# Continuous Panel Models with Time Dependent Parameters

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**Abstract**

Panel data are modeled as dynamic structural equations in continuous time (stochastic differential equations). The continuously moving latent state vector is mapped to an observable discrete time series (or panel) with the help of a measurement equation including errors of measurement (continuous-discrete state space model). Therefore the approach is able to handle data with irregularly observed waves, missing values and arbitrarily interpolated exogenous influences. In order to model development and growth models, the system parameter matrices are assumed to be time dependent.

## 1 Introduction

Continuous time dynamical systems do not seem to be suitable for longitudinal studies in psychology, sociology and economics, since data are mostly collected in panel waves with large time distances (monthly, quarterly, annually etc.). On the other hand, we usually feel that the phenomenon is evolving in continuous time. Both points of view can be unified if we model the system in continuous time, but the measurement model in discrete time. This kind of model is well known in econometrics (Bergstrom, 1976, 1988), finance (Black and Scholes, 1973, Merton, 1990) and engineering (Jazwinski, 1970), but has also been proposed in psychology and the social sciences (Coleman, 1968, Möbus and Nagl, 1983, Singer, 1986, 1992, Oud et al., 1993).

The so called *continuous-discrete state space model* is very flexible and can cover complicating factors in panel analysis such as measurement error, missing data, redefinition of variables and irregular sampling intervals (cf. Engel and Reinecke, 1996, ch. 1, Singer, 1995, 1996).

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Sometimes, due to structural changes or development processes, it must be assumed that the structural parameters of a model change over time (Oud et al., 1993, Oud and Jansen, 1996), but these authors used discrete time models or an indirect estimation method for continuous time systems which was proved to be incorrect (see Hamerle et al., 1991, 1993 and the references cited therein). Moreover, it is claimed that constant parameter models show stationarity, which is not the case because of initial conditions and exogenous variables.

Another example is the modelling of bond prices, where the price at maturity is fixed. A convenient model for this is the Brownian bridge which can be written as an Itô equation with time dependent coefficients (cf. Ball and Torous, 1983 and example 2).

This paper attempts to specify and estimate parameter varying linear continuous time panel models with discrete sampling, missing data and exogenous influences from arbitrary sampling schemes. For the general nonlinear case see Singer, 1996.

# 2 Continuous/discrete state space models with time varying coefficients

We discuss the *continuous/discrete state space model*

$$dy_n(t) = f(y_n(t), t, \psi)dt + g(y_n(t), t, \psi)dW_n(t) \tag{1}$$

where measurements $z_{ni}$ are taken at times $\{t_0, t_1, \dots, t_T\}$ and $t_0 \leq t \leq t_T$:

$$z_{ni} = h(y_n(t_i), t_i, \psi) + \epsilon_{ni}. \tag{2}$$

In (1), $W_n(t)$ denotes $r$-dimensional Wiener processes independent for different panel units $n$ and the units are described by the $p$-dimensional state vectors $y_n(t)$, $n = 1, \dots, N$. They fulfil a system of stochastic differential equations in the sense of Itô (cf. Arnold, 1974) with random initial condition

$$y_n(t_0) \sim N(\mu_n(\psi), \Sigma_n(\psi)) \text{ independently distributed.}$$

In (2), $\epsilon_{ni} \sim N(0, R(t_i, \psi))$ is a $k$-dimensional discrete time white noise process (measurement error). Parametric estimation is based on the $u$-dimensional parameter vector $\psi$.

In order to simplify the approach, a linear vector field (drift) $f(y_n(t), t, \psi) = A(t, \psi) * y_n(t) + b(t, \psi)$ and state independent diffusion coefficient $g(y_n(t), t, \psi) = G(t, \psi)$ is specified. Therefore the solution process $y_n(t)$ does not depend on the definition of the stochastic integral $\int g dW$ (cf. Arnold, 1974, ch. 10). Furthermore, we assume

linear measurements $h(y_n(t), t, \psi) = H(t, \psi)y_n(t) + d(t, \psi)$ where $H(t, \psi)$ are time dependent factor loadings.
The linear specification can be applied to nonlinear systems by Taylor expansions of $f$ and $h$, leading to an extended Kalman filter (EKF) in (14-23). This approach is discussed in Singer, 1996.
In order to model exogenous influences, $A$, $b$, $G$, $\mu$, $\Sigma$ and $H$, $d$, $R$ are assumed to depend on regressor variables $x_n(t)$, i.e. $A(t, \psi) = A(t, x_n(t), \psi)$ etc.
Thus we have the general linear time dependent specification

$$\begin{aligned} dy_n(t) &= A(t, x_n(t), \psi)y_n(t)dt + b(t, x_n(t), \psi)dt + G(t, x_n(t), \psi)dW_n(t) && (3) \\ y_n(t_0) &\sim N(\mu(x_n(t_0), \psi), \Sigma(x_n(t_0), \psi)) \\ z_{ni} &= H(t_i, x_n(t_i), \psi)y_n(t_i) + d(t_i, x_n(t_i), \psi) + \epsilon_{ni} && (4) \end{aligned}$$

To simplify notation only the $t$ dependence of the parameter matrices will be displayed. Formerly, a special case of (3, 4), the linear state space model with constant coefficients

$$\begin{aligned} dy_n(t) &= Ay_n(t)dt + Bx_n(t)dt + GdW_n(t) && (5) \\ z_{ni} &= Hy_n(t_i) + Dx_n(t_i) + \epsilon_{ni} && (6) \end{aligned}$$

was estimated (Singer, 1993, 1995).

# 3 Exact Discrete Models with Fundamental Matrices

In order to compute the likelihood function of system (3, 4), we must express the probability distribution of states in terms of discretely measured data $z_n(t_i)$. Usually this problem is solved by deriving a so called *exact discrete model* from the system equation (3), which is valid at the times of measurement. We obtain (cf. Arnold, 1974, ch. 8):

$$y_n(t) = \Phi(t, t_0)y_n(t_0) + \int_{t_0}^{t} \Phi(t, s)b(s)ds + \int_{t_0}^{t} \Phi(t, s)G(s)dW_n(s), \quad (7)$$

where $\Phi(t, t_0)$ is a fundamental (transition) matrix solving the deterministic equation

$$d/dt\, \Phi(t, t_0) = A(t)\Phi(t, t_0);\ \Phi(t_0, t_0) = I_p. \quad (8)$$

Choosing $t = t_{i+1}$ and $t_0 = t_i$ we obtain a difference equation (AR(1) process)

$$y_n(t_{i+1}) = \Phi(t_{i+1}, t_i)y_n(t_i) + \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, s)b(s)ds + \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, s)G(s)dW_n(s) \quad (9)$$

From (9) it is seen that the conditional distribution of $y_n(t_{i+1})$ given $y_n(t_i)$ is Gaussian since the stochastic integral contains a deterministic function $\Phi(t_{i+1}, s)G(s)$. In the case with constant coefficients, we can express $\Phi$ explicitly as

$$\Phi(t_{i+1}, t_i) = \exp(A(t_{i+1} - t_i)) := \exp(A\Delta t_i) \quad (10)$$

It is well known that the matrix exponential contains the elements of $A$ in a very complicated nonlinear way (cf. Phillips, 1976[25]). This fact has been leading to a controversy whether it is possible to obtain $\hat{A}$ from a knowledge (estimate) of $\Phi$ (cf. Hamerle et al., 1991, 1993). Even in the univariate case $p = 1$ the explicit form of the fundamental matrix contains a time integral

$$\Phi(t_{i+1}, t_i) = \exp(\int_{t_i}^{t_{i+1}} A(s)ds) \quad (11)$$

leading to nonlinear equations for $\psi$. At any rate, the functional form of $\Phi$ implies nonlinear restrictions involving transcendental functions which have to be incorporated into the estimation procedure. In the general multivariate case, we can write $\Phi$ as

$$\Phi(t_{i+1}, t_i) = T \exp(\int_{t_i}^{t_{i+1}} A(s)ds), \quad (12)$$

where $T$ is the Wick time ordering operator yielding a time ordered matrix exponential ($TA(t)A(s) = A(s)A(t)$ if $t < s$). This is necessary since $A(t)$ in general does not commute with $A(s)$ even in simple cases (cf. example 1). Although (12) is an explicit expression, it must be worked out using Taylor series involving time ordered multiple integrals (see, e.g. Abrikosov et al., 1963, p. 47 ff). Therefore it seems difficult to compute the likelihood function of the exact discrete state space model (*prediction error decomposition*; Schweppe, 1965 )

$$L_\psi(z) = \prod_{i=0}^{T-1} \prod_{n=1}^{N} |2\pi\Gamma_{n,i+1|i}|^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2} \operatorname{tr}[\Gamma_{n,i+1|i}^{-1} \nu_{n,i+1}\nu'_{n,i+1}]\right\} p(z_{n0}), \quad (13)$$

where $z = \{z_{ni}| \, n = 1, \ldots, N, \, i = 0, \ldots, T\}$, $\nu_{n,i+1} = z_{n,i+1} - H_{n,i+1}y_{n,i+1|i} - d_{n,i+1}$ is the *innovation*, $\Gamma_{n,i+1|i} = \operatorname{Var}(z_{n,i+1}|z_{ni}, \ldots, z_{n0}) =$ is the innovation covariance and $p(z_{n0})$ is the distribution of the inital state $z_n(t_0)$. Here, $H(t_i, x_n(t_i), \psi)$ is abbreviated as $H_{ni}$ etc.

More explicitly, the innovation $\nu$ is given in terms of the one step predictor

$$y_{n,i+1|i} = \Phi(t_{i+1}, t_i) y_{n,i|i} + \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, s) b(s) ds \tag{14}$$

and prediction error

$$P_{n,i+1|i} = \Phi(t_{i+1}, t_i) P_{n,i|i} \Phi(t_{i+1}, t_i)' + \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, s) G(s) G(s)' \Phi(t_{i+1}, s)' ds. \tag{15}$$

The measurement updates (a posteriori estimates) in the Kalman filter are given explicitly as

$$\begin{aligned}
K_{n,i+1} &= P_{n,i+1|i} H'_{n,i+1} (H_{n,i+1} P_{n,i+1|i} H'_{n,i+1} + R_{n,i+1})^{-} && (16)\\
y_{n,i+1|i+1} &= y_{n,i+1|i} + K_{n,i+1} \nu_{n,i+1} && (17)\\
P_{n,i+1|i+1} &= P_{n,i+1|i} - K_{n,i+1} H_{n,i+1} P_{n,i+1|i} && (18)\\
\nu_{n,i+1} &= z_{n,i+1} - H_{n,i+1} y_{n,i+1|i} - d_{n,i+1} && (19)\\
\Gamma_{n,i+1|i} &= H_{n,i+1} P_{n,i+1|i} H'_{n,i+1} + R_{n,i+1} && (20)
\end{aligned}$$

The filter must be initialized with the conditional expectations

$$\begin{aligned}
y_{n,0|0} &= E[y_n(t_0) | z_n(t_0)] && (21)\\
P_{n,0|0} &= \mathrm{Var}[y_n(t_0) | z_n(t_0)] && (22)
\end{aligned}$$

(e.g. Liptser and Shiryayev, 1978, ch. 13) using the parameters $\mu(x_n(t_0), \psi) = E[y_n(t_0)]$ and $\Sigma(x_n(t_0), \psi) = \mathrm{Var}[y_n(t_0)]$. Thus, we can compute the likelihood function and the ML estimator $\hat{\psi}$ if we are able to evaluate the integrals (14, 15) involving the transition matrix $\Phi$.

# 4 Practical Computation of the Likelihood Function

As in the case of constant coefficients (Singer, 1995), we solve this problem by using a slightly different discretization scheme (note that the exact discrete model does not involve any approximation)

$$\begin{aligned}
y_n(t_{j+1}) &= \Phi(t_{j+1}, t_j) y_n(t_j) + \int_{t_j}^{t_{j+1}} \Phi(t_{j+1}, s) b(s) ds + && (23)\\
&\quad + \int_{t_j}^{t_{j+1}} \Phi(t_{j+1}, s) G(s) dW_n(s) &&\\
z_{nj} &= H_{nj} y_n(t_j) + d_{nj} + \epsilon_{nj}, && (24)
\end{aligned}$$

$j = 0, \ldots, J = (t_T - t_0)/\delta t$, where $\delta t$ is an arbitrary discretization interval so small that all times of measurement $t_i$ can be expressed as multiples of $\delta t$, i.e. $t_i = t_0 + j_i \delta t$ (cf. Singer, 1995). Then, however, we must introduce a missing data treatment in (24) which can be achieved by dropping rows in $H_{nj}$, $d_{nj}$ and $\epsilon_{nj}$ if some component in $z_{nj}$ is missing.

This procedure opens up the possibility of completely irregular sampling for each unit $n$ and each component $k'$ of response variables $z_{nj}$. For example, we can combine times series of different time lags to form a vector time series with irregular sampling and missing data.

A further advantage of the representation (23, 24) is in the treatment of exogenous variables $x_n(t)$. Usually these are only measured at certain time points. In order to evaluate the functional

$$\int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, s) b(s, x_n(s)) ds \tag{25}$$

we must know the values for times $t \in [t_i, t_{i+1}]$. Usually, $x_n(t)$ is interpolated between the function values $x_n(t_i)$ using simple interpolation formulas such as step functions, polygonal lines or parabolas (cf. Phillips, 1976[24]). In the case of constant coefficients, $\Phi(t_{i+1}, s) = \exp(A(t_{i+1} - s))$ can be inserted into integral (25) together with $b(s, \hat{x}_n(s))$ where $\hat{x}_n(s)$ is the interpolating function. The resulting explicit formulas are complicated matrix functions, however (cf. Bergstrom, 1976, 1984, 1990, Phillips, 1976, Sargan, 1976). In the case of time dependent $A(t)$ the evaluation of (25) would involve the time ordered exponential.

All these problems can be circumvented if we let $\delta t \to 0$ in (23, 24) and linearize

$$\begin{aligned} \Phi(t_{j+1}, t_j) &\approx I + A(t_j)\delta t & (26)\\ \int_{t_j}^{t_{j+1}} \Phi(t_{j+1}, s) b(s) ds &\approx b(t_j)\delta t & (27)\\ \int_{t_j}^{t_{j+1}} \Phi(t_{j+1}, s) G(s) dW_n(s) &\approx G(t_j)\delta W_n(t_j). & (28) \end{aligned}$$

Then, the scheme (23) generates the exact discrete model since it is an Euler-Maruyama approximation of stochastic differential equation (3) (cf. Rümelin, 1982). With this approach we do not need to evaluate the transition matrix explicitly or calculate the functional involving the exogenous variables. Therefore more sophisticated interpolation schemes such as splines can be used (cf. Singer, 1995), The linearized $\Phi(t_{j+1}, t_j)$ motivates the formula

$$\Phi(t_{i+1}, t_i) = \lim_{\delta t \to 0} T \prod_{j=j_i}^{j_{i+1}} I + A(t_j)\delta t \tag{29}$$

where later times stand to the left of earlier times, which can be used to compute the reduced form (discrete time) parameters.
As in former work, the maximum likelihood estimates $\hat{\psi}$ are computed iteratively with the help of a quasi Newton algorithm using BFGS secant updates (see Dennis and Schnabel, 1983)

$$\psi_{k+1} = \psi_k + F_k^{-1} s_k \tag{30}$$

where $F_k$ is a secant update and $s_k$ is the score $\partial/\partial\psi_k \, l_{\psi_k}(z); l = \log L$. In later work it is hoped to present analytical scores, but here we use numerical derivatives as in Jones, 1984, Jones and Tryon, 1987, or Harvey and Stock, 1985. The BFGS algorithm (30) is started with $F_0 = I_u$ and updates are computed from the score. Only at the end of the iteration the observed Fisher information $J = -\partial^2/\partial\psi^2 \, l_\psi(z)$ is computed once to obtain asymptotic covariances of $\hat{\psi}$.

# 5 Examples

## 5.1 Growth model with time dependent rates and disturbances

To illustrate the method, a bivariate growth model with changing growth rates was simulated and estimated. The computations were done using Mathematica (Wolfram Research, 1992).
We specify a time independent matrix of growth rates

$$A_0 = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} \tag{31}$$

and a linearly changing part

$$A_1(t) = \begin{bmatrix} \alpha\, t & 0 \\ 0 & \beta\, t \end{bmatrix}. \tag{32}$$

The drift matrix is the sum $A(t) = A_0 + A_1(t)$. We compute the commutator of $A(t)$ with $A(s)$

$$A(t)A(s) - A(s)A(t) = \begin{bmatrix} 0 & (-\alpha+\beta)\,\lambda_{12}\,(s-t) \\ (-\alpha+\beta)\,\lambda_{21}\,(-s+t) & 0 \end{bmatrix}, \tag{33}$$

which shows that the time ordering operator $T$ must be used in calculating the transition matrix $\Phi$. The parameters in $A(t)$ were chosen as $\psi = \{\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \alpha, \beta\} = \{.1, -.01, .02, .3, .01, -.02\}$ which means that the rate $A_{11}$ is increasing whereas the rate $A_{22}$ is decreasing with time (see figure 3). Furthermore, we specify a time dependent diffusion matrix $G = \mathrm{Diag}(0.01\,t, 0.02\,t)$ in order to model increasing noise during the development process. Of course, the parameters in $G$ could be estimated by including them in $\psi$. The measurement model is given by $H(t) = I_2$, $d(t) = 0$ and $R = \mathrm{Diag}(0.001, 0.001)$.
In order to test BFGS algorithm (30) a small simulation study was performed with a sample size of $n = 10$ and $M = 50$ replications. Data were simulated in a time interval $0 \leq t \leq 10$ with discretization interval $\delta t = 0.1$, but afterwards irregularly sampled at times $\tau = \{0, 1, 2, 3, 5, 6, 7, 8, 8.5, 9, 9.5, 10\}$. Figure 1 shows $n = 10$ simulated trajectories of $z_n(t)$ starting with an initial value of $z_n(t_0) = [1, 2]' + \epsilon_{n0}$. In figure 3 the time dependence of the rates $A_{ij}$ is plotted. First the eigenvalues of $A(t)$ are real, but at a certain time they suddenly become conjugate complex, which means that oscillations of the system are possible (figure 4). This example again demonstrates the problems when using an indirect estimation method as proposed by several authors (cf. Coleman, 1968, Sinha and Lastman, 1982, Arminger, 1986, Oud et al., 1993) and criticized in Hamerle et al., 1991, Singer, 1992 . The identification problem arising from complex eigenvalues of $A$ cannot be overcome by claiming that in development processes no oscillations occur: a pair of real eigenvalues can suddenly jump to the complex plane although the respective matrices only differ by a small amount $||A(t) - A(t + \Delta t)||$. In contrast, the direct estimation approach via $\psi$ can ensure identification because of restrictions (the special form of $\Phi$ is correctly specified and $G$ is diagonal, e.g.) and leads to correct estimates. The result of the simulation study is shown in table 1 and figure 5. The histograms and kernel density estimates (using Gauss kernels) show some deviations from normality due to small sample size ($N = 10, T = 12$), but means are near to the true values which are covered by approximate 95% confidence intervals. The BFGS algorithm converges after about 15 iterations, where $||\psi_{k+1} - \psi_k||_\infty < .001$.

## 5.2 Brownian bridge and bond prices

The Brownian bridge is a Wiener type process where initial and final values $B(t_0)$ and $B(t_1)$ are fixed. Usually, these values are set to zero, and $t_0 = 0, t_1 = 1$. It is a simple model for bond prices, which have a fixed payoff at maturity $t_1$, but random variation because of fluctuations in interest rates (cf. Ball and Torous, 1983). When a bond is emitted, its value sligthly deviates from its maturity price in order to avoid arbitrage. A standard Brownian bridge can be written as $B(t) =$

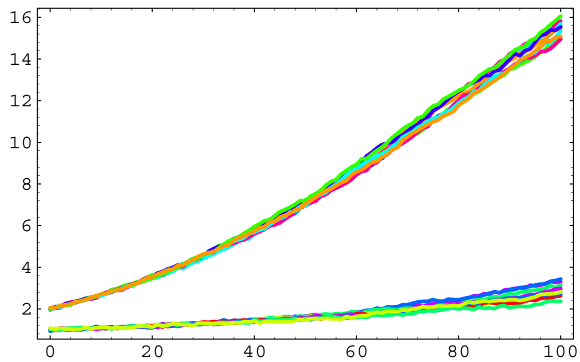


Figure 1: Simulated trajectories of growth model, first and second component $z_{n1}(t), z_{n2}(t)$ (see text).

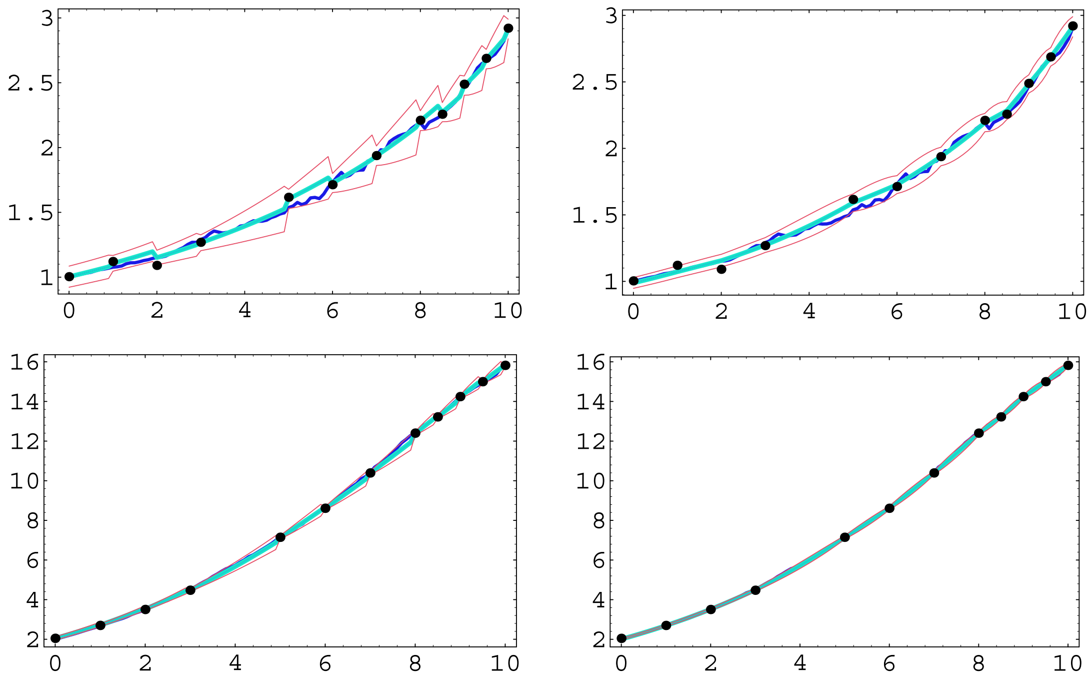


Figure 2: Filtered (left) and smoothed trajectories and 99% HPD confidence intervals for growth model.

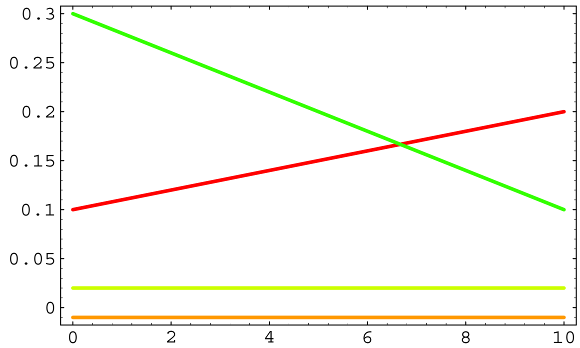


Figure 3: Time dependence of growth rates $A_{ij}(t)$.

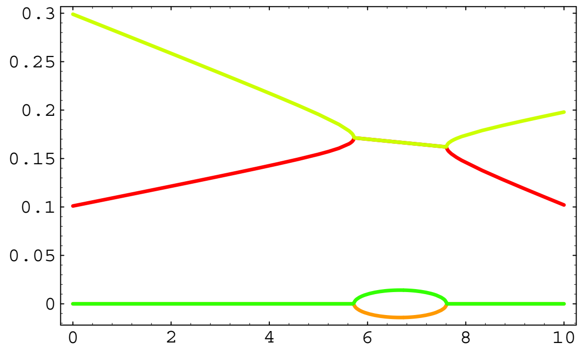


Figure 4: Real and imaginary parts of the eigenvalues of $A(t)$ as a function of time. First they are real, but suddenly they get complex (oscillations) and finally return to the real line.

| | true | mean | std |
|---|---|---|---|
| $k$ | | 15.2 | 1.16619 |
| $\lambda_{11}$ | 10. | 8.82621 | 2.19069 |
| $\lambda_{12}$ | -1. | -0.568103 | 0.923164 |
| $\lambda_{21}$ | 2. | 3.18632 | 3.72381 |
| $\lambda_{22}$ | 30. | 29.5543 | 1.42541 |
| $\alpha$ | 1. | 0.868005 | 0.373103 |
| $\beta$ | -2. | -1.97443 | 0.0954185 |

Table 1: Means and standard deviations of $M = 50$ replications of $\hat{\psi}$ (in percent); $k$ is the number of iterations per sample to achieve convergence.

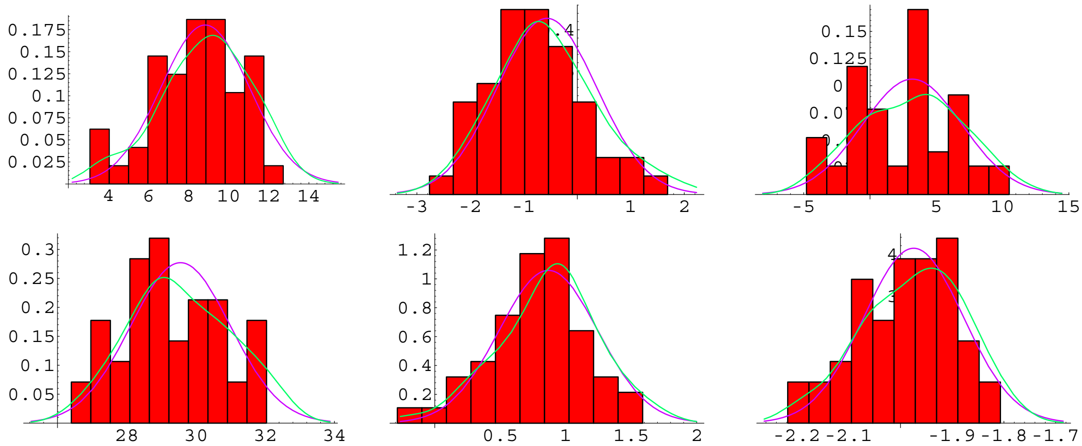

Figure 5: Histograms, fitted normal density and kernel density estimates of replicated ML estimates for growth model.

$W(t) - tW(1), 0 \leq t \leq 1$. A representation, which is more suitable for the methods as developed formerly, is in the form of an Itô equation

$$dB(t) = \frac{B(t)}{t-1} dt + dW(t);\ B(0) = 0. \tag{34}$$

More generally, we can write

$$dB(t) = \alpha \frac{B(t) - B_1}{t - t_1} dt + \sigma dW(t);\ B(t_0) = B_0, \tag{35}$$

where general conditions and drift and variance (volatility) parameters were introduced. Since $B(t)$ can assume negative values, the logarithm of the bond price should be modeled. Figure 6 shows simulated trajectories of a Brownian bridge with $t_0 = 0, t_1 = 1, \delta t = 1/200$ and $B(t_0) = 1, B(t_1) = 2$. We assumed that only one trajectory can be measured at irregular times $\{0, 0.05, 0.1, 0.15, 0.25, 0.5, 1.\}$ without measurement errors. Figure 7 shows data, filtered and smoothed trajectories and 99% HPD intervals $E[B(t)|z] \pm 2.58\sqrt{\mathrm{Var}[B(t)|z]}$ using the true $\psi$. The ML estimate is obtained after 6 iterations starting from the true values $\psi = \{\alpha, \sigma\} = \{1, 1\}$ as $\{1.26279(0.892963), 0.825131(0.248366)\}$ (asymptotic standard deviations in parantheses).

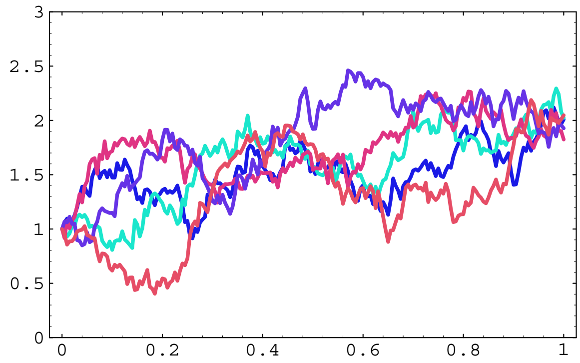


Figure 6: Simulated trajectories of a generalized Brownian bridge.

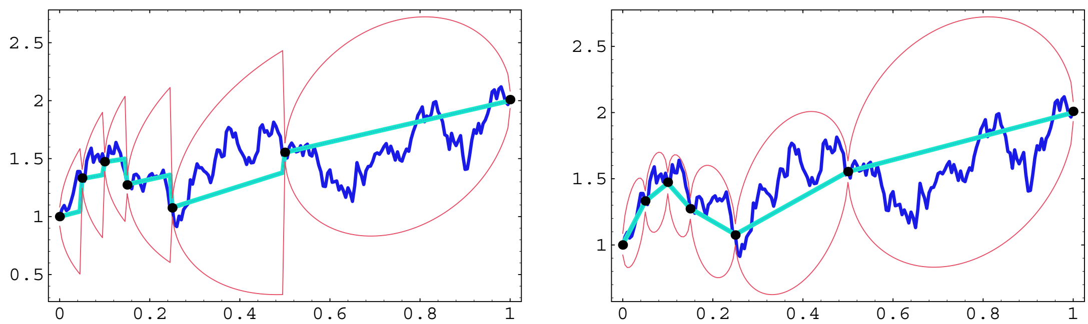


Figure 7: Brownian bridge: filtered and smoothed estimates from discrete measurements and 99% HPD confidence intervals.

## 6 Conclusion

We have shown how to compute ML estimates of a general time dependent stochastic differential equation system from irregularly sampled panel data. Random effects, stochastic exogenous variables and flow data are included by extending the latent state vector $y_n(t)$ with appropriate components. Analytical score functions can be computed by generalizing the approach in Singer, 1995.

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