Conic division algebras

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Let me begin by working my way through the various definitions that will be needed in order to understand the questions I am going to ask and the answers I am going to provide.

Throughout I let F be a field of arbitrary characteristic; in particular, *I do not exclude characteristic* 2. All algebras over F are non-associative F-algebras, always different from zero, possibly infinite-dimensional, and always containing an identity element.

1. Basic concepts.

My first target will be

1.1. Conic algebras. By a *conic algebra* over F I mean an F-algebra C which admits a quadratic form $n_C\colon C \to F$, necessarily unique and called the *norm* of C, such that

$$x^2 - t_C(x)x + n_C(x)1_C = 0.$$

Heret_C is the *trace* of C, i.e., the linear form defined by $t_C(x) := (\partial n_C)(1_C, x)$, where ∂n_C is the symmetric bilinear form belonging to the quadratic form n_C, so $(\partial n_C)(x, y) = n_C(x + y) - n_C(x) - n_C(y)$.

Given a conic algebra C with norm n_C and trace t_C over F, we always have $n_C(1_C) = 1$, $t_C(1_C) = 2$ and write $x^* := t_C(x)1_C - x$ for the *conjugation* of C, which is a linear map of period two that fixes the identity element but in general is not an algebra involution.

1.2. Examples of conic algebras. (a) The base field F, with norm given by $n_F(\alpha) = \alpha^2$.

(b) Any two-dimensional F-algebra C, with norm given by $n_C(x) := \det(L_x)$ in terms of the left multiplication by x in C.

(c) Any *pre-composition algebra* C over F: there exists a quadratic form $n\colon C \to F$ which

(i) *permits composition*: $n(xy) = n(x)n(y)$,

(ii) is *non-degenerate*: $n(x) = (\partial n)(x, y) = 0$ for all y implies $x = 0$).

Then C is conic with norm $n_C = n$. By a theorem of Kaplansky [7], C is a pre-composition algebra if and only if either

(iii) C/F is a purely inseparable field extension of characteristic 2 and *exponent at most* 1, so $C^2 \subseteq F$,

or

(iv) C is a *composition algebra* in the sense that it is finite-dimensional and the symmetric bilinear form ∂n_C is non-degenerate.

In case (iv), C has dimension $1, 2, 4$ or 8, and the following subcases are relevant.

(v) $\dim_F(C) = 2$: C is commutative associative.

(vi) $\dim_F(C) = 4$: C is associative but not commutative; it is called a *quaternion algebra*.

(vii) $\dim_F(C) = 8$: C is *alternative* (i.e., the *associator* $[x, y, z] := (xy)z - x(yz)$ is alternating) but not associative; it is called an *octonion algebra*.

My second target will be

1.3. Division algebras. By a *division algebra* over F I mean an F-algebra D such that for all $u, v \in D$, $u \neq 0$, the equations $ux = v$ and $yu = v$ have unique solutions $x, y \in D$. A division algebra D over F has no zero divisors, so for all $x, y \in D$ the relation $xy = 0$ implies $x = 0$ or $y = 0$. Conversely, an F-algebra D without zero divisors is a division algebra provided it is *locally finite-dimensional* in the sense that every subalgebra on a finite number of generators is finite-dimensional over F.

My aim in this talk will be to indicate constructions of conic algebras which are also division algebras. Classical examples are the real division algebras

(i) $\mathbb{C}$ of complex numbers,

(ii) $\mathbb{H}$ of Hamiltonian quaternions,

(iii) $\mathbb{O}$ of Graves-Cayley octonions.

In the general set-up, it is important to note a

1.4. Necessary condition. Let C be a conic algebra over F.

(i) If C is a division algebra, then its norm is anisotropic: $n_C(x) \neq 0$ for all $x \neq 0$.

This follows immediately from the fact that division algebras have no zero divisors, combined with the relation $xx^* = n_C(x)1_C$. However, the converse of (i) does not hold. Counter examples exist in abundance and will be presented in due course. On the other hand, the converse of (i) does hold if you are willing to throw in an additional hypothesis:

(ii) If C is a pre-composition algebra with n_C anisotropic, then C is a division algebra.

Indeed, the hypotheses imply that C has no zero divisors and, by Kaplansky's theorem, C is either a purely inseparable field extension of characteristic 2 and exponent at most 1 or finite-dimensional, so 1.3 applies.

2. The ordinary Cayley-Dickson construction.

Most conic division algebras encountered in this talk will be derived from the Cayley-Dickson construction. Let me remind you of its

2.1. Main ingredients. The *input* of the Cayley-Dickson construction consists of a conic F-algebra B and a non-zero scalar $\mu \in F$. The *output* is a conic F-algebra

$$\mathrm{Cay}(B, \mu) = B \oplus Bj$$

that lives on the vector space direct sum of two copies of B as above under a multiplication I won't even bother to write down explicitly. Suffice it to say that the norm of C is given by the formula

$$n_C(u + vj) = n_B(u) - \mu n_B(v) \tag{1}$$

in terms of the norm of B. In other words, the norm of C is the orthogonal sum of two copies of the norm of B, the second summand being adorned with a scalar factor $-\mu$. We may therefore speak of the *orthogonal Cayley-Dickson construction* in the present context.

2.2. Invariance under the Cayley-Dickson construction. Unfortunately, useful properties of conic algebras that remain invariant under the Cayley-Dickson construction are in short supply. Well known examples are

(i) flexibility, that is, the identity $(xy)x = x(yx)$, and

(ii) the property of the conjugation to be an algebra involution.

Less well known is

(iii) the property of the norm to be a Pfister quadratic form.

This is important because Pfister forms are a topic in the algebraic theory of quadratic forms that is exceedingly well understood. Rather than going into further details, let me just remark that

(iv) the norms of composition algebras are Pfister,

(v) if B is a conic F-algebra whose norm is Pfister (e.g., an octonion algebra) and $\mu \in F$ is a non-zero scalar, then the norm of $C := \mathrm{Cay}(B, \mu)$ is anisotropic if and only if the scalar μ is not a norm of B.

On the negative side,

(vi) the property of the norm to permit composition is (of course) *not* preserved by the Cayley-Dickson construction.

More precisely, this failure is nicely captured by the formula

$$\begin{aligned} n_C\big((u_1 + v_1 j)(u_2 + v_2 j)\big) = {} & n_C(u_1 + v_1 j) n_C(u_2 + v_2 j) \\ & - \mu \partial n_B([u_1, u_2, v_2], v_1), \end{aligned} \tag{2}$$

where B is an octonion algebra over F, $\mu \in F$ is non-zero and $C = \mathrm{Cay}(B, \mu)$.

2.3. Sedenion algebras. By a *sedenion algebra* over F I mean an F-algebra C such that there exist an octonion algebra B over F and a non-zero scalar $\mu \in F$ satisfying $C \cong \mathrm{Cay}(B, \mu)$. By 2.1, sedenion algebras are conic of dimension 16. In view of 2.2 (v) it is also clear how to construct, at least in principle, sedenion algebras whose norms are anisotropic. The question of whether these algebras will automatically become conic division algebras is, however, another matter since by (2) they are no longer composition algebras. In fact, this question has a negative answer in an important special case.

2.4. The physicists' darling: the real sedenions. The real sedenion algebra so much loved by physicists (see, e.g, [10], [3], [8], [9] for details) arises from the real octonion division algebra $\mathbb{O}$ as $\mathbb{S} := \mathrm{Cay}(\mathbb{O}, -1)$[1]. By (1), its norm is positive definite, hence anisotropic. But, of course, $\mathbb{S}$ cannot be a division algebra because, thanks to the Bott-Kervaire-Milnor theorem [4, Kap. 10, § 2], finite-dimensional real division algebras exist only in dimensions $1, 2, 4, 8$. (Some people even seem to have gone one step further by claiming in book form that the only finite-dimensional real division algebras up to isomorphism are $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$, but this is palpably false.) Since the Bott-Kervaire-Milnor theorem is basically a deep result from algebraic topology, it seems desirable to ask for purely algebraic reasons why $\mathbb{S}$ cannot be a division algebra. The answer to this question is given by

2.5. Theorem. *Let B be an octonion algebra over F, $\mu \in F$ a non-zero scalar and $C := \mathrm{Cay}(B, \mu)$.*

(a) *If $\mathrm{char}(F) = 2$, then C is a division algebra if and only if μ is not the norm of an element of B.*

(b) (Brown [2, Theorem 3]) *If $\mathrm{char}(F) \neq 2$, then C is a division algebra if and only if μ is not the norm of an element of B and $-\mu$ is not the norm of a trace zero element of B.* □

2.6. Corollary. *The algebra $\mathbb{S} = \mathrm{Cay}(\mathbb{O}, -1)$ of real sedenions has zero divisors.*

Proof. In Theorem 2.5 (b), we have $F = \mathbb{R}$, $B = \mathbb{O}$ and $\mu = -1$, hence $-\mu = 1$. But the restriction of n_B to the trace zero elements of B gives a positive definite quadratic form of dimension 7 which, therefore, represents 1. □

Brown has used a combination of his theorem and the Cayley-Dickson construction to exhibit conic division algebras of arbitrarily high dimension over iterated Laurent series fields. These examples, a variant of which over rational function fields has been presented by Flaut [5], can be put into the more general and systematic framework of

3. λ-valued algebras.

λ-valued algebras are for local fields what absolute valued algebras in the sense of Albert [1] (see also Rodriguez Palacios [11]) are for the real numbers. Let me begin by saying a few words about

3.1. Local fields. By a *local field* I mean a field F together with a *discrete valuation*, i.e., a map $\lambda\colon\ F \to \mathbb{Z} \cup \{\infty\}$ which is

(i) surjective,

(ii) *definite*, so $\lambda(\alpha) = \infty \Longleftrightarrow \alpha = 0$,

(iii) *multiplicative*, so $\lambda(\alpha\beta) = \lambda(\alpha) + \lambda(\beta))$,

(iv) *ultra-metrically sub-additive*, so $\lambda(\alpha + \beta) \geq \min\{\lambda(\alpha), \lambda(\beta)\}$,

Moreover, I require that

(v) F is complete with respect to the topology induced by λ.

I then write

(vi) $\mathfrak{o} := \{\alpha \in F \mid \lambda(\alpha) \geq 0\} \subseteq F$ for the *valuation ring*,

[1] I am indebted to Nils Waßmuth for having (re)drawn my attention to this algebra.

(vii) $\mathfrak{p} := \{\alpha \in F \mid \lambda(\alpha) > 0\} \subseteq \mathfrak{o}$ for the *valuation ideal*,

(viii) $\bar{F} = \mathfrak{o}/\mathfrak{p}$ for the *residue field*, and

(ix) $\alpha \mapsto \bar{\alpha}$ for the natural map from $\mathfrak{o}$ to $\bar{F}$.

For the rest of this section, I fix a local field F, the discrete valuation λ and all the rest being understood.

3.2. The concept of a λ-valued algebra. By a *λ-valued algebra* over F I mean an F-algebra C satisfying the following conditions.

(i) C is conic and finite-dimensional.

(ii) The norm of C is an anisotropic Pfister form. Since Springer [13], basically, this condition is known to imply

$$\lambda\big(n_C(x+y)\big) \geq \min\big\{\lambda\big(n_C(x)\big), \lambda\big(n_C(y)\big)\big\}.$$

(iii) $\lambda(n_C(xy)) = \lambda(n_C(x)) + \lambda(n_C(y))$.

It follows from (ii), (iii) that λ-valued algebras have no zero divisors, hence by (i) are division algebras.

3.3. Examples and comments. Composition division algebras over F are clearly λ-valued since their norms permit composition. Conversely, let C be a λ-valued algebra over F that is *not* a composition algebra. Then the norm n_C does not permit composition, but the map $\lambda \circ n_C$ does. Moreover, turning things around a little bit, condition 3.2 (iii) can be written in the form $\lambda(n_C(xy)) = \lambda(n_C(x)n_C(y))$, and so one can also say that the failure of n_C to permit composition is not detected by λ. This looks like a pretty far-fetched phenomenon but, in fact, turns out to be quite common.

3.4. Elementary properties. Let C be a λ-valued F-algebra. From 3.2 (ii), (iii) we immediately deduce that

(i) $\mathfrak{o}_C := \big\{x \in C \mid \lambda\big(n_C(x)\big) \geq 0\big\} \subseteq C$ is an $\mathfrak{o}$-subalgebra and

(ii) $\mathfrak{p}_C := \big\{x \in C \mid \lambda\big(n_C(x)\big) > 0\big\} \subseteq \mathfrak{o}_C$ is an ideal.

Moreover, one can show that

(iii) $\bar{C} := \mathfrak{o}_C/\mathfrak{p}_C$ is a finite-dimensional conic algebra over $\bar{F}$ such that

$$\dim_{\bar{F}}(\bar{C}) = \begin{cases} \dim_F(C) & \text{, in which case we say } C \text{ is } \textit{unramified}\text{, or} \\ \frac{1}{2}\dim_F(C) & \text{, in which case we say } C \text{ is } \textit{ramified}. \end{cases}$$

Time does not permit me to disentangle all the various possibilities under which the Cayley-Dickson construction starting from an *unramified* λ-valued algebra and a non-zero scalar as input will again lead to a λ-valued algebra. Instead, let me just settle with stating the following result.

3.5. Theorem. (Garibaldi-Petersson [6, Corollary 10.14]) *Suppose $\bar{F}$ has characteristic 2 and let B be a flexible unramified λ-valued algebra over F such that $\bar{B}/\bar{F}$ is a purely inseparable field extension of exponent at most 1. Let $\mu \in \mathfrak{o}$ and suppose $\bar{\mu} \notin \bar{B}^2$. Then $C := \mathrm{Cay}(B, \mu)$ is a flexible unramified λ-valued F-algebra such that $\bar{C} = \bar{B}(\sqrt{\bar{\mu}})$.* $\square$

3.6. Corollary. *Suppose F has characteristic zero, $\bar{F}$ has characteristic 2, and $[\bar{F} : \bar{F}^2] = \infty$. For any integer $n \geq 1$ there are scalars $\mu_1, \dots, \mu_n \in \mathfrak{o}^\times$ such that*

$$C_n := \mathrm{Cay}(F; \mu_1, \dots, \mu_n)$$

is an unramified λ-valued algebra of dimension 2^n over F and

$$\bar{C}_n = \bar{F}(\sqrt{\bar{\mu}_1}, \dots, \sqrt{\bar{\mu}_n})$$

is a purely inseparable field extension of exponent 1. □

4. The non-orthogonal Cayley-Dickson construction.

In my constructions of conic division algebras so far, I have become increasingly prone to taking advantage of pathologies in characteristic 2. In this (final) section of my talk, I will bring this tendency to its logical conclusion by assuming from the very beginning that F be a field *of characteristic* 2.

4.1. The main ingredients. The *input* of the non-orthogonal Cayley-Dickson construction consists of

(i) a purely inseparable field extension K/F, possibly of infinite degree, but necessarily of exponent at most 1, so $K^2 \subseteq F$;

(ii) a scalar $\mu \in F$, possibly equal to zero.

(iii) an F-linear form $s\colon K \to F$ which is *unital* in the sense that $s(1) = 1$.

The *output* is a unital F-algebra

$$\mathrm{Cay}(K; \mu, s) = K \oplus Kj$$

living on the vector space direct sum of two copies of K as above under a multiplication the details of which you don't really want to know.

Just for the record, here they are: the multiplication

(iv) makes $K \subseteq C$ a unital subalgebra through the first summand, and

(v) satisfies the multiplication rules

$$\begin{aligned}(vj)u &= \big(s(u)v + uv\big) + (uv)j,\\ u(vj) &= \big(s(uv)1 + s(u)v + s(v)u + uv\big) + (uv)j,\\ (v_1j)(v_2j) &= \big(s(v_1v_2)1 + s(v_1)v_2 + s(v_2)v_1 + (1+\mu)v_1v_2\big)\\ &\quad + \big(s(v_1v_2)1 + s(v_1)v_2 + v_1v_2\big)j.\end{aligned}$$

Instead, we record the following result.

4.2. Proposition. (Garibaldi-Petersson [6, Prop. 4.4]) $C := \mathrm{Cay}(K; \mu, s)$ *is a flexible conic F-algebra with norm $n_C\colon C \to F$ given by*

$$n_C(u + vj) = u^2 + s(uv) + \mu v^2.$$

Moreover, if K/F has finite degree, then n_C is a Pfister form. □

4.3. The Artin-Schreier map. In order to answer the question of when the non-orthogonal Cayley-Dickson construction leads to division algebras, we need an additional ingredient. Let K, μ, s be as in 4.1 (i)−(iii). Then

$$\wp_{K,s}\colon\ K \longrightarrow F, \quad u \longmapsto \wp_{K,s}(u) := u^2 + s(u)$$

is called the associated *Artin-Schreier map.* Note that, as the notation suggests, the Artin-Schreier map does not depend on μ.

4.4. Theorem. (Garibaldi-Petersson [6, Theorem 6.4]) *With the notation and assumptions of* 4.1 *and* $C := \mathrm{Cay}(K; \mu, s)$*, the following conditions are equivalent.*

(i) *C is a conic division algebra.*

(ii) *The norm of C is anisotropic.*

(iii) $\mu \notin \mathrm{Im}(\wp_{K,s})$.

□

4.5. Corollary. *For a Pfister quadratic form over F to be the norm of a conic division algebra it is necessary and sufficient that it be anisotropic.*

Proof. The condition is clearly necessary, by 1.4. Conversely, suppose Q is an anisotropic Pfister form over F. Then it can be shown that Q has the form described in Prop. 4.2. Hence Theorem 4.4 applies. □

4.6. Concluding remarks. (a) The preceding corollary fails in characteristic not 2. For example, fixing $n > 3$, there is a unique anisotropic real Pfister form of dimension 2^n, but no real conic division algebra of that dimension, thanks to the Bott-Kervaire-Milnor theorem.

(b) In view of the results presented in this talk, one might be tempted to suspect that a flexible conic algebra over a field of characteristic 2 is a division algebra if and only if its norm is anisotropic. However, *this is not true.* Easy counter examples exist in dimension 3.

4.7. An open question. In view of the examples known from the literature, it is a natural question to ask whether *the dimension of a conic division algebra over an arbitrary field is always a power of two.* The answer to this question is not known. I have shown a long time ago [12] that the dimension of a conic division algebra over a field of characteristic not two is always even but that's a far cry from being a power of two.

Suppose the preceding question has a negative answer. Then a counter example of the least possible dimension would be a conic division algebra of dimension 6. Such a creature certainly does not exist over the reals, but does it exist over the field $\mathbb{Q}$ of rational numbers?

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