Fast Analytic Option Valuation with GARCH

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Abstract

This paper introduces a new method for pricing European style call options with GARCH models. The resulting pricing formula is an explicit function of the model parameters, current spot asset price, exercise price and time to maturity. The method under consideration is remarkably fast because no numerical integration procedures are involved. Errors, resulting from approximation and series expansion, are surveyed by Monte-Carlo methods and turn out to be negligible.

Keywords: GARCH model; *Gram-Charlier* type A series; Cumulant generating function.

JEL Class.: G13

1 Introduction

Since large-scale trading on derivatives started in the early 1970's, valuation of contingent claims is a subject to intense scientific research and discussion. The seminal work of Black and Scholes (1973) and Merton (1973) and also of Cox, Ross, and Rubinstein (1979) provided the first risk-neutral pricing models. One shortcoming of these original models is the assumption of a fixed volatility which has proven violated in real markets. Many modifications and innovations have been suggested since then. Amongst the most prominent ones, the continuous time bivariate diffusion model of Heston (1993), accounting for stochastic volatility, and the extended *Black-Scholes* model of Dumas, Fleming, and Whaley (1998), incorporating implied volatility. Still one problem remains in such diffusion models, the volatility is not observable. Recent research activity is focused on that problem (see Aihara and Bagchi, 2000; Dragulescu and Yakovenko, 2002; Cvitanic, Liptser, and Rozovskii, 2006; Aït-Sahalia and Kimmel, 2007).

This problem is not inherent in the discrete time autoregressive conditional heteroscedastic models introduced by Engle (1982) and Bollerslev (1986). These models are extraordinary successful, and countless variants and generalizations exist. Amongst the most important, the ARCH-M model of Engle, Lilien, and Robbins (1987), the EGARCH model of Nelson (1991) and the GJR-GARCH

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model of Glosten, Jagannathan, and Runkle (1993). An excellent review of theoretical properties of ARCH models and their application in finance is Gouriéroux (1997).

The first to unify both approaches was Duan (1995), who derived a locally risk-neutral measure (LRNVR), which enabled him to calculate option prices as discounted expectations under the LRNVR measure within a GARCH model context. These expectations cannot be expressed analytically, which means they had to be evaluated by Monte-Carlo methods at first. Formidable progress has been accomplished on that issue in short time. The cornerstones are the empirical martingale simulation (EMS) by Duan and Simonato (1998), a *Markov*-chain approximation method (Duan and Simonato, 2001), the neural network approach of Hanke (1997), a lattice construction method by Ritchken and Trevor (1999) and finally an analytical approximation formula by Duan, Gauthier, and Simonato (1999). At about the same time Heston and Nandi (1997, 2000) introduced a new GARCH model, and the associated risk-neutral parameter transformations, whose characteristic function remains in a log-linear form. They derived recursions for the involved terms and finally provided an analytic expression in the *Fourier*-domain. The capabilities and limitations of GARCH models regarding option pricing are discussed recently in some detail (see Kallsen and Taqqu, 1998; Härdle and Hafner, 2000; Christoffersen and Jacobs, 2004; Hsieh and Ritchken, 2005).

The papers of Duan et al. (1999) and Heston and Nandi (1997, 2000) are the most important basis for the research conducted in this paper. Therefore, the ideas behind their approaches should be reviewed briefly. Duan et al. (1999) calculated the first four moments of the cumulative log-return distribution by evaluating and approximating exceedingly complex moment equations. Gained this moments, the log-return distribution is approximated by a *Gram-Charlier* type A expansion around the normal distribution. They provided an option pricing formula similar to the *Black-Scholes*-formula, including higher order correction terms. The benefit of their method is the instantaneous access to the option price, once the moments have been calculated. The drawback is the complicated moment calculation procedure, involving extensive multiple sums. Additionally, moments of non-integer order occur, which have to be approximated by *Taylor*-expansion.

Heston and Nandi (1997, 2000) designed a particular GARCH model with a preserving log-linear characteristic function and they were able to provide iterative expressions for the terms involved. Remarkably, those recursions were derived for an arbitrary GARCH(p, q) order model. Because the characteristic function is the *Fourier*-transform of the cumulative log-return density, the resulting expression has to be integrated. This has to be done numerically, either by adaptive methods or by quadrature. In most cases these integrals converge rapidly. Nevertheless, numerical integration is inconvenient because it still requires a noticeable amount of computation time.

The method introduced in this paper in a sense merges both ideas. Departing from Heston and Nandi's GARCH$(1, 1)$ model it is shown that the cumulant generating function of the log-return distribution has a preserving linear form. Then some approximations are suggested, resulting in an explicit expression for

the cumulant generating function. By computing derivatives of this expression approximations for the cumulants are obtained, which themselves are incorporated into a *Gram-Charlier* type A series expansion. Finally, the option price is calculated from this series expansion. This method is very fast, roughly 10^{19} options can be valuated per second on an usual personal computer, and, as will become evident, very accurate.

The paper is organized as follows. Section 2 introduces the model and provides the formal derivations of all building blocks involved in computation of the option price. In section 3 the approximations conducted are analyzed in different stress test scenarios with extended Monte-Carlo simulation studies. Section 4 provides an empirical survey of the performance of the pricing formula in different real markets. Section 5 draws conclusions and summarizes the findings.

2 Formal Derivation of the Pricing Formula

In what follows, the generic term GARCH model is used for all extensions of the original model, introduced by Bollerslev (1986).

2.1 The GARCH Model

The model used for pricing European call options is the GARCH(1,1) case of the more general model proposed by Heston and Nandi (1997, 2000)

$$\log\left[\frac{S_t}{S_{t-1}}\right] = r + \lambda h_t + \sqrt{h_t}z_t \tag{1a}$$

$$h_t = \omega + \alpha\left(z_{t-1} - \gamma\sqrt{h_{t-1}}\right)^2 + \beta h_{t-1}, \tag{1b}$$

with $\omega, \alpha, \beta > 0$ and $z_t \sim N(0,1)$. The variance equation (1b) is in fact a nonlinear asymmetric (NAGARCH) configuration (cf. Engle and Ng, 1993). The process remains stationary with finite first and second moment, if $\beta + \alpha\gamma^2 < 1$. Heston and Nandi proved that in the limit $\Delta t \to 0$ this model approaches the continuous time bivariate diffusion model of Heston (1993) with covariance $E[dW_t^1 dW_t^2] = dt$. Further, the variance process converges weakly to the square-root process of Feller (1951), and also Cox, Ingersoll, and Ross (1985). Therefore, it has a strong theoretical rationale. Furthermore, recent research indicates that models beyond GARCH(1,1) with leverage are not superior regarding option valuation (Christoffersen and Jacobs, 2004).

The corresponding model under local risk neutralization reads

$$\log\left[\frac{S_t}{S_{t-1}}\right] = r + \lambda^* h_t + \sqrt{h_t}z_t^* \tag{2a}$$

$$h_t = \omega + \alpha\left(z_{t-1}^* - \gamma^*\sqrt{h_{t-1}}\right)^2 + \beta h_{t-1}, \tag{2b}$$

with

$$\lambda^* = -\frac{1}{2}, \ \gamma^* = \gamma + \lambda + \frac{1}{2} \ \text{ and } \ z_t^* = z_t + \left(\lambda + \frac{1}{2}\right)\sqrt{h_t}.$$

Under the risk neutral measure Q, z_t^* is $N(0,1)$-distributed. For a proof see Heston and Nandi (2000) and also Duan (1995). The next step, according to Heston and Nandi (1997, 2000), is deriving the conditional moment generating function of the logarithm of the spot asset price.

2.2 Moment Generating Function

Let $x_t = \log[S_t]$, then the conditional moment generating function of x_T is

$$M_t(k) = E_t\big[e^{kx_T}\big], \tag{3}$$

where $E_t[\ldots]$ is a short form for the expectation value, conditioned on the information available at time t, $E[\ldots|\mathcal{F}_t]$.

Proposition 2.1
The conditional moment generating function of x_T takes the log-linear form

$$M_t(k) = \exp[kx_t + A_t + h_{t+1}B_t], \tag{4a}$$

with

$$A_t = A_{t+1} + kr + \omega B_{t+1} - \frac{1}{2}\log[1 - 2\alpha B_{t+1}] \tag{4b}$$

$$B_t = k(\lambda + \gamma) - \frac{\gamma^2}{2} + \beta B_{t+1} + \frac{\frac{1}{2}(k-\gamma)^2}{1 - 2\alpha B_{t+1}} \tag{4c}$$

and initial conditions

$$A_{T-1} = kr \ and \ B_{T-1} = \lambda k + \frac{k^2}{2}.$$

A proof of proposition 2.1 is given in appendix B.1. It follows the layout of the more general proof for the GARCH(p,q) model in Heston and Nandi (1997).

2.3 Cumulant Generating Function of Total Log-Return

The total spot asset log-return is $\log[S_T/S_t] = x_T - x_t$. Because the cumulant generating function is the logarithm of the moment generating function, one obtains the conditional cumulant generating function of $x_T - x_t$ by elementary calculus

$$\mathcal{C}_t(k) = A_t + h_{t+1}B_t. \tag{5}$$

Proposition 2.2
Under the risk neutral measure Q, the following approximation for the conditional cumulant generating function of the total log-return $x_T - x_0$, given the information set $\mathcal{F}_0$, holds

$$\mathcal{C}_0(k) = A_0 + h_1 B_0 \tag{6a}$$

with

$$A_0 = Tkr + \frac{\omega + \alpha}{1 - b}(aT - B_0), \ B_0 = a \cdot \frac{1 - b^T}{1 - b} \tag{6b}$$

and

$$a = \frac{1}{2}(k^2 - k), \ b = \beta + \alpha(k - \gamma^*)^2. \tag{6c}$$

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The proof of proposition 2.2 is provided in appendix B.2. In order to calculate the cumulants, one has to compute the derivatives of the cumulant generating function at $k = 0$. The n-th cumulant is given by

$$\kappa_n = \left. \frac{d^n \mathcal{C}_0(k)}{dk^n} \right|_{k=0}. \tag{7}$$

Define the persistence of volatility shocks as $\rho = \beta + \alpha \gamma^{*2}$, then for example the first cumulant is

$$\kappa_1 = \mu = \left. \frac{d\mathcal{C}_0(k)}{dk} \right|_{k=0} = rT - \frac{h_1(1 - \rho^T)}{2(1 - \rho)} + \frac{\left(\frac{1 - \rho^T}{2(1-\rho)} - \frac{T}{2} \right)(\omega + \alpha)}{1 - \rho}.$$

For reference the first four cumulants are given in appendix A.1.

2.4   Series Expansion of the Log-Return Density

The purpose of all efforts conducted so far, was to gain analytical expressions for the cumulants of the total log-return distribution. These expressions may look intimidating but they are quite explicit and can be calculated very fast. Now the log-return probability density can be expressed as series expansion.

It seems natural to expand the unknown density around the normal density because one can expect a stronger influence of the central limit theorem with increasing T. Two possible series expansions are available, the *Gram-Charlier*-series of type A (cf. Cramér, 1957, chap. 17.6) and the *Edgeworth*-expansion (cf. Petrov, 1987). Both expansions are asymptotically equivalent, however, only the *Edgeworth*-expansion is an asymptotic expansion in the proper sense, i.e. the error of the partial sum is controlled by the last included expansion term. For an excellent treatment on this subject see Blinnikov and Moessner (1998) and also the classical references Cramér (1957, chap. 17.6-7) and Feller (1966, chap. 16).

Let $\phi(z)$ denote the standard normal probability density function. Further, define the n-th *Hermite*-polynomial as

$$\mathrm{He}_n(z) = \frac{(-1)^n}{\phi(z)} \frac{d^n \phi(z)}{dz^n}. \tag{8}$$

For given cumulants $\kappa_1 = \mu, \kappa_2 = \sigma^2, \ldots, \kappa_N$, the *Gram-Charlier* A expansion around the normal distribution for the standardized random variable $z = \frac{x - \mu}{\sigma}$ reads

$$p(z) \approx \phi(z) \left(1 + \sum_{n=3}^{N} \frac{\kappa_n}{n! \, \sigma^n} \mathrm{He}_n(z) \right). \tag{9}$$

While the *Gram-Charlier*-series is organized according to the degree of the *Hermite*-polynomials, the *Edgeworth*-expansion groups terms along the powers of σ. An explicit formula for the *Edgeworth*-series is given in Blinnikov and Moessner (1998)

$$p(z) \approx \phi(z) \left(1 + \sum_{n=1}^{N-2} \sigma^n \sum_{\{k_m\}} \mathrm{He}_{n+2r}(z) \prod_{m=1}^{n} \frac{1}{k_m!} \left(\frac{\kappa_{m+2}}{(m+2)! \, \sigma^{2m+2}} \right)^{k_m} \right),$$

$$\tag{10a}$$

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with the set $\{k_m\}$ of all non-negative integer solutions of the *Frobenius-Diophantine*-problem

$$k_1 + 2k_2 + \ldots + Nk_N = N \tag{10b}$$

and $r = k_1 + k_2 + \ldots + k_N$. In the following the option price will be calculated using the *Gram-Charlier* type A expansion up to $N = 4$, to cover the first four moments of the log-return distribution. At the same time the *Hermite*-polynomials involved are only of order four also, protecting against artifacts like excessive non-positivity of the density approximation, in case of divergence of the series. The asymptotic *Edgeworth*-expansion will be used to survey the order of the approximation error.

Proposition 2.3
The fair price of an European style call option with exercise price K and time to maturity T is given by

$$C(S_0, T) = S_0 \Phi(d_1) - e^{-rT} K \Phi(d_2) + S_0 \phi(d_1) e^{\mu + \frac{\sigma^2}{2} - rT} (c_3 + c_4), \tag{11a}$$

with

$$d_1 = d_2 + \sigma \,,\; d_2 = \frac{\log[S_0/K] + \mu}{\sigma} \tag{11b}$$

and

$$c_3 = \frac{\kappa_3}{3!} \cdot \frac{\sigma - d_2}{\sigma^2} \,,\; c_4 = \frac{\kappa_4}{4!} \cdot \frac{d_1^2 - 1 - 3\sigma d_2}{\sigma^3}, \tag{11c}$$

with ϕ and Φ denoting the standard normal probability density and cumulative distribution function, respectively, and the cumulants κ_n according to (7).

The proof is provided in appendix B.3. It is similar to that referenced in Duan et al. (1999) but corrects a calculation error in their proof, which is also inherent in Duan, Gauthier, Sasseville, and Simonato (2004).

3 Monte-Carlo Analysis

The ability to calculate a correct option price is primarily determined by the quality of the approximated log-return distribution. Therefore, this feature is analyzed in a Monte-Carlo simulation study. Two Scenarios are of particular interest. First, a parameter setup is chosen that deliberately generates bad conditions for the proposed approximations. Second, a setup with high volatility persistence is created in order to investigate the series approximation under significant deviation from the normal distribution. An analysis similar to that of Duan et al. (1999) is conducted to survey the effects.

3.1 Alpha Stress Test Scenario

In conducting a stress test for the analytical approximation method, the following parameter values are chosen

$$\omega = 2.5 \times 10^{-4} \,,\; \alpha = 10^{-3} \,,\; \beta = 0.4 \,,\; \gamma^* = 10 \text{ and } r = 2 \times 10^{-4}.$$

This is an extreme setting in two respects. First, the parameter α is much greater than one would expect in practical applications. Because the approximations in proposition 2.2 involve a small constant α, this configuration should create structural stress on the suggested method. Second, the steady state daily volatility under the proposed set of parameters is $\sqrt{h_\infty} = 5\%$. This is also a rather extreme value. In most cases the long term volatility will be considerably smaller.

Figure 1 shows the total log return distribution for different times T. Initial values were chosen to make $h_1 = 2h_\infty$ hold. The histogram density is generated by a Monte-Carlo simulation of 100.000 paths. The solid line is the *Gram-Charlier* A expansion of the unknown density, which virtually coincides with the *Edgeworth*-expansion in this particular case. Therefore, the *Edgeworth*-expansion terms are used to roughly estimate the error of the partial sum. According to (10a) and (10b), the *Edgeworth*-expansion up to terms of order $\mathcal{O}(\sigma^{-4})$ is

$$ p(z) \approx \phi(z) \left(1 + \frac{1}{3!\,\sigma^3} \kappa_3 \mathrm{He}_3(z) + \frac{1}{4!\,\sigma^4} \left(\kappa_4 \mathrm{He}_4(z) + \frac{\kappa_3^2}{3} \mathrm{He}_6(z) \right) \right). \qquad (12) $$

Figure 2 left shows The absolute value of the particular expansion terms for the total log-return at $T = 100$. Notice that not the exact error is estimated, but only its order. Thus, by including forth moment terms the approximation error at $T = 100$ is roughly of order $\mathcal{O}(10^{-2})$. In figure 2 right, the maximum absolute value of the expansion terms is plotted as function of time. It is clearly seen that for small T the error induced by including the forth order term is
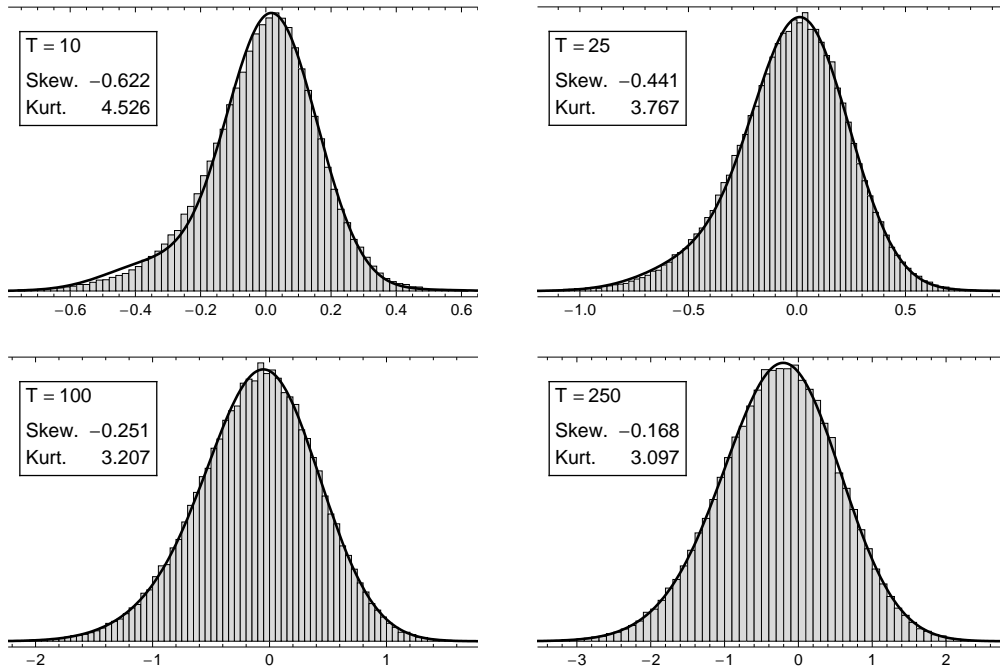

Figure 1: Total Log-Return Distribution – *Gram-Charlier* A Expansion (Solid) and Monte-Carlo Simulation with 100.000 Replications
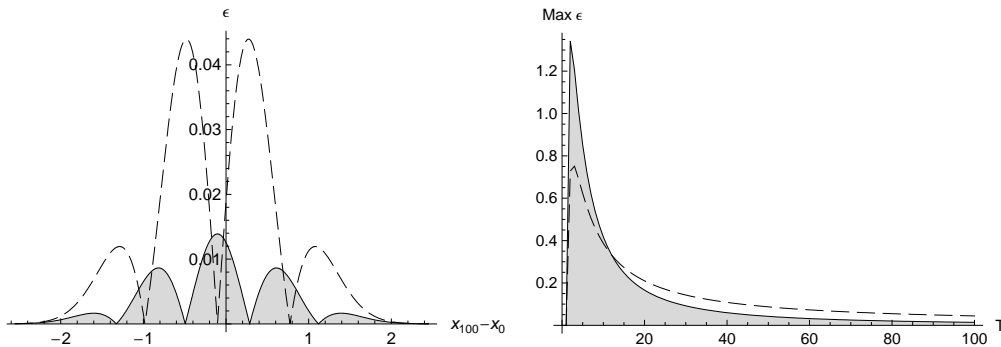
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Figure 2: Errors Induced by Including Edgeworth Expansion Terms – First Term
Dashed, Second Term Solid

greater than that induced by the third order expansion term. This is due to the divergence of the *Edgeworth*-series because of strong deviation from the normal distribution. As T increases, influences of the central limit theorem grow stronger and convergence takes place.

The Edgeworth expansion allows for error control, so that theoretically expansion terms should be included as long as the error is reduced. Practically, it is often sufficient to involve four moments. In this case the *Gram-Charlier* A series is preferable. It performs nearly identical to the corresponding *Edgeworth*-
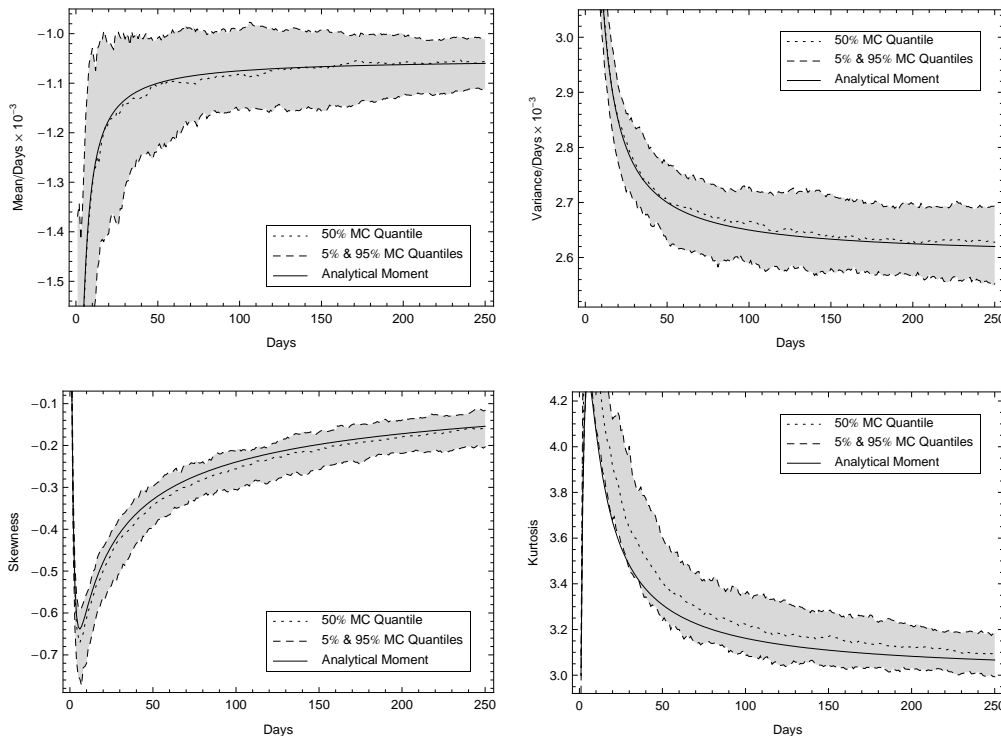

Figure 3: Analytical Moments and High Precision Monte-Carlo Simulation with
1 Million Replications

series, but includes only *Hermite*-polynomials of order four. This is particularly beneficial in case of divergence because artifacts like locally negative density approximations and other numerical problems are mostly avoided.

In order to asses the quality of the moment approximation, a high precision Monte-Carlo study was conducted. One million trajectories were simulated and grouped in 100 equally sized blocks, from which quantiles of the moment distributions were calculated. The results are shown in figure 3. Obviously the moment structure of the total log-return is represented extremely well, even with rather extreme parameter values. This is an encouraging result, indicating a very precise representation of the log-return distribution as already suggested in figure 1. Thus, the analytical pricing formula is not compromised by a large α.

3.2 High Volatility Persistence Scenario

The second test scenario is designed to generate high volatility persistence in the log-return series. The following parameter configuration is used

$$\omega = 2.5 \times 10^{-7},\ \alpha = 10^{-6},\ \beta = 0.5,\ \gamma^* = 700 \text{ and } r = 2 \times 10^{-4}.$$

The volatility persistence resulting from this parameter set is $\rho = 99\%$, which results in an almost integrated GARCH process. Again, the initial values are chosen to meet the requirements of $h_1 = 2h_\infty$. High volatility persistence causes the log-return density to deviate significantly from the normal density. Reverting influences of the central limit theorem become effective very slowly and
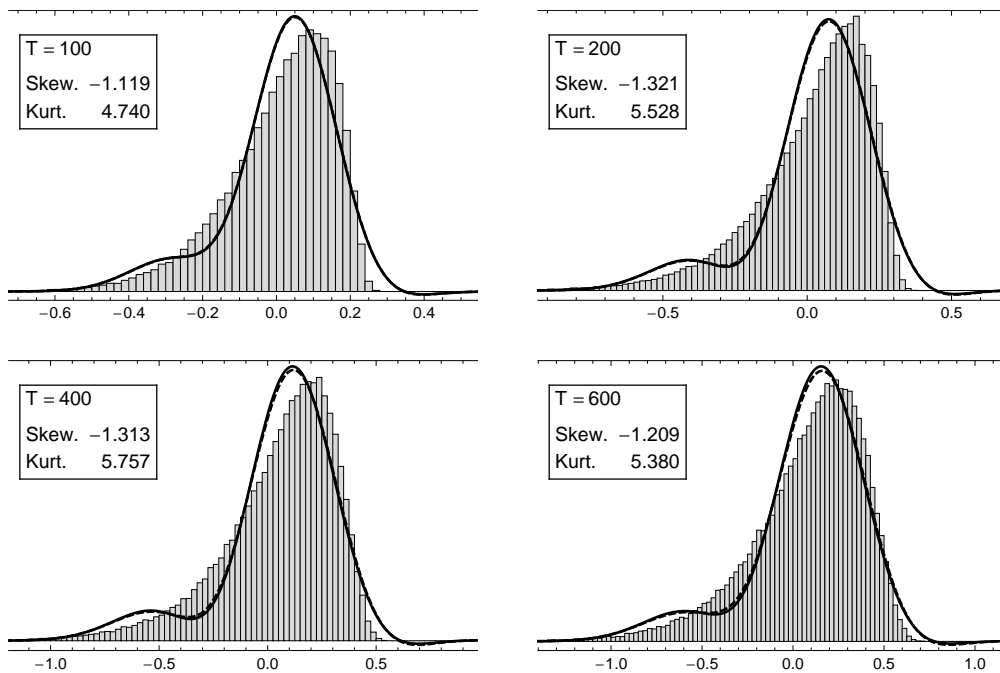

Figure 4: Total Log-Return Distribution – *Gram-Charlier* A Series (Solid) & Edgeworth Expansion (Dashed) and Monte-Carlo Simulation with 100.000 Replications

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delayed. Therefore, the total log return density should exhibit excessive skewness and kurtosis, even for large T. Figure 4 illustrates the effects for different times. The histogram density is generated from a Monte-Carlo simulation with 100.000 paths. The *Gram-Charlier* A series is indicated by a solid line, the *Edgeworth*-expansion dashed. Because the amount of skewness is noticeably larger than zero, the additional *Edgeworth* term takes effect. Nevertheless both expansions still nearly coincide.

The approximation quality is reduced significantly compared against the previous scenario. Artifacts, caused by series divergence, like bimodality and locally negative regions, are now immanent. The latter can be resolved by an appropriate non-negativity condition but caution is recommended, because the moment structure is altered by manipulating the density approximation. Two questions have to be answered. First, should the fourth moment term be abandoned in order to reduce quality degeneration of the series approximation? Second, is the moment structure represented sufficiently to calculate a correct option price?

The answer to the first question is provided by reviewing the *Edgeworth*-expansion terms. Figure 5 provides the required information, organized as in figure 2. Interestingly the maximum error induced by inclusion of the fourth order term is always smaller than that of the third order term. Both terms are roughly of the same order but it can be inferred that at least no harm is done by expanding up to fourth order.

The moment representation itself is again investigated with a high precision Monte-Carlo simulation study. As in the previous scenario, one million trajectories have been generated and grouped in 100 equally sized blocks. Figure 6 shows the results. The first two moments are tracked with remarkable precision. However, the higher moments seem to be slightly underestimated. Notice that the shaded areas are only 90% high probability areas. Further, the density functions of the higher moments seem to be skewed themselves. This is due to the strong deviation of the log-return distribution from normal. Slightly underestimating the higher moments might eventually turn out beneficial because artifacts due to divergence of the series expansion are compensated naturally in
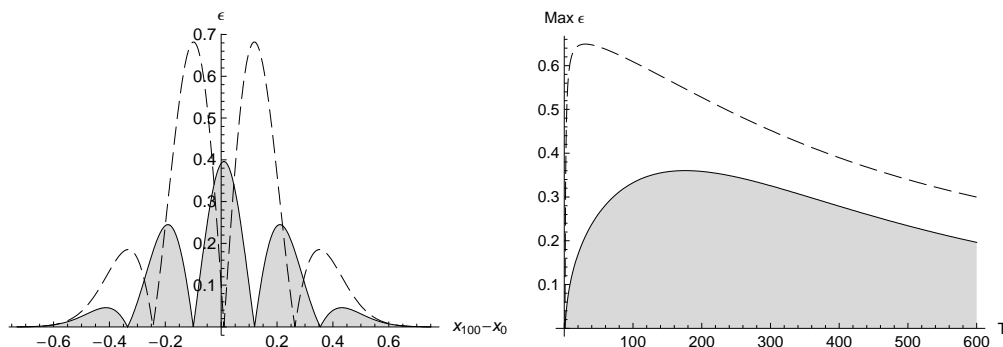

Figure 5: Errors Induced by Including Edgeworth Expansion Terms – First Term Dashed, Second Term Solid
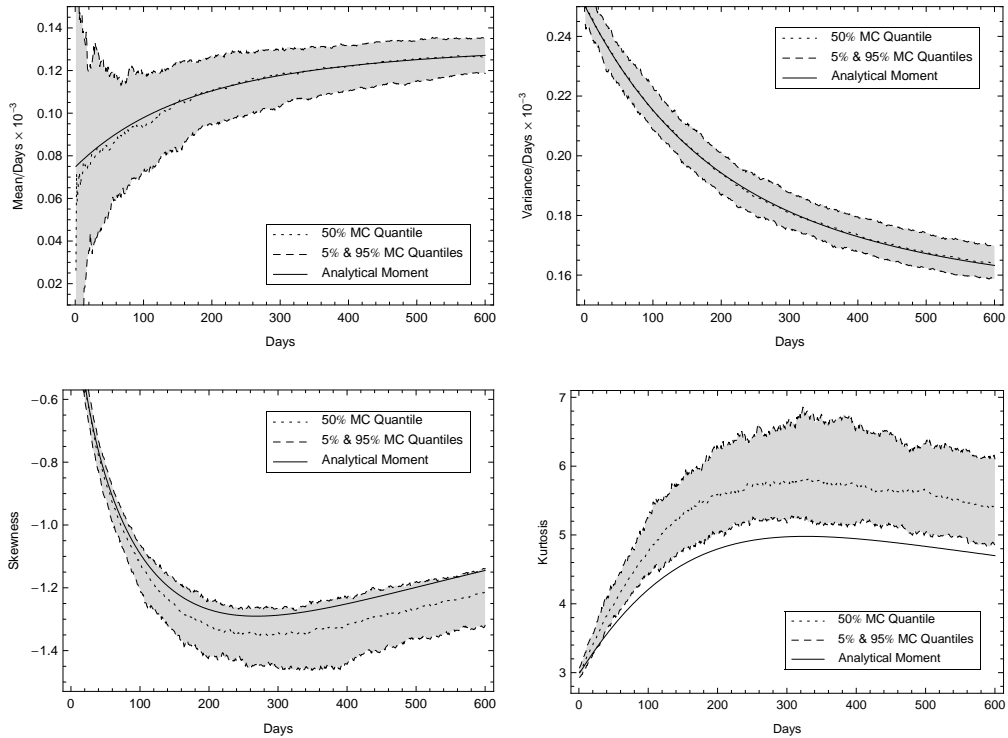
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Figure 6: Analytical Moments and High Precision Monte-Carlo Simulation with 1 Million Replications

that way. Even though the moment representation seems satisfactory, considering all facts, the final question is yet not answered.

In order to asses the quality of the log-return density approximation, and hence the validity of the suggested pricing formula, different options have to be evaluated analytically and by simulation. This was done for four different times to maturity and a quasi continuous spectrum from deep out of the money to deep into the money options with strike price $K = 100$. The comparison criterion is the relative pricing error defined by

$$\text{Rel. Pricing Error} = \frac{C_{\text{Anal.}}}{C_{\text{MC}}} - 1.$$

In order to get illustrative Monte-Carlo confidence bands, five thousand paths were generated for each single configuration of moneyness and time to maturity. Figure 7 shows the results. The relative pricing error by analytical evaluation is indicated by diamonds, connected by a solid line. Sample options with price $C < 0.5$ were abandoned because they generate large relative errors by a small divider. The same precaution was taken in Duan et al. (1999). The 95% Monte-Carlo confidence regions are indicated gray, bordered by circles.

Obviously, the misspricing is not severe. Options at and in the money are valuated quite well, out of the money options seem to be slightly underpriced. Roughly interpolation of the relative pricing error leads to the conclusion that the analytical pricing formula still operates within sufficient limits.
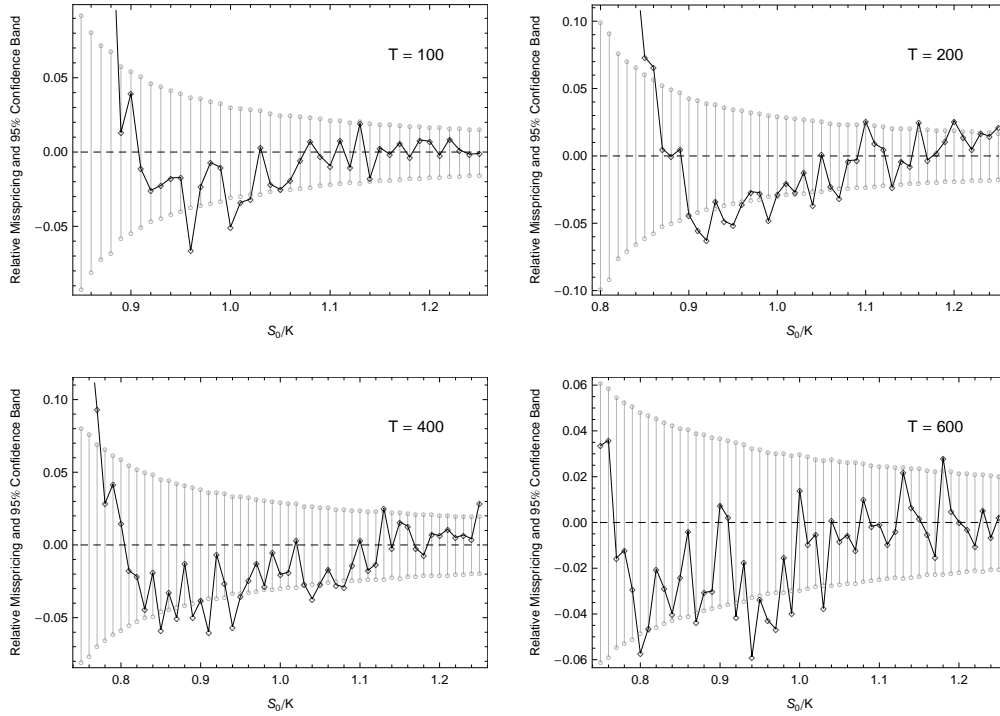
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Figure 7: Relative Pricing Error – Analytical Solution (Black) and 95% Monte-Carlo Confidence Area (Gray)

4 Pricing Options in Real Markets

In order to asses the performance of the new pricing formula in real markets, the GARCH model (1a), (1b) has to be estimated under the physical measure P. This can be accomplished by maximum likelihood. Table 1 shows the estimation results for some selected index markets. Four years of historical data, ranging from Sep. 01, 2004 to Aug. 31, 2008, were included in the estimation procedure, resulting in a total of 1007 observations. The annual risk free rate of return was

Maximum-Likelihood Estimation							
	Parameter Estimates						
Index	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$	Ann. Vol.	Pers.
S&P 500	4.51e-7	1.24e-6	0.73	445.3	0.13	13.82%	97.79%
	(2.17e-7)	(4.16e-7)	(0.07)	(138.2)	(3.64)		
Dow Jones	2.17e-7	1.52e-6	0.80	340.2	-0.19	13.47%	97.60%
	(2.49e-7)	(4.31e-7)	(0.04)	(87.54)	(3.80)		
Hang Seng	9.35e-18	9.87e-6	0.87	81.67	2.28	20.15%	93.92%
	(1.34e-6)	(1.68e-6)	(0.02)	(14.69)	(2.30)		
DAX	4.07e-6	3.11e-6	0.65	296.5	3.82	15.79%	92.80%
	(8.46e-7)	(8.22e-7)	(0.07)	(73.56)	(3.16)		

Table 1: Maximum-Likelihood Parameter Estimation under Physical Measure P

Index Option Valuation with GARCH

Index	Maturity	Moneyness	Gearing	Obs.	BSC	GARCH	MC &	(Std.)
	Option Features			**Option Price**				
S&P 500	15	−3.84%	196.63	0.04	0.022	0.03	0.029	(0.001)
S&P 500	15	0.08%	52.05	0.165	0.131	0.155	0.156	(0.002)
S&P 500	15	7.84%	12.28	0.715	0.706	0.712	0.715	(0.003)
S&P 500	100	−9.72%	80.49	0.105	0.081	0.068	0.064	(0.002)
S&P 500	100	−1.88%	25.30	0.345	0.29	0.303	0.311	(0.004)
S&P 500	100	13.79%	6.66	1.325	1.332	1.365	1.366	(0.008)

Index	Maturity	Moneyness	Gearing	Obs.	BSC	GARCH	MC &	(Std.)
	Option Features			**Option Price**				
Dow Jones	41	−8.54%	227.85	0.03	0.014	0.016	0.014	(0.001)
Dow Jones	41	2.32%	24.17	0.325	0.305	0.341	0.349	(0.003)
Dow Jones	41	15.34%	6.48	1.225	1.256	1.26	1.272	(0.005)
Dow Jones	189	−8.54%	39.86	0.195	0.193	0.184	0.187	(0.004)
Dow Jones	189	0.15%	16.26	0.485	0.482	0.503	0.512	(0.006)
Dow Jones	189	8.83%	8.47	0.935	0.958	1.0	1.005	(0.008)

Index	Maturity	Moneyness	Gearing	Obs.	BSC	GARCH	MC &	(Std.)
	Option Features			**Option Price**				
Hang Seng	106	−6.77%	21.75	0.79	0.627	0.547	0.543	(0.010)
Hang Seng	106	2.94%	11.21	1.58	1.427	1.40	1.401	(0.016)
Hang Seng	106	22.35%	4.24	4.26	4.27	4.295	4.30	(0.023)
Hang Seng	294	−35.88%	52.19	0.325	0.312	0.205	0.202	(0.009)
Hang Seng	294	2.95%	7.74	2.31	2.329	2.263	2.272	(0.029)
Hang Seng	294	27.21%	3.41	5.31	5.561	5.58	5.593	(0.039)

Index	Maturity	Moneyness	Gearing	Obs.	BSC	GARCH	MC &	(Std.)
	Option Features			**Option Price**				
DAX	28	−5.04%	174.97	0.365	0.402	0.249	0.238	(0.006)
DAX	28	0.37%	37.42	1.72	1.646	1.582	1.623	(0.019)
DAX	28	8.87%	10.56	6.12	6.009	6.084	6.12	(0.031)
DAX	91	−12.83%	223.12	0.285	0.447	0.277	0.264	(0.010)
DAX	91	−2.01%	27.3	2.365	2.302	2.217	2.258	(0.032)
DAX	93	10.45%	7.85	8.20	7.773	7.956	7.943	(0.054)

Table 2: Valuation Results for various European Style Call Options

set to $r = 5\%$. Standard errors are indicated in parentheses. The last two columns give the annualized long term volatility associated with the parameter estimates, and the persistence of volatility shocks $\rho = \beta + \alpha\gamma^2$. Obviously, conditions are not as hostile as in the test scenarios of section 3.

An extended empirical analysis of the capabilities of the single lag GARCH model (1a), (1b) regarding option valuation is provided in Heston and Nandi (2000) and Hsieh and Ritchken (2005). The authors proved that it outperforms the *Black-Scholes* model, even if volatility is updated along the specific implied volatility and that a significant portion of the volatility smile can be explained. Hence, the focus here is on demonstrating that option prices can be calculated correctly within the corresponding GARCH model context.

A collection of various index options with different moneyness and times to maturity is presented in table 2. The features columns include time to maturity, moneyness, which is defined as $1 - \text{strike/spot}$, and the gearing, which is the elasticity of the option price regarding changes in the spot asset price. The gearing implicitly contains the ratio of the option. The price columns present the observed option price, given as arithmetic mean of the Bid/Ask-spread, the *Black-Scholes* price with volatility estimated by historical values, the analytical option price, calculated as suggested in this paper and the Monte-Carlo simulated discounted risk-neutral expectation with its standard deviation parenthesized. 10.000 paths were simulated for each MC price in table 2, which results in a good tradeoff between computation time and accuracy.

Attention should not be focused primarily on the misspricing of the GARCH method and the *Black-Scholes* model, respectively, because very few data is reported here. For a detailed empirical analysis see Heston and Nandi (1997, 2000) and Hsieh and Ritchken (2005). At this point it is more important to verify that the analytically computed option price is correct within the suggested GARCH model setup. As seen in section 3, this cannot unconditionally be taken for granted, because approximations and series expansions are involved in the calculation. Therefore, the third from last and the second last column should be compared closely, considering a potential confidence region, indicated by a multiple of the standard deviation in the last column. Obviously, the calculated option prices are very precise. There is no systematic error induced by a particular feature of the option. Hence, deterioration of the log-return density approximation due to divergence of the *Gram-Charlier* A series seems not to be a severe problem in practical applications. Both, S&P 500 and Dow Jones Industrial Average index are high volatility persistent, as reported in table 1. The analytical price falls well into a trusted region around the Monte-Carlo simulated option prices.

5   Conclusions

A new method for valuating European style call options analytically within a GARCH framework was introduced. The procedure avoids excessive moment calculations and numerical integration and is thus very fast. The suggested approximation makes calculation speed independent of the properties of the option, in particular of the time to maturity, which is the major determinant in Monte-Carlo approaches. The computing time, required for pricing one single option, is roughly 10^{-19} seconds on an usual personal computer, which is effectively instantaneous. The consequences are potentially far reaching because one is able to valuate large option portfolios online. Hence, such portfolios become accessible to risk evaluation and scenario analysis inside a closed GARCH model context.

Extensive Monte-Carlo analysis was done to investigate the impact of adverse parameter configurations. Two specific scenarios were analyzed, one designed to compromise the approximations contained in the pricing formula, another to cause deterioration of the series expansion by high volatility persis-

tence. It turned out that the option pricing formula still provides satisfactory results, even under extreme conditions. Finally a collection of index options from different markets were valuated. The parameter estimates revealed that the conditions in real markets are more convenient than in the test scenarios. Not surprisingly, the analytical pricing formula performed well. There is no noticeable systematic misspricing of any kind.

Appendix A

A.1 Cumulants of Total Log-Return

$$\kappa_1 = \mu = \left.\frac{d\mathcal{C}_0(k)}{dk}\right|_{k=0} = rT - \frac{h_1(1-\rho^T)}{2(1-\rho)} + \frac{\left(\frac{1-\rho^T}{2(1-\rho)} - \frac{T}{2}\right)(\omega+\alpha)}{1-\rho}$$

$$\kappa_2 = \sigma^2 = \left.\frac{d^2\mathcal{C}_0(k)}{dk^2}\right|_{k=0} = \frac{h_1(1-\rho^T)}{1-\rho} - \frac{4\alpha\left(\frac{1-\rho^T}{2(1-\rho)} - \frac{T}{2}\right)(\omega+\alpha)\gamma^*}{(1-\rho)^2}$$

$$- h_1\left(\frac{2T\alpha\rho^{T-1}\gamma^*}{1-\rho} - \frac{2\alpha(1-\rho^T)\gamma^*}{(1-\rho)^2}\right)$$

$$+ \frac{(\omega+\alpha)\left(T - \frac{1-\rho^T}{1-\rho} + \frac{2T\alpha\rho^{T-1}\gamma^*}{1-\rho} - \frac{2\alpha(1-\rho^T)\gamma^*}{(1-\rho)^2}\right)}{1-\rho}$$

$$\kappa_3 = \left.\frac{d^3\mathcal{C}_0(k)}{dk^3}\right|_{k=0} = 3h_1\left(\frac{2T\alpha\rho^{T-1}\gamma^*}{1-\rho} - \frac{2\alpha(1-\rho^T)\gamma^*}{(1-\rho)^2}\right)$$

$$- \frac{6\alpha(\omega+\alpha)\gamma^*\left(T - \frac{1-\rho^T}{1-\rho} + \frac{2T\alpha\rho^{T-1}\gamma^*}{1-\rho} - \frac{2\alpha(1-\rho^T)\gamma^*}{(1-\rho)^2}\right)}{(1-\rho)^2} + 3(\omega+\alpha)\left(\frac{1-\rho^T}{2(1-\rho)} - \frac{T}{2}\right)$$

$$\times \left(\frac{2\alpha}{(1-\rho)^2} + \frac{8\alpha^2\gamma^{*2}}{(1-\rho)^3}\right) - \frac{3}{2}h1\left(-\frac{8T\alpha^2\rho^{T-1}\gamma^{*2}}{(1-\rho)^2} + (1-\rho^T)\left(\frac{2\alpha}{(1-\rho)^2} + \frac{8\alpha^2\gamma^{*2}}{(1-\rho)^3}\right)\right)$$

$$- \frac{2T\alpha\rho^{T-1} + 4(T-1)T\alpha^2\rho^{T-2}\gamma^{*2}}{1-\rho}\Bigg) + \frac{\omega+\alpha}{1-\rho}\left(3\left(-\frac{2T\alpha\rho^{T-1}\gamma^*}{1-\rho} + \frac{2\alpha(1-\rho^T)\gamma^*}{(1-\rho)^2}\right)\right.$$

$$+ \frac{3}{2}\left(-\frac{8T\alpha^2\rho^{T-1}\gamma^{*2}}{(1-\rho)^2} + (1-\rho^T)\left(\frac{2\alpha}{(1-\rho)^2} + \frac{8\alpha^2\gamma^{*2}}{(1-\rho)^3}\right) - \frac{2T\alpha\rho^{T-1} + 4(T-1)T\alpha^2\rho^{T-2}\gamma^{*2}}{1-\rho}\right)\Bigg)$$

$$\kappa_4 = \left.\frac{d^4\mathcal{C}_0(k)}{dk^4}\right|_{k=0} = 6(\omega+\alpha)\left(T - \frac{1-\rho^T}{1-\rho} + \frac{2T\alpha\rho^{T-1}\gamma^*}{1-\rho} - \frac{2\alpha(1-\rho^T)\gamma^*}{(1-\rho)^2}\right)\left(\frac{2\alpha}{(1-\rho)^2} + \frac{8\alpha^2\gamma^{*2}}{(1-\rho)^3}\right)$$

$$+ 4(\omega+\alpha)\left(\frac{1-\rho^T}{2(1-\rho)} - \frac{T}{2}\right)\left(-\frac{24\alpha^2\gamma^*}{(1-\rho)^3} - \frac{48\alpha^3\gamma^{*3}}{(1-\rho)^4}\right) + 6h_1\left(-\frac{8T\alpha^2\rho^{T-1}\gamma^{*2}}{(1-\rho)^2} + (1-\rho^T)\right.$$

$$\times \left(\frac{2\alpha}{(1-\rho)^2} + \frac{8\alpha^2\gamma^{*2}}{(1-\rho)^3}\right) - \frac{2T\alpha\rho^{T-1} + 4(T-1)T\alpha^2\rho^{T-2}\gamma^{*2}}{1-\rho}\Bigg) - 2h_1\left(6T\alpha\rho^{T-1}\gamma^*\left(\frac{2\alpha}{(1-\rho)^2}\right.\right.$$

$$\left.+ \frac{8\alpha^2\gamma^{*2}}{(1-\rho)^3}\right) + \frac{6\alpha\gamma^*\left(2T\alpha\rho^{T-1} + 4(T-1)T\alpha^2\rho^{T-2}\gamma^{*2}\right)}{(1-\rho)^2} + (1-\rho^T)\left(-\frac{24\alpha^2\gamma^*}{(1-\rho)^3} - \frac{48\alpha^3\gamma^{*3}}{(1-\rho)^4}\right)$$

$$+ \frac{12(T-1)T\alpha^2\rho^{T-2}\gamma^* + 8(T-2)(T-1)T\alpha^3\rho^{T-3}\gamma^{*3}}{1-\rho}\Bigg) - \frac{8\alpha(\omega+\alpha)\gamma^*}{(1-\rho)^2}\left(3\left(-\frac{2T\alpha\rho^{T-1}\gamma^*}{1-\rho}\right.\right.$$

$$\left.+ \frac{2\alpha(1-\rho^T)\gamma^*}{(1-\rho)^2}\right) + \frac{3}{2}\left(-\frac{8T\alpha^2\rho^{T-1}\gamma^{*2}}{(1-\rho)^2} + (1-\rho^T)\left(\frac{2\alpha}{(1-\rho)^2} + \frac{8\alpha^2\gamma^{*2}}{(1-\rho)^3}\right)\right.$$

$$\left.\left.- \frac{2T\alpha\rho^{T-1} + 4(T-1)T\alpha^2\rho^{T-2}\gamma^{*2}}{1-\rho}\right)\right) + \frac{\omega+\alpha}{1-\rho}\left(6\left(\frac{8T\alpha^2\rho^{T-1}\gamma^{*2}}{(1-\rho)^2} - (1-\rho^T)\right.\right.$$

$$\times \left(\frac{2\alpha}{(1-\rho)^2} + \frac{8\alpha^2\gamma^{*2}}{(1-\rho)^3}\right) + \frac{2T\alpha\rho^{T-1} + 4(T-1)T\alpha^2\rho^{T-2}\gamma^{*2}}{1-\rho}\Bigg) + 2\left(6T\alpha\rho^{T-1}\gamma^*\left(\frac{2\alpha}{(1-\rho)^2}\right.\right.$$

$$\left.+ \frac{8\alpha^2\gamma^{*2}}{(1-\rho)^3}\right) + \frac{6\alpha\gamma^*\left(2T\alpha\rho^{T-1} + 4(T-1)T\alpha^2\rho^{T-2}\gamma^{*2}\right)}{(1-\rho)^2} + (1-\rho^T)\left(-\frac{24\alpha^2\gamma^*}{(1-\rho)^3} - \frac{48\alpha^3\gamma^{*3}}{(1-\rho)^4}\right)$$

$$\left.\left.+ \frac{12(T-1)T\alpha^2\rho^{T-2}\gamma^* + 8(T-2)(T-1)T\alpha^3\rho^{T-3}\gamma^{*3}}{1-\rho}\right)\right)$$

Appendix B

B.1 Proof of Proposition 2.1

By iterating expectations the following relation holds

$$M_t(k) = E_t\big[M_{t+1}(k)\big] = E_t\big[\exp[kx_{t+1} + A_{t+1} + h_{t+2}B_{t+1}]\big].$$

Inserting model equations (1a) and (1b) shows

$$M_t(k) = E_t\Big[\exp\big[k(x_t + r + \lambda h_{t+1} + \sqrt{h_{t+1}}z_{t+1}) + A_{t+1} + \omega B_{t+1}$$
$$+ B_{t+1}\big(\alpha(z_{t+1} - \gamma\sqrt{h_{t+1}})^2 + \beta h_{t+1})\big]\Big].$$

After suitable rearrangement of terms, one obtains

$$M_t(k) = E_t\Big[\exp\Big[k(x_t + r) + A_{t+1} + \omega B_{t+1}$$
$$+ \alpha B_{t+1}\Big(z_{t+1} - \Big(\gamma - \frac{k}{2\alpha B_{t+1}}\Big)\sqrt{h_{t+1}}\Big)^2$$
$$+ \Big(k(\lambda + \gamma) + \beta B_{t+1} - \frac{k^2}{4\alpha B_{t+1}}\Big)h_{t+1}\Big]\Big].$$

The random variable z_{t+1} does not depend on the information set $\mathcal{F}_t$ and therefore, the expectation becomes unconditional. Using

$$E\big[e^{a(z+b)^2}\big] = e^{-\frac{1}{2}\log[1-2a] + \frac{ab^2}{1-2a}}$$

for arbitrary a and b, and equating terms in the L.H.S. and R.H.S results in the recursion formulae (4b) and (4c).

For $t = T - 1$, h_T is known and x_T is normally distributed. Therefore

$$M_{T-1}(k) = \exp\Big[kx_{T-1} + k(r + \lambda h_T) + \frac{k^2}{2}h_T\Big]$$

holds. Identifying terms shows

$$A_{T-1} = kr \text{ and } B_{T-1} = \lambda k + \frac{k^2}{2},$$

which concludes the proof. $\square$

B.2 Proof of Proposition 2.2

Under the risk neutral measure Q the parameter substitutions $\lambda^* = -\frac{1}{2}$ and $\gamma^* = \gamma + \lambda + \frac{1}{2}$ take place. Therefore, B_t changes to

$$B_t = \frac{1}{2}(k^2 - k) - \frac{1}{2}(k - \gamma^*)^2 + \beta B_{t+1} + \frac{\frac{1}{2}(k - \gamma^*)^2}{1 - 2\alpha B_{t+1}}$$
$$= \frac{1}{2}(k^2 - k) + \beta B_{t+1} + \frac{\alpha(k - \gamma^*)^2 B_{t+1}}{1 - 2\alpha B_{t+1}}$$
$$= B_{T-1} + \beta B_{t+1} + \frac{\alpha(k - \gamma^*)^2}{\frac{1}{B_{t+1}} - 2\alpha}.$$

The first step in obtaining an explicit formula is to show that $2\alpha B_{t+1}$ remains reasonably small in a vicinity of $k = 0$ for all t. Because 2α is constant, the order of B_{t+1} has to be analyzed. This is no trivial task, because B is a function of k.

In order to calculate cumulants one has to compute derivatives of the cumulant generating function at $k = 0$. Thus, the behavior of B in an ε-vicinity of $k = 0$ is of interest. Define the left limit vicinity $[0, \varepsilon]$ of $k = 0$, with $0 < \varepsilon \ll 1$. Then obviously, $B_{T-1} < 0$ holds and B_t is a monotonically increasing function of t inside this ε-vicinity. It follows

$$B_t = B_{T-1} + \beta B_{t+1} + \frac{\alpha(\varepsilon - \gamma^*)^2}{\frac{1}{B_{t+1}} - 2\alpha}$$

$$> B_{T-1} + \beta B_{t+1} + \frac{\alpha(\varepsilon - \gamma^*)^2}{\frac{1}{B_{t+1}}},$$

and thus

$$|B_t| < \left| B_{T-1} + \left(\beta + \alpha(\varepsilon - \gamma^*)^2\right) B_{t+1} \right|.$$

This recursion can be iterated in order to obtain an explicit expression. One obtains

$$|B_t| < |B_{T-1}| \sum_{j=0}^{T-t-2} \left(\beta + \alpha(\varepsilon - \gamma^*)^2\right)^j + |B_{T-1}| \left(\beta + \alpha(\varepsilon - \gamma^*)^2\right)^{T-t-1}$$

$$= |B_{T-1}| \sum_{j=0}^{T-1-t} \left(\beta + \alpha(\varepsilon - \gamma^*)^2\right)^j.$$

If the stationarity condition $\beta + \alpha\gamma^* < 1$ under the risk-neutral measure Q holds, it follows immediately that $\beta + \alpha(\varepsilon - \gamma^*)^2 < 1$ for all positive $\gamma^* \geq \frac{\varepsilon}{2}$. These are very mild conditions, because the leverage parameter can be expected positive and ε may be chosen arbitrarily small.

Because B_t is a monotonically increasing, non-positive function of t inside the ε-vicinity, $|B_t|$ is monotonically decreasing. Hence, $|B_t|$ approaches its maximum in the limit $t \to -\infty$. Using the stationarity condition and the restriction for γ^*, one obtains

$$\sup|B| = \lim_{t \to -\infty} |B_t| < \frac{|B_{T-1}|}{1 - \beta - \alpha(\varepsilon - \gamma^*)^2} = \frac{1}{2} \cdot \frac{\varepsilon - \varepsilon^2}{1 - \beta - \alpha(\varepsilon - \gamma^*)^2}$$

$$< \frac{1}{2} \cdot \frac{\varepsilon}{1 - \beta - \alpha\gamma^{*2}} = c \cdot \varepsilon,$$

with constant c. This result indicates that B_t is of order $\mathcal{O}(\varepsilon)$ for all t. Thus, two conclusions can be drawn. First, $2\alpha B_{t+1}$ is of order $\mathcal{O}(\varepsilon)$ as well inside the ε-vicinity of $k = 0$. Second, in calculating derivatives, the limit $\varepsilon \to 0$ has to be computed. In that limit terms of $\mathcal{O}(\varepsilon^2)$ vanish. Therefore, terms of $\mathcal{O}(\alpha^2 B_{t+1}^2)$ may be neglected in the following series expansions.

The second half of the proof starts by expanding the denominator in the R.H.S. of the initial equation for B_t into a geometric series. By only including terms of order $\mathcal{O}(\alpha B_{t+1})$, one obtains

$$\begin{aligned} B_t &= \frac{1}{2}(k^2 - k) - \frac{1}{2}(k - \gamma^*)^2 + \beta B_{t+1} + \frac{1}{2}(k - \gamma^*)^2(1 + 2\alpha B_{t+1}) \\ &= \frac{1}{2}(k^2 - k) + \big(\beta + \alpha(k - \gamma^*)^2\big)B_{t+1} \\ &= a + b \cdot B_{t+1}, \end{aligned}$$

with

$$a = \frac{1}{2}(k^2 - k) \quad \text{and} \quad b = \beta + \alpha(k - \gamma^*)^2.$$

Using the boundary condition at $T-1$ one can calculate B_0 in a successive way

$$\begin{aligned} B_0 &= a \sum_{t=0}^{T-2} b^t + b^{T-1} \cdot \frac{1}{2}(k^2 - k) = a \sum_{t=0}^{T-2} b^t + b^{T-1} \cdot a = a \sum_{t=0}^{T-1} b^t \\ &= a \cdot \frac{1 - b^T}{1 - b}. \end{aligned}$$

Now turning to equation (4b), which is identical under both measures, the physical one and the risk-neutral. The logarithmic term in the R.H.S. can be expanded into a *MacLaurin*-series. Neglecting terms of $\mathcal{O}(\alpha^2 B_{t+1}^2)$ yields

$$\begin{aligned} A_0 &= (T - 1)kr + kr + (\omega + \alpha) \sum_{t=1}^{T-1} B_t = Tkr + \frac{\omega + \alpha}{1 - b} \cdot \sum_{t=1}^{T-1} a\big(1 - b^{T-t}\big) \\ &= Tkr + \frac{\omega + \alpha}{1 - b}\Big(a(T - 1) - a \sum_{t=1}^{T-1} b^t\Big) = Tkr + \frac{\omega + \alpha}{1 - b}\Big(aT - a \sum_{t=0}^{T-1} b^t\Big) \\ &= Tkr + \frac{\omega + \alpha}{1 - b}\big(aT - B_0\big), \end{aligned}$$

which proves proposition 2.2. $\quad\square$

B.3 Proof of Proposition 2.3

Let z_T be the standardized total log-return under the risk neutral measure Q. Following the idea of Jarrow and Rudd (1982), the probability density function of $-z_T$ can be expanded around the standard normal density by

$$p(z) \approx \phi(z)\Big(1 - \frac{\kappa_3}{3!\,\sigma^3}\mathrm{He}_3(z) + \frac{\kappa_4}{4!\,\sigma^4}\mathrm{He}_4(z)\Big).$$

Because of $S_T = S_0 e^{\mu + \sigma z_T}$, the following equivalence holds

$$S_T \geq K \quad \Leftrightarrow \quad -z_T \leq \frac{\log[S_0/K] + \mu}{\sigma} = d_2$$

and the fair price of the call option is

$$e^{-rT}E_0^Q\big[\max[S_T - K, 0]\big] = e^{-rT}E_0^Q\left[\max\big[S_0 e^{\mu - \sigma(-z_T)} - K, 0\big]\right]$$

$$= e^{-rT}\int_{-\infty}^{d_2}\big(S_0 e^{\mu - \sigma z} - K\big)p(z)dz$$

$$\approx e^{-rT}\int_{-\infty}^{d_2}\big(S_0 e^{\mu - \sigma z} - K\big)\phi(z)dz$$

$$- \frac{\kappa_3}{3!\,\sigma^3}e^{-rT}\int_{-\infty}^{d_2}\big(S_0 e^{\mu - \sigma z} - K\big)(z^3 - 3z)\phi(z)dz$$

$$+ \frac{\kappa_4}{4!\,\sigma^4}e^{-rT}\int_{-\infty}^{d_2}\big(S_0 e^{\mu - \sigma z} - K\big)(z^4 - 6z^2 + 3)\phi(z)dz.$$

In order to evaluate these integrals, two important relations are established in advance. First, from

$$e^{-rT}E_0^Q[S_T] = e^{-rT}\int_{-\infty}^{\infty} S_0 e^{\mu - \sigma z}\left(1 - \frac{\kappa_3}{3!\,\sigma^3}\mathrm{He}_3(z) + \frac{\kappa_4}{4!\,\sigma^4}\mathrm{He}_4(z)\right)\phi(z)dz = S_0$$

it follows that

$$e^{\mu + \frac{\sigma^2}{2}}\left(1 + \frac{\kappa_3}{3!} + \frac{\kappa_4}{4!}\right) = e^{rT}.$$

Second, with the definition of the *Hermite*-polynomials and $d_1 = d_2 + \sigma$, one obtains

$$\int_{-\infty}^{d_2} K\mathrm{He}_n(z)\phi(z)dz = -K\mathrm{He}_{n-1}(d_2)\phi(d_2)$$

$$= -\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d_1 - \sigma)^2}K\mathrm{He}_{n-1}(d_2)$$

$$= -\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2 + d_1\sigma - \frac{1}{2}\sigma^2}K\mathrm{He}_{n-1}(d_2)$$

$$= -S_0 e^{\mu + \frac{\sigma^2}{2}}\mathrm{He}_{n-1}(d_2)\phi(d_1).$$

Now, using relation two, computation of the integrals yields

$$e^{-rT}\int_{-\infty}^{d_2}(\ldots)\phi(z)dz = e^{-rT + \mu + \frac{\sigma^2}{2}}S_0\Phi(d_1) - e^{-rT}K\Phi(d_2)$$

$$e^{-rT}\int_{-\infty}^{d_2}(\ldots)(z^3 - 3z)\phi(z)dz = -e^{-rT + \mu + \frac{\sigma^2}{2}}S_0\sigma\big((2\sigma - d_1)\phi(d_1) + \sigma^2\Phi(d_1)\big)$$

$$e^{-rT}\int_{-\infty}^{d_2}(\ldots)(z^4 - 6z^2 + 3)\phi(z)dz = e^{-rT + \mu + \frac{\sigma^2}{2}}S_0\sigma\big((d_1^2 - 1 - 3\sigma d_2)\phi(d_1) + \sigma^3\Phi(d_1)\big).$$

Collecting terms and considering relation one completes the proof of proposition 2.3. $\square$

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