Albert algebras arising from central simple quartic Jordan algebras are reduced.

Holger P. Petersson
Fakultät für Mathematik und Informatik
FernUniversität in Hagen
D-58084 Hagen, Germany
Email: holger.petersson@FernUni-Hagen.de

1. Introduction. A method of constructing cubic Jordan algebras out of quartic ones, akin to the approach of Okubo [4, 5] and Faulkner [2] for finding symmetric compositions inside associative algebras of degree 3, has been devised by Allison-Faulkner [1]. Their key results (as far as the scope of the present note is concerned) read as follows. Throughout we let k be a field of characteristic not 2 or 3,

2. Theorem. (Allison-Faulkner [1, Theorem 5.4]) *Let J be a separable Jordan algebra of degree 4 over k, write t for the (generic) trace of J and*

$$J_0 := \{x \in J \mid t(x) = 0\}$$

for the linear hyperplane of trace zero elements in J. Suppose we are given an element $e \in J_0$ satisfying $t(e^3) \neq 0$. Then there is a unique way of making J_0 into a separable Jordan algebra of degree 3 over k having unit element e and generic norm given by

$$N_0(x) = \frac{t(x^3)}{t(e^3)} \qquad (x \in J_0).$$

This Jordan algebra will be denoted by $\mathrm{Zer}(J, e)$. □

3. Corollary. (Allison-Faulkner [1, Corollary 5.5]) *Let (A, τ) be a central simple associative algebra of degree 8 with symplectic involution over k and $e \in A$ a τ-symmetric element of trace zero satisfying $t(e^3) \neq 0$, where t stands for the generic trace of A. Then*

$$\mathrm{Zer}(A, \tau, e) := \mathrm{Zer}\big(H(A, \tau), e\big)$$

is an Albert algebra over k. □

In a recent preprint, working in a much more general context, S. Pumplün [7] has described it as unclear whether all Albert algebras over k can be obtained in the manner described by Corollary 3. Using standard techniques, it is, however, possible to derive the following result.

4. Theorem. *Let (A, τ) be a central simple associative algebra of degree 8 with symplectic involution over k and $e \in A$ a τ-symmetric element of trace zero satisfying $t(e^3) \neq 0$, where t stands for the generic trace of A. Then the Albert algebra $\mathrm{Zer}(A, \tau, e)$ is reduced.*

Proof. We put $J := \mathrm{Zer}(A, \tau, e)$ and make use of its cohomological mod-3-invariant belonging to $H^3(k, \mathbb{Z}/3\mathbb{Z})$, see Rost [8], Garibaldi-Merkurjev-Serre [3] or Petersson-Racine [6] for details. Accordingly, if J is a division algebra, so is $J \otimes_k l$, for any finite algebraic field extension l/k of degree prime to 3. Now the division algebra belonging to the Brauer class of A has degree 2^n, $0 \leq n \leq 3$. Hence A has a separable splitting field whose degree is a power of 2. By what has just been observed, we may therefore assume that (A, τ) is split, i.e., $A = \mathrm{Mat}_8(k)$ and τ is the symplectic involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}$$

in terms of 2×2-blocks of 4×4-matrices. It follows that $H(A, \tau)$ consists of the matrices

$$\begin{pmatrix} a & b \\ c & a^t \end{pmatrix} \qquad (a \in \mathrm{Mat}_4(k), b, c \in \mathrm{Skew}_4(k)).$$

Picking a non-zero nilpotent $a \in \mathrm{Mat}_4(k)$ of index at most 3, we conclude that

$$x := \begin{pmatrix} a & 0 \\ 0 & a^t \end{pmatrix} \in H(A, \tau)$$

has trace zero and satisfies $x^3 = 0$, hence $N_J(x) = 0$. Thus J is not a division algebra. □

References

[1] B.N. Allison and J.R. Faulkner, *A Cayley-Dickson process for a class of structurable algebras*, Trans. Amer. Math. Soc. **283** (1984), no. 1, 185–210. MR 735416 (85i:17001)

[2] J.R. Faulkner, *Finding octonion algebras in associative algebras*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1027–1030. MR 931729 (89g:17003)

[3] S. Garibaldi, A. Merkurjev, and J-P. Serre, *Cohomological invariants in Galois cohomology*, University Lecture Series, vol. 28, American Mathematical Society, Providence, RI, 2003. MR 1999383 (2004f:11034)

[4] S. Okubo, *Octonion as traceless* 3×3 *matrices via a flexible Lie-admissible algebra*, Hadronic J. **1** (1978), no. 5, 1432–1465. MR 516785 (80j:17001b)

[5] Susumu Okubo, *Pseudo-quaternion and pseudo-octonion algebras*, Hadronic J. **1** (1978), no. 4, 1250–1278. MR 510100 (80a:17004)

[6] Holger P. Petersson and Michel L. Racine, *An elementary approach to the Serre-Rost invariant of Albert algebras*, Indag. Math. (N.S.) **7** (1996), no. 3, 343–365. MR 99j:17045

[7] S. Pumplün, *A construction method for Albert algebras over algebraic varieties*, Preprint, 2011.

[8] M. Rost, *A* (mod 3) *invariant for exceptional Jordan algebras*, C. R. Acad. Sci. Paris Sér. I Math. **313** (1991), no. 12, 823–827. MR 1138557 (92j:19002)