H'(t_{i+1})\Gamma(t_{i+1}|t_i)^{-1} && (46)
\end{aligned}$$

- $K(t_{i+1}|t_i)$ is the **Kalman gain**,
- $Z(t_{i+1}|t_i)$ is the **optimal predictor** of the measurement Z_{i+1},
- $\nu(t_{i+1}|t_i)$ is the **prediction error**
- $\Gamma(t_{i+1}|t_i)$ is the **prediction error covariance matrix**.

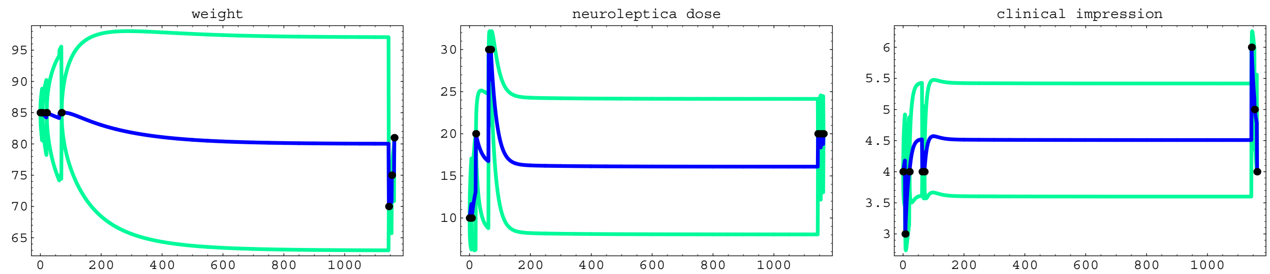

Figure 5: Filtered estimates of y_1 = weight (kg), y_2 = neuroleptica dose (mg), y_3 = clinical impression [2 (better),...,8 (worse)]. Female, age 48, ICD diagnosis F20. Interval [0,1163] days.

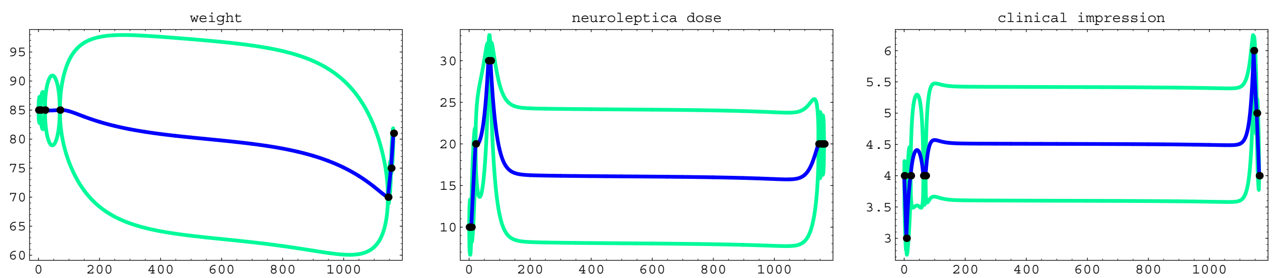

Figure 6: Same person. Interval [0,1163] days. Smoothed estimates with data points and 67%-HPD confidence intervals.

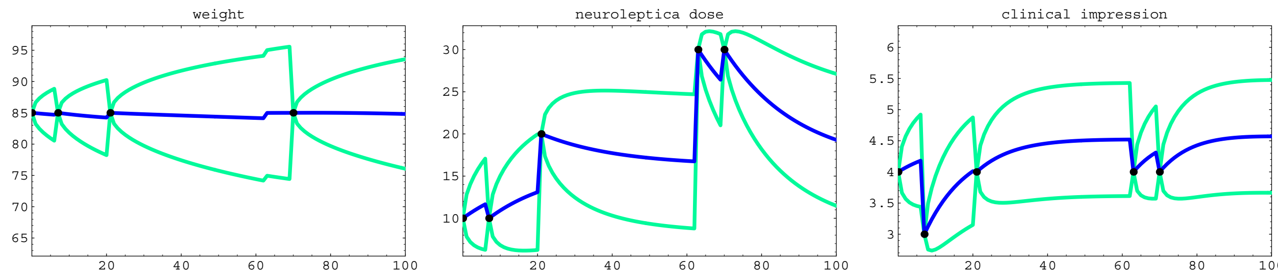

Figure 7: Filtered estimates of y_1 = weight (kg), y_2 = neuroleptica dose (mg), y_3 = clinical impression [2 (better),...,8 (worse)]. Female, age 48, ICD diagnosis F20. Interval [0,100]. The weight is missing at time point $t = 63$, but corrected due to the measurements of dose and clinical impression at the same time.

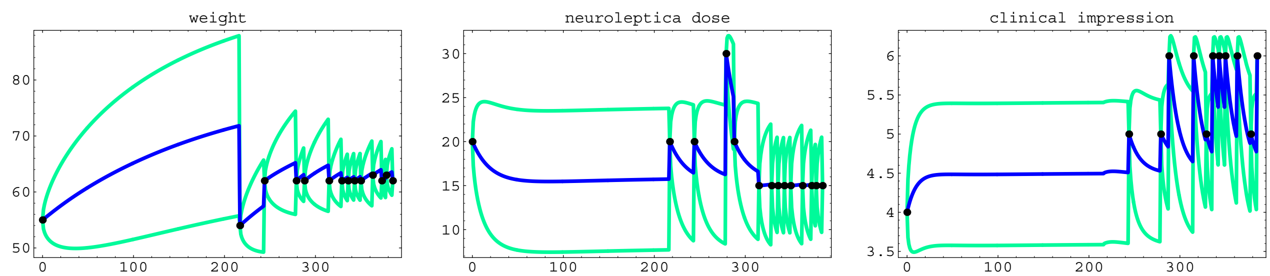

Figure 8: Filtered estimates of y_1 = weight (kg), y_2 = neuroleptica dose (mg), y_3 = clinical impression [2 (better),...,8 (worse)]. Female, age 48, ICD diagnosis F20. Interval [0,386]days.

3.4 SEM approach

SEM-EDM

(cf. Oud et al., 1993, Oud and Jansen, 2000)

$$\eta_n = B\eta_n + \Gamma X_n + \zeta_n \tag{47}$$
$$Y_n = \Lambda\eta_n + \tau X_n + \epsilon_n \tag{48}$$

(**deterministic** X_n; stochastic ξ_n are absorbed in η_n)

$$B = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ A_0^* & 0 & 0 & \dots & 0 \\ 0 & A_1^* & 0 & \dots & 0 \\ \vdots & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & A_{T-1}^* & 0 \end{bmatrix}; \quad X_n = \begin{bmatrix} 1 \\ x_{n0} \\ x_{n1} \\ \vdots \\ x_{nT} \end{bmatrix} : (T+2)q \times 1 \tag{49}$$

$$\Gamma = \begin{bmatrix} \mu & 0 & 0 & \dots & 0 & 0 \\ 0 & B_0^* & 0 & \dots & 0 & 0 \\ 0 & 0 & B_1^* & \dots & 0 & 0 \\ \vdots & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & B_{T-1}^* & 0 \end{bmatrix} : (T+1)p \times (T+2)q \tag{50}$$

$$b_{ni}^* = [\int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, s)B(s,\psi)ds]x_{ni} \tag{51}$$
$$:= B_i^* x_{ni}. \tag{52}$$

Likelihood function

$$l = -\frac{N}{2}(\log|\Sigma_y| + \mathrm{tr}[\Sigma_y^{-1}(M_y + CM_xC' - M_{yx}C' - CM_{xy})]), \tag{53}$$

$$\begin{aligned} E[Y_n] &= [\Lambda(I-B)^{-1}\Gamma + \tau]X_n := CX_n && (54)\\ \Sigma_y &= \mathrm{Var}(Y_n) = \Lambda(I-B)^{-1}\Sigma_\zeta(I-B)^{-T}\Lambda' + \Sigma_\epsilon. && (55) \end{aligned}$$

Moment matrices

$$\begin{aligned} M_y &= Y'Y : (T+1)p \times (T+1)p && (56)\\ M_x &= X'X : (T+2)k \times (T+2)k && (57)\\ M_{yx} &= Y'X : (T+1)p \times (T+2)k && (58)\\ Y' &= [Y_1, ...Y_N] : (T+1)p \times N && (59)\\ X' &= [X_1, ...X_N] : (T+1)k \times N && (60) \end{aligned}$$

- $\eta_n' = [Y_{0n}', ..., Y_{Tn}']$: sampled trajectory (for panel unit n)
- ΓX_n **deterministic** intercept term.
- Essential: SEM (and KF) software permits the **nonlinear parameter restrictions of the EDM** (30–32).
- SEM (47–48) with **arbitrary nonlinear parameter restrictions Mathematica program SEM**, Singer, 2004; public domain)

3.5 Comparision of the approaches

- KF computes the likelihood recursively for the data $Z^t = \{Z_0, ...Z_t\}$,
  conditional distributions $p(Z_{t+1}|Z^t)$ are updated step by step,
  SEM representation utilizes joint distribution of $\{Z_0, ...Z_T\}$.
- KF can work online; new data update conditional moments and likelihood
  SEM uses batch of data $Z = \{Z_0, ...Z_T\}$ with dimension $(T + 1)k$.
  KF only involves the data point $Z_t : k \times 1$
  invert matrices of order $k \times k$ (prediction error covariance).
  SEM must invert the matrices $\text{Var}(Y) : (T + 1)k \times (T + 1)k$ and $B : (T + 1)p \times (T + 1)p$ in each likelihood computation.
  Serious problems: long data sets $T > 100$, not for short panels.
- KF: conditionally Gaussian case $p(Z_{t+1}|Z^t)$ is still Gaussian
  joint distribution of $Z = \{Z_0, ...Z_T\}$ not Gaussian any more.
- KF approach: easily generalized to nonlinear systems (extended Kalman filter EKF)
  transition probabilites are still approximately conditionally Gaussian.

3.6 Comparision of the approaches (continued)

- SEM approach: more familiar to many scientists used to work with LISREL.
  Early days of SEM modeling: only linear restrictions
  Nonlinear likelihood easily programmed and maximized
  using Mathematica, SAS/IML etc.
- Filtered estimates of latent states:
  computed recursively by the KF (the conditional moments)
  smoothed trajectories: (fixed interval) smoother algorithm.
  SEM approach: conditional expectations $E[\eta|Y]$ and $\mathrm{Var}[\eta|Y]$
  matrices of order $(T+1)k \times (T+1)k$ are involved.
- Missing data:
  KF: process data $z_n(t_i) : k \times 1$
  for each time point and panel unit.
  missing data treatment: measurement update
  dropping missing entries in the matrices.
  SEM: individual likelihood approach

4 Conclusion

- Continuous time approaches to time series and panel analysis:
  many theoretical and practical advantages.
- More fundamental level
- Requirements for data sampling: very low
  (no regular panel waves; missing data permitted).
- Application of such models was hampered by the facts:
  - the model is more complex
    (different intervals for the state dynamics and the measurements)
  - standard software cannot implement the restrictions.
- Using LSDE (KF approach) or nonlinear SEM software like Mx or SEM
  obtain exact ML estimates of the fundamental causal actions.
- My opinion: Kalman filter (KF) is preferrable.
  The KF is the recursive, most direct and efficient implementation of the continuous/discrete state space model.

5 Software

- LSDE (1991; SAS/IML) $\Longrightarrow$ SDE (end 2005; Mathematica/C):
  Stochastic Differential Equations
  - Graphics, Simulation, Filtering, ML estimation
  - Arbitrary interpolation of exogenous variables
  - Arbitrary sampling intervals (persons and variables)
  - Missing data
  - Linear Systems:
    Kalman Filter (KF)
    Score with analytic derivatives
  - Nonlinear Systems:
    Extended Kalman Filter (EKF)
    Second Order Nonlinear Filter (SNF)
    Local Linearization (LL)
- SEM (2004; Mathematica):
  - ML estimation
  - Arbitrary nonlinear parameter restrictions
  - Deterministic (X_n) and stochastic (ξ_n) exogenous variables
  - SDE module (EDM)

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