The Dickson conditions for conic algebras

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1. Conic algebras. In this short paper, we let k be an arbitrary commutative ring. A (non-associative) algebra C over k is said to be *conic* if it contains a unit element and the following conditions are fulfilled.

(i) C is projective as a k-module.

(ii) 1_C is a unimodular element of C.

(iii) There exists a quadratic form $n\colon C \to k$ such that

$$x^2 - t(x)x + n(x)1_C = 0 \qquad (t := \partial n(1_C, -))$$

for all $x \in A$.

In (i) we *do not* not require that C be finitely generated as a k-module. Condition (ii) means that there is a linear form λ on C satisfying $\lambda(1_C) = 1$, equivalently, that $k1_C \subseteq C$ is a free submodule of rank 1 and a direct summand at the same time. In (iii), we denote by

$$\partial n\colon C \times C \longrightarrow k, \quad (x, y) \longmapsto \partial n(x, y) := n(x + y) - n(x) - n(y)$$

the bilinear form associated with n. We mostly write $n(x, y) := \partial n(x, y)$ if there is no danger of confusion. By [6, Prop. 2.1.3], conditions (i)−(iii) above determine the quadratic form n uniquely; it is called the *norm* of C and written as n_C, while $t_C := \partial n_C(1_C, -)$ is called the *trace* of C. Aside from our hypotheses on the module structure, conic algebras are the same as *generic degree* 2 *algebras* in the sense of McCrimmon [4]. They also agree with Loos's notion [3] of conic algebras, except that he confines himself to algebras that are *finitely generated* projective as modules.

2. Dickson's theorem. It has been shown by Dickson [2] that a unital algebra A over a field F of characteristic not 2 is conic if and only if it satisfies what we tentatively call the *Dickson condition*: for all $x \in A$, the elements $1_A, x, x^2$ are linearly dependent over F, equivalently, x^2 is a linear combination of 1_A and x. We wish to extend this theorem not only to fields of arbitrary characteristic but, in fact, to arbitrary commutative rings in place of F.

To this end, we return to our base ring k and recall that conic algebras are stable under base change: if C is a conic k-algebra and $R \in k$-**alg**, then C_R is a conic R-algebra, with norm (resp. trace) obtained from the norm (resp. trace) of C by extending scalars from k to R. It therefore follows from Dickson's theorem that the Dickson condition is invariant under base field extensions provided F has characteristic not 2. In fact, as we shall see in Remark 4 below, this holds true for all fields F except $F = \mathbb{F}_2$. We may therefore conclude that the only strategy to salvage Dickson's theorem in all characteristics consists in replacing the ordinary Dickson condition by the *strict* Dickson condition, i.e., by the validity of the ordinary one in every base field extension. As will be seen in due course, this strategy turns out to be successful.

Before stating our main result, we recall a notation introduced by Loos, e.g. in [3]. Let C be a unital k-algebra. Then we put $C^{\cdot} := C/k1_C$ as k-modules and write $x \mapsto x^{\cdot}$ for the natural map from C to $C^{\cdot}$. Note that C is (finitely generated) projective as a k-module if and only if $C^{\cdot}$ is, and if $C^{\cdot}$ is free, so is C.

3. Theorem. *Let C be a unital algebra over k containing 1_C as a unimodular vector and making $C^{\cdot}$ either free or finitely generated projective as a k-module. Then C is a conic k-algebra if and only if it satisfies the strict Dickson condition: for all $x \in C_R$, $R \in k$-**alg**, the element x^2 is an R-linear combination of 1_{C_R} and x in C_R.*

Proof. By hypothesis, the vector $1_C \in C$ is unimodular, so we find a submodule $M \subseteq C$ such that

$$C = k1_C \oplus M \tag{1}$$

as a direct sum of k-modules. Since the assignment $x \mapsto x^2$ defines a (homogeneous) quadratic map $M \to C$ over k, projection to the direct summands $k1_C \cong k$ and M of the decomposition (1) gives quadratic maps $n\colon M \to k$ and $s\colon M \to M$ such that

$$x^2 = s(x) - n(x)1_C \qquad (x \in M). \tag{2}$$

In particular, n is a quadratic form over k that will eventually become the norm (restricted to M) of the prospective conic k-algebra C. On the other hand, by the Dickson condition, we also have $s(x) \in kx$ for $x \in C$, and we would like to think of $s(x)$ as $t(x)x$ with $t(x) \in k$ becoming the trace of x in the prospective conic k-algebra C. But since the annihilator of x (in k), i.e.,

$$\mathrm{Ann}(x) := \{\alpha \in k \mid \alpha x = 0\},$$

may not be zero, we don't even know at this stage how to make $t(x)$ a quantity that is well defined, let alone a linear form. For this reason, we bring the hypotheses on the module structure of $C^{\cdot}$ and the *strict* Dickson condition into play.

Let us first reduce to the case that $C^{\cdot}$ is a free k-module. If not, C is finitely generated projective by hypothesis, so there exists a finite family of elements $f \in k$ that generate the unit ideal in k and make C_f a free k_f-module of finite rank, for each f. Assuming the free case has been settled, it follows that all C_f are conic, and uniqueness of the norm in conic algebras ensures that the norms n_{C_f} glue to give a quadratic form $n\colon C \to k$. It is then clear that C is a conic algebra with norm n.

For the rest of the proof, we may therefore assume that $M \cong C^{\cdot}$ is a free k-module, with basis $(e_i)_{i\in I}$. We let $\mathbf{T} = (\mathbf{t}_i)_{i\in I}$ be a family of independent variables and write $k[\mathbf{T}]$ for the corresponding polynomial ring. For a *non-empty finite* subset $E \subseteq I$, we consider the submodule

$$M^E := \sum_{i\in E} ke_i \subseteq M, \tag{3}$$

which is a direct summand, and the element

$$\mathbf{x}^E := \sum_{i\in E} e_i \otimes \mathbf{t}_i \in M^E_{k[\mathbf{T}]} \subseteq M_{k[\mathbf{T}]}. \tag{4}$$

We claim the annihilator of $\mathbf{x}^E$ in $k[\mathbf{T}]$ is zero; indeed, for $f(\mathbf{T}) \in k[\mathbf{T}]$ the relation $f(\mathbf{T})\mathbf{x}^E = 0$ implies $\sum e_i \otimes (\mathbf{t}_i f(\mathbf{T})) = 0$, hence $\mathbf{t}_i f(\mathbf{T}) = 0$ for all $i \in E$ and then $f(\mathbf{T}) = 0$ since $\mathbf{t}_i$ is not a zero divisor in $k[\mathbf{T}]$. Invoking the strict Dickson condition, we therefore find a unique polynomial $g(\mathbf{T}) \in k[\mathbf{T}]$ such that

$$s(\mathbf{x}^E) = g(\mathbf{T})\mathbf{x}^E.$$

Specializing this relation with an additional variable $\mathbf{u}$ to $\mathbf{uT}$, we obtain $g(\mathbf{uT})\mathbf{ux}^E = s(\mathbf{ux}^E) = \mathbf{u}^2 s(\mathbf{x}^E) = (\mathbf{u}g(\mathbf{T}))\mathbf{ux}^E$, so the polynomial $g(\mathbf{T}) \in k[\mathbf{T}]$ is homogeneous of degree 1. Thus there exists a unique linear form $t^E\colon M^E \to k$ satisfying the relation $s(x) = t^E(x)x$ for all $x \in M^E$ because any such t^E will be converted into the linear homogeneous polynomial $g(\mathbf{T})$ after extending scalars from k to $k[\mathbf{T}]$. Here the uniqueness condition implies that the linear forms t^E as E varies over the non-empty finite subsets of I glue to give a linear form $t\colon M \to k$ such that $s(x) = t(x)x$ for all $x \in M$. Combining with (2), we obtain the relation

$$x^2 - t(x)x + n(x)1_C = 0 \tag{5}$$

for all $x \in M$. We now extend t, n as given on M to all of C by

$$t(\alpha 1_C + x) = 2\alpha + t(x), \quad n(\alpha 1_C + x) = \alpha^2 + \alpha t(x) + n(x) \qquad (\alpha \in k,\ x \in M)$$

and then conclude from a straightforward computation that (5) holds for all $x \in C$. Hence C is a conic algebra. □

4. Remark. In the language of the present note, McCrimmon [4, Part C, 6.1.3] has proved the following. *A unital algebra C over a field k satisfying the Dickson condition is conic if either* (a) k *contains more than two elements, or* (b) $k = \mathbb{F}_2$ *and A has no zero divisors.*

We wish to understand how this result fits into our own approach to the subject, particularly into Thm. 3. To this end, we consider the cubic polynomial law

$$g\colon A \longrightarrow \bigwedge^3 A$$

defined by

$$g_R(x) := 1_A \wedge x \wedge x^2 \in \bigwedge^3 (A_R) = (\bigwedge^3 A)_R \qquad (x \in A_R,\ R \in k\text{-}\mathbf{alg})$$

and note that C satisfies the Dickson condition if and only if $g_k\colon A \to \bigwedge^3 A$ is zero as a set map, while it satisfies the strict Dickson condition if and only if $g\colon A \to \bigwedge^3 A$ is zero as a (cubic) polynomial law. Combining this observation with [5, Thm. 7] (or arguing directly), we conclude from Thm. 3 that for C to satisfy the Dickson condition without satisfying the strict Dickson condition, i.e., without being a conic algebra, it is necessary that $k = \mathbb{F}_2$. This settles clause (a) of McCrimmon's result. Clause (b), however, is a different matter, where McCrimmmon's proof actually shows that a unital algebra without zero divisors over $\mathbb{F}_2$ satisfying the Dickson condition has dimension at most 2. Here is a variant of this result in a slightly modified setting.

5. Proposition. *Let k be a perfect field of characteristic 2 and suppose C is a conic k-algebra without nilpotent elements other than zero. Then C has dimension at most 2.*

Proof. It suffices to show $\mathrm{Ker}(t_C) = k1_C$. Since $t_C(1_C) = 2 = 0$, we clearly have $k1_C \subseteq \mathrm{Ker}(t_C)$. To prove equality, let $x \in C \setminus k1_C$. Then $k[x]$ is a unital 2-dimensioanl k-algebra, so following Bourbaki [1, III §2 Prop. 2] it is either étale, or an inseparable (quadratic) field extension, or the algebra of dual numbers. The second possibility is ruled out by the property of k being perfect, the third by the absence of nilpotent elements in C. Thus $k[x]$ is a quadratic étale k-algebra, whose trace, which is just the trace of C restricted to $k[x]$, therefore has kernel $k1_C$. This and $x \in k[x] \setminus k1_C$ implies $t_C(x) \neq 0$. □

It would be nice to know whether the preceding proposition generalizes to algebras satisfying the Dickson condition. In any event, the question remains whether the Dickson condition and the strict Dickson condition over fields are equivalent. The answer is no. Here is an example.

6. Example. By what we have seen in Remark 4, the choice $k = \mathbb{F}_2$ is forced upon us. The algebra C we are going to construct lives on the vector space direct sum $k1_C \oplus M$, where M is a two-dimensional k-vector space, with basis e_1, e_2. We make C into a unital k-algebra by stipulating that 1_C be its identity element and $e_1^2 = e_2^2 = e_2e_1 = 0$, $e_1e_2 = e_1 + e_2$. Since C has dimension 3 the cubic polynomial law g of Remark 4 is actually a cubic form $g\colon C \to k$. For $r_0, r_1, r_2 \in R$, $R \in k\text{-}\mathbf{alg}$ and $x = r_1e_1 + r_2e_2 \in C_R$, the computation

$$\begin{aligned} 1_C \wedge (r_01_C + x) \wedge (r_01_C + x)^2 = 1_C \wedge x \wedge x^2 &= 1_C \wedge (r_1e_1 + r_2e_2) \wedge \big((r_1r_2)(e_1 + e_2)\big) \\ &= (r_1^2r_2 - r_1r_2^2)(1_C \wedge e_1 \wedge e_2) \end{aligned}$$

shows

$$g_R(r_01_C + x) = r_1r_2(r_1 + r_2) = x^tS_Rx^2 \qquad (S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}),$$

so by [5, Thm. 7] (or by direct inspection) the set map $g_k\colon C \to k$ is zero but g itself is not. This means that C satisfies the Dickson condition but not the strict Dickson condition, hence by Thm. 3 cannot be a conic algebra.

References

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