Random Block Operators

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Abstract We study fundamental spectral properties of random block operators that are common in the physical modelling of mesoscopic disordered systems such as dirty superconductors. Our results include ergodic properties, the location of the spectrum, existence and regularity of the integrated density of states, as well as Lifshits tails. Special attention is paid to the peculiarities arising from the block structure such as the occurrence of a robust gap in the middle of the spectrum. Without randomness in the off-diagonal blocks the density of states typically exhibits an inverse square-root singularity at the edges of the gap. In the presence of randomness we establish a Wegner estimate that is valid at all energies. It implies that the singularities are smeared out by randomness, and the density of states is bounded. We also show Lifshits tails at these band edges. Technically, one has to cope with a non-monotone dependence on the random couplings.

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1 Introduction

Random block operators play an important role in the mathematical modelling of superfluid fermions in a random environment and are thus relevant for mesoscopic disordered quantum systems such as dirty superconductor devices. They arise in the Bogoliubov-de Gennes equation

$$\begin{pmatrix} H & B \\ B^* & -\bar{H} \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = E \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \tag{1.1}$$

that is, the eigenvalue problem for the quasi-particle (or excitation) states $\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ in a mean-field approximation of BCS theory [6]. Without loss of generality we have assumed that the chemical potential equals zero in (1.1). The 'particle' and 'hole' components ψ_+ and ψ_- of the quasi-particle state belong to the single-particle Hilbert space $\mathcal{H}$. The self-adjoint single-particle Hamiltonian $H = H^*$ and the so-called pair potential or gap function B are linear operators on $\mathcal{H}$. The overbar in (1.1) denotes complex conjugation.

Following Altland and Zirnbauer [2] one can classify all block operators that arise in (1.1) according to their behaviour with respect to time-reversal and spin-rotation symmetry. In this paper we will focus on random block operators of the form

$$\mathbb{H} := \begin{pmatrix} H & B \\ B & -H \end{pmatrix} \tag{1.2}$$

with both H and B self-adjoint. This choice corresponds to symmetry class CI of [2] and describes physical systems for which both time-reversal and spin-rotation symmetry hold. Since the Bogoliubov-de Gennes equation results from a mean-field approximation, the expressions for the operators H and B should be determined from self-consistency requirements. For disordered systems the discrete Anderson model in d dimensions,

$$H := \Delta + V \quad \text{on } \mathcal{H} = \ell^2(\mathbb{Z}^d), \tag{1.3}$$

is a generally accepted effective description for this, see e.g. [20]. (Choosing $\mathbb{H}$ as a random matrix from a suitable ensemble would be another [2].) Here,

$$(\Delta\psi)(j) := \sum_{i \in \mathbb{Z}^d : |i-j|=1} \psi(i) \tag{1.4}$$

for all $j \in \mathbb{Z}^d$ and all $\psi \in \ell^2(\mathbb{Z}^d)$ is the centred discrete Laplacian and the random potential V amounts to multiplication by independent and identically distributed, real-valued random variables $\{V(j)\}_{j \in \mathbb{Z}^d}$ according to $(V\psi)(j) := V(j)\psi(j)$.

The form of the gap operator B, which should also be determined by self-consistency, depends on the pairing mechanism. For s-wave (a.k.a. conventional) superconductors B is a multiplication operator in position space $\ell^2(\mathbb{Z}^d)$. Homogeneous s-wave superconductors are described by a multiple of the identity operator, $B = \beta\,\mathbb{1}$ with a self-consistently determined parameter $\beta > 0$. Disordered s-wave superconductors are often described by an effective random multiplication operator $B = b$ [6, 20, 22]. Here $(b\psi)(j) := b(j)\psi(j)$, where $\{b(j)\}_{j \in \mathbb{Z}^d}$ are independent and identically distributed real-valued random variables. In addition, the $b(j)$'s are often required to be independent of the $V(j)$'s. Our main results in Sects. 5 and 6 will be proved in precisely this setting.

Non-diagonal gap operators B occur in the modelling of $d_{x^2-y^2}$-wave superconductors. For example, the momentum-dependent interaction of Cooper pairs leads to $B =$
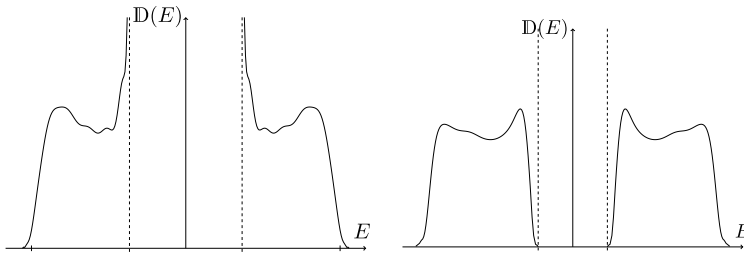
Fig. 1 Comparison of the density of states of a random block operator $\mathbb{H}$ without (*left*) and with (*right*) randomness in the off-diagonal blocks B. The singularities at the inner band edges are smoothed out by the randomness in B

$\beta(\Delta_x^{(1)} - \Delta_y^{(1)})$ for homogeneous superconductors in two dimensions [1, 7], where $\beta > 0$ and $\Delta_{x/y}^{(1)}$ denotes the one-dimensional centred discrete Laplacian in x-, resp. y-direction. Since diagonal disorder has also been used to model inhomogeneous d-wave superconductors [20, 22], our results may also be of relevance for these materials.

The plan of this paper is as follows. In Sect. 2 we study basic spectral features of block operators $\mathbb{H}$ that are of the general form (1.2). Among others we establish the existence of a robust spectral gap of $\mathbb{H}$ in Proposition 2.10. We interpret the robustness of the gap in the context of Anderson's theorem [3, 4] in Remark 2.11(iii).

Section 3 briefly discusses an important special case of (1.2), namely constant off-diagonal blocks $B = \beta \mathbf{1}$. This serves to expose a typical phenomenon in the absence of disorder: the density of states of $\mathbb{H}$ suffers from an inverse square-root singularity at the inner band edges in every dimension $d \in \mathbb{N}$. The singularity is robust in the sense that it always shows up unless the density of states of H vanishes at energy zero—the location of the chemical potential.

We introduce the main objects of this paper, ergodic random block operators, in Sect. 4. The basic spectral consequences of ergodicity are also explored there. This includes the location of the almost-sure spectrum, as well as the definition and self-averaging of the integrated density of states of $\mathbb{H}$.

In Sect. 5 we show that the density of states exists for suitable ergodic random block operators and that it is bounded. This is the content of Theorem 5.1, the main result of this paper, and follows from a Wegner estimate. Figure 1 compares this situation to the one with the singularity at the inner band edges for a constant $B = \beta \mathbf{1}$ as in Sect. 3. Randomness in the off-diagonal blocks smooths out the singularities. We stress that the Wegner estimate of Theorem 5.1 holds for a block random Schrödinger operator with a sign-indefinite single-site potential of mean zero. In contrast, for ordinary (i.e. non-block) random Schrödinger operators such a result is still missing despite a lot of recent efforts [14, 19].

Finally, Sect. 6 establishes Lifshits tails for the integrated density of states of $\mathbb{H}$ at the inner band edges. Theorem 6.1 embodies the second main result presented here.

The Wegner estimate and the Lifshits tail we establish in this paper allow to prove spectral and dynamical localisation for random block operators by performing a multi-scale analysis. The details [8] will be given elsewhere.

2 Basic Properties of Block Operators

In this section we explore some fundamental properties of rather general self-adjoint block operators

$$\mathbb{H} := \begin{pmatrix} H & B \\ B & -H \end{pmatrix} \tag{2.1}$$

on the Hilbert space $\mathcal{H}^2 := \mathcal{H} \oplus \mathcal{H}$. We equip $\mathcal{H}^2$ with the scalar product

$$\langle\!\langle \Psi, \Phi \rangle\!\rangle := \langle \psi_1, \varphi_1 \rangle + \langle \psi_2, \varphi_2 \rangle, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ stands for the Hilbert space scalar product of $\mathcal{H}$ and $\Psi := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, $\Phi := \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathcal{H}^2$. We write $\|\!|\Psi|\!\| := (\|\psi_1\|^2 + \|\psi_2\|^2)^{1/2}$ for the induced norm.

Later we will be mainly interested in the case where the self-adjoint operator H is a discrete Schrödinger operator on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ and B is a multiplication operator with some real-valued function b on $\mathbb{Z}^d$. In this situation both operators H and B are frequently bounded, so that $\mathbb{H}$ is unambiguously well defined as a self-adjoint operator. However, we can treat unbounded operators as well, as one can read off from the following assertion.

Proposition 2.1

 (i) *Let H, B be self-adjoint, assume that $\mathrm{dom}(B) \cap \mathrm{dom}(H)$ is a core for H and that $\mathrm{dom}(|H|^{1/2}) \subseteq \mathrm{dom}(B)$. Then $\mathbb{H}$ is essentially self-adjoint on $(\mathrm{dom}(B) \cap \mathrm{dom}(H)) \oplus (\mathrm{dom}(B) \cap \mathrm{dom}(H))$.*
 (ii) *Let H be self-adjoint, let B be symmetric and H-bounded with bound strictly smaller than one. Then $\mathbb{H}$ is self-adjoint on $\mathrm{dom}(H) \oplus \mathrm{dom}(H)$.*
(iii) *Let B be self-adjoint, let H be symmetric and B-bounded with bound strictly smaller than one. Then $\mathbb{H}$ is self-adjoint on $\mathrm{dom}(B) \oplus \mathrm{dom}(B)$.*

Proof The assertions are special cases of Thm. 2.6.6 and Prop. 2.3.6 in [18]. ☐

Without further mentioning we will assume in the rest of this paper that at least one of the three situations described by Proposition 2.1 applies, thereby ensuring self-adjointness of $\mathbb{H}$. Next we compile some basic structural properties of the spectrum of $\mathbb{H}$.

Lemma 2.2 *The operators $\mathbb{H}$ and $\mathbb{H}' := \begin{pmatrix} B & H \\ H & -B \end{pmatrix}$ are unitary equivalent.*

Proof Define the unitary involution $\mathbb{U}_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, then $\mathbb{H}' = \mathbb{U}_1 \mathbb{H} \mathbb{U}_1{}^*$. ☐

Lemma 2.3 *The spectrum of $\mathbb{H}$ is symmetric around 0, i.e.*

$$\mathrm{spec}(\mathbb{H}) = -\mathrm{spec}(\mathbb{H}). \tag{2.3}$$

In particular, if $\mathbb{H}\Psi = E\Psi$ for some $E \in \mathbb{R}$ and $\Psi = (\psi_1, \psi_2)^T \in \mathcal{H}^2$, then $\mathbb{H}\widetilde{\Psi} = -E\widetilde{\Psi}$, where $\widetilde{\Psi} = (\psi_2, -\psi_1)^T$.

Proof We define the unitary transformation $\mathbb{U}_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathcal{H}^2$, which obeys $\mathbb{U}_2^{-1} = -\mathbb{U}_2$, and observe $\mathbb{U}_2 \mathbb{H} \mathbb{U}_2^* = -\mathbb{H}$. ☐

Remarks 2.4

(i) It follows that the spectrum of $\mathbb{H}^2$ has multiplicity at least 2, except possibly at zero, and that

$$\text{spec}(\mathbb{H}) = \{E \in \mathbb{R} \mid E^2 \in \text{spec}(\mathbb{H}^2)\}. \tag{2.4}$$

(ii) The anti-symmetry of $\mathbb{H}$ under the transformation $\mathbb{U}_2$ is known as 'particle-hole' symmetry.

Lemma 2.5 *The operator $\mathbb{H}^2$ is given by*

$$\mathbb{H}^2 = \begin{pmatrix} H^2 + B^2 & [H, B] \\ -[H, B] & H^2 + B^2 \end{pmatrix} \tag{2.5}$$

and unitarily equivalent to $K_- \oplus K_+$ on $\mathcal{H}^2$, where

$$K_\pm := H^2 + B^2 \pm \mathrm{i}[H, B]. \tag{2.6}$$

Proof This follows from an explicit computation and the observation that

$$\mathbb{U}_3 \mathbb{H}^2 \mathbb{U}_3^* = \begin{pmatrix} K_- & 0 \\ 0 & K_+ \end{pmatrix} \tag{2.7}$$

for the unitary $\mathbb{U}_3 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$. □

Remark 2.6 Introducing the annihilation operator $a := H - \mathrm{i}B$ on $\mathcal{H}$, we can write $K_+ = aa^*$ and $K_- = a^*a$. Thus, the spectra of K_+ and K_- differ at most by $\{0\}$. In fact, we have

$$\begin{pmatrix} K_- & 0 \\ 0 & K_+ \end{pmatrix} = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}^2 \quad \text{and} \quad \mathbb{U}_3 \mathbb{H} \mathbb{U}_3^* = \mathrm{i} \begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix}. \tag{2.8}$$

A direct calculation also shows the next assertion.

Lemma 2.7 *Suppose there exists a unitary involution $U = U^* = U^{-1}$ on $\mathcal{H}$ such that $HU + UH = 0$ and $[B, U] = 0$. Then $\mathbb{U} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & U \\ 1 & -U \end{pmatrix}$ is unitary on $\mathcal{H}^2$ and $\mathbb{U}\mathbb{H}\mathbb{U}^* = \begin{pmatrix} H+UB & 0 \\ 0 & H-UB \end{pmatrix}$.*

Later we will use the following particularisation of Lemma 2.7.

Corollary 2.8 *Assume $B \equiv b$ is the maximal self-adjoint multiplication operator by the function $b : \mathbb{Z}^d \to \mathbb{R}$ and Δ the centred discrete Laplacian as defined in (1.4). Then the operator $\mathbb{H} = \begin{pmatrix} \Delta & b \\ b & -\Delta \end{pmatrix}$ is unitarily equivalent to $H_+ \oplus H_-$, where $H_\pm := \Delta \pm (-1)^j b$.*

Proof Choose $U := (-1)^j$, that is

$$(U\psi)(j) := (-1)^j \psi(j) := (-1)^{\sum_{k=1}^d j_k} \psi(j) \tag{2.9}$$

for all $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$ and $\psi \in \ell^2(\mathbb{Z}^d)$, and apply Lemma 2.7. □

Remarks 2.9

(i) It is essential for the validity of Corollary 2.8 that Δ contains no diagonal terms.

(ii) Lemma 2.5 and Corollary 2.8 imply that $\mathbb{H}^2$ is both unitarily equivalent to $K_- \oplus K_+$ and also to $H_+^2 \oplus H_-^2$. We have $U K_+ U = K_-$ and $U H_+ U = -H_-$. Note also that none of the operators H_+^2 or H_-^2 alone is unitarily equivalent to K_+ or K_-.

(iii) From Corollary 2.8 we can infer some information on the location of the spectrum of discrete Schrödinger operators with periodic potentials. For example, if b is the constant function equal $\beta \in \mathbb{R}$, then the corresponding operators $H_\pm$ are unitarily equivalent. Hence $\mathrm{spec}(H_+) = \mathrm{spec}(H_-) = \mathrm{spec}(\mathbb{H})$. Thus it will follow from Proposition 2.10 below, that the interval $]-\beta, \beta[$ is a spectral gap for the operator $\Delta + (-1)^j \beta$.

Finally, we are concerned with locating the spectrum of general $\mathbb{H}$ and with the occurrence of spectral gaps that arise from the special block structure.

Proposition 2.10

(i) $\mathrm{spec}(\mathbb{H}) \subseteq [-\|H\| - \|B\|, \|H\| + \|B\|]$.

(ii) *Suppose there exists $\lambda \geqslant 0$ such that $H \geqslant \lambda 1$. Then*

$$\mathrm{spec}(\mathbb{H}) \cap]-\lambda, \lambda[= \varnothing. \tag{2.10}$$

(iii) *Suppose there exists $\beta \geqslant 0$ such that $B \geqslant \beta 1$. Then*

$$\mathrm{spec}(\mathbb{H}) \cap]-\beta, \beta[= \varnothing. \tag{2.11}$$

(iv) *Suppose there exist $\lambda, \beta \geqslant 0$ such that $H \geqslant \lambda 1$ and $B \geqslant \beta 1$. Then*

$$\mathrm{spec}(\mathbb{H}) \cap]-(\lambda^2 + \beta^2)^{1/2}, (\lambda^2 + \beta^2)^{1/2}[= \varnothing. \tag{2.12}$$

Remarks 2.11

(i) The endpoints of the interval in Proposition 2.10(i) are sharp upper and lower bounds for the maximum and minimum of $\mathrm{spec}(\mathbb{H})$. This can be seen by choosing B as the multiplication operator by the function $b = (-1)^j$ in Corollary 2.8.

(ii) The statements of Proposition 2.10(ii), (iii) and (iv) remain true if one replaces H and B by $-H$ and $-B$ in the assumptions.

(iii) According to Proposition 2.10(iii), $\mathbb{H}$ has always a spectral gap of size at least 2β, no matter what H is like. This statement can be interpreted in the context of Anderson's theorem [3, 4]: Anderson argued that adding non-magnetic impurities to an s-wave superconductor should have only little (or at best no) effect on the gap operator B. In view of Proposition 2.10(iii), this implies stability of the gap under non-magnetic impurity doping and thus leads to the experimentally observed insensitivity of superconductivity in this case. As an aside we mention that the situation is totally different when doping $d_{x^2-y^2}$-wave superconductors by non-magnetic impurities. Here it is known that disorder leads to the breaking of Cooper pairs.

Proof of Proposition 2.10 The statement in (i) follows from Lemma 2.5 and $\|K_\pm\| \leqslant (\|H\| + \|B\|)^2$.

To prove (iv) we define $\widetilde{H} := H - \lambda \geqslant 0$ and $\widetilde{B} := B - \beta \geqslant 0$. Then we get for the operators $K_\pm$ in Lemma 2.5

$$K_\pm = \widetilde{H}^2 + \widetilde{B}^2 \pm \mathrm{i}[\widetilde{H}, \widetilde{B}] + \lambda^2 + 2\lambda\widetilde{H} + \beta^2 + 2\beta\widetilde{B} \geqslant \lambda^2 + \beta^2, \tag{2.13}$$

where we used that the linear terms in λ and β are manifestly non-negative, as is the sum of the first three terms on account of Remark 2.6. Hence, $\mathbb{H}^2 \geqslant \lambda^2 + \beta^2$.

The proofs of (ii) and (iii) are simpler and proceed along the very same lines. $\qquad\square$

Part (ii) of Proposition 2.10 allows for a generalisation towards different diagonal blocks.

Lemma 2.12 *Let H_1 be a self-adjoint operator on $\mathcal{H}$ such that the block operator $\mathbb{H}_1 := \left(\begin{smallmatrix} H & B \\ B & -H_1 \end{smallmatrix}\right)$ is self-adjoint on $\mathcal{H}^2$. Suppose there exists $\lambda \geqslant 0$ such that both $H \geqslant \lambda$ and $H_1 \geqslant \lambda$. Then*

$$\operatorname{spec}(\mathbb{H}_1) \cap \,]-\lambda, \lambda[= \varnothing. \tag{2.14}$$

Proof We define $\widetilde{H} := H - \lambda \geqslant 0$ and $\widetilde{H}_1 := H_1 - \lambda \geqslant 0$. An explicit computation gives

$$\mathbb{H}_1^2 = \begin{pmatrix} H^2 + B^2 & HB - BH_1 \\ BH - H_1 B & H_1^2 + B^2 \end{pmatrix} = \begin{pmatrix} \lambda^2 + 2\lambda\widetilde{H} & 0 \\ 0 & \lambda^2 + 2\lambda\widetilde{H}_1 \end{pmatrix} + \begin{pmatrix} \widetilde{H} & B \\ B & -\widetilde{H}_1 \end{pmatrix}^2$$
$$\geqslant \lambda^2 \mathbb{1}. \tag{2.15}$$

$\qquad\square$

3 The Case of Constant Diagonal B

As a warm-up and in order to expose some typical features we consider first the simple case where $B = \beta\,\mathbb{1}$ is a constant multiple of the identity. In particular, the operators H and B commute. As a consequence, the spectral theory of $\mathbb{H}$ can be reduced to the diagonalisation of a 2×2-matrix and the spectral theory of H.

Proposition 3.1 *Let H be a self-adjoint operator on $\mathcal{H}$, let $\beta \in \mathbb{R} \setminus \{0\}$ and consider*

$$\mathbb{H} := \begin{pmatrix} H & \beta\mathbb{1} \\ \beta\mathbb{1} & -H \end{pmatrix} \tag{3.1}$$

on $\mathcal{H}^2$. Then

(i) $\operatorname{spec}(\mathbb{H}) = \{E_\pm := \pm(E^2 + \beta^2)^{1/2} : E \in \operatorname{spec}(H)\}$.
(ii) *If $H\varphi = E\varphi$ for some $E \in \operatorname{spec}(H)$ and $\varphi \in \mathcal{H}$, then $\mathbb{H}\widetilde{\Phi}_\pm = E_\pm \widetilde{\Phi}_\pm$ for the non-normalised vector $\widetilde{\Phi}_\pm := \left(\begin{smallmatrix} \varphi \\ \beta^{-1}(E_\pm - E)\varphi \end{smallmatrix}\right) \in \mathcal{H}^2$.*
(iii) *Fix also $E \in \mathbb{R}$. Then there exists a constant $C \equiv C_{E,\beta} > 0$ such that if $\|(H - E)\varphi\| \leqslant \varepsilon$ for some $\varphi \in \mathcal{H}$ and $\varepsilon > 0$, then*

$$\|(\mathbb{H} - E_\pm)\widetilde{\Phi}_\pm\| \leqslant C\varepsilon. \tag{3.2}$$

Proof Part (i) is a corollary of Lemma 2.5, Part (ii) follows from an explicit computation. As to Part (iii) we observe

$$\mathbb{H}\widetilde{\Phi}_\pm = \begin{pmatrix} H\varphi + (E_\pm - E)\varphi \\ \beta\varphi - \beta^{-1}(E_\pm - E)H\varphi \end{pmatrix} = E_\pm \widetilde{\Phi}_\pm + \begin{pmatrix} (H - E)\varphi \\ -\beta^{-1}(E_\pm - E)(H - E)\varphi \end{pmatrix}. \tag{3.3}$$

This implies

$$\|(\mathbb{H} - E_\pm)\widetilde{\Phi}_\pm\|^2 \leqslant \varepsilon^2 + \beta^{-2}(E_\pm - E)^2 \varepsilon^2 =: C^2 \varepsilon^2. \tag{3.4}$$
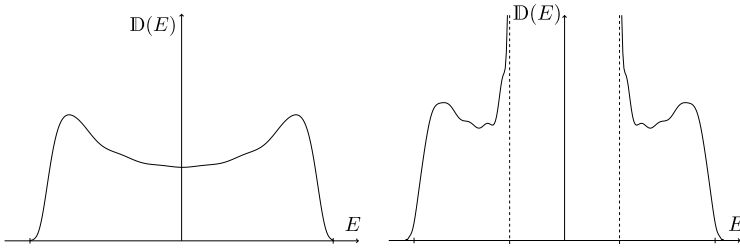
$\qquad\square$

Fig. 2 Comparison of the density of states of a random block operator $\mathbb{H}$ with $B = 0$ (*left*) and $B = \beta 1$ (*right*)

Remark 3.2 We know already from Proposition 2.10(iii) that the spectrum of $\mathbb{H}$ as given by (3.1) has always a gap between $-|\beta|$ and $|\beta|$ no matter how the spectrum of H looks like. For an interpretation of this in the context of Anderson's theorem, see Remark 2.11(iii).

We can also compute the density of states of $\mathbb{H} = \begin{pmatrix} H & \beta 1 \\ \beta 1 & -H \end{pmatrix}$, given the operator H possesses a density of states. The corresponding result is stated and proven here, even though we postpone the formal definition of the integrated density of states and of its Lebesgue derivative, the density of states, to the next section.

Proposition 3.3 *If H possesses an absolutely continuous integrated density of states with Lebesgue derivative D, then the integrated density of states of $\mathbb{H}$ is also absolutely continuous with Lebesgue derivative given by*

$$\mathbb{D}(E) = \frac{|E|}{\sqrt{E^2 - \beta^2}} \big[D(\sqrt{E^2 - \beta^2}) + D(-\sqrt{E^2 - \beta^2}) \big] \tag{3.5}$$

for a.e. $E \in \mathrm{spec}(\mathbb{H}) \subseteq \mathbb{R} \backslash]-|\beta|, |\beta|[$, and $\mathbb{D}(E) = 0$ for $E \notin \mathrm{spec}(\mathbb{H})$.

Remark 3.4 Observe that in the case considered here the density of states $\mathbb{D}$ of $\mathbb{H}$ has a square-root singularity at the inner band edges $\pm\beta$ whenever $D(0+) \neq 0$, see the right panel of Fig. 2. This is typically the case, if 0 lies in the interior of the spectrum of a random Schrödinger operator H [9, 10, 21]. The square-root singularity may remind the reader of the van Hove singularity in dimension $d = 1$. In the present case the nature of the singularity is independent of the dimension d, however.

Proof By assumption we have for every interval $A_0 \subseteq \mathbb{R}$

$$\int_{A_0} dE_0 \, D(E_0) = \lim_{L \to \infty} \frac{\#\{\text{eigenvalues of } H^{(L)} \text{ lying in } A_0\}}{|\Lambda_L|}, \tag{3.6}$$

where $H^{(L)}$ is a self-adjoint finite-volume restriction of H to the cube Λ_L centred about the origin and containing $|\Lambda_L| = L^d$ points for $L \in \mathbb{N}$ odd. In particular, the spectrum of $H^{(L)}$ is discrete and $H^{(L)}$ converge to H in the limit $L \to \infty$. Now, for a given interval $A \subseteq [0, \infty[$ we define

$$A_0 := \Big\{ E_0 \in \mathbb{R} : \sqrt{E_0^2 + \beta^2} \in A \Big\} \tag{3.7}$$

and

$$\mathbb{H}^{(L)} := \begin{pmatrix} H^{(L)} & \beta 1 \\ \beta 1 & -H^{(L)} \end{pmatrix}. \tag{3.8}$$

Proposition 3.1 implies

$$\#\{\text{eigenvalues of } H^{(L)} \text{ lying in } A_0 = \#\{\text{eigenvalues of } \mathbb{H}^{(L)} \text{ lying in } A\}. \tag{3.9}$$

On the other hand, we have the identity

$$\int_{A_0} \mathrm{d}E_0 \, D(E_0) = \int_A \mathrm{d}E \, \mathbb{D}(E), \tag{3.10}$$

which results from the change-of-variables $E_0 = \sqrt{E^2 - \beta^2}$, $E > 0$, on $A_0 \cap [0, \infty[$ and $E_0 = -\sqrt{E^2 - \beta^2}$, $E > 0$, on $A_0 \cap]-\infty, 0]$. Altogether we have

$$\int_A \mathrm{d}E \, \mathbb{D}(E) = \lim_{L \to \infty} \frac{\#\{\text{eigenvalues of } \mathbb{H}^{(L)} \text{ lying in } A\}}{|\Lambda_L|}. \tag{3.11}$$

The same argument applies if $A \subseteq]-\infty, 0[$. Thus, (3.11) generalises to arbitrary Borel sets $A \subseteq \mathbb{R}$, proving the assertion. $\qquad\square$

4 Ergodic Properties of Random Block Operators

From now on we are concerned with random block operators. We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a block-operator-valued random variable

$$\mathbb{H} : \omega \mapsto \mathbb{H}_\omega := \begin{pmatrix} H_\omega & B_\omega \\ B_\omega & -H_\omega \end{pmatrix}, \tag{4.1}$$

where $\mathbb{H}_\omega$ is densely defined on $\mathcal{H}^2 = \ell^2(\mathbb{Z}^d) \oplus \ell^2(\mathbb{Z}^d)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Throughout we will assume that (at least) one of the situations of Proposition 2.1 applies to $\mathbb{H}_\omega$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ so that self-adjointness is ensured.

We say that the random block operator $\mathbb{H}$ is *ergodic* (w.r.t. $p\mathbb{Z}^d$-translations), if there exists a period $p \in \mathbb{N}^d$ and an ergodic group of measure-preserving transformations $\{\tau_j\}_{j \in p\mathbb{Z}^d}$ on Ω such that $\mathbb{H}$ fulfils the covariance relation $\mathbb{U}_j \mathbb{H}_\omega \mathbb{U}_j^* = \mathbb{H}_{\tau_j(\omega)}$ for every $\omega \in \Omega$ and for every $j \in p\mathbb{Z}^d = \times_{k=1}^d (p_k \mathbb{Z})$, where $\mathbb{U}_j := \begin{pmatrix} U_j & 0 \\ 0 & U_j \end{pmatrix}$ and U_j is the unitary translation operator on $\ell^2(\mathbb{Z}^d)$, that is $U_j \varphi := \varphi(\cdot - j)$ for every $\varphi \in \ell^2(\mathbb{Z}^d)$.

Consequently, standard results [5, 11, 12, 16] imply the existence of a non-random closed set $\Sigma \subseteq \mathbb{R}$, the *a.s. spectrum* of $\mathbb{H}$, such that $\mathrm{spec}(\mathbb{H}_\omega) = \Sigma$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Analogous statements hold for the components in the Lebesgue decomposition of the spectrum.

Next we introduce the central quantity of this paper which measures the density of spectral values of $p\mathbb{Z}^d$-ergodic random block operators $\mathbb{H}$.

Definition 4.1 The (non-random) right-continuous, non-decreasing function $\mathbb{N} : \mathbb{R} \to [0, 1]$, defined by

$$\mathbb{N}(E) := \frac{1}{2\,|\Lambda_0|} \, \mathbb{E}\big[\mathrm{tr}_{\mathcal{H}^2}[\chi_{\Lambda_0} \chi_{]-\infty, E]}(\mathbb{H})]\big] \tag{4.2}$$

for all $E \in \mathbb{R}$, is called *integrated density of states* of $\mathbb{H}$. Here we introduced the elementary cell $\Lambda_0 := \{j \in \mathbb{Z}^d : 0 \leqslant j_k < p_k \text{ for all } k = 1, \ldots, d\}$, $\mathbb{E}$ stands for the probabilistic expectation on Ω, the trace extends over $\mathcal{H}^2$ and the notation χ_{Λ_0} refers to the multiplication operator $\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$ where M is multiplication by the indicator function of Λ_0 on $\ell^2(\mathbb{Z}^d)$.

As we shall see from the next lemma, the definition of $\mathbb{N}$ is justified by Birkhoff's ergodic theorem. For $L \in \mathbb{N}$ odd, we denote by $\Lambda_L(j)$ the cube centred about $j \in \mathbb{Z}^d$ and containing $|\Lambda_L(j)| = L^d$ many points of $\mathbb{Z}^d$. We also write $\Lambda_L := \Lambda_L(0)$.

Lemma 4.2 *Let $\mathbb{H}$ be the random, $\mathbb{P}$-a.s. self-adjoint block matrix operator* (4.1), *which is ergodic w.r.t. $p\mathbb{Z}^d$-translations. Then there exists a set $\Omega_0 \subseteq \Omega$ of full probability, $\mathbb{P}(\Omega_0) = 1$, such that*

$$\mathbb{N}(E) = \lim_{L \to \infty} \frac{1}{2L^d} \, \mathrm{tr}_{\mathcal{H}^2} \left[\chi_{\Lambda_L(j)} \chi_{]-\infty, E]}(\mathbb{H}_\omega) \right] \tag{4.3}$$

for every $E \in \mathbb{R}$, every $\omega \in \Omega_0$ and every $j \in \mathbb{Z}^d$. Moreover, the set of growth points of $\mathbb{N}$ is given by the a.s. spectrum of $\mathbb{H}$.

Proof This is fully analogous to [13] or Sect. 5.1 in [12]. □

Mostly we will be interested in more specific random block operators whose diagonal blocks are given by the discrete random Schrödinger operator of the Anderson model in d dimensions and whose off-diagonal blocks are i.i.d. random multiplication operators. For simplicity and ease of presentation we dispense with the presence of magnetic fields. More precisely, let $U_0 : \mathbb{Z}^d \to \mathbb{R}$ be a periodic potential with period $p \in \mathbb{N}^d$ and define the bounded periodic background Hamiltonian $H_0 := -\Delta + U_0$ on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$. Consider two independent sequences of independent real-valued random variables $(V(j))_{j \in \mathbb{Z}^d}$ and $(b(j))_{j \in \mathbb{Z}^d}$ on Ω such that all the $V(j) : \omega \mapsto V_\omega(j)$ are identically distributed with law μ_V and all the $b(j) : \omega \mapsto b_\omega(j)$ are identically distributed with law μ_b, and such that the V's are independent of the b's. For the sake of technical simplicity we assume that the supports of the measures μ_V and μ_b are bounded subsets of $\mathbb{R}$. For each elementary event $\omega \in \Omega$ the functions $V_\omega : j \mapsto V_\omega(j)$ and $b_\omega : j \mapsto b_\omega(j)$ give rise to corresponding multiplication operators. Then the random operators

$$H : \omega \mapsto H_\omega := H_0 + V_\omega \tag{4.4}$$

and

$$b : \omega \mapsto b_\omega \tag{4.5}$$

are $\mathbb{P}$-a.s. bounded and self-adjoint on $\ell^2(\mathbb{Z}^d)$. They are both ergodic w.r.t. $p\mathbb{Z}^d$-translations. From standard results, see e.g. [5, 11, 12, 16], we know that their spectra are $\mathbb{P}$-almost surely given by

$$\mathrm{spec}(H) = \mathrm{spec}(H_0) + \mathrm{supp}(\mu_V) \tag{4.6}$$

and

$$\mathrm{spec}(b) = \mathrm{supp}(\mu_b). \tag{4.7}$$

The corresponding random block operator

$$\mathbb{H} : \omega \mapsto \mathbb{H}_\omega := \begin{pmatrix} H_\omega & b_\omega \\ b_\omega & -H_\omega \end{pmatrix}, \tag{4.8}$$

is also ergodic and $\mathbb{P}$-a.s. bounded and self-adjoint on $\mathcal{H}^2$. The next lemma roughly locates its a.s. spectrum.

Lemma 4.3 *Let $\mathbb{H}$ be the random block operator* (4.8). *Then we have $\mathbb{P}$-a.s. the inclusions*

$$\left\{ \pm \sqrt{E^2 + \beta^2} : E \in \mathrm{spec}(H), \beta \in \mathrm{supp}(\mu_b) \right\} \subseteq \mathrm{spec}(\mathbb{H}) \subseteq [-r, r] \qquad (4.9)$$

where $r := \sup_{E \in \mathrm{spec}(H)} |E| + \sup_{\beta \in \mathrm{supp}(\mu_b)} |\beta|$.

Remark 4.4 Since $\sqrt{E^2 + \beta^2} < E + \beta$, if both $E, \beta > 0$, we suspect from recalling Remark 2.11(i) that even the left inclusion in (4.9) might be a strict one in certain cases. This is indeed true, as can be seen by choosing $H = -\Delta$ to be deterministic and μ_b a symmetrical distribution w.r.t. reflection at zero. In this situation we may apply Corollary 2.8 and get $\mathbb{P}$-a.s.

$$\mathrm{spec}(\mathbb{H}) = \bigcup_{\kappa = \pm} \mathrm{spec}(H_\kappa) = \mathrm{spec}(\Delta) + \mathrm{supp}(\mu_b). \qquad (4.10)$$

Combining Lemma 4.3 with Proposition 2.10(iv) we infer

Corollary 4.5 *Assume that $\inf \mathrm{spec}(H) \geqslant 0$, $\inf \mathrm{supp}(\mu_b) \geqslant 0$ and*

$$\lambda_\pm := \pm\sqrt{[\inf \mathrm{spec}(H)]^2 + [\inf \mathrm{supp}(\mu_b)]^2} > 0. \qquad (4.11)$$

Then λ_- and λ_+ are the endpoints of the open gap interval which separates the positive and negative parts of the a.s. spectrum of $\mathbb{H}$.

Proof of Lemma 4.3 While the right inclusion follows immediately from Lemma 2.10(i) and (4.6), the left inclusion is based on a typical Weyl-sequence argument. Let $(\varphi_n)_{n \in \mathbb{N}}$ be a Weyl sequence of normalised vectors for H_0 and some fixed energy $E_0 \in \mathrm{spec}(H_0)$. Since H_0 is a bounded operator we may assume without loss of generality that each $\varphi_n \neq 0$ has compact support. The Borel-Cantelli lemma implies the existence of a set Ω_0 of full probability, $\mathbb{P}(\Omega_0) = 1$, such that for every $\omega \in \Omega_0$, for every length $L > 0$, for every $\varepsilon > 0$, for every $v \in \mathrm{supp}\, \mu_V$ and every $\beta \in \mathrm{supp}\, \mu_b$ there exist $k \in \mathbb{Z}^d$ with

$$|V_\omega(j) - v| < \varepsilon \quad \text{and} \quad |b_\omega(j) - \beta| < \varepsilon \quad \text{for all } j \in \Lambda_L(k). \qquad (4.12)$$

Now fix also $v \in \mathrm{supp}\, \mu_V$ and $\beta \in \mathrm{supp}\, \mu_b$. We define

$$E_\pm := \pm\sqrt{(E_0 + v)^2 + \beta^2}, \qquad (4.13)$$

and, if $\beta \neq 0$, the vector

$$\Phi_n^\pm := \mathcal{N}_n \begin{pmatrix} \varphi_n \\ \beta^{-1}(E_\pm - E_0 - v)\varphi_n \end{pmatrix}, \qquad (4.14)$$

where the constant $\mathcal{N}_n$ ensures proper normalisation $\|\Phi_n^\pm\| = 1$. If $\beta = 0$, we set $\Phi_n^+ := \begin{pmatrix} \varphi_n \\ 0 \end{pmatrix}$ and $\Phi_n^- := \begin{pmatrix} 0 \\ \varphi_n \end{pmatrix}$. For every $n \in \mathbb{N}$ we have the estimate

$$\||(\mathbb{H}_\omega - E_\pm)\Phi_n^\pm\||$$

$$\leqslant \left\|\left|\begin{pmatrix} V_\omega - v & b_\omega - \beta \\ b_\omega - \beta & -(V_\omega - v) \end{pmatrix} \Phi_n^\pm \right\|\right| + \left\|\left|\left[\begin{pmatrix} H_0 + v & \beta \\ \beta & -(H_0 + v) \end{pmatrix} - E_\pm \right]\Phi_n^\pm \right\|\right|$$

$$=: T_n^{(1)} + T_n^{(2)} \tag{4.15}$$

and the identity

$$\left(T_n^{(1)}\right)^2 = \left\langle\!\left\langle \Phi_n^\pm \begin{pmatrix} (V_\omega - v)^2 + (b_\omega - \beta)^2 & 0 \\ 0 & (V_\omega - v)^2 + (b_\omega - \beta)^2 \end{pmatrix} \Phi_n^\pm \right\rangle\!\right\rangle. \tag{4.16}$$

For every $n \in \mathbb{N}$ and every $\varepsilon > 0$ we exploit the freedom to shift the support of φ_n such that it lies inside some large enough box $\Lambda_L(k)$ for which (4.12) holds. This implies

$$\left(T_n^{(1)}\right)^2 \leqslant 2\varepsilon^2 \langle\!\langle \Phi_n^\pm, \Phi_n^\pm \rangle\!\rangle = 2\varepsilon^2. \tag{4.17}$$

On the other hand, given any $\varepsilon > 0$, we infer from the Weyl-sequence property of $(\varphi_n)_{n \in \mathbb{N}}$ for H_0 and Proposition 3.1 that there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geqslant n_\varepsilon$ the estimate $T_n^{(2)} \leqslant C\varepsilon$ holds with some constant $C > 0$ depending on E_0, v and β. This proves $\||(\mathbb{H}_\omega - E_\pm)\Phi_n^\pm\|| \leqslant (C + \sqrt{2})\varepsilon$, and hence $(\Phi_n^\pm)_{n \in \mathbb{N}}$ is a Weyl sequence for $\mathbb{H}$ and $E_\pm$. $\quad\square$

In order to make manifest the interpretation of the integrated density of states $\mathbb{N}$ as an eigenvalue counting function we have to introduce appropriate finite-volume restrictions of the block operator $\mathbb{H}$.

Definition 4.6 For $\mathbb{H}$ as in (4.8) and a finite cube $\Lambda_L \subset \mathbb{Z}^d$ we introduce the *Dirichlet* and *Neumann* restrictions

$$\mathbb{H}_D^{(L)} := \begin{pmatrix} H_D^{(L)} & b \\ b & -H_D^{(L)} \end{pmatrix} \quad \text{and} \quad \mathbb{H}_N^{(L)} := \begin{pmatrix} H_N^{(L)} & b \\ b & -H_N^{(L)} \end{pmatrix} \tag{4.18}$$

of $\mathbb{H}$ to the $2|\Lambda_L|$-dimensional Hilbert space $\mathcal{H}_L^2 := \ell^2(\Lambda_L) \oplus \ell^2(\Lambda_L)$. We also introduce the *Dirichlet-bracketing* and *Neumann-bracketing* restrictions

$$\mathbb{H}_+^{(L)} := \begin{pmatrix} H_D^{(L)} & b \\ b & -H_N^{(L)} \end{pmatrix} \quad \text{and} \quad \mathbb{H}_-^{(L)} := \begin{pmatrix} H_N^{(L)} & b \\ b & -H_D^{(L)} \end{pmatrix}. \tag{4.19}$$

In the above we have used the Dirichlet, resp. Neumann, restriction

$$H_{D/N}^{(L)} := -\Delta_{D/N}^{(L)} + U_0 + V \tag{4.20}$$

of H to $\ell^2(\Lambda_L)$, and all multiplication operators are to be interpreted as canonical restrictions.

Remarks 4.7

(i) The Neumann Laplacian on $\ell^2(\Lambda_L)$ is also called *graph Laplacian* or *combinatorial Laplacian*. It is defined by

$$(-\Delta_N^{(L)}\psi)(j) := \sum_{k \in \Lambda_L : |j-k|=1} [\psi(j) - \psi(k)] \tag{4.21}$$

for every $\psi \in \ell^2(\Lambda_L)$ and every $j \in \Lambda_L$. The Dirichlet Laplacian can be represented as the perturbation

$$-\Delta_D^{(L)} := -\Delta_N^{(L)} + 2\Gamma^{(L)}, \qquad (4.22)$$

where $\Gamma^{(L)}$ is the multiplication operator given by

$$(\Gamma^{(L)}\psi)(j) := \psi(j)\,\#\{k \in \mathbb{Z}^d \setminus \Lambda_L : |k - j| = 1\} \qquad (4.23)$$

for every $j \in \Lambda_L$. In other words, $\Gamma^{(L)}$ lives only on the outermost layer of Λ_L and multiplies by the number of missing neighbours at every point.

(ii) The finite-volume restrictions $-\Delta_N^{(L)}$ and $-\Delta_D^{(L)}$ are defined such as to obey Dirichlet-Neumann bracketing. Consequently we obtain the chains of inequalities

$$\mathbb{H}_-^{(L)} \leqslant \mathbb{H}_N^{(L)} \leqslant \mathbb{H}_+^{(L)} \quad \text{and} \quad \mathbb{H}_-^{(L)} \leqslant \mathbb{H}_D^{(L)} \leqslant \mathbb{H}_+^{(L)} \qquad (4.24)$$

for every $\Lambda_L \subset \mathbb{Z}^d$ and the Dirichlet-Neumann bracketing properties

$$\mathbb{H}_-^{(\Lambda)} \oplus \mathbb{H}_-^{(\Lambda')} \leqslant \mathbb{H}_-^{(\Lambda \cup \Lambda')} \quad \text{and} \quad \mathbb{H}_+^{(\Lambda)} \oplus \mathbb{H}_+^{(\Lambda')} \geqslant \mathbb{H}_+^{(\Lambda \cup \Lambda')} \qquad (4.25)$$

for all disjoint $\Lambda, \Lambda' \subset \mathbb{Z}^d$ (in obvious abuse of our notation).

The desired interpretation of $\mathbb{N}$ follows from

Lemma 4.8 *Given the random block operator $\mathbb{H}$ from* (4.8) *and any $E \in \mathbb{R}$, we define the random, finite-volume eigenvalue counting function*

$$\mathbb{N}_X^{(L)}(E) := \frac{1}{2\,|\Lambda_L|}\, \mathrm{tr}_{\mathcal{H}_L^2}\left[\chi_{]-\infty,E]}(\mathbb{H}_X^{(L)})\right], \qquad (4.26)$$

where X symbolises any self-adjoint restriction such that

$$\mathbb{H}_-^{(L)} \leqslant \mathbb{H}_X^{(L)} \leqslant \mathbb{H}_+^{(L)}. \qquad (4.27)$$

Then

(i) *there is a set $\Omega_0 \subseteq \Omega$ of full probability, $\mathbb{P}(\Omega_0) = 1$, such that*

$$\mathbb{N}(E) = \lim_{L \to \infty} \mathbb{N}_{X,\omega}^{(L)}(E) \qquad (4.28)$$

for every $\omega \in \Omega_0$, every continuity point $E \in \mathbb{R}$ of $\mathbb{N}$ and every boundary condition X satisfying (4.27).

(ii) *In addition we have the bounds*

$$\mathbb{E}\left[\mathbb{N}_+^{(L)}(E)\right] \leqslant \mathbb{N}(E) \leqslant \mathbb{E}\left[\mathbb{N}_-^{(L)}(E)\right] \qquad (4.29)$$

for every finite cube Λ_L and every $E \in \mathbb{R}$.

Proof The arguments proceed as in the standard random Schrödinger case, see e.g. [5, 11–13, 16]. □

5 Random Diagonal B: Boundedness of the Density of States

We have seen in Sect. 3 that the *density* of states of $\mathbb{H}$ with off-diagonal operators B that are a constant multiple of the identity can have a singularity, even if the density of states of H is bounded. In this section we consider random operators $\mathbb{H}$ of the form (4.8) with diagonal disorder in all blocks. We will prove a Wegner estimate for such models. It implies that the density of states of $\mathbb{H}$ exists and is bounded provided H or b is bounded away from zero and the distribution of the random variables is absolutely continuous with a suitably regular Lebesgue density. Technically, one has to cope with a non-monotone dependence on the random potential: the single-site potential that enters $\mathbb{H}$ is sign-indefinite and has mean zero. In contrast, for ordinary (i.e. non-block) random Schrödinger operators with such single-site potentials it is not known how to prove a comparable Wegner estimate. Existing Wegner estimates involve either higher powers of the volume [19] or yield only log-Hölder continuity of the integrated density of states [14].

Theorem 5.1 *Consider the random block operator*

$$\mathbb{H} : \omega \mapsto \mathbb{H}_\omega = \begin{pmatrix} H_\omega & b_\omega \\ b_\omega & -H_\omega \end{pmatrix} \tag{5.1}$$

where H and b are given as in (4.4) *and* (4.5). *Assume that at least one of the following two conditions is satisfied*:

(H) *there exists $\lambda > 0$ such that $H \geqslant \lambda 1$ holds $\mathbb{P}$-a.s. and μ_V is absolutely continuous with a piecewise continuous Lebesgue density ϕ_V of bounded variation and compact support,*
(B) *there exists $\beta > 0$ such that $b \geqslant \beta 1$ holds $\mathbb{P}$-a.s. and μ_b is absolutely continuous with a piecewise continuous Lebesgue density ϕ_b of bounded variation and compact support.*

Then the integrated density of states $\mathbb{N}$ of $\mathbb{H}$ is Lipschitz continuous and has a bounded density $\mathbb{D} := d\mathbb{N}/dE$. In particular, we have for Lebesgue-almost all $E \in \mathbb{R}$

$$\mathbb{D}(E) \leqslant 2 \, \frac{|E| + 1}{\lambda} \, \|\phi_V\|_{\mathrm{BV}}, \tag{5.2}$$

if Condition (H) *applies, and*

$$\mathbb{D}(E) \leqslant 2 \, \frac{|E| + 1}{\beta} \, \|\phi_b\|_{\mathrm{BV}}, \tag{5.3}$$

if Condition (B) *applies. Here, $\|f\|_{\mathrm{BV}}$ denotes the total variation of $f : \mathbb{R} \to \mathbb{C}$.*

Remarks 5.2

(i) The density of states $\mathbb{D}$ is an even function on account of Lemma 2.3.
(ii) We recall that, typically, $\mathbb{D}$ exhibits a singularity at the inner band edges for constant off-diagonal blocks $b = \beta 1 > 0$, see the left panel of Fig. 1. Case (B) of Theorem 5.1 implies that, regardless of H, this singularity is smeared out by the disorder. In particular, $\mathbb{D}$ remains bounded.
(iii) Figure 3 displays a typical density of states for random block operators in the case (H) of Theorem 5.1. Here, $\mathbb{D}$ is not only bounded but even vanishes at the inner band edges, because it exhibits a Lifshits tail, see Sect. 6.
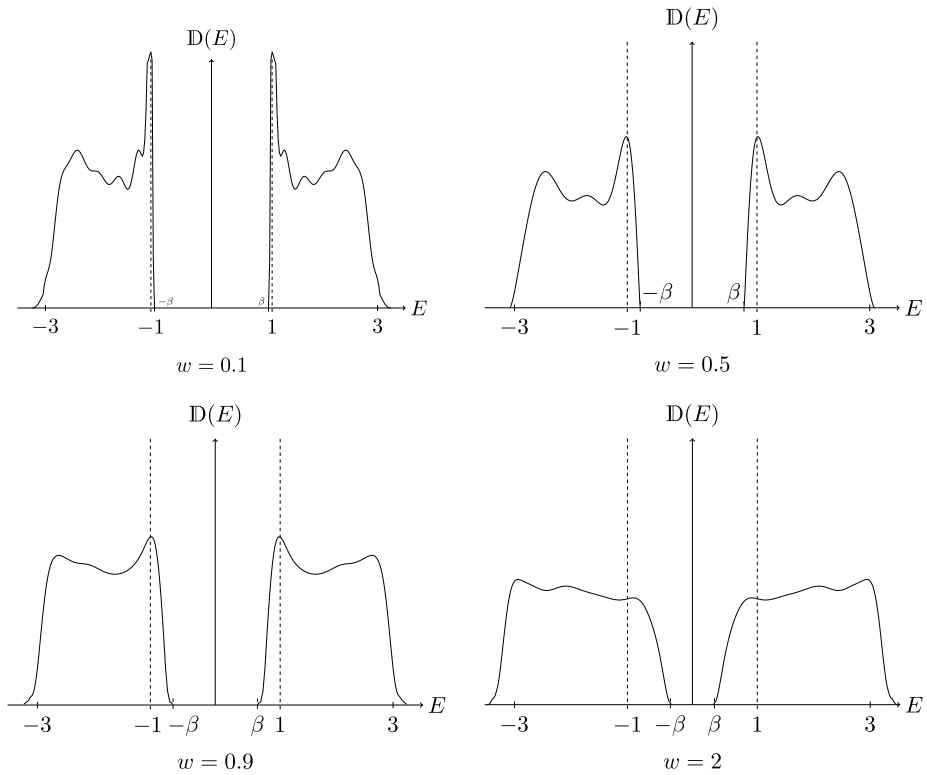
Fig. 3 Assuming $b_\omega(j) = 1 + w\, \widetilde{b}_\omega(j)$ with $\widetilde{b}_\omega(j)$ distributed uniformly in the interval $[-0.5, 0.5]$ we display the dependence of the density of states on the disorder parameter w (Assumption (B) of Theorem 5.1 is satisfied for $w = 0.1, 0.5, 0.9$)

Proof of Theorem 5.1 As compared to the proof of the Wegner estimate for the standard Anderson model, there are essentially two modifications necessary here. They have been isolated in the subsequent Lemmas 5.3 and 5.4.

Lemma 2.2 tells us that the roles of the diagonal operator H and of the off-diagonal operator b are interchangeable. Thus it suffices to prove only the case (H). Moreover, because of Lemma 2.3 we will restrict ourselves to energies $E \geqslant 0$ without loss of generality. In fact, using Condition (H) and Proposition 2.10(ii) it then suffices to consider $E \geqslant \lambda$, which we will do from now on. We denote by $\mathbb{H}^{(L)}(\mathbf{V}) := \mathbb{H}_D^{(L)}$ the Dirichlet restriction (4.18) of the operator $\mathbb{H}$ to the cube Λ_L. In this notation we make explicit the dependence of the operator on the random variables $\mathbf{V} := \{V_j\}_{j \in \Lambda_L}$ (we prefer to write V_j instead of $V(j)$ in this proof and the subsequent lemmas) and suppress the dependence on the entries of the off-diagonal blocks. Furthermore, we write $E_n(\mathbf{V}) := E_n(\mathbb{H}^{(L)}(\mathbf{V}))$ for the nth eigenvalue of $\mathbb{H}^{(L)}(\mathbf{V})$, where the eigenvalues are ordered by magnitude and repeated according to multiplicity. Finally, we fix $\varepsilon \in \,]0, \min\{\lambda, 1\}/3[$ and consider a switch function ϱ, i.e. $\varrho \in C^1(\mathbb{R})$ is non-decreasing with $0 \leqslant \varrho \leqslant 1$, $\varrho(\eta) = 1$ for $\eta > \varepsilon$ and $\varrho(\eta) = 0$ for $\eta < -\varepsilon$. Then

$$0 \leqslant \chi_{]E-\varepsilon, E+\varepsilon]}(\eta) \leqslant \varrho(\eta - E + 2\varepsilon) - \varrho(\eta - E - 2\varepsilon) \tag{5.4}$$

for all $\eta \in \mathbb{R}$, whence

$$
\mathrm{tr}_{\mathcal{H}_L^2} \left[\chi_{]E-\varepsilon, E+\varepsilon]} \left(\mathbb{H}^{(L)}(\mathbf{V}) \right) \right]
$$

$$
\leqslant \sum_{n=1}^{2|\Lambda_L|} \left[\varrho \left(E_n(\mathbf{V}) - E + 2\varepsilon \right) - \varrho \left(E_n(\mathbf{V}) - E - 2\varepsilon \right) \right]
$$

$$
= - \sum_{n=1}^{2|\Lambda_L|} \int_{E-2\varepsilon}^{E+2\varepsilon} \frac{\partial}{\partial \eta} \varrho \left(E_n(\mathbf{V}) - \eta \right) \mathrm{d}\eta
$$

$$
= \sum_{n=1}^{2|\Lambda_L|} \int_{E-2\varepsilon}^{E+2\varepsilon} \varrho' \left(E_n(\mathbf{V}) - \eta \right) \mathrm{d}\eta. \tag{5.5}
$$

The chain rule tells us that

$$
\sum_{j \in \Lambda_L} \frac{\partial}{\partial V_j} \varrho \left(E_n(\mathbf{V}) - \eta \right) = \varrho' \left(E_n(\mathbf{V}) - \eta \right) \sum_{j \in \Lambda_L} \frac{\partial E_n(\mathbf{V})}{\partial V_j}. \tag{5.6}
$$

Since $H^{(L)}(\mathbf{V}) \geqslant \lambda > 0$ by Condition (H), we conclude from this identity and Lemma 5.3

$$
\varrho' \left(E_n(\mathbf{V}) - \eta \right) \leqslant \frac{E_n(\mathbf{V})}{\lambda} \sum_{j \in \Lambda_L} \frac{\partial}{\partial V_j} \varrho \left(E_n(\mathbf{V}) - \eta \right)
$$

$$
\leqslant \frac{E+1}{\lambda} \sum_{j \in \Lambda_L} \frac{\partial}{\partial V_j} \varrho \left(E_n(\mathbf{V}) - \eta \right) \tag{5.7}
$$

for all $n \in \mathbb{N}$ and all $\eta \in [E - 2\varepsilon, E + 2\varepsilon]$. We note that the last inequality uses $3\varepsilon < \min\{\lambda, 1\}$. This guarantees that only those n with $E_n(\mathbf{V}) \in]0, E+1[$ contribute and that the j-sum is non-negative for these n by Lemma 5.3.

Since the random variables $\{V_j\}_{j \in \Lambda_L}$ are independent and have the same individual distribution μ_V, the expectation is just integration with respect to the product of these distributions. Thus (5.5) and (5.7) imply

$$
\mathbb{E} \left\{ \mathrm{tr}_{\mathcal{H}_L^2} \left[\chi_{]E-\varepsilon, E+\varepsilon]} \left(\mathbb{H}_D^{(L)} \right) \right] \right\}
$$

$$
\leqslant \frac{E+1}{\lambda} \sum_{j \in \Lambda_L} \int_{E-2\varepsilon}^{E+2\varepsilon} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\partial}{\partial V_j} \sum_{n=1}^{2|\Lambda_L|} \varrho \left(E_n(\mathbf{V}) - \eta \right) \mathrm{d}\mu_V(V_j) \right)
$$

$$
\times \left(\prod_{k \in \Lambda_L : k \neq j} \mathrm{d}\mu_V(V_k) \right) \mathrm{d}\eta. \tag{5.8}
$$

Since, in general, the function

$$
V_j \mapsto F(V_j) := \sum_{n=1}^{2|\Lambda_L|} \varrho \left(E_n(\mathbf{V}) - \eta \right) = \mathrm{tr}_{\mathcal{H}_L^2} \left[\varrho \left(\mathbb{H}^{(L)}(\mathbf{V}) - \eta \right) \right] \tag{5.9}
$$

is non-monotone in its argument for given $\eta \in \mathbb{R}$ and $V_k \in \mathbb{R}$, $k \neq j$, we deviate from the standard reasoning at this point. Clearly $F \in C^1(\mathbb{R})$ by analytic perturbation theory. Moreover changing V_j amounts to a rank-2-perturbation of $\mathbb{H}^{(L)}(\mathbf{V})$. Thus we have

$|F(v) - F(v')| \leqslant 2$ for all $v, v' \in \mathbb{R}$ and we can apply Lemma 5.4 to (5.8). This gives

$$\mathbb{E}\big\{ \operatorname{tr}_{\mathcal{H}_L^2} \big[\chi_{]E-\varepsilon, E+\varepsilon]}(\mathbb{H}_D^{(L)}) \big] \big\} \leqslant 8\varepsilon \, L^d \|\phi_V\|_{\mathrm{BV}} \frac{E+1}{\lambda}. \tag{5.10}$$

Finally, the assertion follows from Lemma 4.28 and dominated convergence. □

We come to the main deterministic tool used in the proof of Theorem 5.1. Given Assumption (H), it ensures that eigenvalues move around strongly enough when the random variables change their values.

Lemma 5.3 *Let $E(\mathbf{V})$ be an eigenvalue of $\left(\begin{smallmatrix} H^{(L)}(\mathbf{V}) & B^{(L)} \\ B^{(L)} & -H^{(L)}(\mathbf{V}) \end{smallmatrix} \right)$. Then*

$$E(\mathbf{V}) \sum_{j \in \Lambda_L} \frac{\partial E(\mathbf{V})}{\partial V_j} \geqslant \inf \operatorname{spec} \big(H^{(L)}(\mathbf{V}) \big). \tag{5.11}$$

Proof Let $\Psi(\mathbf{V}) \equiv \Psi = \left(\begin{smallmatrix} \psi_1 \\ \psi_2 \end{smallmatrix} \right) \in \mathcal{H}_L^2$ be an eigenvector corresponding to the eigenvalue $E(\mathbf{V}) \equiv E$, normalised according to $\langle\!\langle \Psi, \Psi \rangle\!\rangle = \langle \psi_1, \psi_1 \rangle + \langle \psi_2, \psi_2 \rangle = 1$. Thus, writing $H^{(L)}(\mathbf{V}) \equiv H^{(L)}$, it satisfies

$$\begin{aligned} H^{(L)}\psi_1 + B^{(L)}\psi_2 &= E\psi_1, \\ B^{(L)}\psi_1 - H^{(L)}\psi_2 &= E\psi_2. \end{aligned} \tag{5.12}$$

By the Feynman-Hellmann theorem (see e.g. [17]) we have

$$\frac{\partial E}{\partial V_j} = |\psi_1(j)|^2 - |\psi_2(j)|^2 \tag{5.13}$$

for every $j \in \Lambda_L$ and consequently, inserting (5.12) we find

$$\begin{aligned} E \sum_{j \in \Lambda_L} \frac{\partial E}{\partial V_j} &= E\langle \psi_1, \psi_1 \rangle - E\langle \psi_2, \psi_2 \rangle \\ &= \langle \psi_1, H^{(L)}\psi_1 + B^{(L)}\psi_2 \rangle - \langle B^{(L)}\psi_1 - H^{(L)}\psi_2, \psi_2 \rangle \\ &= \langle \psi_1, H^{(L)}\psi_1 \rangle + \langle \psi_2, H^{(L)}\psi_2 \rangle \\ &\geqslant \inf \operatorname{spec} \big(H^{(L)} \big). \end{aligned} \tag{5.14}$$

□

The next lemma deals with the problem of the non-monotonous dependence of the cumulative eigenvalue counting function on the random potential. It is here where we have to assume a suitable regularity of the distribution.

Lemma 5.4 *Let $\phi : \mathbb{R} \to \mathbb{C}$ be piecewise continuous, of bounded variation and have compact support. Let $F \in C^1(\mathbb{R})$ and assume the existence of a constant $a > 0$ such that $|F(x) - F(y)| \leqslant a$ for all $x, y \in \mathbb{R}$. Then we have*

$$\left| \int_{\mathbb{R}} F'(x)\, \phi(x)\, \mathrm{d}x \right| \leqslant a \, \|\phi\|_{\mathrm{BV}}. \tag{5.15}$$

Proof Step 1: We prove the claim for step functions

$$\phi_N = \sum_{\nu=1}^{N} \xi_\nu \chi_{]x_{\nu-1}, x_\nu]} = \sum_{\nu=1}^{N} (\xi_\nu - \xi_{\nu-1}) \chi_{]x_{\nu-1}, x_N]} \tag{5.16}$$

where $N \in \mathbb{N}$, $\xi_\nu \in \mathbb{C}$ for all $\nu \in \{1, \ldots, N\}$, $\xi_0 := 0$ and $-\infty < x_0 < x_1 < \cdots < x_N < \infty$. For such ϕ_N we get

$$\left| \int_{\mathbb{R}} F'(x) \phi_N(x) \, dx \right| = \left| \sum_{\nu=1}^{N} (\xi_\nu - \xi_{\nu-1}) [F(x_N) - F(x_{\nu-1})] \right|$$

$$\leqslant a \sum_{\nu=1}^{N} |\xi_\nu - \xi_{\nu-1}| \leqslant a \, \|\phi_N\|_{\mathrm{BV}}. \tag{5.17}$$

Step 2: For general ϕ as in the lemma, the claim (5.15) follows from a uniform approximation of ϕ by step functions of the form $\phi_N := \sum_{\nu=1}^{N} \phi(x_\nu) \chi_{]x_{\nu-1}, x_\nu]}$. Indeed, for every given $\varepsilon > 0$ we can choose discretisation points x_ν, $\nu = 1, \ldots, N$ and $N \in \mathbb{N}$ such that $\|\phi - \phi_N\|_\infty \leqslant \varepsilon$, because ϕ is piecewise uniformly continuous. Therefore $\int_{\mathbb{R}} F'(x) \phi(x) \, dx = \lim_{N \to \infty} \int_{\mathbb{R}} F'(x) \phi_N(x) \, dx$ for a suitable sequence of step functions ϕ_N, because F' is bounded on the support of ϕ. This, the bound from Step 1 and $\|\phi_N\|_{\mathrm{BV}} \leqslant \|\phi\|_{\mathrm{BV}}$ finish the proof. □

6 Random Diagonal *B*: Lifshits Tails

It is a striking fact that the integrated density of states of random Schrödinger operators grows only exponentially slowly in the vicinity of fluctuation band edges. This behaviour is called *Lifshits tail* [15].

In Theorem 6.1 we provide a result for random block operators $\mathbb{H}$ of the form (4.8) which limits the growth of the integrated density of states $\mathbb{N}$ at energy λ, provided $H \geqslant \lambda > 0$ exhibits a Lifshits tail at energy λ. We emphasise that Theorem 6.1 is only interesting in the case where $\pm \lambda$ coincide with the endpoints of the spectral gap of $\mathbb{H}$ around zero. According to Corollary 4.5 this always happens if $0 \in \mathrm{supp}(\mu_b)$.

Theorem 6.1 *Consider the random block operator*

$$\mathbb{H} : \omega \mapsto \mathbb{H}_\omega = \begin{pmatrix} H_\omega & b_\omega \\ b_\omega & -H_\omega \end{pmatrix} \tag{6.1}$$

where H and b are given as in (4.4) *and* (4.5) *and let* $\mathbb{N}$ *be its integrated density of states. Suppose in addition that* $\lambda := \inf \mathrm{spec}(H) > 0$ *and that there exists constants* $\alpha, \gamma > 0$ *such that for all sufficiently small* $\varepsilon > 0$

$$\mathbb{P}\left[\inf \mathrm{spec}(H_N^{(L_\varepsilon)}) \leqslant \lambda + \varepsilon \right] \leqslant \mathrm{e}^{-\gamma \varepsilon^{-\alpha}} \tag{6.2}$$

where L_ε *is a sequence of diverging lengths as* $\varepsilon \downarrow 0$ *with* $\lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha/d} L_\varepsilon$ *exists and lies in* $]0, \infty[$. *Then the estimate*

$$\limsup_{\varepsilon \downarrow 0} \frac{\ln \left| \ln[\mathbb{N}(\lambda + \varepsilon) - \mathbb{N}(\lambda)] \right|}{\ln \varepsilon} \leqslant -\alpha \tag{6.3}$$

holds.

Remarks 6.2

(i) Assumption (6.2) is the statement which is typically proven for a random Schrödinger operators H of the form (4.4) when establishing the upper bound

$$\limsup_{\varepsilon \downarrow 0} \frac{\ln \big| \ln[N(\lambda + \varepsilon)] \big|}{\ln \varepsilon} \leqslant -\alpha \tag{6.4}$$

for a Lifshits tail with Lifshits exponent $\alpha = d/2$ of the integrated density of states N at the lower spectral edge λ. In other words, if $0 \in \mathrm{supp}(\mu_b)$, then Theorem 6.1 says that the growth of $\mathbb{N}$ near the lower edge of the positive a.s. spectrum of $\mathbb{H}$ is no faster than the growth of N near the bottom of the a.s. spectrum of H. Note that the bound on $\mathbb{N}$ holds independently of b.

(ii) An analogous statement to Theorem 6.1 holds at the upper edge of the negative spectrum of $\mathbb{H}$.

Proof of Theorem 6.1 We claim that for all energies $E \geqslant 0$ and all finite cubes Λ_L the estimate

$$\mathbb{N}(E) - \mathbb{N}(0) \leqslant \mathbb{E}\big[\mathbb{N}_-^{(L)}(E) - \mathbb{N}_-^{(L)}(0)\big] \tag{6.5}$$

holds. This follows from Dirichlet-Neumann bracketing (4.29) and

$$\mathbb{N}(0) = \frac{1}{2} = \mathbb{E}\big[\mathbb{N}_N^{(L)}(0)\big] = \mathbb{E}\big[\mathbb{N}_-^{(L)}(0)\big]. \tag{6.6}$$

The first two equalities in (6.6) are based on Lemma 2.3 and that zero is not in the spectrum. In order to see the last equality in (6.6) we view $\mathbb{H}_-^{(L)} = \mathbb{H}^{(L)}(1)$ as an analytic perturbation of $\mathbb{H}_N^{(L)} = \mathbb{H}^{(L)}(0)$, where

$$\mathbb{H}^{(L)}(a) := \mathbb{H}_N^{(L)} - 2a \begin{pmatrix} 0 & 0 \\ 0 & \Gamma^{(L)} \end{pmatrix} \tag{6.7}$$

for $a \in \mathbb{R}$ and $\Gamma^{(L)}$ was introduced in (4.22). Analytic perturbation theory tells us that the eigenvalues of $\mathbb{H}^{(L)}(a)$ depend continuously on the parameter a. On the other hand, we infer from Lemma 2.12 that zero lies in an open spectral gap of $\mathbb{H}^{(L)}(a)$ of size at least 2λ for every $a \in [0, 1]$. Therefore $\mathbb{H}^{(L)}(0)$ must have exactly as many positive (negative) eigenvalues as $\mathbb{H}^{(L)}(1)$, and the third equality in (6.6) holds.

Now let $\varepsilon > 0$ and observe

$$\mathbb{E}\big[\mathbb{N}_-^{(L)}(\lambda + \varepsilon) - \mathbb{N}_-^{(L)}(0)\big] \leqslant \mathbb{P}\big[\mathrm{spec}(\mathbb{H}_-^{(L)}) \cap \,]0, \lambda + \varepsilon] \neq \varnothing\big]. \tag{6.8}$$

By contradiction we conclude from Lemma 2.10(ii) that the event in the probability on the r.h.s. of (6.8) implies the event $\inf \mathrm{spec}(H_N^{(L)}) \leqslant \lambda + \varepsilon$. Hence, we get

$$\mathbb{N}(\lambda + \varepsilon) - \mathbb{N}(0) \leqslant \mathbb{P}\big[\inf \mathrm{spec}(H_N^{(L)}) \leqslant \lambda + \varepsilon\big], \tag{6.9}$$

and the claim follows from (6.2). $\qquad \square$

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