A Mathematical View on Voting and Power

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Abstract. In this article we describe some concepts, ideas and results from the mathematical theory of voting. We give a mathematical description of voting systems and introduce concepts to measure the power of a voter. We also describe and investigate two-tier voting systems, for example the Council of the European Union. In particular, we prove criteria which give the optimal voting weights in such systems.

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1. Introduction

A voting system is characterized by a set V of voters and a collection of rules specifying the conditions under which a proposal is approved. Examples for the set V of voters comprise e. g. the citizens of voting age in a country, the members of a parliament, the representatives of member states in a supranational organization or the colleagues in a hiring committee at a university. On any proposal to the voting system the voters may vote 'yes' or 'no'. Here and in the following we exclude the possibility of abstentions [1].

Probably the most common voting rule is 'simple majority': A proposal is approved if more than half of the voters vote in favor of the proposal. This voting rule is implemented in most parliaments and committees. For some special proposals the voting rules may ask for more than half of the voters supporting the proposal. For example a two-third majority could be required for an amendment of the constitution. In such cases we speak of a 'qualified majority'.

In most countries with a federal structure (e. g. the USA, India, Germany, Switzerland, ...) the legislative is composed of two chambers, one of which represents the states of the federation. In these cases the voters are the members of one of the chambers, a typical voting rule would require a simple majority in both chambers. However, the voting rules can be more complicated than this. For example in the USA both the President and the Vice-President are involved in the legislative process, the President by his right to veto a bill, the Vice-President as the President of the Senate (and tie-breaker in the Senate). In Germany the state chamber (called 'Bundesrat') can be overruled by a qualified majority of the Bundestag for certain types of laws, so called 'objection bills'.

[1] Voting systems with abstentions are considered in [11],[4] and [3] for example.

In the Senate, the state chamber of the US legislative system, each state is represented by two senators. Since every senator has the same influence on the voting result ('one senator, one vote'), small states like Wyoming have the same influence in the Senate as large states like California. This is different in the Bundesrat, Germany's state chamber. Here the state governments have a number of seats (3 through 6) depending on the size of the state (in terms of population). The representatives of a state can cast their votes only as a block, i. e. votes of a state can't be split. This is a typical example of a weighted voting system: The voters (here: the states) have different weights, i. e. a specific number of votes.

Another example of a weighted voting system is the Board of Governors of the International Monetary Fund. Each member state represented by a Governor has a number of votes depending on the 'special drawing rights' of that country. For example the USA has a voting weight of 421962 (= 16.74 % of the total weight) while Tuvalu has 756 votes, equivalent to 0.03 %.

The Council of the European Union used to be a weighted voting system before the eastern extension of the European Union in 2004. Since then there is a more complicated voting system for the Council composed of two (or even three) weighted voting procedures.

In section 2 of this paper we present a mathematical description of voting systems in general and specify our considerations to weighted voting systems in section 3. In section 4 we discuss the concept of voting power. Section 5 is devoted to a description of our most important example, the Council of the European Union. Section 6 presents a treatment of two-layer (or two-tier) voting systems. In such systems (e. g. the Council of the EU) representatives of states make decisions as members of a council. We raise and discuss the question of a fair representation of the voters in the countries when the population of the states is different in size. Finally, section 7 presents a systematic probabilistic treatment of the same question.

The first four sections of this paper owe much to the excellent treatments of the subject in [12], [32] and [34]. In a similar way, section 5 relies in part on [12].

2. A Mathematical Formalization

In real life voting systems are specified by a set a rules which fix conditions under which a proposal is approved or rejected. Here are a couple of examples.

Example 1. (1) The 'simple majority rule': A proposal is accepted if more than half of the voters vote 'yes'. More formally: If the voting body has N members then a proposal is approved if (and only if) the number Y of 'yes-votes' satisfies $Y \geq (N+1)/2$. (Recall that we neglect the possibility of abstentions.)

(2) The 'qualified majority rule': A number of votes of at least $r\,N$ is needed, N being the number of voters and r a number in the interval $(1/2, 1]$. Such a qualified majority is typically required for special laws, in particular for amendments to the constitution, for example with $r = 2/3$.
A simple majority rule is a special example of a qualified majority rule, with the choice $r = \frac{1}{2}\,(1 + \frac{1}{N})$.

(3) The 'unanimity rule': A proposal is approved only if *all* voters agree on it. This is a special case of (2), namely for $r = 1$.

(4) The 'dictator rule': A proposal is approved if a special voter, the dictator 'd', approves it.

(5) Many countries have a 'bicameral parliament', i. e. the parliament consists of two chambers, for example the 'House of Representatives' and the 'Senate' in the USA, the 'Bundestag' and the 'Bundesrat' in Germany.

One typical voting rule for a bicameral system is, that a bill needs a majority in both chambers to become law. This is, in deed, the case in Italy, where the chambers are called 'Camera dei Deputati' and 'Senato della Repubblica'. The corresponding voting rules in the USA and in Germany are more complicated and we are going to comment on such systems later.

(6) The UN Security Council has 5 permanent and 10 nonpermanent members. The permanent members are China, France, Russia, the United Kingdom and the USA. A resolution requires 9 affirmative votes, including all votes of the permanent members (veto powers).

The set of voters together with a set of rules constitute a 'voting system'. A convenient mathematical way to formalize the set of rules is to single out which sets of voters can force an affirmative decision by their positive votes.

Definition 2. A *voting system* is a pair $(V, \mathcal{V})$ consisting of a (finite) set V of voters and a subset $\mathcal{V} \subset \mathcal{P}(V)$ of the system of all subsets of V.

Subsets of V are called *coalitions*, the sets in $\mathcal{V}$ are called *winning coalitions*, all other coalitions are called *losing*.

The set $\mathcal{V}$ consists of exactly those sets of voters that win a voting if they all agree with the proposal at hand.

Example 3. (1) In a parliament the set V of voters consists of all members of the parliament. Under simple majority rule, the winning coalitions are those which comprise more than half of the members of the parliament.

(2) If a body V decides according to the unanimity rule, the only winning coalition is V itself, thus $\mathcal{V} = \{V\}$.

(3) In a bicameral parliament consisting of chambers, say, H (for 'House') and S (for 'Senate'), the set of voters consists of the union $H \cup S$ of the two

chambers. If a simple majority in both chambers is required a coalition M is winning if M contains more than half of the members of H *and* more than half of the members of S.
In more mathematical terms: $V = H \cup S$ (as a rule with $H \cap S = \emptyset$)
A coalition $A \subset V$ is winning, if

$$|A \cap H| \ > \frac{1}{2}\,|H| \qquad \text{and} \qquad |A \cap S| \ > \frac{1}{2}\,|S|$$

where $|M|$ denotes the number of elements in the set M.

(4) The voters in the UN Security Council are the permanent and the nonpermanent members. A coalition is winning if it comprises *all* of the permanent members *and* at least four nonpermanent members.

Later we'll discuss two rather complicated voting systems: the federal legislative system of the USA and the Council of the European Union.

In the following, we will *always* make the following assumption:

Assumption 4. If $(V, \mathcal{V})$ is a voting system we assume that:

(1) The set of *all* voters is always winning, i. e. $V \in \mathcal{V}$.
'If all voters support a proposal, it is approved under the voting rules.'

(2) The empty set $\emptyset$ is never winning, i. e. $\emptyset \notin \mathcal{V}$.
'If nobody supports a proposal it should be rejected.'

(3) If a set A is winning ($A \in \mathcal{V}$) and A is a subset of B, then B is also winning.
'A winning coalition stays winning if it is enlarged.'

Remark 5. If $(V, \mathcal{V})$ is a voting system (satisfying the above assumptions) then we can reconstruct the set V from the set $\mathcal{V}$, in fact V is the biggest set in $\mathcal{V}$. Therefore, we will sometimes call $\mathcal{V}$ a voting system without explicit reference to the underlying set of voters V.

Some authors also require that if $A \in \mathcal{V}$ then $\complement A := V \setminus A \notin \mathcal{V}$. Such voting systems are called *proper*. As a rule, real world voting systems are proper. In the following, unless explicitly stated otherwise, we may allow improper voting systems as well.

Now, we introduce two methods to construct new voting systems from given ones. These concepts are implemented in many real world systems.

We start with the method of intersection. The construction of intersection of voting systems can be found in practice frequently. Many bicameral parliamentary voting systems are the intersections of the voting systems of the two constituting chambers.

Definition 6. We define the *intersection* $(W, \mathcal{W})$ of two voting systems $(V_1, \mathcal{V}_1)$ and $(V_2, \mathcal{V}_2)$ by

$$\begin{aligned} W &:= V_1 \cup V_2 \\ \mathcal{W} &:= \{M \subset W \mid M \cap V_1 \in \mathcal{V}_1 \quad \text{and} \quad M \cap V_2 \in \mathcal{V}_2\}. \end{aligned} \tag{1}$$

We denote this voting system by $(V_1 \cup V_2, \mathcal{V}_1 \wedge \mathcal{V}_2)$ or simply by $\mathcal{V}_1 \wedge \mathcal{V}_2$.

Colloquially speaking: A coalition in $\mathcal{V}_1 \wedge \mathcal{V}_2$ is winning if 'it' is winning both in $\mathcal{V}_1$ and in $\mathcal{V}_2$. This construction can, of course, be done with more than two voting systems.

In an analogous way, we define the union of two voting systems.

Definition 7. We define the *union* $(W, \mathcal{W})$ of two voting systems $(V_1, \mathcal{V}_1)$ and $(V_2, \mathcal{V}_2)$ by

$$\begin{aligned} W &:= V_1 \cup V_2 \\ \mathcal{W} &:= \{M \subset W \mid M \cap V_1 \in \mathcal{V}_1 \quad \text{or} \quad M \cap V_2 \in \mathcal{V}_2\}. \end{aligned} \tag{2}$$

We denote this voting system by $(V_1 \cup V_2, \mathcal{V}_1 \vee \mathcal{V}_2)$ or simply by $\mathcal{V}_1 \vee \mathcal{V}_2$.

We end this section with a formalization of the US federal legislative system.

Example 8 (US federal legislative system)**.** We discuss the US federal system in more details. For a bill to pass Congress a simple majority in both houses (House of Representatives and Senate) is required. The Vice President of the USA acts as a tie-breaker in the senate. Thus a bill (at this stage) requires the votes of 218 out of the 435 representatives and 51 of the 100 senators or of 50 senators and the Vice President. If the President signs the bill it becomes law. However, if the President vetoes the bill, the presidential veto can be overruled by a two-third majority in both houses.

To formalize this voting system we define (with $|A|$ denoting the number of elements of the set A):

(1) The House of Representatives

Representatives	$R := \{R_1, \ldots, R_{435}\}$
Coalitions with simple majority	$\mathcal{R}_1 := \{A \subset R \,\big|\, \lvert A\rvert \geq 218\}$
Coalitions with two-third majority	$\mathcal{R}_2 := \{A \subset R \,\big|\, \lvert A\rvert \geq 290\}$

(2) The Senate

Senators	$S := \{S_1, \ldots, S_{100}\}$
Coalitions with at least half of the votes	$\mathcal{S}_0 := \{A \subset S \,\big|\, \lvert A\rvert \geq 50\}$
Coalitions with simple majority	$\mathcal{S}_1 := \{A \subset S \,\big|\, \lvert A\rvert \geq 51\}$
Coalitions with two-third majority	$\mathcal{S}_2 := \{A \subset S \,\big|\, \lvert A\rvert \geq 67\}$

(3) The President

$$V_p \ := \ \{P\} \qquad\qquad \mathcal{V}_p = \{V_p\}$$

(4) The Vice President

$$V_v \ := \ \{VP\} \qquad\qquad \mathcal{V}_v = \{V_v\}$$

Above the President (denoted by 'P') and the Vice President ('VP') constitute their own voting systems in which the only non empty coalitions are those containing the President and the Vice President respectively.

With these notations the federal legislative system of the USA is given by the set V of voters:

$$V \ = \ R \ \cup \ S \ \cup \ V_p \ \cup \ V_v$$

and the set $\mathcal{V}$ of winning coalitions:

$$\mathcal{V} \ = \ \Big(\mathcal{R}_1 \wedge \mathcal{S}_1 \wedge \mathcal{V}_p\Big) \ \vee \ \Big(\mathcal{R}_1 \wedge \mathcal{S}_0 \wedge \mathcal{V}_v \wedge \mathcal{V}_p\Big) \vee \Big(\mathcal{R}_2 \wedge \mathcal{S}_2\Big)$$

3. Weighted Voting Systems

Definition 9. A voting system $(V, \mathcal{V})$ is called a *weighted voting system* if there is a function $w : V \rightarrow [0, \infty)$, called the *voting weight*, and a number $q \in [0, \infty)$, called the *quota*, such that

$$A \in \mathcal{V} \quad \Leftrightarrow \quad \sum_{v \in A} w(v) \ \geq \ q$$

Notation 10. For a coalition $A \subset V$ we set $w(A) \ := \ \sum_{v \in A} w(v)$, so a coalition A is winning if $w(A) \geq q$.

We also define the *relative quota* of a weighted voting system by $r := \frac{q}{w(V)}$. Consequently, A is winning if $w(A) \geq r\, w(V)$.

Remark 11. If $(V, \mathcal{V})$ is a voting system given by the weight function w and the quota q, then for any $\lambda > 0$ the weight function $w'(v) = \lambda\, w(v)$ together with the quota $q' = \lambda\, q$ define the same voting system.

Examples 12. (1) A simple majority voting system is a weighted voting system (with trivial weights). The weight function can be chosen to be identically equal to 1 and the quota to be $\frac{N+1}{2}$ where N is the number of voters. The corresponding relative quota is $r = \frac{1}{2}\,(1 + \frac{1}{N})$.

(2) A voting system with unanimity is a weighted voting system. For example one may choose $w(v) = 1$ for all $v \in V$ and $q = |V|$ (or $r = 1$).

(3) In the German Bundesrat (the state chamber in the German legislative system) the states ('Länder' in German) have a number of votes depending on their population (in a sub-proportional way). Four states have 6 votes, one state 5, seven states have 4 votes and four states have 3 votes. Normally, the quota is 35, which is just more than half of the total weight[2], for amendments to the constitution (as well as to veto certain types of propositions) a quota of 46 (two-third majority) is needed.

(4) The 'Council of the European Union' (also known as the 'Council of Ministers') is one of the legislative bodies of the EU (the other being the 'European Parliament'). In the Council of Ministers each member state of the EU is represented by one person, usually a Minister of the state's government. In the history of the EU, the voting system of the Council was changed a few times, typically in connection with an enlargement of the Union. Until 2003 the voting rules were given by weighted voting systems.

From 1995 to 2003 the EU consisted of 15 member states with the following voting weights:

Country	Votes	Country	Votes
France	10	Greece	5
Germany	10	Austria	4
Italy	10	Sweden	4
United Kingdom	10	Denmark	3
Spain	8	Finland	3
Belgium	5	Ireland	3
Netherlands	5	Luxembourg	2
Portugal	5		

The quota was given by $q = 62$, corresponding to a relative quota of 79 %. Such quotas, well above 50 %, were (and are) typical for the Council of the EU. This type of voting system is called a 'qualified majority' in the EU jargon.

After 2004, with the eastern extension of the EU, the voting system was defined in the Treaty of Nice, establishing a 'threefold majority'. This voting procedure consists of the intersection of three weighted voting system. After this the Treaty of Lisbon constituted a voting system known as the 'double majority'. There is a transition period between the latter two systems from 2014 through 2017. We discuss these voting systems in more detail in Section 5.

(5) The Board of Governors of the International Monetary Fund makes decisions according to a weighted voting system. The voting weights of the member

[2] In a sense this is a simple majority of the weights. Observe, however, that in this paper we use term 'simple majority rule' only for systems with identical weight for all voters.

countries are related to their economic importance, measured in terms of 'special drawing rights'. The quota depends on the kind of proposal under considerations. Many proposals require a relative quota of 70 %. Proposals of special importance require a quota of even 85 % which makes the USA a veto player in such cases (the USA holds more than 16 % of the votes).

(6) The voting system of the UN Security Council does not seem to be a weighted one on first glance. In fact, the way it is formulated does not assign weights to the members. However, it turns out that one can find weights and a quota which give the same winning coalitions. Thus, according to Definition 9 it is a weighted voting system. For example, if we assign weight 1 to the non-permanent members and 7 to the permanent members and set the quota to be 39, we obtain a voting system which has the same winning coalition as the original one. Consequently these voting systems are the same.

The last example raises the questions: Can all voting systems be written as *weighted* voting systems? And, if not, how can we know, which ones can?

The first question can be answered in the negative by the following argument.

Theorem 13. *Suppose* $(V, \mathcal{V})$ *is a weighted voting system and* A_1 *and* A_2 *are coalitions with* $v_1, v_2 \notin A_1 \cup A_2$. *If both* $A_1 \cup \{v_1\} \in \mathcal{V}$ and $A_2 \cup \{v_2\} \in \mathcal{V}$ *then* $A_1 \cup \{v_2\} \in \mathcal{V}$ or $A_2 \cup \{v_1\} \in \mathcal{V}$ *(or both).*

This property of a voting system is called 'swap robust' in [32]. There and in [34] the interested reader can find more about this and similar concepts.

Proof: Suppose w and q are a weight function and a quota for $(V, \mathcal{V})$.
By assumption

$$w(A_1) + w(v_1) \;\geq\; q \qquad \text{and} \qquad w(A_1) + w(v_1) \;\geq\; q$$

Thus

$$\Big(w(A_1) + w(v_2)\Big) \;+\; \Big(w(A_2) + w(v_1)\Big) \;\geq\; 2\,q$$

It follows that at least one of the summands has to be equal to q or bigger, hence

$$A_1 \cup \{v_2\} \in \mathcal{V} \qquad \text{or} \qquad A_2 \cup \{v_1\} \in \mathcal{V}$$

□

From the theorem above we can easily see that, as a rule, bicameral are not weighted voting systems.

Suppose for example, the voting system consists of two disjoint chambers V_1 (the 'house') and V_2 (the 'senate') with $N_1 = 2n_1 + 1$ and $N_2 = 2n_2 + 1$ members, respectively, with $n_1, n_2 \geq 1$. Let us assume furthermore, that a proposal passes if there is a simple majority rule (by definition with equal voting weight) in both chambers.

So, a coalition of $n_1 + 1$ house members and $n_2 + 1$ senators is winning. Let C be a coalition of n_1 house members and n_2 senators and let h_1 and h_2 be two (different) house members not in C and, in a similar way, let s_1 and s_2 be two (different) senators not in C .

Set $A_1 = C \cup \{h_1\}$ and $A_2 = C \cup \{s_2\}$. Then $A_1 \cup \{s_1\}$ and $A_2 \cup \{h_2\}$ are winning coalitions, since they both contain $n_1 + 1$ house members and $n_2 + 1$ senators. However, $A_1 \cup \{h_2\}$ and $A_2 \cup \{s_1\}$ are both losing: The former coalition contains only n_2 senators, the latter only n_1 house members.

The above reasoning can be generalized easily (see, for example, Theorem 20).

We have seen that swap robustness is a necessary condition for weightedness. But, it turns out swap robustness is *not* sufficient for weightedness. A counterexample (amendment to the Canadian constitution) is given in [32].

There is a simple and surprising combinatorial criterion for weightedness of a voting system which is a generalization of swap robustness, found by Taylor and Zwicker [33], (see also [32] and [34]).

Definition 14. Let $A_1, \ldots, A_K$ be subsets of (a finite set) V.
A sequence $B_1, \ldots, B_K$ of subsets of V is called a *rearrangement* (or *trade*) of $A_1, \ldots, A_K$ if for every $v \in V$

$$|\{k \mid v \in A_k\}| \;=\; |\{j \mid v \in B_j\}|$$

where $|M|$ denotes the number of elements of the set M.

In other words: From the voters in $A_1, \ldots, A_K$ we form new coalitions $B_1, \ldots, B_K$, such that a voter occurring r times in the sets A_k occurs the same number of times in the sets B_j.

For example, the sequence $B_1 = \{1, 2, 3, 4\}, B_2 = \emptyset, B_3 = \{2, 3\}, B_4 = \{2\}$ is a rearrangement of $A_1 = \{1, 2\}, A_2 = \{2, 3\}, A_3 = \{3\}, A_4 = \{2, 4\}$.

Definition 15. A voting system $(V, \mathcal{V})$ is called *trade robust*, if the following property holds for any $K \in \mathbb{N}$:

If $A_1, \ldots, A_K$ is a sequence of winning coalition, i.e. $A_k \in \mathcal{V}$ for all k, and if $B_1, \ldots, B_K$ is a rearrangement of the $A_1, \ldots, A_K$ then at least one of the B_k is winning.

$(V, \mathcal{V})$ is called *M-trade robust*, if the above conditions holds for all $K \leq M$.

Theorem 16 (Taylor and Zwicker)**.** *A voting system is weighted if and only if it is trade robust.*

It is straight forward to prove that any weighted voting system is trade robust. One can follow the idea of the proof of Theorem 13. The other direction of the assertion is more complicated, and more interesting. The proof can be found in [33] or in [34]. In fact, these authors show that every $2^{2^{|V|}}$-trade robust voting system is weighted.

We have seen that there are voting system which can not be written as weighted system. However, it turns out that *any* voting system is an *intersection* of weighted voting systems.

Theorem 17. *Any voting system* $(V, \mathcal{V})$ *is the intersection of weighted voting systems* $(V, \mathcal{V}_1)$, $(V, \mathcal{V}_2)$, $\ldots$, $(V, \mathcal{V}_M)$.

Proof: For any losing coalition $L \subset V$ we define a weighted voting system (on V) by assigning the weight 1 to all voters *not* in L, the weight 0 to the voters *in* L and setting the quota to be 1. Denote the corresponding voting system by $(V, \mathcal{V}_L)$.

Then the losing coalition in this voting systems are exactly L and its subsets. The winning coalitions are those sets K with

$$K \cap (V \setminus L) \;\neq\; \emptyset$$

Then

$$\mathcal{V} \;=\; \bigwedge_{L \subset V;\, L \notin \mathcal{V}} \mathcal{V}_L \tag{3}$$

In deed, if K is winning in *all* the $\mathcal{V}_L$, then K is not losing in $(V, \mathcal{V})$, so $K \in \mathcal{V}$. On the other hand, if $K \in \mathcal{V}$ it is not a subset of a losing coalition by monotonicity, hence $K \in \mathcal{V}_L$ for all losing coalitions L. □

Definition 18. The *dimension* of a voting system $(V, \mathcal{V})$ is the smallest number M, such that $(V, \mathcal{V})$ can be written as an intersection of M weighted voting systems.

It is usually not easy to compute the dimension of a given voting system. For example, the exact dimension of the voting system of the Council of the European Union according to the Lisbon Treaty is unknown. Kurz and Napel [21] prove that its dimension is at least 7.

In a situation of a system divided into 'chambers' we have the following results.

Example 19. Suppose $(V_i, \mathcal{V}_i), i = 1, \ldots, M$ are voting systems with unanimity rule. Then the $(V_1 \cup \ldots \cup V_M, \mathcal{V}_1 \wedge \ldots \wedge \mathcal{V}_M)$ is a weighted voting system, i. e. the intersection of unanimity voting systems has dimension one. In deed, the composed system is a unanimous voting system as well and hence is weighted (see Example 12.2).

Theorem 20. *Let* $(V_1, \mathcal{V}_1), (V_2, \mathcal{V}_2), \ldots, (V_M, \mathcal{V}_M)$ *be simple majority voting systems with pairwise disjoint* V_i.

If for all i *we have* $|V_i| \geq 3$ *then the dimension of*

$$(V, \mathcal{V}) := (V_1 \cup \ldots \cup V_M, \mathcal{V}_1 \wedge \ldots \wedge \mathcal{V}_M)$$

is M.

Proof: First, we observe that for each i there is a losing coalition L_i and voters $\ell_i, \ell_i' \in V_i \setminus L_i$, such that $L_i \notin \mathcal{V}_i$, but $L_i \cup \{\ell_i\} \in \mathcal{V}_i$ and $L_i \cup \{\ell_i'\} \in \mathcal{V}_i$.

Suppose there were weighted voting systems $(U_1, \mathcal{U}_1), \ldots, (U_K, \mathcal{U}_K)$ with $K < M$ such that their intersection is $(V, \mathcal{V})$. We consider the coalitions

$$K_i = V_1 \cup \ldots \cup V_{i-1} \cup L_i \cup V_{i+1} \cup \ldots \cup V_M$$

Then all K_i are losing in $(V, \mathcal{V})$, since L_i is losing in $\mathcal{V}_i$ by construction. Hence, for each K_i there is a j, such that K_i is losing in $(U_j, \mathcal{U}_j)$. Since $K < M$ there is a $j \leq K$ such that two different K_i are losing coalitions in $(U_j, \mathcal{U}_j)$, say K_p and K_q with $p \neq q$.

Now, we exchange two voters between K_p and K_q, more precisely we consider

$$\begin{aligned} K'_p &:= \Big(K_p \setminus \{\ell_q\}\Big) \cup \{\ell_p\} \\ \text{and} \quad K'_q &:= \Big(K_q \setminus \{\ell_p\}\Big) \cup \{\ell_q\} \end{aligned}$$

By construction, both K'_p and K'_q are *winning* coalitions in $(V, \mathcal{V})$ and hence in $(U_j, \mathcal{U}_j)$. But this is impossible since K'_p and K'_q arise from two losing coalitions by a swap of two voters and $(U_j, \mathcal{U}_j)$ is weighted, hence swap robust. $\square$

Example 21. The US federal legislative system (see Example 8) is not a weighted voting system due to two independent chambers (House and Senate). Moreover, it has two components (President and Vice President) with unanimity rule, and, as we defined it, it contains a *union* of voting systems. So, our previous results on dimension do not apply. It turns out, that it has dimension 2 (see [32]).

4. Voting Power

Imagine two countries, say France and Germany, plan to cooperate more closely by building a council which decides upon certain questions previously decided by the two governments. The members of the council are the French President and the German Chancellor.

The German side suggests that the council members get a voting weight proportional to the population of the corresponding country. So, the French President would have a voting weight of 6, the German Chancellor a weight of 8, corresponding to a population of about 60 millions and 80 millions respectively. 'Of course', for a proposal to pass one would need more than half of the votes.

It is obvious that the French side would not agree to these rules. No matter how the French delegate will vote in this council, he or she will never ever affect the outcome of a voting! The French delegate is a 'dummy player' in this voting system.

Definition 22. Let $(V, \mathcal{V})$ be a voting system.
A voter $v \in V$ is called a *dummy player* (or dummy voter) if for any winning coalition A which contains v the coalition $A \setminus \{v\}$, i.e. the coalition A with v removed, is still winning.

One might tend to believe that dummy players will not occur in real world examples. Surprisingly enough, they do.

Example 23 (Council of EEC)**.** In 1957 the 'Treaty of Rome' established the European Economic Community, a predecessor of the EU, with Belgium, France, Germany, Italy, Luxembourg and the Netherlands as member states. In the Council of the EEC the member states had the following voting weights.

Country	Votes
Belgium	2
France	4
Germany	4
Italy	4
Luxembourg	1
Netherlands	2

The quota was 12.

In this voting system Luxembourg is a dummy player! In deed, the minimal winning coalitions consist of either the three 'big' countries (France, Germany and Italy) or two of the big ones and the two medium sized countries (Belgium and the Netherlands). Whenever Luxembourg is a member of a winning coalition, the coalition is also winning if Luxembourg defects. This voting system was in use until 1973.

From these examples we learn that there is no *immediate* way to estimate the power of a voter from his or her voting weight. For instance, in the above example Belgium is certainly more than twice as powerful as Luxembourg. Whatever 'voting power' may mean in detail, a dummy player will certainly have *no* voting power.

In the following we'll try to give the term 'voting power' an exact meaning. There is no doubt that in a mathematical description only *certain aspects* of power can be modelled. For example, aspects like the art of persuasion, the power of the better argument or external threats will not be included in those mathematical concepts.

In this section we introduce a method to measure power which goes back to Penrose [27] and Banzhaf [2]. It is based on the definition of power as the ability of a voter to change the outcome of a voting by his or her vote. Whether my vote 'counts' depends on the behavior of the other voters. We'll say that a voter v is 'decisive' for a *losing* coalition A if A becomes winning if v joins the coalition, we call v 'decisive' for a *winning* coalition if it becomes losing if v leaves this coalition. More precisely:

Definition 24. Suppose $(V, \mathcal{V})$ is a voting system. Let $A \subset V$ be a coalition and $v \in V$ a voter.

(1) We call v *winning decisive* for A if $v \notin A$, $A \notin \mathcal{V}$ and $A \cup \{v\} \in \mathcal{V}$.
We denote the set of all coalitions for which v is winning decisive by

$$\mathcal{D}^+(v) \;:=\; \{A \subset V \mid A \notin \mathcal{V};\; v \notin A;\; A \cup \{v\} \in \mathcal{V}\,\} \tag{4}$$

(2) We call v *losing decisive* for A if $v \in A$, $A \in \mathcal{V}$ and $A \setminus \{v\} \notin \mathcal{V}$.
We denote the set of all coalitions for which v is losing decisive by

$$\mathcal{D}^-(v) \ := \ \{A \subset V \mid A \in \mathcal{V};\ v \in A;\ A \setminus \{v\} \in \mathcal{V}\ \} \tag{5}$$

(3) We call v *decisive* for A if v is winning decisive or losing decisive for A.
We denote the set of all coalitions for which v is decisive by

$$\mathcal{D}(v) \ := \ \mathcal{D}^+(v) \cup \mathcal{D}^-(v)\,. \tag{6}$$

The 'Penrose-Banzhaf Power' for a voter v is defined as the portion of coalitions for which v is decisive. Note that for a voting system with N voters there are 2^N (possible) coalitions.

Definition 25. Suppose $(V, \mathcal{V})$ is a voting system, $N = |V|$ and $v \in V$. We define the *Penrose-Banzhaf power* $PB(v)$ of v to be

$$PB(v) \ = \ \frac{|\mathcal{D}(v)|}{2^N}$$

Remark 26. The Penrose-Banzhaf power associates to each voter v a number $PB(v)$ between 0 and 1, in other words PB is a function $PB : V \to [0,1]$. It associates with each voter the fraction of coalitions for which the voter is decisive.

If we associate to each coalition the probability $\frac{1}{2^N}$, thus considering all coalitions as equally likely, then $PB(v)$ is just the probability of the set $\mathcal{D}(v)$. Of course, one might consider other probability measure $\mathbb{P}$ on the set of all coalitions and define a corresponding power index by $\mathbb{P}\big(\mathcal{D}(v)\big)$. We will discuss this issue later.

If a coalition A is in $\mathcal{D}^-(v)$ then $A \cup \{v\}$ is in $\mathcal{D}^+(v)$ and if A is in $\mathcal{D}^+(v)$ then $A \setminus \{v\}$ is in $\mathcal{D}^-(v)$. This establishes a one-to-one mapping between $\mathcal{D}^+(v)$ and $\mathcal{D}^-(v)$. It follows that

$$|\mathcal{D}^+(v)| \ = \ |\mathcal{D}^-(v)| \ = \ \frac{1}{2}\,|\mathcal{D}(v)|\,. \tag{7}$$

This proves:

Proposition 27. *If $(V, \mathcal{V})$ is a voting system with N voters and $v \in V$ then*

$$PB(v) \ = \ \frac{|\mathcal{D}^+(v)|}{2^{N-1}} \ = \ \frac{|\mathcal{D}^-(v)|}{2^{N-1}} \tag{8}$$

We also define a normalized version of the Penrose-Banzhaf power.

Definition 28. If $(V, \mathcal{V})$ is a voting system with Penrose-Banzhaf power $PB : V \to [0,1]$ then we call the function $NPB : V \to [0,1]$ defined by

$$NPB(v) \ := \ \frac{PB(v)}{\sum_{w \in V} PB(w)}$$

the *Penrose-Banzhaf index* or the *normalized Penrose-Banzhaf power*.

The Pernose-Banzhaf index quantifies the share of power a voter has in a voting system.

Proposition 29. *Let $(V, \mathcal{V})$ be a voting system with Penrose-Banzhaf power PB and Penrose-Banzhaf index NPB.*

(1) *For all $v \in V$: $0 \leq PB(v) \leq 1$ and $0 \leq NPB(v) \leq 1$. Moreover,*

$$\sum_{v \in V} NPB(v) = 1 \,. \tag{9}$$

(2) *A voter v is a dummy player if and only if $PB(v) = 0$ ($\Leftrightarrow NPB(v) = 0$).*

(3) *A voter v is a dictator if and only if $NPB(v) = 1$.*

As an example we compute the Penrose-Banzhaf power and the Penrose-Banzhaf index for the Council of the EEC.

Country	Votes	PB	NPB
Belgium	2	3/16	3/21
France	4	5/16	5/21
Germany	4	5/16	5/21
Italy	4	5/16	5/21
Luxembourg	1	0	0
Netherlands	2	3/16	3/21

For small voting bodies (as for the above example) it is possible to compute the power indices with pencil and paper, but for bigger systems one needs a computer to do the calculations. For example, the programm IOP 2.0 (see [5]) is an excellent tool for this purpose.

For a parliament with N members and equal voting weight (and any quota) it is clear that the Penrose-Banzhaf *index* $NPB(v)$ is $\frac{1}{N}$ for any voter v. This follows from symmetry and formula (9).

It is instructive (and useful later on) to compute the Penrose-Banzhaf *power* in this case.

Theorem 30. *Suppose $(V, \mathcal{V})$ is a voting system with N voters, voting weight one and simple majority rule.*
Then the Penrose-Banzhaf power $PB(v)$ is independent of the voter v and

$$PB(v) \;\approx\; \frac{2}{\sqrt{2\pi}} \, \frac{1}{\sqrt{N}} \qquad \text{as } N \to \infty \,. \tag{10}$$

Remark 31. By $a(N) \approx b(N)$ as $N \to \infty$ we mean that $\lim_{N\to\infty} \frac{a(N)}{b(N)} = 1$

Theorem 30 asserts that the Penrose-Banzhaf power in a body with simple majority rule is roughly inverse proportional to the *square-root* of the number N of voters and not to N itself as one might guess at a first glance. So, in a system with four times as much voters, the Penrose-Banzhaf power of a voter is one half ($= \frac{1}{\sqrt{4}}$) of the power of a voter in the smaller system. The reason is that there are much more coalitions of medium size (with about $N/2$ participants) than coalitions of small or large size. This fact will be important later on!

The proof is somewhat technical and can be omitted by readers who are willing to accept the theorem without proof.

Proof: We treat the case of odd N, the other case being similar. So suppose $N = 2n + 1$. A voter v is decisive for a losing coalition A if and only if A contains exactly n voters (but not v).

There are $\binom{2n}{n}$ such coalitions. Hence, by (10)

$$BP(v) \;=\; \frac{1}{2^{2n}} \binom{2n}{n} . \tag{11}$$

Now, we use Stirling's formula to estimate $\binom{2n}{n}$. Stirling's formula asserts that

$$n! \;\approx\; n^n \, e^{-n} \sqrt{2\pi n} \qquad \text{as } n \to \infty \,.$$

Thus, as $N \to \infty$ we have:

$$\begin{aligned} \binom{2n}{n} \;&\approx\; \frac{(2n)^{2n} \, e^{-2n} \; 2 \sqrt{\pi n}}{n^{2n} \, e^{-2n} \; 2 \, \pi \, n} \\ &=\; \frac{2^{2n}}{\sqrt{\pi}\sqrt{n}} \end{aligned}$$

so

$$BP(v) \;\approx\; \frac{2}{\sqrt{2\pi}} \, \frac{1}{\sqrt{N}}$$

□

Instead of using *decisiveness* as a basis to measure power one could use the voter's *success*. A procedure to do so is completely analogous to the considerations above: We count the number of 'times' a voter agrees with the result of the voting ('is successful').

Definition 32. Suppose $(V, \mathcal{V})$ is a voting system. For a voter $v \in V$ we define

the set of *positive success*

$$\mathcal{S}_{+}(v) \;=\; \{A \in \mathcal{V} \mid v \in A\} \tag{12}$$

the set of *negative success* by

$$\mathcal{S}_-(v) \;=\; \{A \notin \mathcal{V} \mid v \notin A\} \tag{13}$$

and the set of *success*

$$\mathcal{S}(v) \;=\; \mathcal{S}_+(v) \cup \mathcal{S}_-(v) \tag{14}$$

Remark 33. If A is the coalition of voters agreeing with a proposal, then $A \in \mathcal{S}_+(v)$ means, the proposal is approved with the consent of v, similarly $A \in \mathcal{S}_-(v)$ means, the proposal is rejected with the consent of v.

Definition 34. The *Penrose-Banzhaf rate of success* $Bs(v)$ is defined as the portion of coalitions such that v agrees with the voting result, more precisely:

$$Bs(v) \;=\; \frac{|\mathcal{S}(v)|}{2^N}$$

where N is the number of voters in V.

Remark 35. For all v we have $Bs(v) \geq 1/2$, in particular, a dummy player v has $Bs(v) = 1/2$.

There is a close connection between the Penrose-Banzhaf power and the Penrose-Banzhaf rate of success.

Theorem 36. *For any voting system* $(V, \mathcal{V})$ *and any voter* $v \in V$

$$Bs(v) \;=\; \frac{1}{2} \;+\; \frac{1}{2}\,PB(v) \tag{15}$$

This is a version of a theorem by Dubey and Shapley [8]. It follows that the success probability of a voter among N voters in a body with simple majority rule is approximately $\frac{1}{2} + \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{N}}$.

The above results makes it essentially equivalent to define voting power via decisiveness or via success. However, equation (15) is peculiar for the special way we count coalitions here. If we don't regard all coalitions as equally likely, (15) is not true in general (see [23]).

5. The Council of the European Union: A case study

In many supranational institutions the member states are represented by a delegate, for example a member of the country's government. Examples of such institutions are the International Monetary Fund, the UN Security Council and the German Bundesrat (for details see Examples 12). Our main example, which we are going to explain in more detail, is the Council of the European Union ('Council of Ministers'). The European Parliament and the Council of Ministers are the two legislative institutions of the European Union.

In the Council of the European Union each state is represented by one delegate (usually a Minister). Depending on the agenda the Council meets in different 'configurations', for example in the 'Agrifish' configuration the agriculture and fishery ministers of the member states meet to discuss questions in their field. In each configuration, every one of the 28 member countries is represented by one member of the country's government. The voting rule in the Council has changed a number of times during the history of the EU (and its predecessors). Until the year 2003 the voting rule was a weighted one. It was common sense that the voting weight of a state should increase with the state's size in terms of population. The exact weights were not determined by a formula or an algorithm but were rather the result of negotiations among the governments. The weights during the period 1958–1973 are given in Example 23, those during the period 1995–2003 are discussed in Example 12 (4). The three, later four, big states, France, Germany, Italy, and the United Kingdom, used to have the same number of votes corresponding to a similar size of their population, namely around 60 millions. After German unification, the German population suddenly increased by about one third.

This fact together with the planned eastern accession of the EU were the main issues at the European Summit in Nice in December 2000. The other big states disliked the idea to increase the voting weight of Germany beyond their own one while the German government pushed for a bigger voting weight for the country. The compromise found after nightlong negotiations 'in smoky back-rooms' was the 'Treaty of Nice'. In mathematical terms, the voting system of Nice is the intersection of three (!) weighted voting systems, each system with the same set of voters (the Ministers), but with different voting rules. In the first system a simple majority of the member states is required. The second system is a weighted voting system the weights of which are the result of negotiations (see Table 1 in the Appendix). In particular the four biggest states obtained 29 votes each, the next biggest states (Spain and Poland) got 27 votes. In the Treaty of Nice two inconsistent quotas are stipulated for the EU with 27 members: At one place in the treaty the quota is set to 255, in another section it is fixed at 258 of the 345 total weight![3] With the accession of Croatia in 2013 the quota was set to 260 of a total weight of 352. In the third voting system, certainly meant as a concession towards Germany, the voting weight is given by the population of the respective country. The quota is set to 64 %. With these rules the Nice procedure is presumably one the most complicated voting system ever implemented in practice.

It is hopeless to analyze this system 'with bare hands'. For example, it is not at all obvious to which extend Germany gets more power through the third voting system, the only one from which Germany can take advantage of its bigger population compared to France, Italy and the UK. One can figure out that the Penrose-Banzhaf power index of Germany is only negligibly bigger than that of the other big states, the difference in Penrose-Banzhaf index between Germany and

[3] The self-contradictory Treaty of Nice was signed and ratified by 27 states.

France (or Italy or the UK) is about 0.000001. If instead of the voting according to population Germany had been given a voting weight of 30 instead of 29, this difference would be more than 1600 times as big! (for more details see Table 1 in the Appendix and the essay [16]).

In 2002 and 2003 the 'European Convention', established by the the European Council and presided by former French President Valéry Giscard d'Estaing, developed a 'European Constitution' which proposed a new voting system for the Council of the EU, the 'double majority'. The double majority system is the intersection of two weighted voting systems, one in which each member state has just one vote, the other with the population of the state as its voting weight. This seems to resemble the US bicameral system ('The Connecticut Compromise'): The House with proportional representation of the states and the Senate with equal votes for all states.

Presumably the reasoning behind the double majority rule is close to the following: On one hand, the European Union is a union of *citizens*. A fair representation of citizens, so the reasoning, would require that each state has a voting weight proportional to its population. On the other hand, the EU is a union of independent *states*, in this respect it would be just to give each state the same weight. The double majority seems to be a reasonable compromise between these two views.

The European Constitution was not ratified by the member states after its rejection in referenda in France and the Netherlands, but the idea of the double majority was adopted in the 'Treaty of Lisbon'. The voting system in the Council, according to the Treaty of Lisbon is 'essentially' the intersection of two weighted voting systems.

In the first voting system ($\mathcal{V}_1$) each representative has one vote (i. e. voting weight =1), the relative quota is 55 %. In the second system ($\mathcal{V}_2$) the voting weight is given by the population of the respective state, the relative quota being 65 %.

Actually, a third voting system ($\mathcal{V}_3$) is involved, in which each state has voting weight 1 again, but with a quota of 25 (more precisely three less than the number of member states). The voting system $\mathcal{V}$ of the Council is given by:

$$\mathcal{V} \;:=\; \big(\mathcal{V}_1 \,\wedge\, \mathcal{V}_2\big) \;\vee\; \mathcal{V}_3$$

In other words: A proposal requires either the consent of 55 % of the states which also represent at least 65 % of the EU population or the approval by 25 states.

The third voting system does not play a big role in practice, but is merely important psychologically as it eliminates the possibility that three big states alone can block a proposal. This rule actually adds 10 winning coalitions to the more than 30 million winning coalitions if only the two first rules were applied.

Compared to the Treaty of Nice the big states like Germany and France gain power by the Lisbon system, others in particular Spain and Poland loose considerably. Not surprisingly, the Polish government under Premier Minister Jarosław Kaczyński objected heavily to the new voting system. They proposed a rule called the 'square root system', under which each state gets a voting weight proportional to the square root of its population. In fact, the slogan of the Polish government was ‘Square Root or Death’.

The perception of this concept in the media as well as among politicians was anything but positive. For example, in a column of the Financial Times [28], one reads: ‘Their [the poles’] slogan for the summit - “the square root or death” - neatly combines obscurity, absurdity and vehemence’ and: ‘Almost nobody else wants the baffling square root system ...’. In terms of the Penrose-Banzhaf indices, the square root system is to a large extend between the Nice and the Lisbon system. The square root system was finally rejected by the European summit.

With three rather different systems under discussion and two of them implemented the question arises: What is a just system? This, of course, is not a mathematical question. But, once the concept of ‘justice’ is clarified, mathematics may help to determine the best possible system.

One way to approach this question is to consider the influence citizens of the EU member states have on decisions of the Council. Of course, this influence is rather indirect by the citizens ability to vote for or against their current government. A reasonable criterion for a just system would be that every voter has the same influence on the Council’s decisions regardless of the country whose citizens he or she is. This approach will be formalized and investigated in the next section.

6. Two-Tier voting systems

In a direct democracy the voters in each country would instruct their delegate in the Council by public vote how to behave in the Council.[4] Thus the voters in the Union would decide in a two-step procedure. The first step is a public vote in each member state, the result of which would determine the votes of the delegates in the Council and hence the final decision. In fact, such a system is (in essence) implemented in the election of the President of the USA through the Electoral College.[5]

Modern democracies are -almost without exceptions- representative democracies. According to the idea of representative democracy, the delegate in the Council of Ministers will act on behalf of the country’s people and is -in principle- responsible to them. Consequently, we will assume idealistically (or naively?) that the delegate in the Council knows the opinion of the voters in her or his country and acts in the Council accordingly. If this is the case we can again regard the decisions of the Council as a two-step voting procedure in which the first step -the public vote- is invisible, but its result is known or at least guessed with some precision by the government and moreover is obeyed by the delegate.

In such a ‘two-tier’ voting system we may speak about the (indirect) influence

[4] Of course, the voters in the Union could also decide directly then, we’ll talk about this in the next section.

[5] As a rule, the winner of the public vote in a state appoints *all* electors of that state. This is different only in Nebraska and Maine.

a voter in one of the member states has on the voting in the Council. Now, we define these notions formally.

Definition 37. Let $(S_1,\mathcal{S}_1),\ldots,(S_M,\mathcal{S}_M)$ be voting systems ('M states') with $S := \bigcup_{i=1}^{M} S_i$ ('the union'). Suppose furthermore that $C = \{c_1,\ldots,c_M\}$ ('Council with delegates of the states') and that $(C,\mathcal{C})$ is a voting system.
For a coalition $A \subset S$ define

$$\Phi(A) \;=\; \{\, c_i \mid A \cap S_i \in \mathcal{S}_i\} \qquad (16)$$

$$\text{and} \qquad \mathcal{S} \;=\; \{A \subset S \mid \Phi(A) \in \mathcal{C}\} \qquad (17)$$

The voting system $(S,\mathcal{S})$ is called the *two-tier voting system* composed of the lower tier voting systems $(S_1,\mathcal{S}_1),\ldots,(S_M,\mathcal{S}_M)$ and the upper tier voting system $(C,\mathcal{C})$. We denote it by $\mathcal{S} = \mathfrak{T}\Big(\mathcal{S}_1,\ldots,\mathcal{S}_M;\mathcal{C}\Big)$.

Example 38. The Council of the EU can be regarded as typical two-tier voting system. We imagine that the voters in each member state decide upon proposals by simple majority vote (e. g. through opinion polls) and the Ministers in the Council vote according to the decision of the voters in the respective country.

We call systems as in the above example 'simple two-tier voting systems', more precisely:

Definition 39. Suppose $(S_1,\mathcal{S}_1),\ldots,(S_M,\mathcal{S}_M)$ and $(C,\mathcal{C})$ with $C = \{c_1,\ldots,c_M\}$ are voting systems. The corresponding two-tier voting system $(S,\mathcal{S})$ with $\mathcal{S} = \mathfrak{T}\Big(\mathcal{S}_1,\ldots,\mathcal{S}_M;\mathcal{C}\Big)$ is called a *simple two-tier voting system* if the set S_i are pairwise disjoint and the $\mathcal{S}_i$ are simple majority voting systems.

We are interested in the voting power exercised indirectly by a voter in one of the states S_i. For the (realistic) case of simple majority voting in the states and arbitrary decision rules in the Council we have the following result. In its original form this result goes back to Penrose [27].

Theorem 40. *Let $(S,\mathcal{S})$ be a simple two-tier voting system composed of $(S_1,\mathcal{S}_1),\ldots,(S_M,\mathcal{S}_M)$ and $(C,\mathcal{C})$ with $C = \{c_1,\ldots,c_M\}$. Set $N_i = |S_i|$, $N = \sum_{i=1}^{M} N_i$ and $N_{\min} = \min_{1\le i\le M} N_i$.*

If PB_i is the Penrose-Banzhaf power of c_i in $\mathcal{C}$, then the Penrose-Banzhaf power $PB(v)$ of a voter $v \in S_k$ in the two-tier voting system $(S,\mathcal{S})$ is asymptotically given by:

$$PB(v) \quad \approx \quad \frac{2}{\sqrt{2\pi N_k}} \quad PB_k \qquad\qquad as \;\; N_{\min} \to \infty \qquad (18)$$

Proof: To simplify the notation (and the proof) we assume that all N_i are odd, say $N_i = 2n_i+1$. The case of even N_i requires an additional estimate but is similar otherwise.

A voter $v \in S_k$ is critical in $\mathcal{S}$ for a losing coalition A if and only if v is critical for the losing coalition $A \cap S_k$ in $\mathcal{S}_k$ *and* the delegate c_k of S_k is critical in $\mathcal{C}$ for the losing coalition $\Phi(A)$.

Under our assumption, for any coalition B in S_i either $B \in \mathcal{S}_i$ or $S_i \setminus B \in \mathcal{S}_i$. [6] So, for each i there are exactly $2^{N_i - 1}$ winning coalitions in $\mathcal{S}_i$ and the same number of losing coalitions. For each coalition $K \subset C$ there are consequently 2^{N-M} different coalitions A in S with $\Phi(A) = K$.

According to Theorem 30 there are $\approx \frac{2}{\sqrt{2\pi N_k}}\, 2^{N_k - 1}$ losing coalition B in $\mathcal{S}_k$ for which v is critical. So, for each losing coalition $K \subset C$ there are approximately

$$\frac{2}{\sqrt{2\pi N_k}}\, 2^{N-M}$$

losing coalitions for which v is critical in $\mathcal{S}_k$.

There are $2^{M-1}\, PB_k$ losing coalitions in C for which c_k is critical, hence there are

$$2^{N-1} PB_k \frac{2}{\sqrt{2\pi N_k}}$$

losing coalitions in $\mathcal{S}$ for which $v \in S_k$ is critical. Thus

$$PB(v) \quad \approx \quad \frac{1}{2^{N-1}}\, 2^{N-1} PB_k \frac{2}{\sqrt{2\pi N_k}} \quad = \quad \frac{2}{\sqrt{2\pi N_k}}\, PB_k$$

□

There is an important -and perhaps surprising- consequence of Theorem 40. In a two-tier voting system as in the theorem it is certainly desirable that all voters in the union have the *same* influence on decisions of the Council regardless of their home country.

Corollary 41 (Square Root Law by Penrose)**.** *If* $(S, \mathcal{S})$ *is a simple two-tier voting system composed of* $(S_1, \mathcal{S}_1), \ldots, (S_M, \mathcal{S}_M)$ *with* $N_i = |S_i|$ *and* $(C, \mathcal{C})$ *with* $C = \{c_1, \ldots, c_M\}$*. Then for large* N_i *we have:*

The Penrose-Banzhaf power $PB(v)$ *in* $\mathcal{S}$ *for a voter* $v \in S_k$ *is independent of* k *if and only if the Penrose-Banzhaf power* PB_i *of* c_i *is given by* $C\sqrt{N_i}$ *for all* i *with some constant* C*.*

Thus, the optimal system (in our sense) is (at least very close to) the 'baffling' system proposed by the Polish government! Making the voting weights proportional to the square root of the population does not give automatically power indices proportional to that square root. However, Wojciech Słomczyński and Karol Życzkowski [30] from the Jagiellonian University Kraków found that in a weighted

[6] For even N_i this is only approximately true. Therefore, the case of odd N_i is somewhat easier.

voting system for the Council in which the *weights* are given by the square root of the population and the relative quota is set at (about) 62 %, the resulting Penrose-Banzhaf index follows the square root law very accurately. This voting system is now known as the *Jagiellonian Compromise*. Despite the support of many scientists (see e. g. [25],[17],[24]), this system was ignored by the vast majority of politicians.

Table 1 in the appendix shows the Penrose-Banzhaf power indices for the Nice system and compares it to the square root law, which is the ideal system according to Penrose. There is a pretty high relative deviation from the square root law. Some states, like Greece and Germany, for example, get much less power than they should, others, like Poland, Ireland and the smaller states gain too much influence. All in all there seems to be no systematic deviation.

The same is done in Table 2 for the Lisbon rules. Under this system, Germany and the small states gain too much power while all medium size states do not get their due share. Assigning a weight proportional to the population is over-representing the big states according to the Square Root Law. In a similar manner, giving all states the same weight is over-representing the small states, if equal representation of all citizens is aimed at. One might hope that the Lisbon rules compensate these two errors. But this is not the case. The Lisbon rules over-represent both very big and very small states, but under-represents all others.

One might hope that the Nice or the Lisbon system may observe the square root law at least approximately if the quota are arranged properly. This is not the case, see [19].

The indices in Table 1 and Table 2 were computed using the powerful program IOP 2.0 by Bräuninger and König [5].

7. A Probabilistic Approach

In this section, we sketch an alternative approach to voting, in particular to the question of optimal weights in two-tier voting systems, namely a probabilistic approach. This section is mathematically more involved than the previous part of this paper, but it also gives, we believe, more insight to the question of a fair voting system.

7.1. Voting Measures and First Examples.

We regard a voting system $(V, \mathcal{V})$ as a system that produces output ('yes' or 'no') to a random stream of proposals. We assume that these proposals are totally random, in particular a proposal and its opposite are equally likely. The proposal generates an answer by the voters, i. e. it determines a coalition A of voters that support it. If A is a winning coalition the voting system's output is 'yes', if A is losing, the output is 'no'.

It is convenient to assume (without loss of generality) that $V = \{1, 2, \dots, N\}$.

We denote the voting behavior the voter i by $X_i \in \{-1,+1\}$. X_i depends, of course, on the proposal ω under consideration. $X_i(\omega) = 1$ means voter i agrees with the proposal, $X_i(\omega) = -1$ means i rejects ω. The voting result of all voters is a vector $X = (X_1, \ldots, X_N) \in \{-1,+1\}^N$. The random input generates a probability distribution $\mathbb{P}$ on $\{-1,+1\}^N$ and thus makes the X_i random variables.

The voting rules associate to each voting vector X a voting outcome: 'Yes' or 'No'.

We assume that the voters act rationally, at least in the sense that they either agree with a proposal or with its opposite, but never with both. Since we regard a proposal and its counterproposal as equally likely rationality implies that the probability $\mathbb{P}$ is invariant under changing all voters' decisions, in the sense of the following definition.

Definition 42. A probability measure $\mathbb{P}$ on $\{-1,+1\}^N$ is called a *voting measure* if

$$\mathbb{P}\big(X_1 = x_1, \ldots, X_N = x_N\big) \;=\; \mathbb{P}\big(X_1 = -x_1, \ldots, X_N = -x_N\big) \tag{19}$$

for all $x_1, x_2, \ldots, x_N \in \{-1, 1\}$.

Remark 43. Definition 42 implies in particular that

$$\mathbb{P}\big(X_i = 1\big) \;=\; \mathbb{P}\big(X_i = -1\big) \;=\; \frac{1}{2} \tag{20}$$

Thus the distribution of any single X_i is already fixed. Note, however that there is still a great deal of freedom to choose the measure $\mathbb{P}$. It is the correlation structure that makes voting measures differ from one another. This correlation structure describes how voters may be influenced by each other or by some other factors like common believes or values, a state ideology, a dominant religious group or other opinion makers.

Notation 44. (1) Given a voting measure $\mathbb{P}$ on $\{-1,+1\}^N$ we use the same letter $\mathbb{P}$ to denote an associated measure on the set of all coalitions given by:

$$\mathbb{P}\big(A\big) \;:=\; \mathbb{P}\big(X_i = 1 \text{ for } i \in A \text{ and } X_i = -1 \text{ for } i \notin A\big) \tag{21}$$

(2) To shorten notation we use the short hand

$$\mathbb{P}_B\big(x_1, x_2, \ldots, x_N\big) \;:=\; \mathbb{P}\big(X_1 = x_1, X_2 = x_2, \ldots, X_N = x_N\big)$$

Example 45. (1) If we assume the voters cast their votes independently of each other we obtain the probability measure we already encountered in Remark 26:

$$\mathbb{P}_B(x_1, \ldots, x_N) \;=\; \frac{1}{2^N} \tag{22}$$

for all $(x_1, \ldots, x_N) \in \{+1,-1\}^N$. We call this voting measure the *independence measure* or the *Penrose-Banzhaf measure*, since it leads to the Penrose-Banzhaf power index.

(2) Another measure which occurs in connection with the Shapley-Shubik power index ([29],[31]) is the measure

$$\mathbb{P}_S(x_1,\ldots,x_N) \;=\; \frac{1}{2^{N+1}}\int_{-1}^{1}(1+p)^k\,(1-p)^{N-k}\,dp \tag{23}$$

where $k=|\{i \mid x_i=1\}|$.

We calll this voting measure the *Shapley-Shubik measure*. We will discuss it and its generalization, the 'common believe measure', below.

(3) Extreme agreement between the voters may be modelled by the voting measure $\mathbb{P}_{\pm 1}$ ('unanimity measure') which is concentrated on $(-1,-1,\ldots,-1)$ and $(1,1,\ldots,1)$, i. e.

$$\mathbb{P}_{\pm1}\big(x_1,x_2,\ldots,x_N\big) \;=\; \begin{cases} \frac{1}{2}, & \text{if } x_i=1 \text{ for all } i;\\ \frac{1}{2}, & \text{if } x_i=-1 \text{ for all } i;\\ 0, & \text{otherwise.}\end{cases} \tag{24}$$

7.2. Basic examples.

Now we introduce two classes of voting measures which we will discuss below in connection with two-tier voting systems.

7.2.1. The Common Believe Measure. The 'common believe' voting measure is a generalization of the Shapley-Shubik measure (see [14]). In this model there is a common believe in a group (e. g. a state). For example, there might be a dominant religion inside the state with a strong influence on the people in certain questions of ethics.

This 'believe' associates to a proposal a probability with which voters inside the group will agree with this proposal. The common believe is a random variables Z with values in the interval $[-1,1]$ and distribution (=measure) μ, i. e. $\mu(I)=\mathbb{P}(Z\in I)$ for any interval I. $Z>0$ models a collective tendency in favor of the proposal at hand. This tendency increases with increasing Z, analogously $Z<0$ means a tendency against the proposal. More precisely, if $Z=\zeta$ then the voters still decide independent of each other, but with a probability

$$p_\zeta \;=\; \frac{1}{2}\big(1+\zeta\big) \qquad \text{for 'yes' and} \tag{25}$$

$$1-p_\zeta \;=\; \frac{1}{2}\big(1-\zeta\big) \qquad \text{for 'no'}. \tag{26}$$

Definition 46. If μ is a probability measure on $[-1,1]$ with $\mu([a,b])=\mu([-b,-a])$ then we call the voting measure $\mathbb{P}_\mu$ on $\{-1,+1\}^N$ defined by

$$\mathbb{P}_\mu\big(A\big) \;:=\; \int_{-1}^{1} P_\zeta\big(A\big)\,d\mu(\zeta) \tag{27}$$

the *common believe voting measure* with collective measure μ.
Here

$$P_\zeta(A) \;:=\; {p_\zeta}^{|A|}\,\big(1-p_\zeta\big)^{N-|A|} \tag{28}$$

is a product measure.

Remark 47. (1) The probability p_ζ defined in (25) is chosen such that the expectation is given by ζ.

(2) The condition $\mu([a,b]) = \mu([-b,-a])$ ensures that $\mathbb{P}_\mu$ is a voting measure.

(3) For $\mu = \delta_0$ we obtain the Penrose-Banzhaf measure $\mathbb{P}_B$.

(4) If μ is the uniform distribution on $[-1,1]$ we recover the Shapley-Shubik measure $\mathbb{P}_S$ of (23).

7.2.2. The Curie-Weiss Model. Finally, we introduce a model which originates in the statistical physics of magnetism. In this original context the meaning of the random variables X_i is the state of an elementary magnet pointing 'up' (for $X_i = 1$) or 'down' ($X_i = -1$). In statistical physics one is interested in describing collective phenomena, in particular alignment, of the magnets. This collective behavior depends on an external parameter, the temperature T. It is common in physics to introduce the 'inverse temperature' $\beta = \frac{1}{T}$. At low β (high temperature) one expects rather random (i. e. almost independent) behavior of the magnets, while for high β one expects that most of the magnets point into the same direction. So, β measures the strength of the (positive) correlation between the magnets, in our case between the voters.

One of the easiest models in physics which actually shows such a behavior is the 'Curie-Weiss' model. We will describe and use this model in the context of voting. While the Common Believe model describes values (or prejudices), the Curie-Weiss measure models the tendency of voters to agree with one another.

Definition 48. For given $\beta \geq 0$ the *Curie-Weiss measure* $\mathbb{P}_\beta$ is the probability measure

$$\mathbb{P}_\beta\big(x_1, x_2, \ldots, x_N\big) \;:=\; Z^{-1}\, e^{\frac{\beta}{2N}\,(\sum_{i=1}^{N} x_i)^2} \tag{29}$$

where $Z = Z_\beta$ is the normalization which makes $\mathbb{P}_\beta$ a probability measure, i. e.

$$Z \;=\; \sum_{x_1,x_2,\ldots,x_N \in \{-1,+1\}} e^{\frac{\beta}{2N}\,(\sum_{i=1}^{N} x_i)^2} \tag{30}$$

Remark 49. We are mainly interested in the behavior of random variables such as $\sum_{i=1}^{N} X_i$ for *large* N. Hence we actually consider *sequences* $\mathbb{P}^{(N)}$ of voting measures on $\{-1,+1\}^N$ and $N = 1, 2, \ldots$. For the independence measure as well as for the common believe measure $\mathbb{P}^{(N)}$ is just the restriction of the corresponding measure on the infinite dimensional space $\{+1,-1\}^{\mathbb{N}}$ to $\{-1,+1\}^N$. Note, however, that the Curie-Weiss measures depend explicitly on the parameter N and are *not* the restrictions of a measure on the infinite dimensional space.

The behavior of random variables distributed to $\mathbb{P}_\beta = \mathbb{P}_\beta{}^{(N)}$ change drastically at the inverse temperature $\beta = 1$. In physical jargon such a phenomenon is called a 'phase transition'.

Theorem 50. *Suppose the random variables $X_1, \cdots, X_N$ are $\mathbb{P}_\beta^{(N)}$-distributed Curie-Weiss random variables and set $m_N = \frac{1}{N}\sum_{i=1}^{N} X_i$ then*

(1) *If $\beta \le 1$ then*

$$m_N \xrightarrow{\mathcal{D}} \delta_0 \tag{31}$$

(2) *If $\beta > 1$ then*

$$m_N \xrightarrow{\mathcal{D}} \frac{1}{2}(\delta_{-m(\beta)} + \delta_{m(\beta)}) \tag{32}$$

where $m(\beta)$ is the unique (strictly) positive solution of

$$\tanh(\beta t) = t \tag{33}$$

Above $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution. and δ_a denotes the Dirac measure at the point $a \in \mathbb{R}$, defined by

$$\delta_a(M) \;:=\; \begin{cases} 1, & \text{if } a \in M; \\ 0, & \text{otherwise.} \end{cases}$$

A proof of the above theorem as well as additional information on the Curie-Weiss model can be found in [9], [35], and [15]

The above classes of voting measures introduced above give rise to power indices and success measures (see [18]). Instead of exploring them in this direction we will discuss their use in the description of two-tier voting systems in the next section.

7.3. Two-tier Systems and Public Vote.

In this section we apply the probabilistic approach to get more insight into two-tier voting systems. In this approach we try to minimize the discrepancy between the voting result in the two-tier system and the voting result in a general public vote.

Typical examples, we try to model, are the Council of the EU or the Electoral College of the USA. We will assume in the following that voters from different states (e. g. member states of the EU) vote independently of each other. However, inside the states we allow correlations between voters.

Throughout this section we assume the following situation:
$(S, \mathcal{S})$ is a simple two-tier voting system composed of $(S_1, \mathcal{S}_1), \ldots, (S_M, \mathcal{S}_M)$ and $(C, \mathcal{C})$ with $C = \{c_1, \ldots, c_M\}$. We set $N_\nu = |S_\nu|$, $N = \sum_{\nu=1}^{M} N_\nu$. The vote of the i^{th} voter in state S_ν will be denoted by $X_{\nu i} \in \{+1, -1\}$.

A public vote in the state S_ν is given by:

$$P_\nu \;=\; \sum_{i=1}^{N_\nu} X_{\nu i}\,. \tag{34}$$

Since we assume the voting system in S_ν is simple majority, a proposal is approved within S_ν if $P_\nu > 0$ and rejected otherwise.

A public vote in S is given by:

$$P \;=\; \sum_{\nu=1}^{M} P_\nu \;=\; \sum_{\nu=1}^{M}\sum_{i=1}^{N_\nu} X_{\nu i} \tag{35}$$

Suppose now, that the voting system in the Council C is given by weights w_ν and a quota q. Define the function χ by:

$$\chi(x) \;=\; \begin{cases} 1, & \text{if } x > 0; \\ -1, & \text{otherwise.} \end{cases} \tag{36}$$

As above, we assume that the delegates in the Council vote according to the public vote in their respective country, i. e. c_ν will vote 1 if $\sum_{i=1}^{N_\nu} X_{\nu i} > 0$. Thus the vote in the Council will be:

$$C \;=\; \sum_{\nu=1}^{M} w_\nu\, \chi\Big(\sum_{i=1}^{N_\nu} X_{\nu i}\Big) \tag{37}$$

We remark that both P and C depend on the proposal in question.

Our goal is to choose the weights w_ν as good as possible. It would be desirable to have $P = C$, however a moment's reflection shows that there is no choice of the weights which give $P = C$ for *all* proposals ω (i. e. for all possible distributions of 'yes' and 'no' among the voters). So, the best we can hope for is to choose w_ν such that the discrepancy between P and C is minimal *in average*.

Definition 51. Let $\mathbb{P}$ be a voting measure and denote the expectation with respect to $\mathbb{P}$ by $\mathbb{E}$. We call the number

$$\Delta\big(w_1,\ldots,w_M\big) \;=\; \Delta_{\mathbb{P}}\big(w_1,\ldots,w_M\big) \;:=\; \mathbb{E}\left(|P-C|^2\right) \tag{38}$$

the *democracy deficit* of the two-tier voting system with respect to the voting measure $\mathbb{P}$.

We call weights $w_1,\ldots,w_M$ *optimal* (with respect to $\mathbb{P}$), if they minimize the function $\Delta_{\mathbb{P}}$.

Just as the power index and the success rate the democracy deficit depends on the choice of a voting measure. The choice of a good voting measure depends on the particular situation as well as the specific goal of our consideration. We'll comment on this point later on.

In the following, we will consider only such voting measures for which voters from different states are independent. The case of correlations of voters across state borders is more complicated. First results in this direction can be found in [22] and [20].

Theorem 52. *Suppose that the voters $X_{\nu i}$ and $X_{\rho j}$ in different states ($\nu \neq \rho$) are independent under the voting measure $\mathbb{P}$.*
Then the democracy deficit $\Delta_{\mathbb{P}}(w_1, \ldots, w_M)$ is minimal if the weights are given by:

$$w_\nu \;=\; \mathbb{E}\Big(\big|\sum_{i=1}^{N_\nu} X_{\nu i}\big|\Big) \tag{39}$$

For the proof see [14].

The quantity $M_\nu = \big|\sum_{i=1}^{N_\nu} X_{\nu i}\big|$ is the margin with which the voters in state S_ν decide, in other words, it is the difference in votes between the winning and the losing part of the voters. So, the representative of S_ν in the Council is actually backed by M_ν voters, not by *all* voters in S_ν. The optimal weight w_ν according to Theorem 52 is thus the expected margin of a decision of the voters in S_ν. We regard this result as rather intuitive.

Theorem 52 tells us that the optimal weight depends on the correlation structure within the states S_ν. The simplest case are uncorrelated (actually independent) voters within the states.

This is modelled by the independence measure, i. e. the Penrose-Banzhaf measure $\mathbb{P}_B$. Since the random variables $X_{\nu i}$ are independent (even within the states), we know by the central limit theorem that the random variables $\frac{1}{\sqrt{N}} \sum_{i=1}^{N_\nu} X_{\nu i}$ converge in law to a standard normal distribution. Thus, we infer (as $N_\nu \to \infty$):

$$w_\nu \;=\; \mathbb{E}\Big(\big|\sum_{i=1}^{N_\nu} X_{\nu i}\big|\Big) \;\approx\; C\,\sqrt{N_\nu}\,. \tag{40}$$

Hence we proved:

Theorem 53. *The optimal weights w_ν for independent voters are proportional to $\sqrt{N_\nu}$ for large N_ν.*

This result is close in spirit to the square root law by Penrose. In fact, Felsenthal and Machover [12] call it the second square-root rule.

Let us now consider the voting measure $\mathbb{P}_\mu$ inside the states and suppose that $\mu \neq \delta_0$. In this case the voters inside a state are not independent. In fact, it is not hard to see that for $i \neq j$

$$\mathbb{E}\Big(X_{\nu i}\, X_{\nu j}\Big) \;=\; \int_{-1}^{1} t^2\, d\mu(t) \;>\; 0\,. \tag{41}$$

We have (see [14])

Theorem 54. *Suppose the collective measure μ is not concentrated in the single point 0, then (as $N_\nu \to \infty$)*

$$\mathbb{E}_\mu\Big(\big|\sum_{i=1}^{N_\nu} X_{\nu i}\big|\Big) \;\approx\; C\,N_\nu\,. \tag{42}$$

So, the optimal weights for the Common Believe Measure are proportional to N_ν.

There is a generalization of this theorem when the collective measure $\mu = \mu_N$ may depend on N. With an appropriate choice of μ_N one can get $w_\nu \sim N^\alpha$ for any $1/2 \le \alpha \le 1$ (for a proof of both the theorem and its generalization see [14]).

These examples suggest that positive correlation between voters and thus collective behavior leads to a higher optimal voting weight in the Council. This conjecture is supported by the following two results.

Theorem 55. *Let $\mathbb{P}_\beta$ be the Curie-Weiss voting measure. Then the optimal weights for the Council are given by*

$$w_\nu \;=\; \mathbb{E}_\mu\Big(\big|\sum_{i=1}^{N_\nu} X_{\nu i}\big|\Big) \;\approx\; \begin{cases} C_\beta\,\sqrt{N}, & \text{for } \beta < 1; \\ C_1\,N^{3/4}, & \text{for } \beta = 1; \\ C_\beta\,N, & \text{for } \beta > 1. \end{cases} \tag{43}$$

For a proof of this theorem see [14] (in combination with [9] or [15]).

There is a common pattern behind the above result which we summarize now.

Theorem 56. *Suppose $\mathbb{P}_\mu$ is a voting measure on the simple two-tier voting system $\mathcal{S} = \mathfrak{T}\Big(\mathcal{S}_1, \ldots, \mathcal{S}_M; \mathcal{C}\Big)$ under which voters in different states are independent and set $\Sigma_N^{(\nu)} = \sum_{i=1}^{N_\nu} X_{\nu i}$.*

(1) *If*

$$\frac{1}{N}\,\Sigma_N^{(\nu)} \;\stackrel{\mathcal{D}}{\Longrightarrow}\; \mu$$

with $\mu \neq \delta_0$, then the optimal weight w_ν is asymptotically given by

$$w_\nu \;\approx\; \int |t|\,d\mu(t)\;N$$

(2) *If*

$$\frac{1}{N}\,\Sigma_N^{(\nu)} \;\stackrel{\mathcal{D}}{\Longrightarrow}\; \delta_0$$

$$\text{and} \qquad \frac{1}{N^\alpha}\,\Sigma_N^{(\nu)} \;\stackrel{\mathcal{D}}{\Longrightarrow}\; \rho$$

with $\rho \neq \delta_0$, then the optimal weight w_ν is asymptotically given by

$$w_\nu \;\approx\; \int |t|\,d\rho(t)\;N^\alpha$$

The above considerations raise the question which voting measure to choose in a particular situation. As usual, the answer is: It depends!

If one wants to describe a particular political situation at a specific time one should try to infer the voting measure from statistical data. Such an approach may be of use, for example, for opinion polls and election forecasts.

A quite different situation occurs if one wants to design a constitution for a political union. Such a concept should be independent of the current political constellation in the states, which is subject to fluctuate, and even the particular states under consideration, as the union might be enlarged in the future. In this case it seems that the best guess is to choose the independence measure.

This particular voting measure has a tendency to perhaps give big states less power than they might deserve under a more realistic voting measure. However, it seems to the author that this ‘mistake’ is less severe than to give big states more power than they ought to have.

8. A Short Outlook

In this paper we have considered various situations when mathematics has something to say about politics. We have shown that real world voting systems, such as the system for the Council of the EU or the Electoral College, are so complex that 'common sense' is simply not enough to understand the system. In fact, mathematical tools are necessary to analyze them. Sometimes, common sense is even misleading. For example, most people tend to believe that a voting weight proportional to the population would be fair for the Council of the EU.

Every day's experience shows that politicians are reluctant to ask scientists (and especially mathematicians) how to do, what they believe is their job, for example how to design a voting system. This empirical fact has certainly various roots. One is, we believe, that mathematicians have only very occasionally made clear that they work on problems with relations to politics. Another reason is that there are considerable cultural differences between the world of mathematics and the world of politics. Whatever the reasons may be, we believe a discussion between politicians and voting theorists would be beneficial to both sides, and to normal voters.

In particular for the voting system in the Council of the EU it would be helpful to contact voting theorists before the next reform. It is striking that politicians agreed at different times on two voting systems with almost opposite defects, while the reasonable system proposed by the Polish government was neglected (if not ridiculed). There was a petition signed by more than 50 scientists from various European countries sent to all governments of the EU member states, which explained the benefits of the square root voting system for the Council of Ministers. Only one of the (then) 25 governments reacted.

We believe that a voting system which can be considered as unjust on a scientific basis will certainly not promote the idea of a unified Europe.

There is also a positive example of interaction between politicians and mathematicians, namely in the fields of fair allocation of seats in a parliament. This is the biproportional apportionment which was implemented by the mathematician Friedrich Pukelsheim for the Swiss Canton Zürich and subsequently for other Swiss Cantons. This system allows a representation in a parliament which is both proportional with respect to parties and with respect to regions. For the biproportional apportionment method in theory and practice see Pukelsheim [26] and references therein.

Appendix

Country	Population	Weight	Power	Square root	Deviation
Germany	15.9	29	7.6	9.1	-16.5
France	13	29	7.6	8.2	-7.3
United Kingdom	12.7	29	7.6	8.1	-6.2
Italy	12	29	7.6	7.9	-3.8
Spain	9.2	27	7.2	6.9	4.3
Poland	7.5	27	7.2	6.2	16.1
Romania	3.9	14	4.2	4.5	-6.7
Netherlands	3.3	13	3.9	4.2	-7.1
Belgium	2.2	12	3.6	3.4	5.9
Greece	2.2	7	2.1	3.3	-36.4
Czech Republic	2.1	12	3.6	3.3	9.1
Portugal	2.1	12	3.6	3.3	9.1
Hungary	1.9	12	3.6	3.2	12.5
Sweden	1.9	10	3	3.1	-3.2
Austria	1.7	10	3	3	0
Bulgaria	1.4	10	3	2.7	11.1
Denmark	1.1	7	2.1	2.4	-12.5
Finland	1.1	7	2.1	2.4	-12.5
Slovakia	1.1	7	2.1	2.4	-12.5
Ireland	0.9	12	3.6	2.2	63.3
Croatia	0.8	7	2.1	2.1	0
Lithuania	0.6	7	2.1	1.7	23.5
Slovenia	0.4	4	1.2	1.5	-20
Latvia	0.4	4	1.2	1.4	-14.3
Estonia	0.3	4	1.2	1.2	0
Cyprus	0.2	4	1.2	0.9	33.3
Luxembourg	0.1	4	1.2	0.8	50
Malta	0.1	3	0.9	0.7	28.6

Table 1: Penrose-Banzhaf power indices for the Nice treaty

Population: in % of the EU population
Square root: Ideal power according to Penrose
Deviation: Difference between actual and ideal power in % of the square root power

Country	Population	Power	Square root	Deviation
Germany	15.9	10.2	9.1	12.2
France	13	8.4	8.2	2.5
United Kingdom	12.7	8.3	8.1	1.6
Italy	12	7.9	7.9	0
Spain	9.2	6.2	6.9	-9.8
Poland	7.5	5.1	6.2	-18.5
Romania	3.9	3.8	4.5	-15.9
Netherlands	3.3	3.5	4.2	-16.4
Belgium	2.2	2.9	3.4	-14.6
Greece	2.2	2.9	3.3	-14.3
Czech Republic	2.1	2.8	3.3	-14
Portugal	2.1	2.8	3.3	-15.6
Hungary	1.9	2.8	3.2	-11.5
Sweden	1.9	2.7	3.1	-13
Austria	1.7	2.6	3	-11.3
Bulgaria	1.4	2.5	2.7	-8.6
Denmark	1.1	2.3	2.4	-3.2
Finland	1.1	2.3	2.4	-2.6
Slovakia	1.1	2.3	2.4	-2.1
Ireland	0.9	2.2	2.2	2.2
Croatia	0.8	2.2	2.1	4.6
Lithuania	0.6	2	17	12.3
Slovenia	0.4	2	1.5	40.9
Latvia	0.4	2	1.4	36.7
Estonia	0.3	1.9	1.2	61.9
Cyprus	0.2	1.8	0.9	95.5
Luxembourg	0.1	1.8	0.8	140
Malta	0.1	1.8	0.7	170.8

Table 2: Penrose-Banzhaf power indices for the Lisbon treaty

Population: in % of the EU population
Square root: Ideal power according to Penrose
Deviation: Difference between actual and ideal power in % of the square root power

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