Composition algebras over commutative rings[1]

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In this lecture I intend to give a survey of what is presently known about composition algebras over arbitrary commutative associative rings of scalars. I doing so, I hope I will be able to convince you that this is an interesting topic indeed, and that there are many difficult problems just waiting to be attacked. Rather than starting off from a formal definition, I prefer to postpone it for a while and to embark instead on an informal tour de force through the fundamental features of

1. Composition algebras over fields.

Due to the informal character of this part of my lecture, I will occasionally be rather sloppy here in phrasing my hypotheses. So for the time being suppose k is a field and C is a composition algebra over k. I am, of course, assuming that you all know what a composition algebra is, although I will demonstrate in moment that in actual fact you do not. Suffice it to say at this stage that a composition algebra over k is a finite-dimensional unital k-algebra having a *norm*, i.e., a non-singular quadratic from permitting composition. Let us begin by recalling some

1.1. Basic facts. Composition algebras over k

- have a unique norm $n = n_C$,
- are quadratic alternative,
- exist only in dimensions $1, 2, 4, 8$,
- are associative iff the dimension is ≤ 4,
- are commutative associative iff the dimension is ≤ 2,
- are invariant under base field extensions, so if K/k is an extension field, then the extended algebra C_K continues to be a composition algebra, over K of course.

Let us now pass to

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1.2. Construction methods. The most popular method of constructing composition algebras is, of course, the

Cayley-Dickson doubling process (abbreviated CD). Its
Input consists of an associative composition algebra B of dimension n over k and an invertible scalar $\mu \in k$, while its
Output is a composition algebra $C = \mathrm{Cay}(B, \mu)$ of dimension $2n$ over k (hence the word *doubling*) containing B as a composition subalgebra.
The most important property enjoyed by this construction is what I call
Completeness. If C is a composition algebra of dimension $2n$ over k, containing B as a composition subalgebra of dimension n, there always exists a scalar $\mu \in k^\times$ such that the inclusion $B \hookrightarrow C$ extends to an isomorphism $\mathrm{Cay}(B, \mu) \xrightarrow{\sim} C$.

Distinctly less popular but also very important is the

Zorn quadrupling process (abbreviated Z). Here the
Input consists of a composition algebra D of dimension $r \leq 2$, forcing D to be commutative associative by 1.1, and a hermitian space of dimension 3 over D having determinant -1, while the
Output is a composition algebra $C = \mathrm{Zor}(D; V, h)$ of dimension $4r$ over k (hence the word *quadrupling*) containing D as a composition subalgebra. This construction enjoys the same completeness property as the Cayley-Dickson doubling process. In the
Special case that $D = k \oplus k$ splits, the Zorn quadrupling process reduces to the construction of the Zorn vector matrix algebra that lives on the space

$$\mathrm{Zor}(k) := \begin{pmatrix} k & k^3 \\ k^3 & k \end{pmatrix}$$

under ordinary matrix multiplication, modified appropriately by the usual vector product in 3-space.

We are now ready for the

1.3. Enumeration of composition algebras. Here we will assume $\mathrm{char}\, k \neq 2$ for simplicity. Then the *key fact* is that compositions algebras over k are enumerated by the Cayley-Dickson doubling process, so they may all be obtained from the base field by an iterated application of this process; in particular, every octonion algebra over k has the form

$$C \cong \mathrm{Cay}(k; \mu_1, \mu_2, \mu_3), \quad \mu_1, \mu_2, \mu_3 \in k^\times.$$

But this enumeration has huge redundancies. For example, the scalars μ_i above may be changed trivially by multiplying them with invertible squares from the base field without changing the isomorphism class of C. Distinctly less trivial is the fact that, for k the rationals, the octonion algebras

$$\mathrm{Cay}(\mathbb{Q}; -\mu_1, -\mu_2, -\mu_3), \quad \mu_1, \mu_2, \mu_3 \in \mathbb{Q}_+^\times$$

are all isomorphic to the unique octonion division algebra over $\mathbb{Q}$. These redundancies make it imperative to address the

1.4. Classification of composition algebras. The key idea for solving this problem is to look for *classifying invariants* of composition algebras which are exhibited by the following fundamental theorem.

Norm Equivalence Theorem. (Jacobson [1958], van der Blij-Springer [1960]) *Composition algebras over k are classified by their norms, so if C, C' are composition algebras over k, then*

$$C \cong C' \Longleftrightarrow n_C \cong n_{C'}.$$

The significance of this result derives from the fact that the exceedingly well developed algebraic theory of quadratic forms over fields can now be applied to composition algebras. For example, we immediately obtain the following

Corollary. *For $n = 2, 4, 8$, there is a unique composition algebra with zero divisors of dimension n over k, namely,*

- $k \oplus k$ *for* $n = 2$,
- $M_2(k)$ *for* $n = 4$,
- $\mathrm{Zor}(k)$ *for* $n = 8$.

It should also be mentioned that the
Key tool for proving the norm equivalence theorem is
Witt cancellation of non-singular quadratic forms q, q_1, q_2 over k:

$$q \perp q_1 \cong q \perp q_2 \Longrightarrow q_1 \cong q_2.$$

In the remainder of this talk, I will now examine the question of how the fundamental features of composition algebras over fields may be extended to

2. Composition algebras over commutative rings.

So from now on, k will be an arbitrary commutative ring. We write

- k-**alg** for the category of unital commutative associative k-algebras, homomorphisms between them always taking 1 into 1,
- $\mathrm{Spec}(k) = \{\mathfrak{p} \mid \mathfrak{p} \subseteq k \text{ is a prime ideal}\}$ for the prime spectrum of k, equipped with the Zariski topology.

For $\mathfrak{p} \in \mathrm{Spec}(k), f \in k$, we write

- $k_\mathfrak{p} = S^{-1}k, S = k - \mathfrak{p}$, for the localization of k at $\mathfrak{p}$, which is a local ring with maximal ideal $\mathfrak{p}_\mathfrak{p}$ and residue field $\kappa(\mathfrak{p}) = k_\mathfrak{p}/\mathfrak{p}_\mathfrak{p} = \mathrm{Quot}(k/\mathfrak{p})$,
- $k_f = S^{-1}k, S = \{1, f, f^2, \dots\}$.

In dealing with composition algebras over fields, we don't have to worry about the underlying module structures: All K-modules, K a field, are free. On the other hand, in dealing with composition algebras over rings, we do have to worry about the underlying module structures and they definitely should not be allowed to be completely arbitrary. We could of course insist that they all be free, but this would be far too restrictive since, as we shall see in due course, there are many natural constructions starting off from a situation where the underlying modules are free, and ending up with a situation where this is no longer the case. It seems that the right balance between avoiding the pitfalls of chaotic counter examples and at the same time allowing for the right amount of flexibility, is struck by looking at

2.1. Finitely generated projective modules. Let M be a k-module and, with $\mathfrak{p}, f$ as before, put

$$M_{\mathfrak{p}} := M \otimes k_{\mathfrak{p}}, \quad M(\mathfrak{p}) := M \otimes \kappa(\mathfrak{p}), \quad M_f := M \otimes k_f.$$

M is said to be *finitely generated projective* if it is a direct summand of a free module of finite rank, so $M \oplus M' \cong k^n$ for some k-module M' and some positive integer n. There are two important characterizations of this concept.

(i) M is finitely generated projective iff, for all $\mathfrak{p} \in \mathrm{Spec}(k)$, $M_{\mathfrak{p}}$ is a free $k_{\mathfrak{p}}$-module of fine rank.
(ii) M is finitely generated projective iff there are finitely many elements $f_1, \ldots, f_m \in k$ such that $k = \sum k f_i$ and M_{f_i} is a free k_{f_i}-module of finite rank for all $i = 1, \ldots, m$.

These characterizations allow us to define the *rank* of M as the continuous (= locally constant) function

$$\mathrm{rk}(M) : \mathrm{Spec}(k) \longrightarrow \mathbb{Z}, \quad \mathfrak{p} \longmapsto [\mathrm{rk(M)}](\mathfrak{p}) := \mathrm{rk}_{k_{\mathfrak{p}}} M_{\mathfrak{p}} = \dim_{\kappa(\mathfrak{p})} M(\mathfrak{p}).$$

We say that M *has rank* n if the function $\mathrm{rk}(M)$ on $\mathrm{Spec}(k)$ is constant equal to n.

Now we are almost ready for the formal definition of composition algebras. There is just one thing missing, namely, the concept of

2.2. Separable quadratic forms. This concept is due to Loos. Let M be a finitely generated projective module over k. A quadratic form $q : M \to k$ is said to be

- *non-singular* if its induced symmetric bilinear form, defined by the expression

  $$q(x, y) = q(x + y) - q(x) - q(y),$$

  is non-singular in the sense that it induces an isomorphism from M onto its dual M^* in the usual way; q is called

- *non-degenerate* if for all $x \in M$ the relations $q(x) = q(x, y) = 0$ for all $y \in M$ imply $x = 0$.

Non-singularity of a quadratic form is a good notion since it is invariant under base change: for every $R \in k$-**alg**, the extended quadratic form $q_R : M_R \to R$ continues to be non-singular over R. Non-degeneracy of a quadratic form, on the other hand, is a bad notion since it is *not* invariant under base change. Therefore we say that q is

- *separable* if the extended quadratic form $q_K : M_K \to K$ is non-degenerate for all *fields* $K \in k$-**alg**.

We are now in a position to formally define

2.3. Composition algebras. In its present form, this concept is also due to Loos. A non-associative algebra C over k is said to be a *composition algebra* if

(i) C contains an identity element,

(ii) C is faithful and finitely generated projective as a k-module, and

(iii) there exists a *norm*, i.e., a separable quadratic form $n : C \to k$ that permits composition: $n(xy) = n(x)n(y)$ for all $x, y \in C$.

The composition algebras you encounter in the literature are invariable based on a concept that either is not invariant under base change or prevents the base ring itself from being a composition algebra unless it contains $\frac{1}{2}$. The present concept avoids these deficiencies: composition algebras are always invariant under base change and k is always a composition algebra. We will now analyze step-by-step to what extend the fundamental features of composition algebras over fields carry over to arbitrary commutative base rings.

2.4. Basic facts. Ditto
A composition algebra C is said to be *non-singular* if its norm has this property. If C has rank n this holds true provided $n > 1$ or k contains $\frac{1}{2}$

2.5. Construction methods.

The absolute Cayley-Dickson doubling process.
Input: Ditto
Output: Ditto
Completeness: Let C be a composition algebra of rank $2n$ over k containing B as a non-singular composition subalgebra of rank n. Then the inclusion $B \hookrightarrow C$ extends to an isomorphism $\mathrm{Cay}(B, \mu) \xrightarrow{\sim} C$ iff

$$B^{\perp} \cap C^{\times} \neq \emptyset. \tag{2.5.1}$$

Condition (2.5.1) is what Bourbaki calls a dangerous bend. As we shall see in due course, it is not automatic, so the question presents itself what to do with the Cayley-Dickson doubling if (2.5.1) fails. An answer will now be given as follows.

For the time being, we fix an associative composition algebra B over k and write ι_B, $x \mapsto \overline{x}$, for its conjugation. The key notion we require is that of

2.6. Hermitian discriminant modules. A right B-module M is said to be *locally free of rank* 1 if $M_{\mathfrak{p}} \cong B_{\mathfrak{p}}$ as right $B_{\mathfrak{p}}$-modules for all $\mathfrak{p} \in \mathrm{Spec}(k)$, or equivalently, if there exist finitely many $f_1, \ldots, f_m \in k$ such that $k = \sum k f_i$ and $M_{f_i} \cong B_{f_i}$ as right B_{f_i}-modules for all $i = 1, \ldots, m$. The analogy to the characterization of finitely generated projective modules over commutative rings is obvious. But while it has been shown by Knus [1991] that locally free right B-modules of rank 1 are always projective, the converse does not hold since B need not be commutative (simply put $B = M_2(k)$ and $M = k^2$ (row space) as a right B-module, which is finitely generated projective but clearly not locally free of rank 1). By a *hermitian discriminant module* over B we men a pair (M, h) consisting of a locally free right B-module M of rank 1 and a non-singular hermitian form $h : M \times M \to B$ that is *central* in the sense that $h(v, v) \in k1_B$ for all $v \in M$, the latter condition being automatic if $\frac{1}{2} \in k$. For example, if M is free as a right B-module, then $(M, h) \cong (B, \langle \mu \rangle_{\mathrm{her}})$ for some $\mu \in k^{\times}$ and conversely. We are now ready for the

2.7. Relative Cayley-Dickson doubling process. The
Input consists of a non-singular associative composition algebra B of over k and a central hermitian discriminant module (M, h) over B. The
Output is the algebra $C = \mathrm{Cay}(B, M, h)$ that lives on the k-module $B \oplus M$ under the multiplication

$$(u \oplus v)(u' \oplus v') := \big(uu' + h(v', v)\big) \oplus (v'.u + v.\overline{u'}).$$

That C is indeed a composition algebra follows from two facts: (i) If $(M, h) = (B, \langle \mu \rangle_{\mathrm{her}})$, $\mu \in k^{\times}$ is a free hermitian discriminant module of rank 1 over B, then $\mathrm{Cay}(B, M, h)$ agrees with the ordinary

Cayley-Dickson doubling $\mathrm{Cay}(B, \mu)$, hence is a composition algebra. (ii) It suffices to check the property of C being a composition algebra locally at every prime $\mathfrak{p} \in \mathrm{Spec}(k)$. Fortunately, we also have
Completeness of the relative Cayley-Dickson doubling: If C is any composition algebra of rank $2n$ over k containing B as a non-singular composition subalgebra of rank n, there exists a hermitian discriminant module (M, h) over B such that the inclusion $B \hookrightarrow C$ extends to an isomorphism $\mathrm{Cay}(B, M, h) \xrightarrow{\sim} C$.

It is a natural question to ask for non-trivial

2.8. Examples of central hermitian discriminant modules. These will be provided by answering the following question.

- Which locally free right B-modules of rank 1 support central hermitian discriminant modules?

To answer this question, we require several auxiliary items.

- *The Picard set.* The set

$$\mathrm{Pic}(B) := \{[M] \mid M \text{ is a locally free right } B\text{-module of rank } 1\}$$

  of isomorphism classes of locally free right B-modules of rank 1 is a pointed set, with base point equal to $[B]$, called the *Picard set* of B; it is even a group, the Picard *group* of B (under $\otimes_B$) if B is commutative.

- *Non-abelian cohomology.* Locally free right B-modules of rank 1 are classified by $H^1(k, \mathbf{G}_{\mathrm{m}B})$, so

$$\mathrm{Pic}(B) = H^1(k, \mathbf{G}_{\mathrm{m}B}) \quad \text{as pointed sets,}$$

  where $\mathbf{G}_{\mathrm{m}B}$ is the group scheme given by

$$\mathbf{G}_{\mathrm{m}B}(R) = B_R^{\times}. \qquad\qquad (\text{for } R \in k\text{-}\mathbf{alg})$$

- *The norm.* The norm of B induces a homomorphism

$$n_B : \mathbf{G}_{\mathrm{m}B} \longrightarrow \mathbf{G}_{\mathrm{m}k}$$

  of group schemes, which in turn gives rise to a homomorphism

$$n_B : \mathrm{Pic}(B) = H^1(k, \mathbf{G}_{\mathrm{m}B}) \longrightarrow H^1(k, \mathbf{G}_{\mathrm{m}k}) = \mathrm{Pic}(k)$$

  of pointed sets. Hence if $M \in \mathrm{Pic}(B)$ is a locally free right B-module of rank 1, then $n_B(M) \in \mathrm{Pic}(k)$ is a *line bundle*, i.e., a finitely generated projective module of rank 1, over k, called the *norm* of B.

- *Hermitian maps into bimodules with involution.* (Loos [1994]) Let (P, ι) be a bimodule with involution over B, so P is a B-bimodule and $\iota : P \to P, p \mapsto \overline{p}$, a k-linear map satisfying $\overline{\overline{p}} = p, \overline{upu'} = \overline{u'}\,\overline{p}\,\overline{u}$ for $u \in B, p \in P$. Then

$$\mathrm{Cent}(P, \iota) = \{p \in P \mid p = \overline{p}, up = pu \ (u \in B)\} \subseteq P$$

is a k-submodule, called the *centre* of (P, ι). For example, if L is a line bundle over k,

$$(P, \iota) = (L \otimes B, \mathbf{1}_L \otimes \iota_B)$$

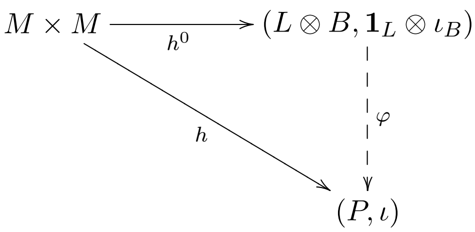
is a B-bimodule with involution satisfying

$$\mathrm{Cent}(L \otimes B, \mathbf{1}_L \otimes \iota_B) \cong L.$$

Now if M is any right B-module, (central non-singular) hermitian maps $M \times M \to (P, \iota)$ make obvious sense.

By gluing local data, we can now prove the following

Theorem. *Let M be a locally free right B-module of rank 1 and put $L := n_B(M)$. Then there exists a non-singular central hermitian map $h^0 : M \times M \to (L \otimes B, \mathbf{1}_L \otimes \iota_B)$, unique up to an invertible factor in k, that is universal in the sense that for any B-bimodule (P, ι) with involution and any central hermitian map $h : M \times M \to (P, \iota)$, there exists a unique homomorphism φ of B-bimodules with involution making a commutative diagram as shown.*

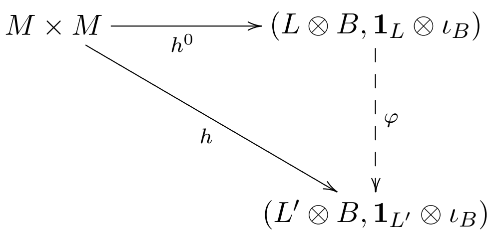


Corollary 1. *For $L' \in \mathrm{Pic}(k)$, the following conditions are equivalent.*

(i) *There exists a non-singular central hermitian map $h : M \times M \to (L' \otimes B, \mathbf{1}_{L'} \otimes \iota_B)$.*

(ii) *$n_B(M) \cong L'$.*

Proof. Since (ii) $\Rightarrow$ (i) follows immediately from the theorem, we are left with (i) $\Rightarrow$ (ii). Setting $L := n_B(M)$, the theorem yields a commutative diagram



and since h, h^0 are both non-singular, φ must be an isomorphism, which in turn induces an isomorphism

$$\varphi : L \cong \mathrm{Cent}(L \otimes B, \mathbf{1}_L \otimes \iota_B) \xrightarrow{\sim} \mathrm{Cent}(L' \otimes B, \mathbf{1}_{L'} \otimes \iota_B) \cong L'.$$

□

Corollary 2. *M supports a non-singular central hermitian form over B iff $n_B(M) \cong k$.* □

We now apply the preceding considerations in the following

Example. Let $B = M_2(k)$ be the algebra of 2-by-2 matrices over k, forcing ι_B to be the symplectic involution

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

Now if N is any finitely generated projective k-module of rank 2, then

$$M := N \oplus N \qquad\qquad \text{(row-2-space over } N)$$

is a locally free right B-module of rank 1 under ordinary matrix multiplication. Put

$$L := \det(M) = \bigwedge^2(N) \in \mathrm{Pic}(k)$$

and define

$$h : M \times M \longrightarrow (L \otimes B, \mathbf{1}_L \otimes \iota_B) = \big(M_2(L), \iota_{M_2(L)}\big)$$

by

$$h\big((v_1, v_2), (w_1, w_2)\big) := \begin{pmatrix} v_2 \wedge w_1 & v_2 \wedge w_2 \\ -v_1 \wedge w_1 & -v_1 \wedge w_2 \end{pmatrix}$$

for $v_i, w_i \in M$, $i = 1, 2$. This may look a bit weird at first, but interchanging the v's and the w's, for instance, amounts to the same as applying the symplectic involution to the right-hand side; indeed, a straightforward verification shows that h is a non-singular central hermitian map, so Corollary 1 implies $L \cong n_B(M)$.

We now come to the

The Zorn quadrupling process.
Input: Ditto
Output: Ditto
Completeness: Ditto
Special case: Suppose (V, h) is a ternary hermitian space over the split algebra $D = k \oplus k$. Notice that the determinant of (V, h) in this generality is a hermitian space of rank 1, so our condition that it be -1 should be phrased here more accurately as

$$\det(V, h) = (k \oplus k, \langle -1 \rangle_{\mathrm{her}}).$$

If this condition is fulfilled, the Zorn quadrupling process leads to a relative version of the Zorn vector matrix algebra in the sense that

$$\mathrm{Zor}(k \oplus k, V, h) \cong \mathrm{Zor}(M, \theta)$$

where

- M is a finitely generated projective k-module of rank 3,
- $\theta : \bigwedge^3(M) \xrightarrow{\sim} k$ is an isomorphism ("volume element"), giving rise to a volume element

$$\theta^{*-1} : \bigwedge^3(M^*) \cong \big(\bigwedge^3(M)\big)^* \xrightarrow{\sim} k^* \cong k,$$

- the *associated vector products*

$$\times_\theta : M \times M \longrightarrow M^*, \quad \times_\theta : M^* \times M^* \longrightarrow M$$

  are defined by

$$\langle w, u \times_\theta v\rangle = \theta(u \wedge v \wedge w), \quad \langle u^* \times v^*, w^*\rangle = \theta^{*-1}(u^* \wedge v^* \wedge w^*),$$

- we have

$$\mathrm{Zor}(M, \theta) = \begin{pmatrix} k & M^* \\ M & k \end{pmatrix}$$

  and

$$\begin{pmatrix} \alpha_1 & v^* \\ v & \alpha_2 \end{pmatrix} \begin{pmatrix} \beta_1 & w^* \\ w & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 + \langle w, v^*\rangle & \alpha_1 w^* + \beta_2 v^* + v \times_\theta w \\ \beta_1 v + \alpha_2 w - v^* \times_\theta w^* & \langle v, w^*\rangle + \alpha_2\beta_2 \end{pmatrix}.$$

We now come to the really sticky points of our extension programme, which begin with the

2.9. Enumeration of composition algebras. Here we can only say

???

These question marks will be illuminated by the following

Examples. (Knus-Parimala-Sridharan[1994]) Let $k = K[X_1, \dots, X_n]$ be the polynomial ring in $n > 1$ variables over a field K of characteristic not 2. Then there are infinitely many octonion algebras over k such that (C^0, n_C^0), the trace zero elements in C equipped with the corresponding restriction of the norm, is indecomposable as a quadratic space. (But observe in parantheses that, thanks to the positive solution to Serre's conjecture, C^0 is free, hence decomposable, as a k-module.) Therefore C contains no composition subalgebras of rank > 1, so in particular, it can be realized neither by the absolute nor by the relative Cayley-Dickson doubling process nor by the Zorn quadrupling process.

Things get even worse when looking at the

2.10. Classification of composition algebras. Here again we can only say

???

The simple reason is that the
key tool for dealing with composition algebras over fields, i.e.,
Witt cancellation, fails miserably not only with arbitrary non-singular quadratic forms, but also with the ones attached to composition algebras. Here is an instructive

Example. For any line bundle M over k,

$$B = \begin{pmatrix} k & M^* \\ M & k \end{pmatrix} = \mathrm{End}_k(k \oplus M)$$

is a quaternion algebra. It is easy to see that $B \cong M_2(k)$ if and only if there exists a line bundle L over k that is generated by two elements and satisfies $M \cong L \otimes L$. Suppose this holds true but M is *not* free. Then

$$\mathbf{h} \perp \mathbf{h} \cong n_{M_2(k)} \cong n_B \cong \mathbf{h} \perp \mathbf{h}_M,$$

where **h** is the ordinary hyperbolic plane and

$$\mathbf{h}_M : M \oplus M^* \longrightarrow k, \quad v \oplus v^* \longmapsto \langle v, v^* \rangle,$$

is the hyperbolic plane twisted by M. But Ottmar has shown me a two-line proof that, if $\mathbf{h} \cong \mathbf{h}_M$, then M would be free, so $\mathbf{h}$ and $\mathbf{h}_M$ cannot be isometric, and Witt cancellation does indeed fail.

2.11. Open problems. It follows from the failure of Witt cancellation that the proof of the norm equivalence theorem that works over fields breaks down over rings. But it *does not* follow that the theorem itself is false over rings. We therefore state the

Norm Equivalence Problem. (NEP) If C, C' are composition algebras over k, then

$$C \cong C' \Longleftrightarrow n_C \cong n_{C'}\,\mathbf{?}$$

Here are a few comments.

- The direction "$\Longrightarrow$" follows trivially from the uniqueness of the norm.
- For $\mathrm{rk}(C) \leq 2$, the answer is easily seen to be yes.
- For $\mathrm{rk}(C) = 4$, the answer has been shown by Knus [1991], using Clifford algebras, to be yes.
- For $\mathrm{rk}(C) = 8$, **???**

Closely related to the norm equivalence problem is what I call the

Isotopy Problem. (IP) If C is a composition algebra over k and $a, b \in C^\times$, then

$$C^{(a,b)} \cong C\,\mathbf{?}$$

To let you appreciate this problem, I will have to remind you of the notion of isotopy, due to McCrimmon [1971]. In doing so, I restrict myself to composition algebras. So let C be a composition algebra over k and $a, b \in C^\times$. Then $C^{(a,b)}$, the *a, b-isotope* of C, lives on the k-module C under the multiplication

$$x \cdot_{a,b} y := (xa)(by). \qquad (x, y \in C)$$

It is a composition algebra over k satisfying

$$1_{C^{(a,b)}} = (ab)^{-1}, \quad n_{C^{(a,b)}} = n_C(ab)n_C, \quad [C^{(a,b)}]^+ = [C^+]^{(ab)}.$$

Concerning the isotopy problem itself, the following comments seem to be in order.

- For $\mathrm{rk}(C) \leq 4$, the answer is yes since C is associative and

  $$L_{ab} : C^{(a,b)} \xrightarrow{\sim} C$$

  is an isomorphism.
- For $\mathrm{rk}(C) = 8$, **???**
- A positive solution to the norm equivalence problem implies a positive solution to the isotopy problem since

  $$L_{ab} : n_{C^{(a,b)}} \xrightarrow{\sim} n_C$$

  is an isometry.

- We may always assume $b = a^{-1}$ since

$$L_{ab} : C^{(a,b)} \xrightarrow{\sim} C^{(a^2b, b^{-1}a^{-2})}$$

  is an isomorphism. We therefore put ${}^aC := C^{(a,a^{-1})}$, which has the same unit, norm, Jordan structure as C. We also have

$${}^a({}^bC)) = {}^{(ab)}C,$$

  and record the observations

- ${}^bC \cong {}^{(a^3b)}C \cong {}^{(aba^2)}C \cong {}^{(ba^6)}C$,
- ${}^bC \cong C$ if $k[b]^\perp \cap C^\times \neq \emptyset$,
- ${}^bC \cong C$ if b has trace 0.

We close the discussion of the isotopy problem by looking at a

Special case. In fact, among the remaining open cases, this should presumably be the easiest one. We put $C = \mathrm{Zor}(k)$ and consider an arbitrary element

$$b = \begin{pmatrix} \beta_1 & v_2 \\ v_1 & \beta_2 \end{pmatrix} \in C^\times.$$

Then we have the following partial affirmative answer to the isotopy problem.

Proposition. *We have ${}^bC \cong C$ if one of the following conditions are fulfilled.*

(i) *β_1 or β_2 is invertible.*

(ii) *v_1 or v_2 is unimodular, so there exists a vector $u \in k^3$ satisfying $u^tv_1 = 1$ or $u^tv_2 = 1$.*

(iii) *There is a unimodular vector $u \in k^3$ such that $u^tv_1 = 0$ or $u^tv_2 = 0$.*

□

Let me close my lecture with another open problem which I regard as very important.

Coordinatization Problem. For $i = 1, 2$, let M_i be a finitely generated projective k-module of rank 3 and $\theta_i : \bigwedge^3(M_i) \xrightarrow{\sim} k$ a volume element. Find conditions in terms of (M_1, θ_1) and (M_2, θ_2) that are necessary and sufficient for the octonion algebras $\mathrm{Zor}(M_1, \theta_1)$ and $\mathrm{Zor}(M_2, \theta_2)$ to be isomorphic.