A joint application of the put-call-parity and importance sampling to variance reduced option pricing

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Abstract: Pricing derivatives with Monte-Carlo simulations involves standard errors that typically decrease at a rate proportional to $N^{-0.5}$ where N is the sample size. Several approaches have been discussed to reduce empirical variance for a given sample size. This paper analyzes the joint application of the put-call-parity approach and importance sampling to non-path-dependent and path-dependent options. Significant variance reduction is observed for in-the-money European and Arithmetic Asian options. For put options, synergies are realized in the sense that the total variance reduction effect achieved by the combined approach is higher than the effect explained by the two standalone approaches.

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1 Introduction

A difficulty related to the pricing of options using Monte-Carlo simulations is the fact that the empirical variance of estimators decreases slowly, typically at a rate $\propto 1/N$ where N is the sample size. Several techniques have been discussed to reduce the empirical variance for a given sample size N [1]. Importance sampling turns out to be a particularly effective variance reduction technique. A transformation of the probability measure serves to increase the number of trajectories contributing to the Monte-Carlo estimator [2, 3, 4, 5].

An alternative approach to variance reduction - at least for in-the-money options - is the application of the put-call-parity. E.g., instead of simulating an in-the-money put the corresponding out-of-the-money call can be simulated. The put price then can be calculated from the put-call-parity yielding a variance reduced estimator [6, 7].

This paper will investigate how the joint application of the put-call-parity and importance sampling can lead to synergies in the simulation of variance reduced Monte-Carlo estimators.

2 Importance sampling

Foundations of importance sampling The method of importance sampling was first introduced to efficiently simulate chain reactions in nuclear reactors [8]. The fundamental idea is the transformation of the probability measure governing the simulation.

The expectation value of a function $h : \mathbb{R}^d \to \mathbb{R}, X \to h(X)$ of a random variable X with probability density p is calculated as

$$\alpha = \mathbb{E}_p \left[h\left(X\right)\right] = \int h\left(x\right)\, p\left(x\right)\, dx. \tag{1}$$

An unbiased Monte-Carlo estimator for α with i.i.d. realizations $X_1, \ldots, X_n$ of the random variable X is

$$\hat{\alpha}_p = \frac{1}{n}\sum_{i=1}^{n} h\left(X_i\right). \tag{2}$$

With any other probability density p' the estimator can be rewritten as

$$\alpha = \int h\left(x\right)\, p\left(x\right)\, dx = \int h\left(x\right)\, \frac{p\left(x\right)}{p'\left(x\right)}\, p'\left(x\right)\, dx = \mathbb{E}_{p'}\left[h\left(X\right)\, \frac{p\left(X\right)}{p'\left(X\right)}\right]. \tag{3}$$

The ratio p/p' is called likelihood ratio or Radon-Nikodým derivative [1, 9].

The unbiased Monte-Carlo estimator

$$\hat{\alpha}_{p'} = \frac{1}{n}\sum_{i=1}^{n} h\left(X_i\right)\,\frac{p\left(X_i\right)}{p'\left(X_i\right)} \tag{4}$$

is distributed with empirical variance

$$\mathrm{Var}_{p'}\left[h\left(X\right)\,\frac{p\left(X\right)}{p'\left(X\right)}\right] = \mathbb{E}_{p'}\left[\left(h\left(X\right)\,\frac{p\left(X\right)}{p'\left(X\right)}\right)^2\right] - \mathbb{E}_{p'}\left[\left(h\left(X\right)\,\frac{p\left(X\right)}{p'\left(X\right)}\right)\right]^2. \tag{5}$$

Importance sampling is based on the minimization of this variance term. In the special case of a non-negative function h, by choosing

$$p'(x) \propto h(x)\,p(x) \tag{6}$$

the variance term in equation (5) vanishes. The product $h(x)\,p(x)$ can be transformed into a new probability density by normalizing. The difficulty is to obtain the normalization factor. For this purpose, the integral

$$\alpha = \int h\left(x\right)\,p\left(x\right)\,dx \tag{7}$$

would have to be calculated. However, the calculation of this quantity was the original problem to be solved in equation (1). Nonetheless, already by approximating the proportionality factor significant variance reduction can be achieved [1].

Importance sampling by adding an additional drift term The importance sampling approach applied in this paper was introduced by Singer (2014) [5] for the multivariate case[2]. Here, the univariate case is derived and the multivariate result is given. The aim is to find a variance reduced estimator for the Feynman-Kac formula

$$C(S_t,t) = \mathbb{E}\left[e^{\int_t^T r(S_\tau,\tau)\mathrm{d}\tau} h(S_T)|S(t)=S_t\right]. \tag{8}$$

Going forward, we will consider the case of a constant interest rate r

$$C(S_t,t) = e^{-r(T-t)}\mathbb{E}\left[h\left(S_T\right)|S(t)=S_t\right] \tag{9}$$

which solves the Black-Scholes differential equation [10]

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} - rC = 0. \tag{10}$$

[2] A similar approach has already been introduced by Melchior and Öttinger (1995) [2].

For the purpose of numerical calculations, equation (9) can be expressed as

$$C(S_t,t) \approx e^{-r(T-t)} \int h\left(S_n\right) p(S_n,\tau_n|S_{n-1},\tau_{n-1}) \times \ldots \times p(S_1,\tau_1|S_t,t)\mathrm{d}S_n \ldots \mathrm{d}S_1 \tag{11}$$

using a finite-dimensional approximation[3] on a grid $\tau_j = t + j\Delta\tau$ with n discretization steps. We set $\tau_n = T$ and consequently $S_n = S_T$.

The random process $S(t)$, i.e. the stock price of the derivative's underlying, is supposed to follow an Itō differential equation of the form

$$dS(t) = f(S(t))\,dt + g(S(t))\,dW(t) \tag{12}$$

where f is a drift parameter, t the time, g a diffusion coefficient and $W(t)$ a Wiener process. Defining $\Delta t = t_2 - t_1$ and $\Delta S = S_2 - S_1$, the transition density between t_1 and t_2 for small Δt is approximately the density of the normal distribution with expectation value $f\,\Delta t$ and variance $g^2\,\Delta t$:

$$\begin{aligned} p\left(S_2,t_2 \mid S_1,t_1\right) &= \frac{1}{\sqrt{2\pi\,g^2\,\Delta t}}\,\exp\left\{-\frac{(\Delta S - f\,\Delta t)^2}{2\,g^2\,\Delta t}\right\} \\ &= \frac{1}{\sqrt{2\pi\,g^2\,\Delta t}}\,\exp\left\{-\frac{1}{2}\,\frac{\Delta S^2}{g^2\,\Delta t} + \frac{\Delta S\,f}{g^2} - \frac{1}{2}\,\frac{f^2\,\Delta t}{g^2}\right\} \end{aligned} \tag{13}$$

This density function solves the Kolmogorov backward equation[4] with the differential operator $\mathcal{L}$:

$$\begin{aligned} \frac{\partial\,p}{\partial t_1} &= -\left[f\,\frac{\partial}{\partial S_1} + \frac{1}{2}\,g^2\,\frac{\partial^2}{\partial S_1^2}\right]\,p \\ &= -\mathcal{L}\,p \end{aligned} \tag{14}$$

For the optimal density $p' = p^{\text{opt}}$ in accordance with equation (6) we define:

$$p^{\text{opt}}\left(S\right) = p\left(S\right)\frac{h\left(S\right)}{\mathbb{E}\left[h\left(S\right)\right]} \tag{15}$$

The resulting underlying is supposed to follow a stochastic differential equation

[3] For details see Appendix B in [5]

[4] For details see [11], chapter 4

similar to equation (12) with modified drift term:

$$dS = f^{\text{opt}}\, dt + g\, dW \tag{16}$$

The diffusion coefficients of equations (12) and (16) coincide because otherwise the Radon-Nikodým derivative would diverge.

Differentiating the optimal density p^{opt} with respect to t and inserting the Kolmogorov backward equation (14) after a lengthy calculation yields

$$\delta f \equiv f^{\text{opt}} - f = \frac{g^2}{C} \frac{\partial C}{\partial S}. \tag{17}$$

In the multivariate case with a scalar C, vectors f and g and a diffusion matrix $\Omega = gg^T$ one obtains the following optimal drift term [5]:

$$\delta f = \Omega \, \frac{\nabla C}{C} \tag{18}$$

The (univariate) optimal stochastic differential equation follows:

$$dS = f^{\text{opt}}\, dt + g\, dW = \left(f + \frac{g^2}{C} \frac{\partial C}{\partial S} \right)\, dt + g\, dW \tag{19}$$

In the Black-Scholes model with $f = rS$ and $g = \sigma S$ this equation can be written as follows:

$$dS = \left(r + \epsilon\, \sigma^2 \right)\, S\, dt + \sigma\, S\, dW \tag{20}$$

The following abbreviation for the option price elasticity was introduced:

$$\epsilon = \frac{S}{C} \frac{\partial C}{\partial S} \tag{21}$$

In equations (19)-(21), C is the Feynman-Kac formula from equation (9). Again the same problem as in equation (7) materializes: To describe the optimal stochastic differential equation, knowledge of C is required. Approaches how to cope with this problem will be discussed in a subsequent paper. In this analysis, the Black-Scholes formula for European options will be used [12]. In the case of simulating European options in the Black-Scholes model, this is the best possible choice (as the Black-Scholes formula yields the analytically exact value). But also for other options like Arithmetic Asian options it represents a useful approximation.

In order to evaluate estimators of the form of equation (4), the Radon-Nikodým derivative must be calculated. Here the 2nd row of equation (13) is used:

$$\begin{aligned} \frac{p}{p^{\text{opt}}} &= \exp\left\{-\frac{\Delta S}{g^2}\left(f^{\text{opt}}-f\right)+\frac{1}{2g^2}\,\Delta t\,\left(f^{\text{opt}2}-f^2\right)\right\} \\ &= \exp\left\{-\left(f^{\text{opt}}-f\right)\,\frac{1}{g^2}\,\left[\Delta S-\frac{1}{2}\,\left(f^{\text{opt}}+f\right)\,\Delta t\right]\right\} \\ &= \exp\left\{-\left(f^{\text{opt}}-f\right)\,\frac{1}{g^2}\,\left[\frac{1}{2}\,\left(f^{\text{opt}}-f\right)\,\Delta t+g\,\Delta W\right]\right\} \end{aligned} \tag{22}$$

By inserting equation (17) the Radon-Nikodým derivative simplifies to

$$\frac{p}{p^{\text{opt}}} = \exp\left\{-\frac{g}{C}\,\frac{\partial C}{\partial S}\,\Delta W-\frac{1}{2}\,\left(\frac{g}{C}\,\frac{\partial C}{\partial S}\right)^2\,\Delta t\right\}. \tag{23}$$

With sample size N and n discretization steps for the Black-Scholes model one obtains the variance reduced Monte-Carlo estimator

$$\begin{aligned} &\hat{C}_{\text{IS}}(S(t),t) \\ &= \frac{1}{N}\sum_{i=1}^{N}\exp\left\{-\sum_{k=0}^{n-1}\left[\sigma\frac{S_{ik}}{C_{ik}}\,\frac{\partial C_{ik}}{\partial S_{ik}}\,\Delta W_{ik}+\frac{1}{2}\,\left(\sigma\frac{S_{ik}}{C_{ik}}\,\frac{\partial C_{ik}}{\partial S_{ik}}\right)^2\,\Delta t\right]\right\} \\ &\qquad\times e^{-r\,(T-t)}\,h\left(S_i\left(T\right)\right) \\ &= \frac{1}{N}\sum_{i=1}^{N}\exp\left\{-\sum_{k=0}^{n-1}\left[\sigma\,\epsilon_{ik}\,\Delta W_{ik}+\frac{1}{2}\,\sigma^2\,\epsilon_{ik}^2\,\Delta t\right]\right\} \\ &\qquad\times e^{-r\,(T-t)}\,h\left(S_i\left(T\right)\right). \end{aligned} \tag{24}$$

Again, the abbreviation (21) has been used. As mentioned, ϵ can be calculated from the Black-Scholes formula [12]. The result for call options is [13]

$$\epsilon_{\text{Call}}\left(S(t),t\right) = \left(1-\frac{Ke^{-r(T-t)}}{S(t)}\frac{\Phi\left(d_2(S(t),t)\right)}{\Phi\left(d_1(S(t),t)\right)}\right)^{-1}. \tag{25}$$

An analog result follows for put options [14]:

$$\epsilon_{\text{Put}}\left(S(t),t\right) = \left(1-\frac{Ke^{-r(T-t)}}{S(t)}\frac{\Phi\left(-d_2(S(t),t)\right)}{\Phi\left(-d_1(S(t),t)\right)}\right)^{-1} \tag{26}$$

Other importance sampling approaches are being discussed in literature [4, 15].

3 Variance reduction with put-call-parities for European and Asian options

European options By simple non-arbitrage arguments it can be shown that for a European put with pay-off function

$$P_T = (K - S_T)^+ \tag{27}$$

the following put-call-parity holds true where P_t is the put price at t and C_t the call price [16]:

$$P_0 = C_0 + Ke^{-rT} - S_0 \tag{28}$$

Asian options Also for Arithmetic Asian put options with pay-off function

$$P_T = \left(K - \bar{S}_T\right)^+ \text{ with } \bar{S}_T = \frac{1}{m}\sum_{i=1}^{m} S_i \tag{29}$$

a put-call-parity holds. $S_m = S_T$ and equidistant S_i are assumed.

With

$$\mu = \left(e^{-\frac{rT(m-1)}{m}} + \ldots + e^{-\frac{rT}{m}} + 1\right) \tag{30}$$

it can be shown that the following relation is required to avoid arbitrage [7]:

$$P_0 = C_0 + Ke^{-rT} - \frac{\mu}{m} S_0 \tag{31}$$

Variance reduction Reider (1994) [6] suggested to apply put-call-parities to variance reduced importance sampling. It can be shown that in-the-money (call/put) options can be estimated more efficiently by first estimating the corresponding (put/call) option with the same parameters and subsequently calculating the required (call/put) value from the put-call-parity [7].

4 Joint application of the put-call-parity and importance sampling

In a previous paper it was suggested that a combined application of the put-call-parity and importance sampling might be particularly attractive [7]. It was discussed that importance sampling in many cases is especially attractive for out-of-the-money options. Therefore, the valuation of an in-the-money (call/put) option could be conducted more efficiently by pricing the corresponding out-of-the-money (put/call) option with importance sampling and then calculating the desired option value from the put-call-parity. This approach will be analyzed and discussed in this paper.

5 Numerical Results

Introductory remarks In the following, prices of a European and an Arithmetic Asian option will be estimated by Monte-Carlo simulations. Variance reduction will be achieved by applying the put-call-parity (PCP) and by importance sampling (IS). The two approaches will also be combined, i.e. importance sampling will be applied to call/put options and then the corresponding put/call option value will be calculated from parities 28 and 31.

In order to compare the combined variance reduction approach (PCP-IS) with the standalone approaches (PCP and IS) a performance ratio will be calculated involving the variance reduction factors (VRF) as follows:

$$\text{Performance ratio} = \frac{\mathrm{VRF}_{\text{PCP-IS MC}}}{\mathrm{VRF}_{\text{PCP MC}} \times \mathrm{VRF}_{\text{IS MC}}} \tag{32}$$

Here, the variance reduction factor is calculated as the ratio between the empirical variance of a benchmark Monte-Carlo estimator and the empirical variance of a variance-reduced Monte-Carlo estimator.

Obviously, whenever the performance ratio exceeds 1, synergies result from combining the two approaches.

European options As shown in figure 1, Monte-Carlo simulations of a European put option in the Black-Scholes model were conducted and compared to the analytical Black-Scholes formula [12]. As expected in the case of a direct Monte-Carlo simulation without application of variance reduction techniques, for the in-the-money price regime the standard error was higher than for out-of-the-money constellations. The same result holds when importance sampling is applied. However, when applying the put-call-parity approach the reverse is the case: in-the-money prices were estimated more accurately both in simulations with and without importance sampling.

More explicitly, the reduction of the empirical variance of the estimators is shown in figure 2. As expected from previous research [7], the put-call-parity approach was most effective for deep-in-the-money put options. The importance sampling

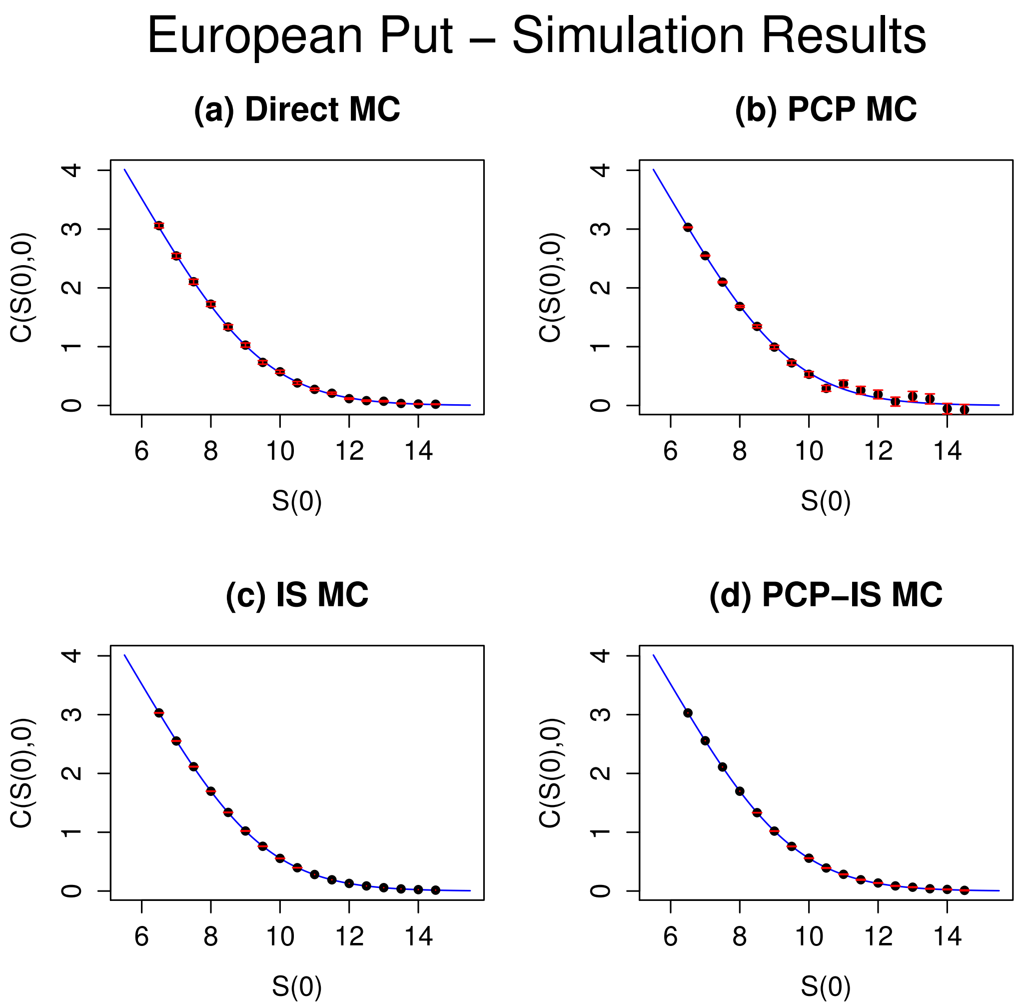

Figure 1: **(a) Direct MC**: Black dots: direct Monte-Carlo simulation of European put with pay-off function (27) with $r = 0.05$, $\sigma = 0.2$, $T = 1$, $K = 10$, $n = 100$ discretization steps and $N = 1,000$ simulated trajectories. Red bars: Standard error of simulated option value. Blue line: Analytic option price calculated from the Black-Scholes formula [12]. **(b) PCP MC**: Black dots: Put prices calculated employing the PCP (28). Required call prices were estimated by Monte-Carlo simulation with the same parameters as in (a). Red bars: as in (a). Blue line: as in (a). **(c) IS MC**: Black dots: Put prices calculated employing the IS approach described in section 2. For purposes of numerical stability the estimated ϵ values calculated from equation (26) have been limited to the interval $[-\beta, \beta]$ with $\beta = 10,000$. Red bars: as in (a). Blue line: as in (a). **(d) PCP-IS MC**: Black dots: Put prices calculated in a combined approach of (b) and (c). First call prices were estimated via importance sampling applying equation (25) with $\beta = 10,000$. Then, put prices were calculated from equation (28). Red bars: as in (a). Blue line: as in (a).

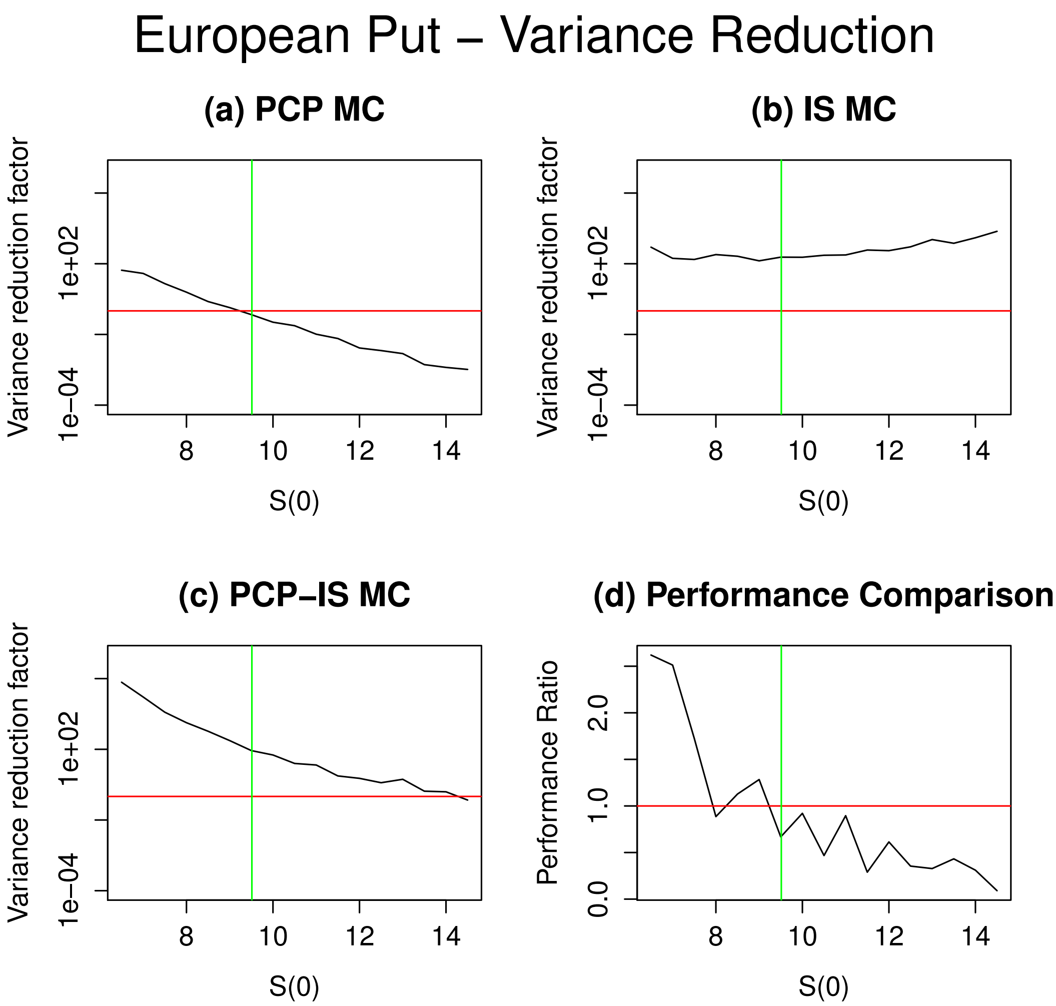

Figure 2: **(a) PCP MC**: Black line: Variance reduction achieved by employing the PCP (28). The variance reduction factor is calculated as the ratio between the empirical variances of the two Monte-Carlo estimators presented in (a) and (b) in figure 1. Red line: Line where variance reduction factor equals one. Green line: Line where $S(0) = Ke^{-rT}$. **(b) IS MC**: Black line: Variance reduction achieved by employing the IS approach described in section 2. Red line: as in (a). Green line: as in (a). **(c) PCP-IS MC**: Black line: Variance reduction achieved by combining the approaches from (a) and (b). Red line: as in (a). Green line: as in (a). **(d) Performance Comparison**: Black line: Performance calculated as the ratio between the variance reduction factor of the combined approach as shown in (c) and the product of the variance reduction factors of the standalone approaches shown in (a) and (b), see equation (32). Values > 1 indicate synergies resulting from the combination of the PCP approach with the IS approach. Red line: Line where the combined approach yields the same variance reduction factor as the product of the two standalone approaches. Green line: as in (a).

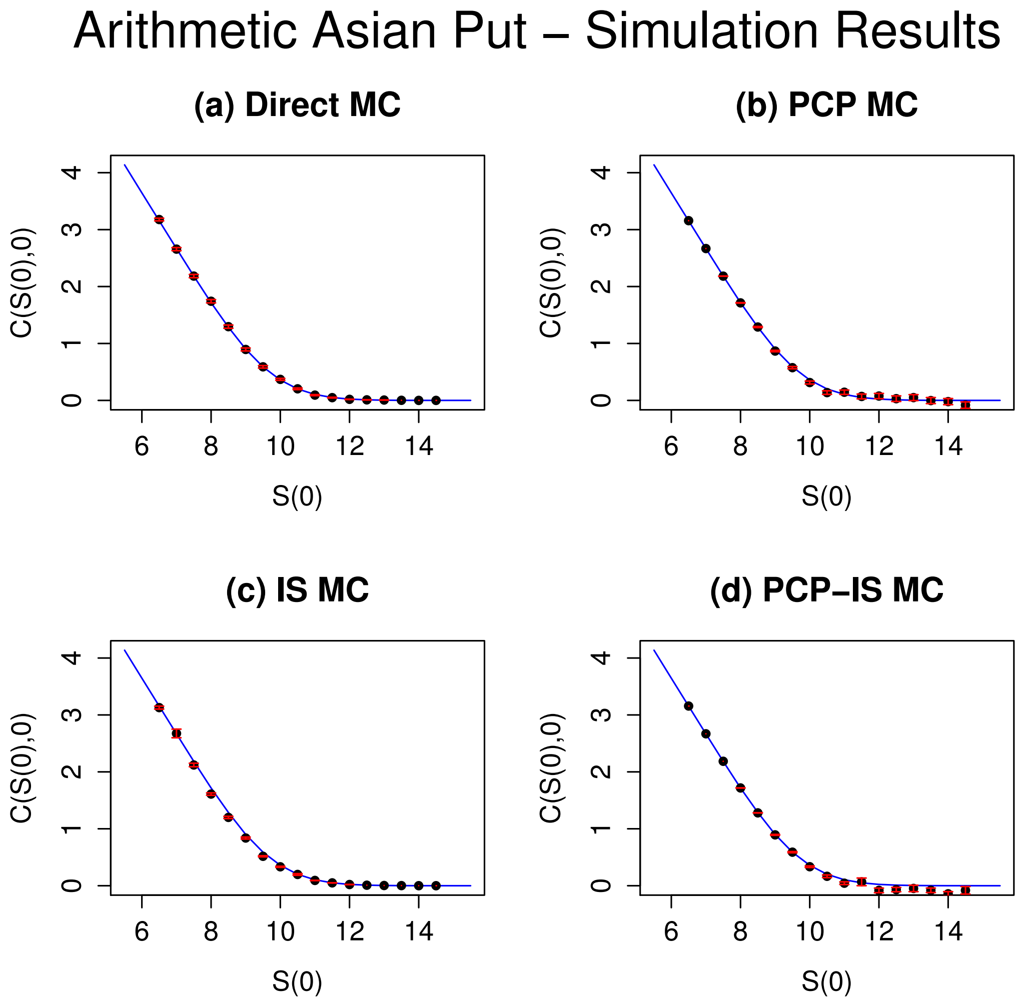

Figure 3: **(a) Direct MC**: Black dots: direct Monte-Carlo simulation of Arithmetic Asian put with pay-off function (29) with $r = 0.05$, $\sigma = 0.2$, $T = 1$, $K = 10$, $n = 100$ discretization steps, $N = 1,000$ simulated trajectories and $m = 10$, i.e. the course trajectory was divided in m equal parts and the final S value of each interval was taken to calculate $\bar{S}$. Red bars: Standard error of simulated option value. Blue line: Option price simulated with the same parameters as before, but with increased number of trajectories $N_{\text{reference}} = 100,000$. **(b) PCP MC**: Black dots: Put prices calculated employing the PCP (31). Required call prices were estimated by Monte-Carlo simulation with the same parameters as in (a). Red bars: as in (a). Blue line: as in (a). **(c) IS MC**: Black dots: Put prices calculated employing the IS approach described in section 2. For purposes of numerical stability the estimated ϵ values calculated from equation (26) have been limited to the interval $[-\beta, \beta]$ with $\beta = 10,000$. Red bars: as in (a). Blue line: as in (a). **(d) PCP-IS MC**: Black dots: Put prices calculated in a combined approach of (b) and (c). First call prices were estimated via importance sampling applying equation (25) with $\beta = 10,000$. Then, put prices were calculated from equation (31). Red bars: as in (a). Blue line: as in (a).

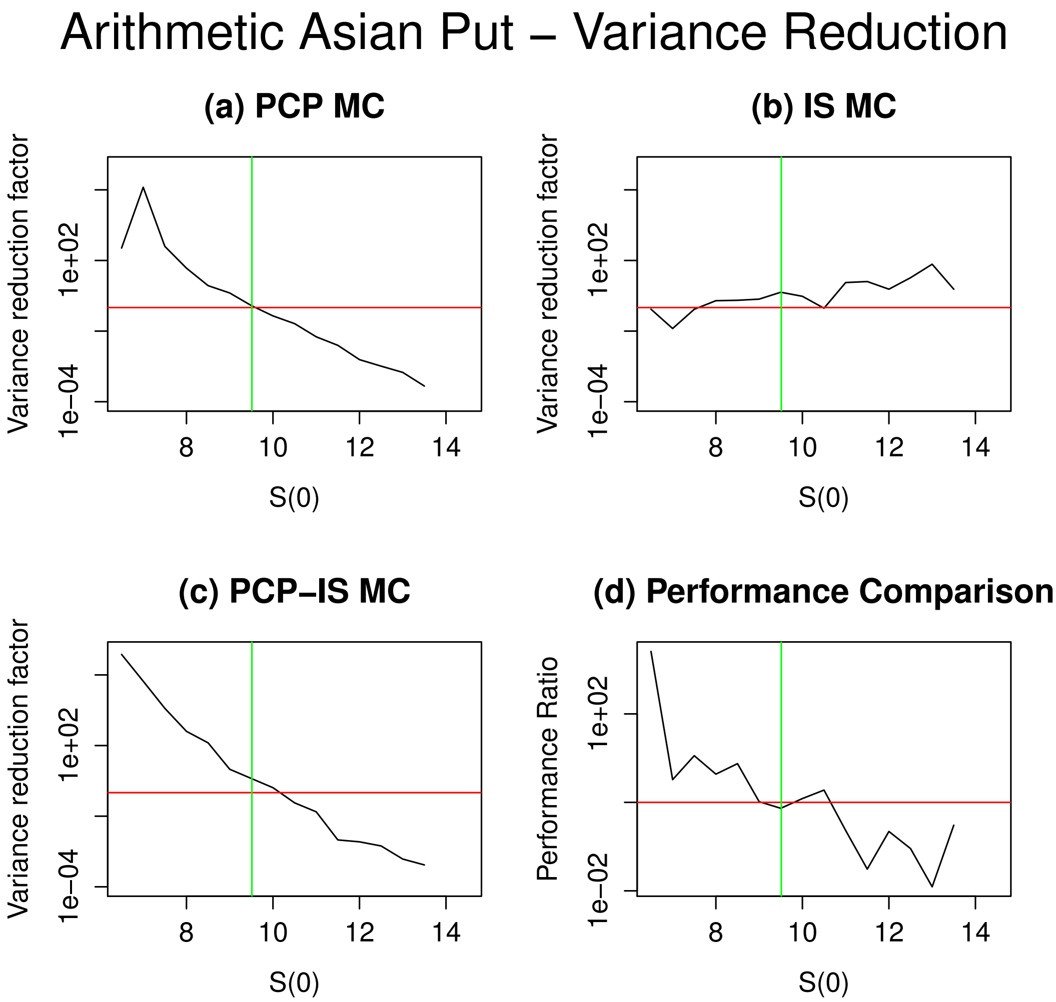

Figure 4: **(a) PCP MC**: Black line: Variance reduction achieved by employing the PCP (31). The variance reduction factor is calculated as the ratio between the empirical variances of the two Monte-Carlo estimators presented in (a) and (b) in figure 3. Red line: Line where the variance reduction factor equals one. Green line: Line where $S(0) = Ke^{-rT}$. **(b) IS MC**: Black line: Variance reduction achieved by employing the IS approach described in section 2. Red line: as in (a). Green line: as in (a). **(c) PCP-IS MC**: Black line: Variance reduction achieved by combining the approaches from (a) and (b). Red line: as in (a). Green line: as in (a). **(d) Performance Comparison**: Black line: Performance calculated as the ratio between the variance reduction factor of the combined approach as shown in (c) and the product of the variance reduction factors of the standalone approaches shown in (a) and (b), see equation (32). Values > 1 indicate synergies resulting from the combination of the PCP approach with the IS approach. Red line: Line where the combined approach yields the same variance reduction factor as the product of the two standalone approaches. Green line: as in (a).

approach was effective on a broad range of underlying values. The combined approach also turned out to work well on a broad range of underlying values, however with decreasing effectiveness.

As described above, performance comparisons were conducted. For small S_0 values, the combined approach of PCP and IS turned out to be particularly effective. E.g., for $S_0/Ke^{-rT} \approx 0.618$ by applying the PCP approach, a variance reduction factor of 53.1 was achieved while the IS approach yielded a factor of 504. Contrastingly, the combined approach achieved a factor of $70,100$. Thus, the variance reduction factor is 2.62 times higher than the product of the reduction factors of the two standalone approaches. Similar synergies have been achieved for other in-the-money underlying values. However, with increasing S_0 synergies decreased and eventually disappeared. For increased S_0 values the IS approach turned out to be more effective than the combined approach.

In the case of call options no synergies were realized (performance ratio < 1). However, the combined approach for in-the-money options still yielded better variance reduction results than any of the two variance reduction approaches considered individually.

Arithmetic Asian options Similar results have been achieved for Arithmetic Asian put options (see figures 3 and 4). Again, the PCP approach worked very well for in-the-money puts. However, the IS approach here partially delivered negative variance reduction results, i.e. a variance increase. However, when combining the approaches, for in-the-money puts again high variance reductions were achieved. E.g., for $S_0/Ke^{-rT} \approx 0.618$ a variance reduction factor of $745,000$ was achieved. The PCP approach alone achieved a variance reduction factor of 337 and IS a variance increase involving a factor of 0.852. Thus, the performance measure yielded a value of $2,590$.

6 Discussion

Variance reduction achieved by the PCP-IS approach The results clearly indicate that combining importance sampling with the put-call-parity approach to variance reduced option pricing is a powerful tool for the pricing of in-the-money options. The presented approach has been applicable both to the non-path-dependent case (European options) and to the path-dependent case (Arithmetic Asian options). Furthermore, the approach is very general in the sense that neither the involved importance sampling approach nor the involved put-call-parities are limited to a specific underlying model (here the Black-Scholes case has been examined).

Further improvement of estimators for Asian options The IS variance reduction for Asian options might be further improved by applying a more accurate estimate of the option price elasticity ϵ. In this analysis, the Black-Scholes formula for European options was applied as a rough estimate. Other approximations are currently being researched and will be presented in a subsequent paper.

Combination of PCP with other importance sampling approaches Combining other authors' approaches to importance sampling with the put-call-parity approach should be further researched as also here, synergies might be obtained, possibly outperforming this paper's approach.

7 Conclusion

Variance reduced Monte-Carlo simulations for European and Arithmetic Asian put options have been conducted. For in-the-money options significant variance reduction by combining importance sampling with the put-call-parity approach has been achieved.

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