

Semicircle law for generalized Curie-Weiss matrix ensembles at subcritical temperature

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Abstract

In [13] Hochstättler, Kirsch, and Warzel showed that the semicircle law holds for generalized Curie-Weiss matrix ensembles at or above the critical temperature. We extend their result to the case of subcritical temperatures for which the correlations between the matrix entries are stronger. Nevertheless, one may use the concept of *approximately uncorrelated* ensembles that was first introduced in [13]. In order to do so one needs to remove the average magnetization of the entries by an appropriate modification of the ensemble that turns out to be of rank 1 thus not changing the limiting spectral measure.

1 Introduction

Hochstättler, Kirsch, and Warzel proved in [13] the semicircle law for ensembles of real symmetric matrices where the upper triangular part is filled by what they called *approximately uncorrelated* random variables [13, Def. 4] (see also Definition 5 below). An important motivation for introducing this notion is that collections of random variables with values in $\{-1, 1\}$ that are distributed according to the Curie-Weiss law at or above the critical temperature are approximately uncorrelated and thus the semicircle law holds for the corresponding matrix ensembles. It is the main goal of the present paper to show how one may use the concept of approximately uncorrelated random variables to prove a semicircle law also for subcritical temperatures.

In order to state our result precisely we need a few definitions. Curie-Weiss random variables ξ_1, \dots, ξ_M , also called spins, take values in $\{-1, 1\}$ with probability

$$\mathbb{P}_\beta^M(\xi_1 = x_1, \dots, \xi_M = x_M) = Z_{\beta, M}^{-1} e^{\frac{\beta}{2M} (\sum_{i=1}^M x_i)^2}$$

where $Z_{\beta, M}$ denotes the normalization constant. By \mathbb{E}_β^M we denote the expectation with respect to \mathbb{P}_β^M .

The parameter $\beta \geq 0$ is interpreted in physics as *inverse temperature*, $\beta = \frac{1}{T}$. Information on the physical meaning of the Curie-Weiss model can be found in [6, 26]. If $\beta = 0$ the random variables ξ_i are independent while for $\beta > 0$ there is a positive correlation between the ξ_i that grows with β . At the *critical inverse temperature* $\beta = 1$ a phase transition occurs. While at and above the critical temperature ($\beta \leq 1$) the average spin $\frac{1}{M} \sum_{i=1}^M \xi_i$ converges in distribution to the Dirac measure δ_0 , the average spin converges to $\frac{1}{2}(\delta_{-m(\beta)} + \delta_{m(\beta)})$ below the critical temperature ($\beta > 1$), where $m = m(\beta) \in (0, 1)$ is called the *average magnetization*. It can be defined as the (unique) strictly positive solution of

$$\tanh(\beta m) = m. \quad (1)$$

For a proof of this fact see e. g. [6] or [15]. The following observation is fundamental for the analysis of [13] and also for the present paper: Curie-Weiss distributed random variables are of *de Finetti type*, i.e. they can be represented as an average of independently distributed random variables. More precisely, for $t \in [-1, 1]$ we denote the probability measures

$$\begin{aligned} P_t^{(M)} &= \bigotimes_{j=1}^M P_t^{(1)} \text{ as the } M\text{-fold product of } P_t^{(1)} \text{ with} \\ P_t^{(1)}(\{1\}) &= \frac{1}{2}(1+t) \text{ and } P_t^{(1)}(\{-1\}) = \frac{1}{2}(1-t). \end{aligned} \quad (2)$$

If M is clear from the context we write P_t instead of $P_t^{(M)}$. By E_t resp. $E_t^{(M)}$ we denote the corresponding expectation. For any function ϕ on $\{-1, 1\}^M$ we have

$$\begin{aligned} \mathbb{E}_\beta^M(\phi(X_1, \dots, X_M)) &= \frac{1}{Z} \int_{-1}^1 E_t(\phi(X_1, \dots, X_M)) \frac{e^{-MF_\beta(t)/2}}{1-t^2} dt, \\ \text{with } F_\beta(t) &:= \frac{1}{\beta} \left(\frac{1}{2} \ln \frac{1+t}{1-t} \right)^2 + \ln(1-t^2), \quad t \in (-1, 1), \end{aligned} \quad (3)$$

and normalization constant $Z := \int_{-1}^1 e^{-MF_\beta(t)/2} \frac{dt}{1-t^2}$.

In this sense \mathbb{P}_β^M is a weighted t average over all $P_t^{(M)}$. This fact can be proved using the so called Hubbard-Stratonovich transformation (see [13] or [15] and references therein). The above considerations motivate the following definition of *generalized Curie-Weiss ensembles* introduced in [13, Def. 29].

Definition 1 Suppose $\alpha > 0$, $\beta \geq 0$ and let $F_\beta, P_t^{(M)}$ be defined as in (3), (2) above. We define a probability measure on \mathbb{R}^{N^2} by

$$\mathbb{P}_{N^2}^{N^\alpha F_\beta} = \int_{-1}^1 P_t^{(N^2)} d\nu_N(t), \quad d\nu_N(t) = \frac{1}{Z_N} \cdot \frac{e^{-N^\alpha F_\beta(t)/2}}{1-t^2} dt,$$

where Z_N is chosen such that the de Finetti measure ν_N becomes a probability measure (see Proposition 8 c) for normalizability of ν_N). The corresponding matrix ensemble $(X_N)_N$ is then defined as follows. Pick $\frac{N(N+1)}{2}$ different components of the $\mathbb{P}_{N^2}^{N^\alpha F_\beta}$ -distributed random vector to fill the upper triangular part of $X_N \in \mathbb{R}^{N \times N}$. The remaining entries $X_N(i, j)$, $1 \leq j < i \leq N$ are then determined by the symmetry $X_N(i, j) = X_N(j, i)$. Observe that $\mathbb{P}_{N^2}^{N^\alpha F_\beta}$ is invariant under permutations of components. Thus the resulting matrix ensemble does neither depend on the choice of the $\frac{N(N+1)}{2}$ components (out of N^2) nor on the order in which the upper triangular part of X_N is filled. The so generated random matrix ensemble $(X_N)_N$ is called a generalized $\mathbb{P}_{N^2}^{N^\alpha F_\beta}$ -Curie-Weiss ensemble.

Remark 2 Note that it is only in the case $\alpha = 2$ that the N^2 real random variables from which the entries of the matrix X_N are chosen are indeed Curie-Weiss distributed. The generalization thus consists of allowing for $\alpha \neq 2$ which still remains in the framework of exchangeable random variables. However, for $\alpha > 2$ one may view the ensemble to be generated by selecting the matrix entries from a collection of N^α Curie-Weiss distributed random spins ± 1 .

Observe that the definition in [13] is even more general than Definition 1. It also allows to replace F_β by more general functions F (cf. Remark 12).

Let \mathbb{P} be the probability measure underlying some matrix ensemble $(A_N)_N$ of real symmetric matrices $A_N \in \mathbb{R}^{N \times N}$. Denote by $\lambda_1 \leq \dots \leq \lambda_N$ the eigenvalues of A_N . Then we call $\sigma_N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}$ the eigenvalue distribution measure of A_N . We say that the *semicircle law holds* for the ensemble $(A_N)_N$, if σ_N converges weakly in probability to σ_{sc} as $N \rightarrow \infty$, i.e. for all bounded continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and for all $\epsilon > 0$ we have

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\left| \int_{-\infty}^{\infty} f(x) d\sigma_N(x) - \int_{-\infty}^{\infty} f(x) d\sigma_{sc}(x) \right| > \epsilon \right) = 0,$$

where the semicircle σ_{sc} is the Borel measure on \mathbb{R} with support $[-2, 2]$ and (Lebesgue-) density $\frac{1}{2\pi} \sqrt{4-x^2}$ at $x \in [-2, 2]$.

The following result is proved in [13].

Theorem 3 (Theorem 31 and Corollary 28 in [13]) Let $(X_N)_N$ be a generalized $\mathbb{P}_{N^2}^{N^\alpha F_\beta}$ -Curie-Weiss ensemble. Then the semicircle law holds for X_N/\sqrt{N} if either

(i) $\beta \in [0, 1)$ and $\alpha \geq 1$ or (ii) $\beta = 1$ and $\alpha \geq 2$
is satisfied.

The purpose of the present paper is to extend this result to subcritical temperatures $\beta > 1$. Our main result reads:

Theorem 4 *Let $(X_N)_N$ be a generalized $\mathbb{P}_{N^2}^{N^\alpha F^\beta}$ -Curie-Weiss ensemble with $\alpha \geq 1$ and $\beta > 1$. Denote by $m = m(\beta)$ the unique strictly positive solution of the equation $\tanh(\beta m) = m$ (cf. (1)). Then the semicircle law holds for $A_N := X_N / \sqrt{N(1 - m^2)}$, i.e. for all bounded continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and for all $\epsilon > 0$ we have*

$$\lim_{N \rightarrow \infty} \mathbb{P}_{N^2}^{N^\alpha F^\beta} \left(\left| \int_{-\infty}^{\infty} f(x) d\sigma_N(x) - \int_{-\infty}^{\infty} f(x) d\sigma_{sc}(x) \right| > \epsilon \right) = 0,$$

where σ_N denotes the eigenvalue distribution measure of A_N .

As mentioned above, Theorem 3 was proved in [13] by introducing the concept of *approximately uncorrelated random variables*. For ensembles of real symmetric matrices this may be formulated as follows (cf. [13, Def. 4]).

Definition 5 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space for an ensemble $(X_N)_N$ of real symmetric $N \times N$ matrices. We say that the entries are approximately uncorrelated if*

$$\left| \mathbb{E} \left(\prod_{\nu=1}^{\ell} X_N(i_\nu, j_\nu) \prod_{\rho=1}^p X_N(u_\rho, v_\rho) \right) \right| \leq \frac{C_{\ell,p}}{N^{\frac{\ell}{2}}} \quad (4)$$

and for every $\ell \in \mathbb{N}$ there is a sequence $a_{\ell,N}$ converging to 0 as $N \rightarrow \infty$ with

$$\left| \mathbb{E} \left(\prod_{\nu=1}^{\ell} X_N(i_\nu, j_\nu)^2 \right) - 1 \right| \leq a_{\ell,N} \quad (5)$$

for all sequences $(i_1, j_1), \dots, (i_\ell, j_\ell)$ which are pairwise disjoint and disjoint to the sequences $(u_1, v_1), \dots, (u_p, v_p)$ with N -independent constants $C_{\ell,p}$. By sequence we mean that all indices $(i_s, j_s), (u_s, v_s)$ may depend on N and that for each value of N they belong to the set $\{(i, j) : 1 \leq i \leq j \leq N\}$.

In [13, Theorem 5] it is shown that conditions (4) and (5) are well suited to apply the original ideas of Wigner [27, 28] and Grenander [12] to prove the semicircle law via the method of moments (see also [2, 3, 18, 21, 22] and the monographs [1, 4, 23, 25]). Matrix ensembles with correlated entries have already been considered in [5, 7, 8, 10, 11, 14, 19, 24]. See [13] and the recent survey [16] for more information on these results.

Theorem 3 follows from [13, Theorem 5] by verifying that under conditions (i) and (ii) the ensemble $\mathbb{P}_{N^2}^{N^\alpha F_\beta}$ has approximately uncorrelated entries. For $\beta > 1$ the situation is different. For example, condition (4) is violated for any $\alpha > 0$, because (cf. [13, proof of Proposition 35])

$$\lim_{N \rightarrow \infty} \mathbb{E}_{N^2}^{N^\alpha F_\beta} (X_N(1, 1)X_N(1, 2)) = m^2 > 0, \quad (6)$$

where, again, m denotes the average magnetization (1). Indeed, by the independence of $X_N(1, 1)$ and $X_N(1, 2)$ with respect to P_t we conclude $E_t(X_N(1, 1)X_N(1, 2)) = E_t(X_N(1, 1))E_t(X_N(1, 2)) = t^2$. Moreover, and this is the crucial difference from the case $\beta \leq 1$, the function F_β has two minimizers $\pm m$ (cf. Proposition 8) so that for large values of N we have $\nu_N \sim \frac{1}{2}(\delta_{-m} + \delta_m)$ from which (6) follows. Proposition 7 stated below can be used to make this argument rigorous.

In order to prove Theorem 4 we use an idea that has already been introduced in the proof of Proposition 32 in [13]. For the moment let us proceed heuristically and assume $\nu_N = \frac{1}{2}(\delta_{-m} + \delta_m)$. Therefore we only need to consider the matrix ensembles related to $P_{\pm m}$. These fall into the class of real symmetric matrices with i.i.d. entries $X_N(i, j), i \leq j$ that are distributed according to $\frac{1}{2}[(1 \pm m)\delta_1 + (1 \mp m)\delta_{-1}]$. Consequently $E_{\pm m}(X_N(i, j)) = \pm m$ and $E_{\pm m}(X_N^2(i, j)) = 1$. Subtracting the mean and dividing by the standard deviation leads to standard Wigner ensembles

$$\tilde{X}_{N, \pm} := \frac{1}{\sqrt{1 - m^2}}(X_N \mp m\mathcal{E}_N), \quad \mathcal{E}_N(i, j) := 1 \quad \text{for all } 1 \leq i, j \leq N. \quad (7)$$

By the classical results of Wigner [27, 28] the eigenvalue distribution measures of $\tilde{X}_{N, \pm}/\sqrt{N}$ converge to the semicircle law σ_{sc} when considered with respect to the probability measures $P_{\pm m}$.

Now we need to relate $\tilde{X}_{N, \pm}$ back to X_N . To this end, observe that $X_N/\sqrt{N(1 - m^2)}$ is a rank 1 perturbation of both $\tilde{X}_{N, \pm}/\sqrt{N}$ and we can therefore expect (see Lemma 11) that the eigenvalue distribution measures of $X_N/\sqrt{N(1 - m^2)}$ converge to the semicircle law as well. What is still missing is an indicator when to replace the $\mathbb{P}_{N^2}^{N^\alpha F_\beta}$ -distributed X_N by $\tilde{X}_{N, +}$ and when by $\tilde{X}_{N, -}$. As we will see by some basic large deviations argument in Proposition 9 the sum of the entries is an efficient choice for that. Define

$$S_N := \sum_{1 \leq i \leq j \leq N} X_N(i, j), \quad \mathcal{X}_{N, +} := 1_{\{S_N > 0\}}, \quad \mathcal{X}_{N, -} := 1_{\{S_N \leq 0\}} = 1 - \mathcal{X}_{N, +}, \quad (8)$$

$$Y_{N, \pm} := \frac{1}{\sqrt{1 - m^2}}(X_N \mp m\mathcal{E}_N)\mathcal{X}_{N, \pm}, \quad Y_N := Y_{N, +} + Y_{N, -}. \quad (9)$$

The core of the proof of Theorem 4 is to show that the entries of Y_N are approximately uncorrelated. Since $Y_N - X_N/\sqrt{1 - m^2}$ has rank 1 our main result is then a consequence of [13, Theorem 5] and Lemma 11.

We also have results on the largest and second largest singular value of generalized Curie-Weiss matrices [17], which complement and improve on [13].

The plan of the paper is as follows. In Section 2 we use Laplace's method to make the starting point of our heuristic argument, $\nu_N \sim \frac{1}{2}(\delta_{-m} + \delta_m)$ for large values of N , precise. The remaining arguments for the proof of our main result Theorem 4 are gathered in Section 3.

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2 Analysis of the de Finetti measures

In this section we state and prove in Lemma 6 the precise version of the formula $\nu_N \sim \frac{1}{2}(\delta_{-m} + \delta_m)$ for the de Finetti measure that we used in the heuristic arguments at the end of the Introduction.

Lemma 6 *Assume $\alpha > 0$, $\beta > 1$ and let $\nu_N \equiv \nu_{N,\alpha,\beta}$ be given as in Definition 1. Recall also the definition of the average magnetization $m \equiv m(\beta)$ in (1). Then the following holds:*

a) *There exist numbers $C, \delta > 0$ such that for all $N \in \mathbb{N}$:*

$$\nu_N\left(\left[-\frac{m}{2}, \frac{m}{2}\right]\right) \leq Ce^{-\delta N^\alpha}$$

b) *For all $\ell \in \mathbb{N}$ there exist $C_\ell > 0$ such that for all $N \in \mathbb{N}$:*

$$\int_{\frac{m}{2}}^1 |t - m|^\ell d\nu_N(t) \leq C_\ell N^{-\frac{\alpha\ell}{2}}$$

c) $\lim_{N \rightarrow \infty} \int_{\frac{m}{2}}^1 (1 + m^2 - 2mt)^\ell d\nu_N(t) = \frac{1}{2}(1 - m^2)^\ell$ for all $\ell \in \mathbb{N}$.

Lemma 6 is proved at the end of this section using Laplace's method. The following version is a special case of [20, Theorem 7.1] (cf. [13, Proposition 24]).

Proposition 7 (Laplace method [20]) *Suppose $F : (-1, 1) \rightarrow \mathbb{R}$ is differentiable, $\phi : (-1, 1) \rightarrow \mathbb{R}$ is continuous and for some $-1 < a < b \leq 1$ we have*

1. $\inf_{t \in [c,b]} F(t) > F(a)$ for all $c \in (a, b)$.

2. As $t \searrow a$ we have

$$F(t) = F(a) + P (t - a)^\kappa + \mathcal{O}((t - a)^{\kappa+1}) \quad (10)$$

$$\phi(t) = Q (t - a)^{\lambda-1} + \mathcal{O}((t - a)^\lambda) \quad (11)$$

where κ, λ and P are positive constants, Q is a real constant, and (10) is differentiable.

3. For $x \in \mathbb{R}$ sufficiently large the function $t \mapsto e^{-x F(t)/2} \phi(t)$ belongs to $L^1(a, b)$.

Then as $x \rightarrow \infty$ the integral $I(x) := \int_a^b e^{-x F(t)/2} \phi(t) dt$ satisfies

$$I(x) \approx \frac{Q}{\kappa} \Gamma\left(\frac{\lambda}{\kappa}\right) \left(\frac{2}{xP}\right)^{\frac{\lambda}{\kappa}} e^{-x F(a)/2}$$

where $A(x) \approx B(x)$ means $\lim_{x \rightarrow \infty} \frac{A(x)}{B(x)} = 1$ and Γ denotes the Gamma function.

Next we summarize those properties of the function F_β that are used in the proof of Lemma 6. In view of Proposition 7 we need to analyze the monotonicity properties of F_β . In addition we provide an estimate that is useful to establish integrability of ν_N near the endpoints $t = \pm 1$.

Proposition 8 Assume $\beta > 1$ and let $F_\beta, m \equiv m(\beta)$ be defined as in (1). Then:

- a) $F_\beta \in C^3(-1, 1)$ is an even function, i.e. $F_\beta(t) = F_\beta(-t)$ for all $t \in (-1, 1)$.
- b) $F'_\beta < 0$ on $(0, m)$, $F'_\beta > 0$ on $(m, 1)$, and $F''_\beta(m) > 0$.
- c) For all $t \in (-1, 1)$ we have $e^{-F_\beta(t)/2} \leq e^{9\beta/2}(1 - |t|)$.

Proof. Statement a) is obvious. Statement b) follows from a straight forward computation for which it is useful to observe that $\ln \frac{1+t}{1-t} = 2 \operatorname{Artanh}(t)$ and $F'_\beta(t) = \frac{2}{\beta(1-t^2)} (\operatorname{Artanh}(t) - \beta t)$. Because of the evenness of F_β it suffices to prove statement c) for $0 \leq t < 1$ only. Set $X := -\ln(1-t) \geq 0$. Since $\ln(1+t) \geq 0$ we have

$$\frac{F_\beta(t)}{2} = \frac{1}{8\beta} (X + \ln(1+t))^2 + \frac{1}{2} (\ln(1+t) - X) \geq \frac{1}{8\beta} X^2 - \frac{1}{2} X.$$

Using in addition that $\frac{1}{8\beta} X^2 - \frac{3}{2} X + \frac{9}{2} \beta \geq 0$ the claim follows. \blacksquare

We now have all ingredients to verify Lemma 6.

Proof (Lemma 6). We begin by evaluating the asymptotic behavior of the norming constant

$$Z_N = \int_{-1}^1 e^{-N^\alpha F_\beta(t)/2} \phi(t) dt \quad \text{with} \quad \phi(t) = \frac{1}{1-t^2}.$$

In order to apply Proposition 7 we need to split the domain of integration into those four regions where F_β is monotone. Due to the evenness of F_β (Proposition 8 a) it suffices to consider the integrals over $[-m, 0]$ and $[m, 1]$. In both cases the parameters for condition 2. of Proposition 7 are $\kappa = 2$, $P = F''_\beta(-m)/2 = F''_\beta(m)/2 > 0$, $\lambda = 1$, and $Q = 1/(1-m^2)$. Conditions 1. and 3. of Proposition 7 are satisfied because of statements b) and c) of Proposition 8. Hence

$$Z_N \approx \frac{2\Gamma(\frac{1}{2})}{1-m^2} \left(\frac{4}{F''_\beta(m)N^\alpha} \right)^{\frac{1}{2}} e^{-N^\alpha F_\beta(m)/2}. \quad (12)$$

Keeping the integrand fixed one may apply Proposition 7 also to the integral over $[-\frac{m}{2}, 0]$ (now $\kappa = 1$, $P = F'_\beta(-m/2)$, $\lambda = 1$, $Q = 1/(1-(m/2)^2)$). We arrive at

$$\int_{-\frac{m}{2}}^{\frac{m}{2}} e^{-N^\alpha F_\beta(t)/2} \frac{dt}{1-t^2} \approx \frac{2\Gamma(1)}{1-(\frac{m}{2})^2} \cdot \frac{2}{F'_\beta(-\frac{m}{2})N^\alpha} e^{-N^\alpha F_\beta(\frac{m}{2})/2} \quad (13)$$

Taking the quotient of (13) and (12) one easily derives statement a) with, say, $\delta = \frac{1}{2}(F_\beta(\frac{m}{2}) - F_\beta(m)) > 0$.

In order to obtain the remaining claims set $\phi_1(t) := |t-m|^\ell/(1-t^2)$ resp. $\phi_2(t) := (1+m^2-2mt)^\ell/(1-t^2)$. Applying Proposition 7 to the intervals $[-m, -\frac{m}{2}]$ and $[m, 1]$ we find

$$\begin{aligned} \int_{\frac{m}{2}}^1 e^{-N^\alpha F_\beta(t)/2} \phi_1(t) dt &\approx \frac{\Gamma(\frac{\ell+1}{2})}{1-m^2} \left(\frac{4}{F''_\beta(m)N^\alpha} \right)^{\frac{\ell+1}{2}} e^{-N^\alpha F_\beta(m)/2} \\ \int_{\frac{m}{2}}^1 e^{-N^\alpha F_\beta(t)/2} \phi_2(t) dt &\approx \frac{\Gamma(\frac{1}{2})}{(1-m^2)^{1-\ell}} \left(\frac{4}{F''_\beta(m)N^\alpha} \right)^{\frac{1}{2}} e^{-N^\alpha F_\beta(m)/2} \end{aligned}$$

and statements b) and c) follow from (12). ■

3 Proof of the Main Result

We begin the proof with large deviations estimates that demonstrate the efficiency of the indicators $\mathcal{X}_{N,\pm}$ defined in (8). They are immediate consequences of Hoeffding's inequality. Nevertheless, we provide a short proof for the convenience of the reader.

Proposition 9 *Let P_t and S_N be defined as in (2), (8). For $a \in (0, 1)$ denote $q_a := -\frac{1}{4} \log(1-a^2) > 0$. Then the following estimates hold true.*

- a) For all $t \in [a, 1]$: $P_t^{(N^2)}(S_N \leq 0) \leq e^{-q_a N^2}$.
- b) For all $t \in [-1, -a]$: $P_t^{(N^2)}(S_N > 0) \leq e^{-q_a N^2}$.

Proof. Since $P_{\pm 1}(S_N = \pm \frac{1}{2}N(N+1)) = 1$ we only need to consider $t \in (-1, 1)$. Define $\lambda(t) := \frac{1}{2} \log \frac{1-t}{1+t}$. Then

$$E_t(e^{\lambda(t)X_N(1,1)}) = \frac{1}{2}e^{\lambda(t)}(1+t) + \frac{1}{2}e^{-\lambda(t)}(1-t) = \sqrt{1-t^2}.$$

For $t \in [a, 1)$ we have $\lambda(t) < 0$. Using in addition that the random variables $X_N(i, j)$, $1 \leq i \leq j \leq N$, are independent and identically distributed with respect to the probability measure P_t we obtain

$$P_t(S_N \leq 0) \leq E_t(e^{\lambda(t)S_N}) = E_t(e^{\lambda(t)X_N(1,1)})^{\frac{N(N+1)}{2}} \leq e^{-qaN^2}.$$

Similarly, $\lambda(t) > 0$ for $t \in (-1, -a]$ and $P_t(S_N > 0) \leq E_t(e^{\lambda(t)S_N}) \leq e^{-qaN^2}$. ■

We are now ready to prove the lemma which is the key for proving our main result, i.e. to show that Y_N defined in (9) has approximately uncorrelated entries.

Lemma 10 *Let $(X_N)_N$ be a generalized $\mathbb{P}_{N^2}^{N^\alpha F^\beta}$ -Curie-Weiss ensemble with $\alpha \geq 1$ and $\beta > 1$ and let Y_N be defined as in (9) (see also (7)). Then $(Y_N)_N$ has approximately uncorrelated entries with respect to the probability measure $\mathbb{P}_{N^2}^{N^\alpha F^\beta}$ (see Definition 5).*

Proof.

Let us first consider (5) and define (see (8), (9))

$$\begin{aligned} G_{N,\pm} &:= \mathcal{X}_{N,\pm} \prod_{\nu=1}^{\ell} Y_N(i_\nu, j_\nu)^2 = \mathcal{X}_{N,\pm} \prod_{\nu=1}^{\ell} (\mathcal{X}_{N,\pm} Y_N(i_\nu, j_\nu))^2 \\ &= \mathcal{X}_{N,\pm} \prod_{\nu=1}^{\ell} Y_{N,\pm}(i_\nu, j_\nu)^2 = (1-m^2)^{-\ell} \mathcal{X}_{N,\pm} \prod_{\nu=1}^{\ell} (X_N(i_\nu, j_\nu) \mp m)^2 \end{aligned}$$

Obviously,

$$\mathbb{E}_{N^2}^{N^\alpha F^\beta} \left(\prod_{\nu=1}^{\ell} Y_N(i_\nu, j_\nu)^2 \right) = \mathbb{E}_{N^2}^{N^\alpha F^\beta} (G_{N,+}) + \mathbb{E}_{N^2}^{N^\alpha F^\beta} (G_{N,-}) \quad \text{and}$$

$$\mathbb{E}_{N^2}^{N^\alpha F^\beta} (G_{N,\pm}) = (1-m^2)^{-\ell} \int_{-1}^1 E_t(\mathcal{X}_{N,\pm} H_{N,\pm}) d\nu_N(t), \quad (14)$$

with $H_{N,\pm} := \prod_{\nu=1}^{\ell} (X_N(i_\nu, j_\nu) \mp m)^2$.

Note that $X_N^2(i_\nu, j_\nu) = 1$ and that $E_t(X_N(i_\nu, j_\nu)) = t$. Since the index pairs $(i_1, j_1), \dots, (i_\ell, j_\ell)$ are assumed to be pairwise disjoint we obtain

$$E_t(H_{N,\pm}) = (1+m^2 \mp 2mt)^\ell. \quad (15)$$

In order to evaluate $\mathbb{E}_{N^\alpha}^{N^2 F^\beta}(G_{N,+})$ we decompose

$$\begin{aligned} \int_{-1}^1 E_t(\mathcal{X}_{N,+} H_{N,+}) d\nu_N &= \int_{-1}^{-\frac{m}{2}} E_t(\mathcal{X}_{N,+} H_{N,+}) d\nu_N \\ &+ \int_{-\frac{m}{2}}^{\frac{m}{2}} E_t(\mathcal{X}_{N,+} H_{N,+}) d\nu_N \\ &- \int_{\frac{m}{2}}^1 E_t(\mathcal{X}_{N,-} H_{N,+}) d\nu_N + \int_{\frac{m}{2}}^1 E_t(H_{N,+}) d\nu_N. \end{aligned} \quad (16)$$

From Proposition 9, from Lemma 6 a), and from the trivial estimate $|H_{N,+}| \leq (1+m)^{2\ell}$ we obtain constants $q, C, \delta > 0$ such that for all $N \in \mathbb{N}$:

$$\begin{aligned} \left| \int_{-1}^{-\frac{m}{2}} E_t(\mathcal{X}_{N,+} H_{N,+}) d\nu_N \right| &\leq (1+m)^{2\ell} e^{-qN^2}, \\ \left| \int_{\frac{m}{2}}^1 E_t(\mathcal{X}_{N,-} H_{N,+}) d\nu_N \right| &\leq (1+m)^{2\ell} e^{-qN^2}, \\ \left| \int_{-\frac{m}{2}}^{\frac{m}{2}} E_t(\mathcal{X}_{N,+} H_{N,+}) d\nu_N \right| &\leq C(1+m)^{2\ell} e^{-\delta N^\alpha}. \end{aligned}$$

Thus the first three summands on the right hand side of (16) all converge to zero as N tends to ∞ . By (14), (15) and Lemma 6 c) we have proved

$$\lim_{N \rightarrow \infty} \mathbb{E}_{N^2}^{N^\alpha F^\beta}(G_{N,+}) = \frac{1}{2}.$$

By analogue arguments the expected value of $G_{N,-}$ can be seen to converge to $\frac{1}{2}$ as well and (5) is established. Estimate (4) can be proved in a similar fashion. To this end redefine

$$H_{N,\pm} := \prod_{\nu=1}^{\ell} (X_N(i_\nu, j_\nu) \mp m) \prod_{\rho=1}^p (X_N(u_\rho, v_\rho) \mp m).$$

The expected value in (4) equals

$$(1-m^2)^{-\frac{\ell+p}{2}} \left(\int_{-1}^1 E_t(\mathcal{X}_{N,+} H_{N,+}) d\nu_N + \int_{-1}^1 E_t(\mathcal{X}_{N,-} H_{N,-}) d\nu_N \right). \quad (17)$$

Observe that

$$E_t(H_{N,\pm}) = (t \mp m)^\ell \cdot Q_\pm(t)$$

with Q_\pm being a polynomial of degree $\leq p$ and $|Q_\pm(t)| \leq (1+m)^p$ for all $t \in [-1, 1]$. Using (16), the arguments thereafter, $|H_{N,+}| \leq (1+m)^{\ell+p}$, and statement b) of Lemma 6 one may establish the existence of constants $C, \delta, q > 0$ such that for all $N \in \mathbb{N}$:

$$\left| \int_{-1}^1 E_t(\mathcal{X}_{N,+} H_{N,+}) d\nu_N \right| \leq (1+m)^{\ell+p} (2e^{-qN^2} + Ce^{-\delta N^\alpha} + C_\ell N^{-\frac{\alpha\ell}{2}}).$$

Clearly, the same estimate holds for $|\int_{-1}^1 E_t(\mathcal{X}_{N,-} H_{N,-}) d\nu_N|$ and (4) is proved since we have assumed $\alpha \geq 1$. ■

The final observation we need is that rank 1 perturbations preserve the weak convergence (in probability) of the eigenvalue distribution measures (cf. [9]).

Lemma 11 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space underlying two ensembles $(A_N)_N, (B_N)_N$ of real symmetric $N \times N$ matrices that satisfy $\text{rank}(A_N(\omega) - B_N(\omega)) \leq 1$ for all $\omega \in \Omega$. Denote by σ_N resp. by μ_N the eigenvalue distribution measures of A_N resp. B_N . Assume furthermore that $(\mu_N)_N$ converges weakly in probability to the semicircle law σ_{sc} . Then $(\sigma_N)_N$ also converges weakly in probability to the semicircle law.*

Proof. The assumption on the rank of $A_N - B_N$ implies that the eigenvalues of the two matrices interlace. Therefore we have for any interval $I \subset \mathbb{R}$ and any $\omega \in \Omega$ that

$$|\#(\text{eigenvalues of } A_N(\omega) \text{ in } I) - \#(\text{eigenvalues of } B_N(\omega) \text{ in } I)| \leq 2. \quad (18)$$

In order to prove Lemma 11 we fix a bounded continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ and numbers $\varepsilon, \gamma > 0$. We have to show that there exists N_0 such that for all $N \geq N_0$ the estimate

$$\mathbb{P}\left(\left|\int f d\sigma_N - \int f d\sigma_{sc}\right| > \varepsilon\right) < \gamma \quad (19)$$

holds. Choose a step function $g = \sum_{i=1}^r \alpha_i \mathcal{X}_{I_i}$ ($\alpha_i \in \mathbb{R}$, the intervals $I_i \subset [-4, 4]$ are pairwise disjoint) such that

$$\sup\{|f(x) - g(x)| : x \in [-4, 4]\} \leq \min\left(\frac{\varepsilon}{6}, 1\right). \quad (20)$$

Write $\int f d\sigma_N - \int f d\sigma_{sc} = \sum_{j=1}^6 \Delta_j$, with

$$\begin{aligned} \Delta_1 &:= \int f d\mu_N - \int f d\sigma_{sc}, & \Delta_2 &:= \int_{-4}^4 (f - g) d\sigma_N, \\ \Delta_3 &:= \int_{-4}^4 g d\sigma_N - \int_{-4}^4 g d\mu_N, & \Delta_4 &:= \int_{-4}^4 (g - f) d\mu_N, \\ \Delta_5 &:= \int_{\mathbb{R} \setminus [-4, 4]} f d\sigma_N, & \Delta_6 &:= - \int_{\mathbb{R} \setminus [-4, 4]} f d\mu_N. \end{aligned}$$

It suffices to show for each $j = 1, \dots, 6$ that there exists N_j such that for all $N \geq N_j$ we have

$$\mathbb{P}(|\Delta_j| > \frac{\varepsilon}{6}) < \frac{\gamma}{6} \quad (21)$$

Indeed, (19) then holds for all $N \geq N_0 := \max\{N_j : j = 1, \dots, 6\}$.

For $j = 2, 4$ estimate (21) follows trivially from (20) with $N_j = 1$. The case $j = 1$ is a consequence of the assumption of Lemma 11. Estimate (18) together with (20) imply

$$|\Delta_3| \leq \sum_{i=1}^r |\alpha_i| \cdot |\sigma_N(I_i) - \mu_N(I_i)| \leq r(\|f\|_\infty + 1) \cdot \frac{2}{N} < \frac{\varepsilon}{6}$$

by choosing N_3 sufficiently large. Next we treat $j = 6$. Using the assumed weak convergence of $(\mu_N)_N$ again and choosing an arbitrary continuous test function $f_0 : \mathbb{R} \rightarrow [0, 1]$ with $f_0|_{[-2,2]} = 0$ and $f_0|_{\mathbb{R} \setminus [-4,4]} = 1$ one may obtain N_6 such that for all $N \geq N_6$:

$$\mathbb{P}\left(\mu_N(\mathbb{R} \setminus [-4, 4]) > \frac{\varepsilon}{12(\|f\|_\infty + 1)}\right) \leq \mathbb{P}\left(\int f_0 d\mu_N > \frac{\varepsilon}{12(\|f\|_\infty + 1)}\right) < \frac{\gamma}{6}. \quad (22)$$

On the one hand this implies $\mathbb{P}(|\Delta_6| > \frac{\varepsilon}{12}) < \frac{\gamma}{6}$ for all $N \geq N_6$. On the other hand we may use (22) to handle Δ_5 . It follows from (18) that

$$|\mu_N(\mathbb{R} \setminus [-4, 4]) - \sigma_N(\mathbb{R} \setminus [-4, 4])| \leq \frac{2}{N} < \frac{\varepsilon}{12(\|f\|_\infty + 1)}$$

for all $N \geq \tilde{N}_5$ with a suitable choice of \tilde{N}_5 . Setting $N_5 := \max(\tilde{N}_5, N_6)$ then completes the proof with the help of (22). ■

After all ingredients have been gathered we may now conclude the proof of our main result.

Proof (Theorem 4). We denote by σ_N resp. μ_N the eigenvalue distribution measures of $A_N := X_N/\sqrt{N(1-m^2)}$, resp. $B_N := Y_N/\sqrt{N}$ (see (9), (7) for the definition of Y_N). Since $(Y_N)_N$ is an approximately uncorrelated scheme of random variables it follows from [13, Theorem 5] that $(\mu_N)_N$ converges weakly in probability to the semicircle law σ_{sc} . Since

$$A_N - B_N = \frac{m}{\sqrt{N(1-m^2)}}(\mathcal{X}_+ - \mathcal{X}_-)\mathcal{E}_N$$

always has rank 1, the claim now follows from Lemma 11. ■

Remark 12 Observe that in the proof of Theorem 4 we did not use the special form of the function F_β but only those properties that were collected in Proposition 8. Hence the result also holds for generalized $\mathbb{P}_{N^2}^{N^\alpha F}$ -Curie-Weiss ensembles (cf. [13, Definition 29]), if $F : (-1, 1) \rightarrow \mathbb{R}$ satisfies all the properties stated in Proposition 8. Of course, condition c) may be replaced by the requirement that $\int_{-1}^1 e^{-xF(t)/2} \frac{dt}{1-t^2} < \infty$ for sufficiently large values of x , because this is the only use of property c).

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