Università degli Studi di Trento
Department of Mathematics

Master Degree in

Mathematics

Random evolution on combinatorial and metric graphs

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October 23, 2019

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Introduction

This work concerns the analysis of diffusion equations on graphs, where the leading operator randomly varies during the time evolution. A graph $G = (V, E)$ is a couple defined by a set V of vertices, which can be connected by some edges in E. Graphs can have complicated structure, but we only deal with those whose number of vertices and edges is finite. In particular, it is possible to distinguish two classes of graphs on which we are going to work. If every edge of G only represents the connection between its endpoints, without any physical meaning, then G is called *combinatorial graph*. On the other hand, we speak of *metric graphs* when all the edges are identified with one-dimensional intervals and then, as the name suggests, we can equip G with the structure of metric space.

These two types of graphs open the way for the research in several areas of mathematics. There is a wide literature where combinatorial graphs are studied with an approach closer to branches like algebra, topology, discrete mathematics. During the last century many geometric properties of graphs were analyzed, in order to find some invariants and to develop a rich and elegant theory of graphs. Several text books can be mentioned, for instance [**11**, **25**].

Additionally, graphs represent an interesting and challenging subject in the field of analysis, to which the present work belongs. In fact, graphs provide an efficient substitute for the classical Euclidean domains on which we define specific matrices or differential operators and then study difference or partial differential equations. In particular, among the various evolution problems on graphs, we have focused on diffusion systems, both in the combinatorial and metric settings.

Concerning the discrete framework, one can define a function on a combinatorial graph G as a map assigning a complex value to each vertex. Hence, in the first chapter of this thesis we are going to present a well-known model which describes the heat diffusion on G. Suppose that the heat is concentrated only at each vertex and possible exchanges occur only between vertices which are connected by some edge. The model of this physical phenomenon, known in literature as *discrete heat equation* on G, is given by an autonomous and deterministic linear system, where the leading operator A is a

$|V| \times |V|$ matrix. More precisely, A is the minus Laplacian $-\mathcal{L}$ associated with G, one of the fundamental matrices defined on a graph. The discrete Laplacian has many nice properties and its spectrum was deeply studied in several works (see, for instance [**21**]).

We review well-known results about the well-posedness and the asymptotic behaviour of the solution of the heat discrete equation. In fact, we can prove that the semigroup generated by the minus Laplacian converges uniformly towards the orthogonal projection P onto the kernel of $\mathcal{L}$. Hence, when the graph environment is connected, namely every two vertices are linked by some path of edges, then P exactly coincides with the orthogonal projection P_0 onto the linear subspace of constant functions on V. Then, given an initial distribution f, the limit heat distribution will be equal to the average of f computed on all the graph: the connectedness of G let the heat reach every vertex. In addition, the structure of G also influences the rate of convergence to the equilibrium, which is equal to the lowest nonzero eigenvalue of $\mathcal{L}$.

In Chapter 2, we investigate whether there are analogous results when the diffusion system is no longer autonomous and deterministic. More precisely, suppose we want to analyze the heat equation on a graph whose set of edges evolves in a random way: obviously this means that the connectivity of the domain varies and so does the possibility of heat exchange between the vertices. In order to describe this situation, in the second chapter we introduce a new dynamical system and we study its random evolution. The notion of *random evolution* was introduced by Griego and Hersh in [**14**, **15**], where the authors considered the selection from a finite number of strongly continuous semigroups by means of a finite-state Markov chain.

In this context, in order to simulate the time evolution of the graph environment on which we study the diffusion, we consider a finite family $\mathcal{C} = \{G_1, \dots, G_N\}$ of graphs sharing the same set of vertices, but with different edges. Additionally, we introduce a Markov chain which takes values exactly in $\mathcal{C}$. Roughly speaking, during every unit time interval we are going to study the heat diffusion on a different graph in $\mathcal{C}$, determined by the jumps of the Markov chain. As we will better understand, this implies that the evolution operator of the model is expressed in terms of a random product of matrices and studying the limit of $S(t)$ for $t \to +\infty$ means dealing with infinite products of them. Regarding this, there are several articles (like [**1**, **9**, **10**, **12**, **16**, **17**]) about the convergence of (random) products of matrices and contractive operators on infinite-dimensional Hilbert spaces.

In the end, we are going to see that it is possible to obtain the same long-time behaviour as in the case of a fixed connected graph under particular conditions involving the stochasticity of the model and the connectivity of graphs in $\mathcal{C}$. In fact, the main result proved is that if the Markov chain is irreducible, then the evolution operator $S(t)$ converges uniformly and – since it is also a stochastic process – almost surely towards the orthogonal projection P_0 onto the constant functions on V. Therefore, even when the graph environment is not constant in time, in this case the limit distribution still converges to the average of the initial datum on all the vertices in V.

These diffusion models can be generalized in the continuous setting of metric graphs. In this case, we can consider G as a family of intervals joined by the vertices in a suitable way. Hence, we can naturally define functions edgewise and in the same way it is possible to introduce differential operators, equipped with some vertex conditions. All the classical results on PDEs and on evolution semigroups are therefore brought in this framework, where an interesting theory about the so-called *quantum graphs* (metric graphs on which differential operators and vertex conditions are defined) is developing. We refer to [**5**, **22**] for a deep presentation on this branch of graph theory.

Therefore, the aim of Chapter 3 is to present all the known classical results about the *continuous heat diffusion equation* on one fixed metric graph: now the heat distribution is to be meant along every edge of the domain. This means introducing and analyzing in detail the Laplacian operator associated with a metric graph – which acts edgewise as a second derivative – and studying a partial differential equation on the network domain equipped with Kirchhoff and continuity's vertex conditions. Also in the infinite-dimensional setting, it turns out that the model is still well-posed. Additionally, we will see that its long-time behaviour is similar and parallel to the one in the discrete heat equation. In fact, we still obtain the uniform covnergence of the evolution semigroup towards P_0, provided that G is connected. In this framework, P_0 is still the orthogonal projection onto the subspace of functions which are now constant on every edge of the metric graph.

Since the asymptotic behaviour of the diffusion problem on a metric graph is quite similar and analogous to the one in the discrete setting, the challenge in Chapter 4 is to find a way of generalizing the random evolution model presented in Chapter 2.

We wondered how the evolution dynamics of the metric graph environment could be meant in this framework. Again, we have built a non autonomous stochastic model, whose architecture does not present big differences from the discrete one.

In fact, we can start again from a finite family $\mathcal{C}$ of N metric graphs: now each one shares the same set of edges, while the configuration of vertices and the way that these are joined is different graph by graph. Also the stochasticity of the model is the same as before: a Markov chain jumps from one to another metric graph at each integer time. In such a way, we still study the heat diffusion on a non autonomous environment and we are still interested in analyzing the convergence of the evolution operator for long times.

The final result we will obtain in Chapter 4 is exactly the metric version of the theorem proved before: as long as we require the irreducibility of the Markov chain, the evolution operator converges uniformly and almost surely to P_0 if and only if the union of the initial graphs in $\mathcal{C}$ is connected.

In fact, we will see that all the tools and theoretical results used in the combinatorial framework still hold: it is only necessary to extend some technical computation to the infinite-dimensional case and everything works well.

On the other hand, one aspect in the discrete setting, the least expected one, will not be obvious to be generalized: the notion of union of graphs. In fact, in the random evolution model in Chapter 2 we are able to deal with this concept because there is a rigorous and well-posed definition of union of graphs, present in all the classical text books. However, the same no longer holds when we leave the combinatorial framework to study metric graphs. In this case, in fact, there is just an intuitive and heuristic idea, but a unique and formal notion of union of metric graphs is missing in the literature.

For this reason, in order to parallelize the results obtained in Chapter 2, a great effort is placed into formalizing the union of metric graphs and to obtain a reasonable definition.

Using this definition, at the end of Chapter 4 we will obtain a main general result about the long-time behaviour of two random evolution models, one on combinatorial and one on metric evolving graphs. As the connectedness of the network plays a funtamental role in the corresponding diffusion problems on a fixed combinatorial or metric graph, here we find a similar situation. In fact, in both settings, the connectedness of the union of graphs in $\mathcal{C}$ provides a necessary and sufficient condition in order that the heat spreads until getting homogenized for long times.

Chapter 1

The heat equation on a combinatorial graph

The aim of this chapter is to introduce the reader to the main notions of Graph Theory, emphasizing those properties we are going to use throughout the thesis.

First of all, we are going to give some definitions about combinatorial graphs, a particular type of graphs which will be often used. Given a combinatorial graph, one can associate to it some matrices. We are going to focus on one of these, the so-called discrete (or combinatorial) Laplace operator.

Furthermore, we are going to analyze the discrete heat equation on a combinatorial graph, which is the starting point of this research. In particular, we are going to present the case when the fixed graph is finite, i.e. it has a finite number of vertices and edges. The well-posedness and the long-time behaviour of the dynamical system associated with the heat equation are well-known (e.g. see [**22**]). We will see that, despite the system is always exponentially stable, the heat limit distribution also depends on the particular structure of the graph.

1.1 Introduction to combinatorial graphs and to the discrete Laplace operator

In this section we give an overview of the theory of graphs, in order to introduce the terminology used in the sequel. There is a huge bibliography on the topics we are going to talk about: we will mainly follow the guidelines of [**5**, **7**, **11**, **21**, **22**, **25**].

A *graph*, denoted by $G = (V, E)$, is a structure defined by a set V of *vertices* (or *nodes*) and a set $E \subset V \times V$ of *edges*. The latters are given by ordered couples of vertices and, in the case of *combinatorial* graphs, they do not necessarily have a physical or geometric meaning. Therefore, it is irrelevant how to draw the edges in this type of graphs: they only represent the possible communication between to vertices.

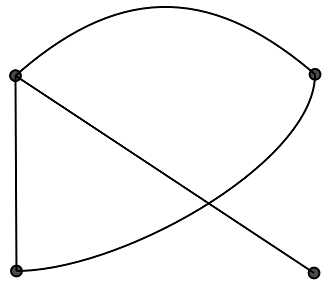


Figure 1.1: An example of graph.

Two vertices $u, v \in V$ are called *adjacent* (denoted by $u \sim v$) if there exists an edge $e \in E$ joining them, namely $e = (u, v)$ or $e = (v, u)$. A vertex and an edge e are said to be *incident* if v corresponds to one of the endpoints of e. Two edges are incident if there exists a vertex which is incident with both of them. Additionally, we denote the set of all the edges with endpoint $v \in V$ by

$$E_v = \{e \in E : e \text{ is incident to } v\}$$

and the cardinality d_v of this set is called *degree* of the vertex v.

We deal with particular classes of combinatorial graphs: more precisely, by "graph" we will always mean a *finite* graph, which has a finite number of edges and vertices. In addition, throughout this chapter and the next one we will focus on *simple* graphs, namely we exclude the presence of loops and/or multiple edges (see Fig. 1.2).

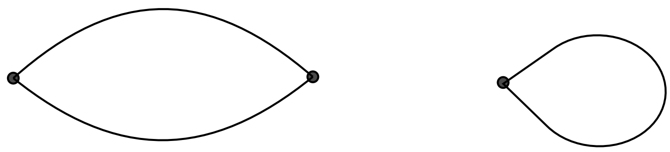


Figure 1.2: On the left, we have an example of graph with two multiple edges which connect the same couple of vertices. On the right, we have a loop, namely an edge where both endpoints coincide.

If we consider two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, whose vertices are enumerated, we shall define the notion of *union* of G_1 and G_2 as the graph given by

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2).$$

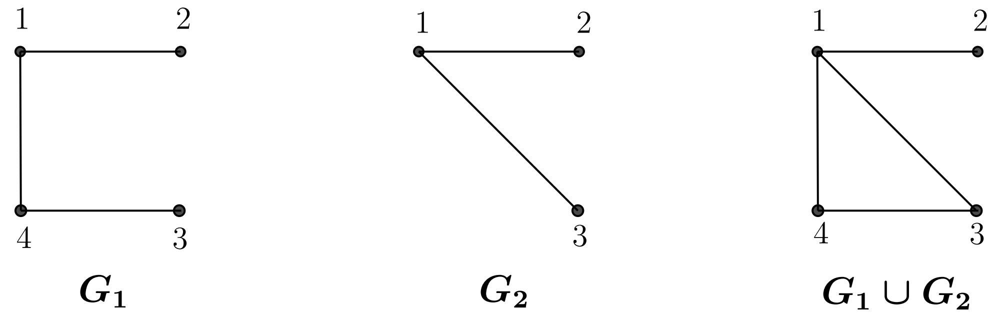


Figure 1.3: Example of union of two graphs.

Given a graph $G = (V, E)$, we can introduce the notion of *subgraph* as a graph $G' = (V', E')$ where both V' and E' are respectively subsets of V and E. Fixed two vertices $v_1, v_N \in V$, a *path* which links v_1 and v_N is a subgraph $\Gamma = (V', E')$, where

$$V' = \{v_1, v_2 \ldots, v_{N-1}, v_N\} \subset V \quad \text{and} \quad E' = \{(v_1, v_2), \ldots, (v_{N-1}, v_N)\} \subset E.$$

A graph G is said to be *connected* if for every couple of vertices u, v there exists a path Γ linking them. Otherwise, a maximal connected subgraph of G is called *component* and it turns out that G can be expressed as disjoint union of their connected components. In the following, we will mention the *connectedness* of a graph when we will mean the property of a graph to be connected.

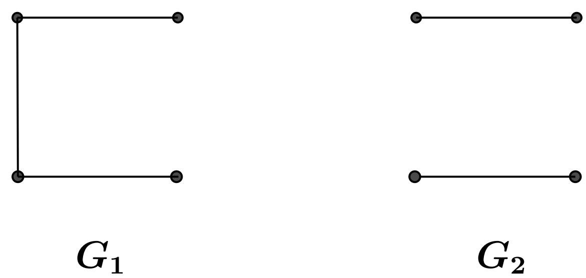


Figure 1.4: G_1 is a connected graph, whereas G_2 is disconnected and given by the disjoint union of two connected components.

On the other hand, when we talk about the connectivity of a graph, we will just refer to how much the vertices are linked to each others by edges.

Given a simple and finite combinatorial graph $G = (V, E)$, we introduce a $|V| \times |V|$

matrix A called *adjacency matrix* and defined as follows:

$$A = (A_{uv}), \qquad A_{uv} := \begin{cases} 1 & \text{if } u \sim v, \\ 0 & \text{if } u \nsim v. \end{cases}$$

The matrix A was deeply studied in the past (see [**8**]): we notice that it is symmetric and, since there are no loops, it has null trace. However, this matrix does not have so many good properties: for instance, one can easily show that it generates a semigroup $\{e^{tA}\}_{t\geq 0}$ which is not contractive and that

$$\lim_{t\longrightarrow+\infty} \|e^{\pm tA}\| = +\infty.$$

For this reason, we need to find a matrix associated to G with nicer properties. We define the *degree matrix* D as the diagonal $|V| \times |V|$ matrix

$$D := \operatorname{diag}(d_v)_{v\in V}.$$

Then, we are finally able to introduce the *discrete Laplace operator* associated with G, given by the $|V| \times |V|$ matrix

$$\mathcal{L} := D - A.$$

Example 1.1. Given the graph $G = (V, E)$ shown in Fig. 1.5, we have

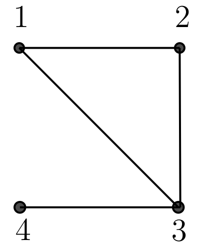


Figure 1.5

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, the associated discrete Laplacian is

$$\mathcal{L} = D - A = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

One immediately notices that the Laplacian inherits the symmetry from A and that the algebraic sum of the elements in every row is always zero.

Given a graph $G = (V, E)$, one can define functions on it just by assigning a certain value at each vertex in V. We denote by

$$C(V) := \{f : V \to \mathbb{C}\} \cong \mathbb{C}^{|V|}$$

the space of all the complex-valued functions on G. By the definition of $C(V)$, they can be meant as $|V|$-dimensional vectors, whose components represent their value at each $v \in V$:

$$\forall f \in C(V) \cong \mathbb{C}^{|V|} : \ f = \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_N) \end{pmatrix}, \qquad \text{where } |V| = N, \text{ for some } N \in \mathbb{N}.$$

We can equip $C(V)$ with the canonical scalar product

$$(f, g) = \sum_{v \in V} f(v)\overline{g(v)}, \qquad f, g \in C(V)$$

and with the induced norm

$$\|f\| = \left(\sum_{v \in V} |f(v)|^2 \right)^{\frac{1}{2}}.$$

Remark 1.2. In many text books, in order to emphasize the structure of Hilbert space or to use a notation which is suitable even if G is not finite, instead of $C(V)$ they refer to the space

$$\ell^2(V) = \{f : V \to \mathbb{C} : \sum_{v \in V} |f(v)|^2 < +\infty\}.$$

Clearly, the above definition agrees with $\mathbb{C}^{|V|}$ when $|V| < \infty$. Therefore, for simplicity we will always use the notation $\mathbb{C}^{|V|}$.

At this point, given a graph $G = (V, E)$ and a function $f \in \mathbb{C}^{|V|}$ defined on it, one can apply the Laplacian $\mathcal{L}$ to f and obtain

$$\begin{aligned}(\mathcal{L}f)(u) &= \sum_{v\in V} \mathcal{L}_{uv} f(v) \\ &= \sum_{v\in V} (D_{uv} - A_{uv}) f(v) \\ &= d_u f(u) - \sum_{v\sim u} f(v) \\ &= \sum_{v\sim u} \left(f(u) - f(v)\right), \qquad u \in V.\end{aligned}$$

We can consider the symmetric sesquilinear form associated with $\mathcal{L}$

$$q(f,g) = \sum_{v\sim u} \left(f(u) - f(v)\right) \overline{\left(g(u) - g(v)\right)}, \qquad f, g \in \mathbb{C}^{|V|} \tag{1.1}$$

and the corresponding quadratic form

$$q(f) = (\mathcal{L}f, f) = \sum_{v\sim u} |f(u) - f(v)|^2 \geq 0, \qquad f \in \mathbb{C}^{|V|}. \tag{1.2}$$

From (1.1) and (1.2) it follows that $\mathcal{L}$ is positive semidefinite and self-adjoint. Therefore, its eigenvalues are real and nonnegative and, in particular, it turns out that 0 is always an eigenvalue. Looking for the eigenfunctions associated with 0, one gets

$$(\mathcal{L}f)(u) = \sum_{v\sim u} \left(f(u) - f(v)\right) = 0,\ u \in V \iff f(u) = f(v),\ \forall v \sim u,\ u \in V. \tag{1.3}$$

Thus, if G is connected, then the only choice is that f is constant on V, 0 is a simple eigenvalue and a possible one-dimensional basis for $\ker \mathcal{L}$ is given by $\mathbb{1} = (1, \ldots, 1)^\top \in \mathbb{C}^{|V|}$. Otherwise, if G is given by the disjoint union of $G^{(1)}, \ldots, G^{(L)}$ connected components, then from (1.3) we deduce that a possible eigenfunction f associated with 0 has to be constant on each $G^{(h)}$, for all $h = 1, \ldots, L$. Then, 0 has multiplicity equal to L (i.e. the number of the connected components) and a possible basis for $\ker \mathcal{L}$ is given by

$$\left\{\mathbb{1}_1, \ldots, \mathbb{1}_L\right\},$$

where for every $h = 1, \ldots, L$

$$\mathbb{1}_h(u) = \begin{cases} 1 & \text{if } u \in V(G^{(h)}), \\ 0 & \text{otherwise,} \end{cases} \qquad u \in V.$$

Until now, we have only considered undirected graphs, but we can assign an orientation on $G = (V, E)$, just by assuming that every edge has an initial and a terminal vertex. Given an oriented graph, we can define the *incidence matrix* ϕ with dimension $|V| \times |E|$ as follows:

$$\phi_{ue} = \begin{cases} 1 & \text{if } v \text{ is a terminal edge for } e, \\ -1 & \text{if } v \text{ is a initial edge for } e, \\ 0 & \text{otherwise}, \end{cases} \qquad u \in V,\ e \in E.$$

Obviously, this matrix depends on the particular orientation imposed on a graph, however one can easily prove (see [**22**], Chapter 2) that the product between ϕ and its transpose matrix is invariant under orientation. In particular, it turns out that the discrete Laplacian is exactly given by

$$\mathcal{L} = \phi\, \phi^{\top}.$$

1.2 The discrete heat equation on a combinatorial graph

Once introduced the basic notions about combinatorial graphs, we shall present one first example of difference equation computed on a discrete network, namely the *discrete heat equation.*

Fixed a graph $G = (V, E)$, assume that the heat is concentrated only at each vertex and not along the edges, which in a combinatorial graph do not have a physical meaning. We denote by

$$u(t, v) \in \mathbb{C}, \qquad t \geq 0,\ v \in V$$

the heat concentration at the time t in the vertex v.

We are going to analyze how the heat distribution on each vertex varies during time, assuming that heat exchanges are possible only between two nodes linked by some edge. Thus, the *discrete heat equation* on G is given by definition as

$$\frac{du}{dt}(t, v) = -\mathcal{L}u(t, v), \qquad t \geq 0,\ v \in V \tag{1.4}$$

and the leading operator driving this evolution problem is the minus Laplace matrix $-\mathcal{L}$ associated with G.

In order to rewrite (1.4) in a more compact form, let us introduce the heat distribution

$u(t)$ at the time t on G as

$$u(t) = \begin{pmatrix} u(t, v_1) \\ \vdots \\ u(t, v_d) \end{pmatrix} \in \mathbb{C}^{\mathrm{d}},$$

supposing that from now on $|V| = d$. Therefore, problem (1.4) can be reformulated as

$$\frac{du}{dt}(t) = -\mathcal{L}u(t), \qquad t \geq 0$$

and, considering the vector

$$u_0 = \begin{pmatrix} u_0(v_1) \\ \vdots \\ u_0(v_d) \end{pmatrix} \in \mathbb{C}^{\mathrm{d}},$$

one obtains the Cauchy problem

$$\begin{cases} \dfrac{du}{dt}(t) = -\mathcal{L}u(t), & t \geq 0, \\ u(0) = u_0. \end{cases} \tag{DHE}$$

When G is a finite graph, (DHE) corresponds to a linear system of d equations. Since $-\mathcal{L}$ is dissipative, namely

$$\mathrm{Re}(-\mathcal{L}f, f) \leq 0, \qquad f \in \mathbb{C}^{\mathrm{d}},$$

then by the Lumer-Phillips Theorem (e.g. Theorem 3.15 of [**13**]) it follows that $-\mathcal{L}$ generates a contractive semigroup given by the family of exponential matrices

$$\left\{e^{-t\mathcal{L}}\right\}_{t\geq 0}.$$

Furthermore, from the theory of dynamical systems, it is well-known that (DHE) is well-posed and that its solution is

$$u(t) = e^{-t\mathcal{L}}u_0.$$

At this point, one can apply the Spectral Theorem and then represent the solution of (DHE) in terms of

$$u(t) = e^{-t\mathcal{L}}u_0 = \sum_{k=1}^{d} e^{-t\lambda_k}(f, e_k)e_k,$$

where $\{e_k\}_{k=1}^d$ is an orthonormal basis of eigenfunctions of the (minus) Laplacian. In the above expression λ_ks are the eigenvalues of $\mathcal{L}$ indexed in increasing order counting the multiplicity. We remark that λ_1 is always 0, whereas $\lambda_2 > 0$ if and only if G is connected.

Let us now recall a well-known result, which we are going to apply several times in the next chapters.

Lemma 1.3. *If $\mathcal{L}$ denotes the discrete Laplacian, then*

$$\ker \mathcal{L} = \operatorname{Fix} e^{-t\mathcal{L}}, \qquad t > 0,$$

where

$$\operatorname{Fix} e^{-t\mathcal{L}} := \{f \in \mathbb{C}^d : \; e^{-t\mathcal{L}} f = f\}$$

is the fixed points set.

Proof. We only show the case when G is connected, so that the kernel of the Laplacian is one-dimensional and the notation is simpler. From this proof, one can easily extend the thesis to the general case.

Being G connected, we have seen that $f \in \ker \mathcal{L}$ if and only if f is constant on V. Let us denote it by

$$f = \alpha e_1, \qquad \alpha \in \mathbb{C},$$

where $e_1 = \frac{1}{|V|}\mathbb{1}$. Thus, by orthogonality, for all $t > 0$ one has

$$e^{-t\mathcal{L}} f = (f, e_1)e_1 + \sum_{k=2}^{d} e^{-t\lambda_k} \underbrace{(f, e_k)e_k}_{=0} = (\alpha e_1, e_1)e_1 = \alpha e_1 = f,$$

then $f \in \operatorname{Fix} e^{-t\mathcal{L}}$.

On the other hand, let $f \in \mathbb{C}$ represented as

$$f = \sum_{k=1}^{d} (f, e_k)e_k.$$

We have that $f \in \operatorname{Fix} e^{-t\mathcal{L}}$ if and only if

$$\sum_{k=1}^{d} e^{-t\lambda_k}(f, e_k)e_k = \sum_{k=1}^{d} e^{-\lambda_k}(f, e_k)e_k,$$

then

$$e^{-t\lambda_k}(f, e_k) = (f, e_k), \qquad \forall k = 1, \dots, d. \tag{1.5}$$

Equality (1.5) holds if and only if $f = \alpha e_1 \in \ker \mathcal{L}$ for some $\alpha \in \mathbb{C}$, otherwise we would get a contradiction. Thus, the proof is complete. $\square$

We are now able to study the long-time behaviour of (DHE) in the following

Theorem 1.4 (**Asymptotic behaviour of (DHE)**). *Let P be the orthogonal projection onto the eigenspace of $\mathcal{L}$ associated with $\lambda = 0$, namely the orthogonal projection onto the kernel of the discrete Laplacian. Then the semigroup generated by $-\mathcal{L}$ uniformly converges towards P, namely*

$$\lim_{t \longrightarrow +\infty} \|e^{-t\mathcal{L}} - P\| = 0,$$

and the convergence rate is given by $\overline{\lambda}$, the lowest nonzero eigenvalue of $\mathcal{L}$.

In particular, it follows that for any initial datum $u_0 \in \mathbb{C}^d$

$$\lim_{t \longrightarrow +\infty} u(t) = \lim_{t \longrightarrow +\infty} e^{-t\mathcal{L}} u_0 = Pu_0.$$

Proof. Let P_k be the orthogonal projection such that

$$P_k f = (f, e_k) e_k, \qquad f \in \mathbb{C}^{\mathrm{d}},\ \forall k :\ \lambda_k \neq 0.$$

Now the exponential matrix of $-\mathcal{L}$ may be written as

$$e^{-t\mathcal{L}} = P + \sum_{k:\lambda_k \neq 0} e^{-t\lambda_k} P_k.$$

Thus, recalling that $\overline{\lambda} > 0$, we have

$$\begin{aligned} \|e^{-t\mathcal{L}} - P\| &= \left| \sum_{k:\lambda_k \neq 0} e^{-t\lambda_k} P_k \right| \\ &\leq \sum_{k:\lambda_k \neq 0} e^{-t\lambda_k} \|P_k\| \\ &\leq \sum_{k:\lambda_k \neq 0} e^{-t\lambda_k} \\ &\leq M e^{-t\overline{\lambda}} \xrightarrow{t \to +\infty} 0, \end{aligned}$$

where $M = \sum_{k:\lambda_k \neq 0} 1 < \infty$. $\square$

Notice that, despite the convergence to equilibrium always holds, the graph environment on which we are studying the heat diffusion has an influence on the behaviour of the solution for long times. First of all, the convergence rate directly depends on G, since we have shown that it is given by the lowest nonzero eigenvalue of the discrete Laplace matrix of G. In turn, this eigenvalue can be estimated by some graph invariants[1] (see,

[1] Following the definition given by [**11**] in Chapter 1, two graphs $G = (V, E)$ and $G' = (V', E')$ are

for instance, [**21**]).

In conclusion, we want to investigate how the limit distribution varies depending on the structure of the combinatorial graph. The previous theorem states that the solution of the discrete heat equation tends to a vector in $\mathbb{C}^{\mathrm{d}}$, given by

$$\lim_{t \longrightarrow +\infty} u(t) = Pu_0,$$

where P is the orthogonal projection onto the kernel of the Laplacian and $u_0 \in \mathbb{C}^{\mathrm{d}}$ is the initial datum. Therefore

1. if G is a connected graph, then 0 is a simple eigenvalue and the one-dimensional $\ker \mathcal{L}$ is spanned by $\mathbb{1}$. Thus, P agrees with the orthogonal projection P_0 onto the subspace of constant functions on the graph:

$$P_0 u_0 := Pu_0 = (u_0, e_1)e_1 = \frac{1}{d}(u_0, \mathbb{1})\ \mathbb{1} = \frac{1}{d}\sum_{v \in V} u_0(v)\ \mathbb{1} \in \mathbb{C}^{\mathrm{d}},$$

   which is indeed a constant function on G. In particular, we notice that the constant value assigned to each vertex

$$\sum_{v \in V} u_0(v)\ \mathbb{1} =: \fint_G u_0$$

   is exactly the average of the initial datum u_0 computed on V: the connectedness of G let the heat spread and reach every vertex;

2. otherwise, when G is not connected and then it is given by the disjoint union of connected components $G^{(1)}, \dots, G^{(L)}$, we have seen that 0 has multiplicity equal to L as an eigenvalue of the Laplacian. As a consequence, $\ker \mathcal{L}$ is spanned by the characteristic function on each connected component

$$\{\mathbb{1}_1, \dots \mathbb{1}_L\}.$$

   Therefore, we get

$$Pu_0 = \sum_{h=1}^{L} \frac{1}{|V(G^{(h)})|}(u_0, \mathbb{1}_h)\ \mathbb{1}_h \in \mathbb{C}^{\mathrm{d}}$$

isomorphic if there exists a bijection $\varphi\colon V \to V'$ such that $e = (u, v) \in E$ then $(\varphi(u), \varphi(v)) \in E'$ for all $u, v \in V$. Therefore, a graph invariant is a graph property which is invariant under isomorphisms.

and

$$Pu_0(v) = \frac{1}{|V(G^{(h)})|} \sum_{w \in V} u_0(w) = \frac{1}{|V(G^{(h)})|} \sum_{w \in V(G^{(h)})} u_0(w) =: \fint_{G^{(h)}} u_0, \qquad v \in V(G^{(h)}).$$

From the above expressions, one deduce that in this situation for long times the heat tends to homogenize on the graph G in a different way as before: it spreads component by component until getting constant and equal to the average of u_0 on each $G^{(h)}$, $h = 1, \dots, L$.

This is the long-time behaviour of this diffusion model, studied on a fixed combinatorial graph. Starting from (DHE), in the next chapter we will modify this framework in order to obtain variations of the system just described comparing it with each others.

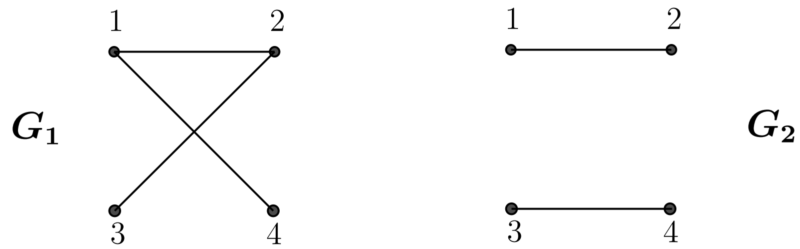


Figure 1.6: Suppose to study two heat diffusion problems on G_1 and on G_2. Given the initial datum $u_0 = (a, b, c, d)^\top \in \mathbb{C}^4$, then the limit heat distribution Pu_0 is different depending on the graph. Since G_1 is connected, then $Pu_0 = P_0 u_0 = (\alpha, \alpha, \alpha, \alpha)$, where $\alpha = (a + b + c + d)/4$. On the other hand, being G_2 disconnected, then we have that $Pf = (\alpha, \alpha, \beta, \beta)$, where $\alpha = (a + b)/2$ and $\beta = (c + d)/2$.

Chapter 2

Random evolution on combinatorial graphs

In this chapter we are going to introduce one of the original problems proposed in this thesis: the heat diffusion on combinatorial graphs. In Chapter 1, we have already shown some results about the discrete heat diffusion equation. Now, we no longer focus on one fixed graph, constant in time, but we are interested in studying what happens when the graph randomly changes during the evolution of the system.

First of all, when we talk about "evolution of graph" we actually mean that the set of its edges varies, while the vertices always remain fixed. Suppose that we want to study the heat diffusion starting from an initial combinatorial graph G. We recall that the set of its edges $E(G)$ plays a fundamental role in this context, since the heat exchange between two vertices is possible only when there exists an edge connecting them. Then, if we decide to "turn on" some edges and to "turn off" some others according to a suitable random law, as a consequence the dynamics of the heat diffusion will change as time goes on.

Therefore, we first intend to find a way of suitably modelling the proposed situation. Additionally, once we will have got a well-posed dynamical system, the following step is to analyze its long-time behaviour, comparing it to the case in the previous chapter.

2.1 Evolution problems with variable coefficients

Before talking about graphs, we first want to illustrate the general idea of the random evolution model. In fact, in this section we introduce some basic notions about evolution equations on randomly evolving structures.

Consider a set $\mathcal{K} = \{A_0, \ldots, A_{N-1}\}$ of $d \times d$ matrices and we choose a sequence

$\{A_{j_k},\ k \in \mathbb{N}\}$ from it. We are going to study the associated evolution problem

$$\frac{d^+u}{dt}(t) = A_{j_k}u(t), \quad t \in [k, k+1),\ k \in \mathbb{N}, \tag{2.1}$$

where $\dfrac{d^+}{dt}$ is the right derivative.

Definition 2.1. A *weak solution* of (2.1) is a function $u \in H^1([0, +\infty); \mathbb{R}^{\mathrm{d}})$ such that (2.1) is satisfied for all $t \geq 0$.

A weak solution admits a unique value at $t = 0$: hence, we can complement (2.1) with an initial value.

Definition 2.2. If $u_0 \in \mathbb{R}^{\mathrm{d}}$, then a *weak solution* of the problem

$$\begin{cases} \dfrac{d^+u}{dt}(t) = A_{j_k}u(t), & t \in [k, k+1),\ k \in \mathbb{N}, \\ u(0) = u_0 \in \mathbb{R}^{\mathrm{d}} \end{cases} \tag{2.2}$$

is a weak solution of (2.1) which additionally satisfies the initial condition $u(0) = u_0$.

In particular, a weak solution of (2.2) satisfies the equation in its integrated form

$$u(t) = u_0 + \sum_{h=0}^{k-1} A_{j_h} \int_h^{h+1} u(s)ds + A_{j_k} \int_k^t u(s)ds, \quad t \in [k, k+1],\ k \in \mathbb{N}.$$

We are now about to find a specific solution of (2.2), in terms of an evolution operator. For all $t \geq 0$ we define recursively an operator family on $\mathbb{R}^{\mathrm{d}}$ by

$$S(t)u_0 := \begin{cases} e^{tA_{j_0}}u_0 & t \in [0, 1], \\ e^{(t-k)A_{j_k}}S(k)u_0 & t \in [k, k+1]. \end{cases} \tag{2.3}$$

First of all, we remark that despite $\{S(t)\}_{t\geq 0}$ can be written in terms of the semigroups generated by matrices in $\mathcal{K}$, it turns out that (2.3) itself does not define a semigroup. In fact, only the first condition

$$S(0) = I$$

holds, but if we take $t = 1$ and $s = \frac{1}{2}$, then

$$S(t+s) = S\left(\frac{3}{2}\right) = e^{\frac{1}{2}A_{j_1}}e^{A_{j_0}} \neq e^{A_{j_0}}e^{\frac{1}{2}A_{j_0}} = S(t)S(s).$$

We know that all semigroups $\{T_i(t)\}_{t\geq 0}$ generated by the matrices A_i are strongly and uniformly continuous, thus we find that $\{S(t)\}_{t\geq 0}$ itself is strongly and uniformly continuous, namely

$$\lim_{\epsilon\downarrow 0} ||S(t+\epsilon)u_0 - S(t)u_0|| = 0, \qquad u_0 \in \mathbb{R}^{\mathrm{d}},\ t \geq 0 \tag{S.C.}$$

and

$$\lim_{\epsilon\downarrow 0} ||S(t+\epsilon) - S(t)|| = 0, \qquad t \geq 0. \tag{U.C.}$$

Furthermore, taking the right derivatives of (2.3) yields

$$\frac{d^+S(t)}{dt}u_0 = \begin{cases} A_{j_0}S(t)u_0 & t \in [0,1) \\ A_{j_k}S(t)u_0 & t \in [k,k+1), \end{cases} \tag{2.4}$$

therefore $\{S(t)\}_{t\geq 0}$ only fails to be differentiable at the instants $t = k$. In fact

$$\begin{aligned} \lim_{t\to(k+1)^-} \frac{dS(t)}{dt} &= A_{j_k}e^{A_{j_k}}S(k), \\ \lim_{t\to(k+1)^+} \frac{dS(t)}{dt} &= A_{j_{k+1}}e^{A_{j_k}}S(k) \end{aligned} \tag{2.5}$$

which can be different, because in general $A_{j_k} \neq A_{j_{k+1}}$. Formula (2.4) shows that for all $u_0 \in \mathbb{R}^{\mathrm{d}}$ the mapping

$$\begin{aligned} u : [0,+\infty) &\longrightarrow \mathbb{R}^{\mathrm{d}} \\ t &\longmapsto S(t)u_0 \end{aligned}$$

is a weak solution of (2.1) and satisfies the initial condition

$$S(0)u_0 = u_0.$$

Now, we also prove the uniqueness of a weak solution of (2.2).

Theorem 2.3. *For every $u_0 \in \mathbb{R}^d$, the evolution equation* (2.1) *with initial data $u(0) = u_0$ has a unique weak solution.*

Proof. We adapt the proof of Proposition II.6.4 in [**13**] to the non autonomous case.

We are going to show that there exists a unique solution of (2.1) on $[0,T]$, for each $T > 0$. We start from $T = 1$ and, due to linearity, it is sufficient to prove that

$$\begin{cases} \dfrac{du}{dt}(t) = A_{j_0}u(t), & t \in [0,1], \\ u(0) = 0 \end{cases} \tag{2.6}$$

has $u(t) \equiv 0$ as unique solution. In fact, let $u : [0,1] \to \mathbb{R}^{\mathrm{d}}$ be a further solution of (2.6), which also solves the integral version of (2.6)

$$
\begin{aligned}
u(t) &= u(0) + A_{j_0} \int_0^t u(r)dr \\
&= A_{j_0} \int_0^t u(r)dr, \qquad t \in [0,1].
\end{aligned}
\tag{2.7}
$$

Take $t \in [0,1]$, then for each $s \in (0,t)$ we get

$$
\begin{aligned}
\frac{d}{ds}\left(S(t-s)\int_0^s u(r)dr\right) &= S(t-s)u(s) - A_{j_0}S(t-s)\int_0^s u(r)dr \\
&= S(t-s)u(s) - S(t-s)A_{j_0}\int_0^s u(r)dr \\
&= S(t-s)u(s) - S(t-s)u(s) = 0,
\end{aligned}
\tag{2.8}
$$

where we used the fact that A_{j_0} commutes with its exponential matrix $S(t-s) = e^{(t-s)A_{j_0}}$. Hence, integration from 0 to t of the expression (2.8) gives

$$
\int_0^t \frac{d}{ds}\left(S(t-s)\int_0^s u(r)dr\right)ds = \int_0^t u(r)dr = 0
$$

and from (2.7) we conclude that $u(t) \equiv 0$ for $t \in [0,1]$.

Now we want to show the uniqueness of (2.1) in $[0,2]$. Suppose that there exist $u : [0,2] \to \mathbb{R}^{\mathrm{d}}$ and $v : [0,2] \to \mathbb{R}^{\mathrm{d}}$ solutions of (2.1) with initial condition $u(0) = v(0) = u_0$. For the previous computations, we can deduce that u and v coincide on the time interval $[0,1]$, namely

$$
u\big|_{[0,1]} \equiv v\big|_{[0,1]}
$$

and we set $u(1) = v(1) = u_1$. Since we just require to have a weak solution, we only need to prove that

$$
u\big|_{[1,2]} \equiv v\big|_{[1,2]}
$$

to ensure the uniqueness on $[0,2]$. If we consider the autonomous problem

$$
\begin{cases}
\dfrac{du}{dt}(t) = A_{j_1}u(t), \quad t \in [1,2], \\
u(1) = u_1,
\end{cases}
\tag{2.9}
$$

we know that $u\big|_{[1,2]}$ and $v\big|_{[1,2]}$ solves (2.9) and this holds if and only if

$$
\tilde{u}(t) := u\big|_{[1,2]}(t+1), \quad \tilde{v}(t) := v\big|_{[1,2]}(t+1), \quad t \in [0,1]
$$

are solutions of

$$\begin{cases} \dfrac{d\tilde{u}}{dt}(t) = A_{j_1}\tilde{u}(t), \quad t \in [0,1], \\ \tilde{u}(0) = u_1. \end{cases} \tag{2.10}$$

At this point, proceeding as in the case $T = 1$, it is clear that (2.10) admits unique solution, thus

$$\tilde{u}(t) = \tilde{v}(t), \quad t \in [0,1]$$

and

$$u\big|_{[1,2]} \equiv v\big|_{[1,2]}.$$

This proves the uniqueness of (2.1) on $[0,2]$ and, by recursion, one can obtain the same on $[0,T]$ for every $T > 0$. Due to the arbitrariness of T, there exists one weak solution of (2.1) for all $t \in [0,+\infty)$. □

In the next section we are going to study the long-time behaviour of $\{S(t)\}_{t\geq 0}$ for the specific model on graphs. We remark that while $S(t)$ is well defined for all $t > 0$, it is a priori not clear whether its limit exists.

In general, when we talk about asymptotic behaviour of bounded operators from a normed vector space $(X, \|\cdot\|)$ to itself, we always have to specify which type of convergence we mean. In the sequel, we will consider

- the *strong convergence* of $S(t)$ towards a bounded operator $M \in \mathcal{B}(X)$, namely

$$\lim_{t\to+\infty} ||S(t)f - Mf||_X = 0, \qquad \forall f \in X;$$

- the *uniform convergence* of $S(t)$ towards a bounded operator $M \in \mathcal{B}(X)$, namely

$$\lim_{t\to+\infty} ||S(t) - M|| = 0,$$

  where $||\cdot||$ is the operator norm.

Remark 2.4. Clearly, uniform convergence always implies the strong one. However, when we work in finite dimension, also the opposite implication is true.

In fact, if $X = \mathbb{R}^{\mathrm{d}}$, let us consider a basis $\{e_k\}_{k=1}^d$. From the strong convergence, we have that for all k and for all $\epsilon > 0$, there exists t_k such that

$$|S(t)e_k - Me_k| < \epsilon \qquad \forall t > t_k.$$

If we now define $t_\epsilon := \max\{t_k,\ k = 1,\dots,d\}$, this implies that

$$|S(t)e_k - Me_k| < \epsilon \qquad \forall t > t_\epsilon,\ \forall k.$$

Due to linearity and thanks to Cauchy-Schwarz inequality, for all $f \in \mathbb{R}^{\mathrm{d}}$ with $|f| \leq 1$ we obtain

$$|S(t)f - Mf| = \left| \sum_{k=1}^{d} f_k (S(t)e_k - Me_k) \right| \leq \left(\sum_{k=1}^{d} |f_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{d} |S(t)e_k - Me_k|^2 \right)^{\frac{1}{2}} \leq \|f\| \sqrt{d} \epsilon.$$

Thus, taking the supremum over $f \in \mathbb{R}^{\mathrm{d}}$ with $|f| \leq 1$

$$\|S(t) - M\| \leq \sqrt{d}\epsilon \qquad \forall t > t_\epsilon$$

which implies the uniform convergence.

Coming back to our model, since $S(n)$ is given by a product of n exponential matrices generated by $A_{j_0}, \ldots, A_{j_{n-1}}$, we can perform an asymptotic study in terms of products like

$$\left\{ \prod_{i=0}^{n} M_{j_i}, \;\; n \in \mathbb{N} \right\} \tag{2.11}$$

where $M_{j_i} = e^{A_{j_i}}$ is a $d \times d$ matrix. Hence, in order to prove that $S(n)$ converges uniformly towards some $d \times d$ matrix $\overline{M}$, we will equivalently show that

$$\prod_{i=0}^{+\infty} M_{j_i} := \lim_{n \to +\infty} \prod_{i=0}^{n} M_{j_i} = \overline{M},$$

where the limit is taken with respect to the matrix norm.

2.2 The discrete heat equation on evolving combinatorial graphs

On the basis of the evolution model just described, we are now ready to introduce a random evolution diffusion model on graphs. In this section, we are going to deal with combinatorial graphs which are finite, simple and unweighted. We can consider one graph G just at first. As already explained, our goal is to analyze how heat spreads on G while their edges randomly changing in time.

In order to model the evolution of network G in an efficient way, we now forget about G itself and start considering a finite collection of graphs $\mathcal{C} = \{G_1, \ldots, G_N\}$ with an arbitrary, but fixed ordering. Roughly speaking, these graphs correspond to all the possible stages that the graph G can reach at a certain time. Therefore, it is reasonable to require that $G_1, \dots, G_N$ have different sets of edges $E(G_1), \dots, E(G_N)$, whereas the set of vertices $V = \{v_1, \dots, v_d\}$ is the same for every graph and represents the structure of the domain which always remains fixed.

Now we want to introduce the stochasticity of this model. We then consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we define a stationary Markov chain $\{j_k\}_{k\geq 0}$ which takes values in the set of graphs $\mathcal{C}$. Hence, we have that for each $k \in \mathbb{N}$

$$\begin{aligned} j_k : \Omega &\longrightarrow \mathcal{C} \\ \omega &\longmapsto j_k(\omega) \end{aligned}$$

is a random variable and it represents the state of the chain after the k-th jump. As we can guess, the time sequence of graphs will be determined by the jumps of this Markov chain.

Finally, we are able to write the system of equations for our model. The idea is to generalize the discrete heat equation (DHE) introduced in the previous chapter and to construct a system with the same form of (2.2). We denote by

$$\mathcal{K} = \{-\mathcal{L}(G_1), \dots, -\mathcal{L}(G_N)\} \tag{2.12}$$

the finite family of discrete Laplace operators associated with the graphs in $\mathcal{C}$ and we consider again

$$u(t) = \begin{pmatrix} u(t, v_1) \\ \vdots \\ u(t, v_d) \end{pmatrix} \in \mathbb{C}^{\mathrm{d}},$$

i.e. the vector of heat concentration at each vertex of G.

Hence, the discrete random evolution (DRE) system has the following form:

$$\begin{cases} \dfrac{d^+u}{dt}(t) = -\mathcal{L}(j_k)u(t), \quad t \in [k, k+1), \ k \in \mathbb{N} \\ u(0) = u_0 \in \mathbb{C}^{\mathrm{d}}, \end{cases} \tag{DRE}$$

where $u_0 \in \mathbb{C}^{\mathrm{d}}$ is the initial heat concentration.

Looking at this system, the main difference with (2.2) can be found in the sequence which determines the evolution of the graphs. We remark that in this random context the model is clearly no longer deterministic as before. In fact, (DRE) is not only non autonomous, but its leading operator $-\mathcal{L}(j_k)$ is also a stochastic process taking values in $\mathcal{K}$. In particular, $\{-\mathcal{L}(j_k)\}_{k\geq 0}$ is still a Markov chain, with the same class structure as $\{j_k\}_{k\geq 0}$. For this reason, we will directly refer to the original Markov chain to talk about the stochasticity of the model.

Despite the stochasticity of (DRE), it is also convenient to analyze every deterministic problem one obtains fixing a realization $\omega \in \Omega$. Hence, we get

$$\begin{cases} \dfrac{d^+u}{dt}(t) = -\mathcal{L}(j_k(\omega))u(t), \quad t \in [k, k+1),\ k \in \mathbb{N} \\ u(0) = u_0 \in \mathbb{C}^{\mathrm{d}} \end{cases} \tag{DRE ω}$$

and this problem is exactly (2.2) with $\mathcal{K}$ defined as in (2.12) and with a deterministic sequence given by $\{j_k(\omega)\}_{k\geq 0}$.

Once introduced the model in a rigorous way, we can now better understand its dynamics. Fig. 2.1 gives a more intuitive idea about what happens when we fix $\omega \in \Omega$ and how the graphs' evolution actually works.

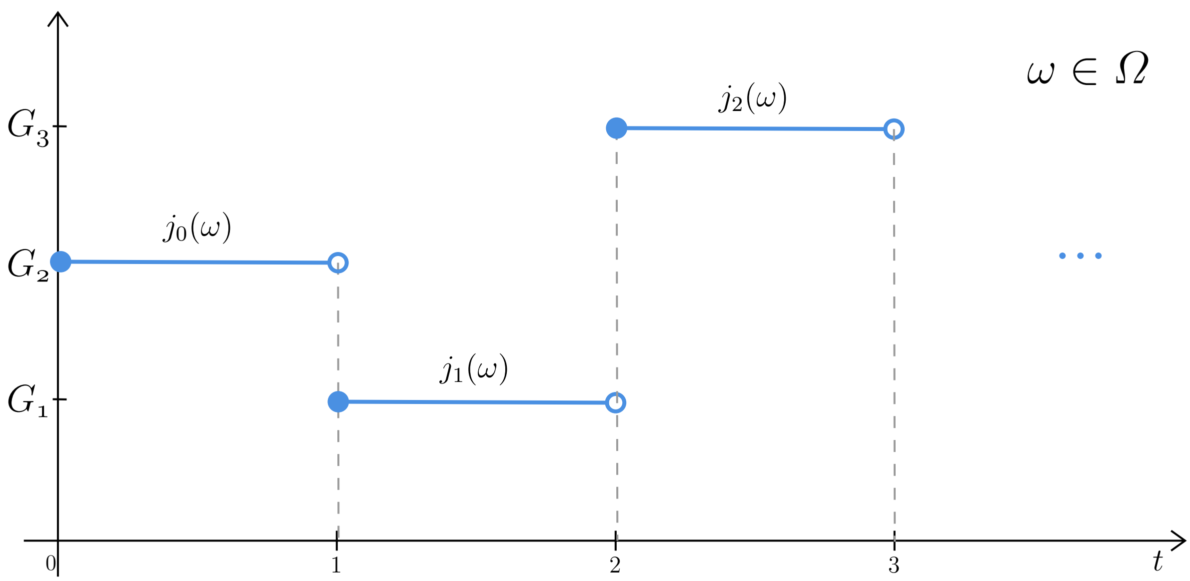


Figure 2.1: Dynamics of (DRE ω).

Suppose that we deal with three graphs in $\mathcal{C} = \{G_1, G_2, G_3\}$ and we start from an initial heat distribution u_0. If we fix a realization $\omega \in \Omega$ of the Markov chain, we have seen that our network will evolve according to a deterministic sequence.

More in detail, for the first time interval $[0, 1]$, we have to study the discrete heat diffusion on the graph given by the initial state $j_0(\omega) = G_2$. After this interval, at $t = 1$ we "freeze" our system and we substitute the current graph with the state $j_1(\omega) = G_1$ reached by the Markov chain after its first jump. Hence, for the next interval $[1, 2]$, (DRE ω) will follow the heat diffusion on the new graph G_1, which represents the second stage of the evolving domain. From $t = 1$ on, the initial condition for the new system is given by the final solution $e^{-\mathcal{L}(j_0(\omega))}u_0$ of the previous one in $t = 1$ and the leading operator is $-\mathcal{L}(j_1(\omega))$. This system will remain unvaried and autonomous until $t = 2$, then the Markov chain will jump into G_3 and we will have to study the evolution on a different network again and so on for all the times.

Using those results proved in the previous section, (DRE) is well-posed and its weak solution is given by

$$u(t) = e^{-(t-k)\mathcal{L}(j_k)}e^{-\mathcal{L}(j_{k-1})}\cdots e^{-\mathcal{L}(j_0)}f, \quad t \in [k, k+1]. \tag{2.13}$$

Let us focus on the evolution operator

$$S(t) = e^{-(t-k)\mathcal{L}(j_k)}e^{-\mathcal{L}(j_{k-1})}\cdots e^{-\mathcal{L}(j_0)}, \quad t \in [k, k+1] \tag{2.14}$$

Due to the dependence on the Markov chain $\{j_k\}_{k\geq 0}$, we have that $\{S(t)\}_{t\geq 0}$ is a stochastic process: in particular it is a random product of matrices which receives one more factor at every integer time according to the Markov chain $\{j_k\}_{k\geq 0}$.

2.2.1 Asymptotics

Once we have proved the existence and uniqueness of the solution, we can finally present the main part of this research: the study of the long-time behaviour of model (DRE). In particular, we are going to refer to the results shown for the discrete heat equation on a single graph G fixed in time. In that framework, we briefly recall that the evolution operator for long times converges to the orthogonal projection onto the kernel of $\mathcal{L}(G)$. This means that, as one can expect, the heat spreads during the time and tends towards the average of u_0 on each connected component of the graph. Hence, as we have already remarked in the previous chapter, the connectedness of G plays a fundamental role in this analysis, because it determines the asymptotic dynamics of the model.

In the new framework of this chapter, where the network is no longer constant in time, there are several aspects we now have to consider. First of all, we deal with a stochastic evolution operator and this obviously implies that we need a convergence notion different from the one used in the deterministic case. In addition, it seems suitable to impose some conditions related to the connectivity of graphs. However, since we no longer work with only one graph, we have to understand which are these conditions are and how much we can weaken them.

First of all, our idea is to study what happens trajectory by trajectory of the Markov chain. More precisely, we are going to study the long-time behaviour of each family of operators $\{S(t,\omega)\}_{t\geq 0}$, for fixed $\omega \in \Omega$. When we study the dynamics of a single trajectory, the evolution operator is clearly a deterministic product and it has the following form:

$$S(t,\omega) = e^{-(t-k)\mathcal{L}(j_k(\omega))} e^{-\mathcal{L}(j_{k-1}(\omega))} \cdots e^{-\mathcal{L}(j_0(\omega))}, \qquad t \in [k, k+1],\ \omega \in \Omega. \tag{2.15}$$

Since each exponential matrix $e^{-t\mathcal{L}(j_k(\omega))}$ is strongly, uniformly continuos and contractive, we find that $\{S(t,\omega)\}_{t\geq 0}$ preserves these properties. Then, denoting by $\{T(t, j_k(\omega))\}_{t\geq 0}$ the semigroup generated by the minus Laplacian $-\mathcal{L}(j_k(\omega))$, we can also rewrite (2.15) as the product

$$S(t,\omega) = T(t-k, j_k(\omega))T(1, j_{k-1}(\omega)) \cdots T(1, j_0(\omega)), \qquad t \in [k, k+1],\ \omega \in \Omega$$

and study its asymptotic behaviour using the notions of strong and uniform convergence recalled in the deterministic case.

In particular, we are going to neglect the trajectories with zero probability $\mathbb{P}$, hence it is convenient to introduce an additional notion of convergence which is referred to a random environment.

Definition 2.5. Let $(X, \|\cdot\|)$ be a normed vector space and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider a stochastic process $\{S(t)\}_{t\geq 0}$ with state space in $\mathcal{B}(X)$ and a (deterministic) bounded operator $M \in \mathcal{B}(X)$. We will say that $S(t)$ *converges* $\mathbb{P}$*-almost surely* towards M, namely

$$\lim_{t\to+\infty} S(t) = M \qquad \mathbb{P} - \text{a.s.},$$

when

$$\mathbb{P}\left(\lim_{t\to+\infty} S(t,\omega) = M\right) = 1.$$

Depending on the different convergence of the bounded operator $S(t,\omega)$, we will say that $S(t)$ *converges uniformly/strongly and* $\mathbb{P}$*-almost surely* towards M.

At this point, we can talk more in detail about our purposes. Imagine that we are analyzing the heat diffusion during the time and we want to guess its distribution when $t \to +\infty$. At every integer time the graph environment on which we study the diffusion changes: for this reason, it seems reasonable to deduce that if we let the graph evolve in a suitable way, the heat will spread on the entire set of vertices V, until its distribution gets homogeneous. From a mathematical point of view, we have seen that this situation corresponds to show that $S(t)$ converges towards P_0, the orthogonal projection onto the subspace of constant functions on G spanned by $\mathbb{1} = (1, \ldots, 1)^\top \in \mathbb{R}^{\mathrm{d}}$.

Then, in a first moment, we are going to deal with the following
Problem (P): Determine whether and under which conditions the evolution operator of (DRE) converges almost surely to the projection P_0, namely

$$\lim_{t\to+\infty} S(t) = P_0 \qquad \mathbb{P} - \text{a.s.}. \tag{$\star$}$$

Regarding the possible conditions under which ($\star$) holds, they can have different nature. Firstly, it seems reasonable to require some hypotheses on the mechanism of graphs' choice which determines the evolution of (DRE). This means that we have to consider some suitable properties related to the Markov chain $\{j_k\}_{k\geq 0}$: in this way, we can get more information about the sequence of matrices which compose the operator $S(t)$. For this purpose, from now on we will require that

<u>Condition (1):</u> the Markov chain $\{j_k\}_{k\geq 0}$ is irreducible.

It is well-known that an irreducible Markov chain with finite state space is positively recurrent when it is aperiodic or when it has period p. Hence, this implies that with probability 1 the chain will visit every graph in $\mathcal{C}$ infinitely many times during the evolution of the system.

In addition, as we have specified before, we certainly need to impose some conditions about the connectivity of the graphs in $\mathcal{C}$. Let us start from the case when all graphs $G_1, \dots, G_N$ are connected.

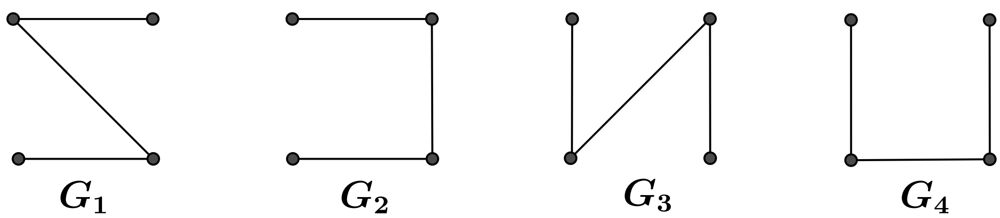


Figure 2.2: All graphs in $\mathcal{C}$ are connected.

In a random evolution like the one described in Fig. 2.2, where at each stage the graph environment is connected, we can intuitively imagine that ($\star$) holds. In fact, we recall that in the discrete heat equation on a fixed graph, its connectedness allows the heat distribution to tend towards the average of u_0 on V. Then, roughly speaking, the same happens even when we jump from a connected graph to another one: there will always be a possible path of edges which let the heat – even though more slowly – reach every vertex in V for long times.

The following Proposition 2.7 proves this conjecture in a more rigorous way. We first present the following

Lemma 2.6. *Let $G = (V, E)$ be a connected graph. Then for all $t > 0$*

$$||Qe^{-t\mathcal{L}}|| < 1,$$

where $\mathcal{L}$ is the Laplace operator associated with G and $Q := I - P_0$ is the orthogonal projection onto the subspace $\langle \mathbb{1} \rangle^{\perp}$ orthogonal to the constant functions.

Proof. By contradiction, suppose that $||Qe^{-t\mathcal{L}}|| = 1$, then there exists $f \in \mathbb{C}^{\mathrm{d}}$, $f \neq 0$ such that

$$Qe^{-t\mathcal{L}} f = f,$$

i.e. f belongs to the fixed points space $\mathrm{Fix}\left(Qe^{-t\mathcal{L}}\right)$. Additionally, we have that $f \in \mathrm{rg}\, Q$, then we can deduce that

$$Qe^{-t\mathcal{L}} f = e^{-t\mathcal{L}} Q f = e^{-t\mathcal{L}} f = f$$

and so $f \in \mathrm{Fix}\left(e^{-t\mathcal{L}}\right) = \ker \mathcal{L}$.

Since G is a connected graph, it is well-known that the kernel of its Laplacian agrees with the subspace of constant functions on V, thus $f = c$, for some constant c. But f is also in $\mathrm{rg}\, Q$ and by definition the latter contains only 0 as unique constant. Then, we conclude that the only possible choice is $f = c = 0$, which is a contradiction.

□

Now, we are finally able to prove

Proposition 2.7. *Let us consider model (DRE) where $\mathcal{C} = \{G_1, \ldots, G_N\}$ is a finite family of connected graphs and $\{j_k\}_{k \geq 0}$ a Markov chain. Then*

$$\lim_{t \to +\infty} \|S(t, \omega) - P_0\| = 0, \qquad \forall \omega \in \Omega. \tag{2.16}$$

Before proving this result, observe that the convergence proposed in (2.16) is not the almost sure one in Definition 2.5: it is clearly stronger, since it holds for any trajectory, without imposing particular conditions on the Markov chain. This suggests that under the assumption of connectedness of all graphs, we are able to prove a result which is more general than ($\star$). It does not matter which sequence of graphs (DRE) follows: $S(t)$ will always uniformly converge to P_0.

Proof. Fixed $\omega \in \Omega$, we consider again

$$Q := I - P_0,$$

whose range is $\langle \mathbb{1} \rangle^\perp$. We can prove the thesis in an equivalent way, showing that

$$\lim_{t \to +\infty} \|QS(t,\omega)\| = 0.$$

First of all, by definition Q is idempotent and commutes with the exponential matrix of every Laplace operator. Hence

$$\begin{aligned} QS(t) &= Qe^{-(t-k)\mathcal{L}(j_k)} e^{-\mathcal{L}(j_{k-1})} \cdots e^{-\mathcal{L}(j_0)} \\ &= Qe^{-(t-k)\mathcal{L}(j_k)} Qe^{-\mathcal{L}(j_{k-1})} \cdots Qe^{-\mathcal{L}(j_0)}, \end{aligned}$$

where we neglect the dependence on ω to simplify the notation.

Thanks to Lemma 2.6, we deduce that each matrix $Qe^{-\mathcal{L}(G_i)}$, $i = 1, \dots, N$ has norm strictly less than 1 and we denote by

$$\delta := \max \left\{ ||Qe^{-\mathcal{L}(G_i)}|| \ \ : \ \ i = 1, \dots, N \right\} < 1.$$

For all $t > 0$, let $k \in \mathbb{N}$ be such that $k \leq t < k+1$. By submultiplicativity of the matrix norm we have

$$\begin{aligned} ||QS(t,\omega)|| &= ||Qe^{-(t-k)\mathcal{L}(j_k)} Qe^{-\mathcal{L}(j_{k-1})} \cdots Qe^{-\mathcal{L}(j_0)}|| \\ &\leq ||Qe^{-(t-k)\mathcal{L}(j_k)}|| \, ||Qe^{-\mathcal{L}(j_{k-1})}|| \cdots ||Qe^{-\mathcal{L}(j_0)}|| \\ &\leq ||Qe^{-\mathcal{L}(j_{k-1})}|| \cdots ||Qe^{-\mathcal{L}(j_0)}|| \\ &\leq \delta^{k-1}. \end{aligned}$$

If $t \to +\infty$, then $k \to +\infty$ and we finally get

$$\lim_{t \to +\infty} \|QS(t,\omega)\| = 0.$$

At this point, we can rewrite the evolution operator as

$$S(t,\omega) = P_0 S(t,\omega) + (I - P_0) S(t,\omega). \tag{2.17}$$

Thanks to the spectral representation of each exponential matrix, for all $i = 1, \dots, N$

$$e^{-t\mathcal{L}(G_i)} = P_0 + \sum_{\lambda_k \neq 0} e^{-t\lambda_k} P_k, \qquad t > 0,$$

where λ_ks are the $k-1$ strictly positive eigenvalues of the Laplacian $\mathcal{L}(G_i)$ and P_ks are the mutually orthogonal eigenprojections. In particular, P_0 is the eigenprojection associated with the simple eigenvalue $\lambda_0 = 0$. Applying P_0 to both sides, due to the properties of these projections we obtain

$$P_0 e^{-t\mathcal{L}(G_i)} = \underbrace{P_0^2}_{P_0} + \sum_{\lambda_k \neq 0} e^{-t\lambda_k} \underbrace{P_0 P_k}_{0} = P_0, \qquad t > 0$$

and, recursively,

$$P_0 S(t,\omega) = P_0 \qquad t > 0.$$

Then, from (2.17) we can write

$$S(t,\omega) = P_0 + QS(t,\omega)$$

and finally conclude that

$$\lim_{t\to+\infty} ||S(t,\omega) - P_0|| = \lim_{t\to+\infty} ||QS(t,\omega)|| = 0,$$

so the proof is complete. □

The result just presented is surely strong, but it only holds under a very restrictive condition too: all the graphs are required to be connected. However, one can wonder what happens when we weaken this hypothesis and admit also non connected graphs in $\mathcal{C}$. May we hope that $S(t,\omega)$ still converges to P_0 in this situation? The answer is affirmative, but the price to pay is that the convergence for all the realizations $\omega \in \Omega$ no longer holds.

In fact, suppose that Condition (1) holds and fix $\omega \in \Omega$ such that

$$j_k(\omega) = \overline{G} \qquad \forall k \geq 0, \tag{2.18}$$

where $\overline{G}$ is some non connected graph in $\mathcal{C}$. In this specific situation (DRE) is just the discrete heat equation on the graph $\overline{G}$ which remains always the same and, as we have seen in the previous chapter, we clearly do not expect that the evolution operator converges to P_0. Fortunately, trajectories like (2.18) have zero probability and this is the reason why we have expressed problem (P) in terms of the almost sure convergence. From now on, the goal is to find a suitable and as weak as possible connectivity's condition that allows us to prove (P).

We start adapting a result of [**3**] in our setting. The authors proved that if the second eigenvalues of the operators $\{A_{j_k}, k \geq 0\}$ driving a certain non-autonomous diffusion equation as (2.1) are bounded from below away from 0, then the solution of (2.2) converges for all initial data u_0 towards the projection of u_0 onto the first eigenspace. In our case, this projection is exactly P_0 and the eigenspace is spanned by $\mathbb{1}$. We are going to refine their method in the following

Proposition 2.8. *Fix $\omega \in \Omega$, consider the sequence $\boldsymbol{j}(\omega) = \{j_k,\ k \in \mathbb{N}\}$ and denote by $\lambda_2(j_k)$ the second lowest eigenvalue of the discrete Laplacian $\mathcal{L}(j_k)$. If*

$$\sum_{k=0}^{+\infty} \lambda_2(j_k) = +\infty, \tag{2.19}$$

then $S(t,\omega)$ strongly (then uniformly) converges towards the projection P_0 onto the constant functions, namely the solution $u(t,\omega)$ converges for all initial datum u_0 to $P_0 u_0 = (u_0, e_1)e_1$, where $e_1 := \frac{1}{\sqrt{d}}(1,\ldots,1)^\top \in \mathbb{R}^d$.

Proof. We are going to adapt the proof of Theorem 5.4 in [**3**].

Throughout this proof we will neglect the dependence on ω for simplicity. We start setting

$$\tilde{u}(t) := u(t) - (u_0, e_1)e_1,$$

so now we have to show that $\tilde{u}(t) \to 0$ as $t \to +\infty$.

We can estimate the Euclidean norm of the vector $\tilde{u}(t) \in \mathbb{C}^{\mathrm{d}}$ by

$$\begin{aligned} ||\tilde{u}(t)||^2 - ||\tilde{u}(0)||^2 &= \int_0^t \frac{d}{dt}||\tilde{u}(s)||^2 ds \\ &= 2\int_0^t (\frac{d\tilde{u}}{dt}(s),\ \tilde{u}(s))ds \\ &= -2\int_0^t (\mathcal{L}(j_{\lfloor s \rfloor})\tilde{u}(s),\ \tilde{u}(s))ds. \end{aligned} \tag{2.20}$$

Applying the Spectral Theorem, we write

$$\tilde{u}(s) = \sum_{h=1}^{d} (\tilde{u}(s),\ e_h(j_{\lfloor s \rfloor}))\ e_h(j_{\lfloor s \rfloor}),$$

where $\{e_h(j_{\lfloor s \rfloor})\}_{h=1}^d$ is a basis of mutually orthogonal eigenvectors of the Laplacian $\mathcal{L}(j_{\lfloor s \rfloor})$

with common first eigenvector $e_1(j_{\lfloor s \rfloor}) = e_1$, $\forall s \geq 0$. Thus by linearity

$$\begin{aligned}
\mathcal{L}(j_{\lfloor s \rfloor})\tilde{u}(s) &= \mathcal{L}(j_{\lfloor s \rfloor}) \left(\sum_{h=1}^{d} (\tilde{u}(s),\ e_h(j_{\lfloor s \rfloor}))\ e_h(j_{\lfloor s \rfloor}) \right) \\
&= \sum_{h=1}^{d} (\tilde{u}(s),\ e_h(j_{\lfloor s \rfloor}))\lambda_h(j_{\lfloor s \rfloor})e_h(j_{\lfloor s \rfloor}) \\
&\leq \lambda_2(j_{\lfloor s \rfloor}) \sum_{h=2}^{d} (\tilde{u}(s),\ e_h(j_{\lfloor s \rfloor}))\ e_h(j_{\lfloor s \rfloor}) \\
&= \lambda_2(j_{\lfloor s \rfloor}) \sum_{h=1}^{d} (\ \tilde{u}(s),\ e_h(j_{\lfloor s \rfloor})\ )\ e_h(j_{\lfloor s \rfloor}) \\
&= \lambda_2(j_{\lfloor s \rfloor})\tilde{u}(s),
\end{aligned}$$

where we used the fact that $(\tilde{u}(s), e_1) = 0\ \forall s \geq 0$. Then from (2.20) we get

$$\begin{aligned}
||\tilde{u}(t)||^2 - ||\tilde{u}(0)||^2 &= -2 \int_0^t (\mathcal{L}(j_{\lfloor s \rfloor})\tilde{u}(s),\ \tilde{u}(s))ds \\
&\leq -2 \int_0^t \lambda_2(j_{\lfloor s \rfloor})||\tilde{u}(s)||^2 ds.
\end{aligned}$$

By Gronwall's Lemma we deduce that

$$||\tilde{u}||^2 \leq ||\tilde{u}(0)||^2\ e^{-2\int_0^t \lambda_2(j_{\lfloor s \rfloor})ds}.$$

Hence, for $t \to +\infty$, we have

$$\int_0^{+\infty} \lambda_2(j_{\lfloor s \rfloor})ds = \sum_{k=0}^{+\infty} \lambda_2(j_k) = +\infty$$

and $||\tilde{u}(t)|| \to 0$. □

The statement we have just proved ensures the convergence towards P_0 provided that (2.19) holds. First of all, we notice that this series of second lowest eigenvalues diverges when the family $\mathcal{C}$ contains only connected graphs. In fact, since $\lambda_2(i) \neq 0$ for all $G_i \in \mathcal{C}$, (2.19) is an infinite sum of strictly positive terms and thus diverges. This means that Proposition 2.8 is another way to prove the convergence result we have already known.

However, Proposition 2.8 is not useless, in fact it ensures that $S(t)$ converges almost surely to P_0 for a more general family $\mathcal{C}$ of graphs. In fact, the following result states that it is sufficient that at least one among $G_1, \ldots, G_N$ is connected.

Corollary 2.9. *Let us consider model (DRE) where $\mathcal{C}$ contains at least one connected* *graph $\tilde{G}$ and let $\{j_k\}_{k\geq 0}$ be an irreducible Markov chain. Then the evolution operator $S(t)$ converges strongly (then uniformly) and almost surely towards the projection P_0.*

Proof. We fix $\omega \in \Omega$ and we recall that, due to the connectedness of $\tilde{G}$, 0 is a simple eigenvalue of $\mathcal{L}(\tilde{G})$: therefore, its second lowest eigenvalue $\lambda_2(\tilde{G})$ is strictly positive. Consider

$$\sum_{k=0}^{+\infty} \lambda_2(j_k) = +\infty, \tag{2.21}$$

whose terms are always positive and given by all the second lowest eigenvalues of the graphs in $\mathcal{C}$

$$\{\lambda_2(G_i),\ \ G_i \in \mathcal{G}\},$$

according to the evolution of (DRE). As we know, when they are associated with connected graphs, some of these eigenvalues are strictly positive, whereas the ones corresponding to disconnected graphs are equal to 0. With probability 1, we can estimate (2.21) as follows:

$$\sum_{k=0}^{+\infty} \lambda_2(j_k) \geq \sum_{h:j_h=G} \lambda_2(\tilde{G}) = +\infty \quad \mathbb{P} - \text{a.s.},$$

where in the last equality we have applied the fact that $\tilde{G}$ is visited infinitely many times by the chain with probability 1. Then, the thesis follows just applying Proposition 2.8.

□

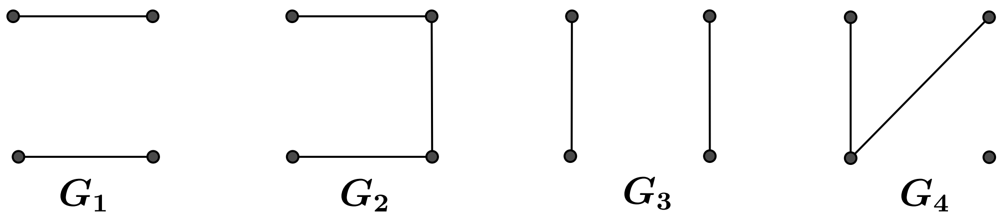


Figure 2.3: Only G_2 is connected, the other graphs are disconnected.

Despite the previous results allow us to prove ($\star$) without requiring a very strong condition on $\mathcal{C}$, we can even do better. In fact we are going to show that it is not necessary that one of the graphs we are dealing with has to be connected, as the following example shows.

Example 2.10. Let $V = \{v_1, v_2, v_3\}$ and we consider $\mathcal{C} = \{G_1, G_2\}$ according to the following picture

Both graphs are clearly non connected, nevertheless we can prove that also in this case $S(t)$ converges almost surely to the orthogonal projection

$$P_0 = \frac{1}{3}J = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Equivalently, as done in the proof of Proposition 2.7, we are going to show that $QS(t) \to 0$ as $t \to +\infty$ where

$$Q = I - P_0 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Suppose that the Markov chain which jumps from G_1 to G_2 is irreducible, thus with probability 1 both graphs are reached infinitely many times. This means that for long times there are two terms composing the product $QS(t)$, both repeated infinitely often as the time goes on, and they are

$$M_1 = Qe^{-\mathcal{L}(G_1)}, \quad M_2 = Qe^{-\mathcal{L}(G_2)}.$$

As we know, both M_1 and M_2 are contractions, but we cannot estimate them as in the case of all connected graphs because it is not true that $\|M_i\| < 1,\ i = 1, 2$. In fact, after easy computations we get

$$M_1 = Qe^{-\mathcal{L}(G_1)} = \begin{pmatrix} \frac{1}{6} + \frac{1}{2e^2} & \frac{1}{6} - \frac{1}{2e^2} & -\frac{1}{3} \\ \frac{1}{6} - \frac{1}{2e^2} & \frac{1}{6} + \frac{1}{2e^2} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

and its eigenvalues are $\lambda_1 = 1$, $\lambda_2 = \frac{1}{e^2}$ and $\lambda_3 = 0$. Thus, denoting by $\rho(M_i)$ the spectral radius of M_i, we have

$$1 \leq \rho(M_1) \leq \|M_1\| \leq 1$$

and this implies that $\|M_1\| = 1$ and in a similar way we obtain $\|M_2\| = 1$, too. However, one can show by an explicit computation that the operators M_1M_2 and M_2M_1 satisfy

$$\|M_1M_2\| = \|M_2M_1\| = \delta \approx 0.413262 < 1.$$

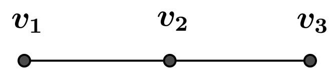
Clearly, due to the irreducibility of the Markov chain, also the sequences M_1M_2 and M_2M_1 are repeated infinitely many times with probability 1. This implies that for $t > 0$ large enough, there exists $k(t) \in \mathbb{N}$ positively diverging as $t \to +\infty$ and, for instance, we get

$$\|QS(t)\| \leq \|M_1M_2\|^{k(t)} = \delta^{k(t)} \xrightarrow{t\to+\infty} 0 \qquad \mathbb{P} - \text{a.s.}.$$

This example suggests that the connectedness of all or some graphs in $\mathcal{C}$ is not a necessary condition in order that $(\star)$ holds. In fact, what is really important is to consider a more general concept of connectivity, which involves all the graphs in $\mathcal{C}$ and at the same time does not require that the single ones are connected. We are going to show that the right hypothesis to be imposed is the following

Condition (2): the union graph $G := (V,\ E_1 \cup \cdots E_N)$ has to be connected.

In view of (2), we now understand why Example 2.10 works. In fact, the union graph of G_1 and G_2 is given by



$\boldsymbol{G_1 \cup G_2}$

which is clearly connected.

Before proving the sufficience of Condition (2), we need three lemmas. The first one characterizes the connectedness of the union graph in terms of the Laplace operators of $G_1, \ldots, G_N$.

Lemma 2.11. *Let $G_1, \ldots, G_N$ be combinatorial graphs and let $G = (V,\ E_1 \cup \ldots \cup E_N)$ their union graph. Then*

$$G \text{ is connected} \iff \bigcap_{i=1}^{N} \ker \mathcal{L}(G_i) = \langle \mathbb{1} \rangle.$$

Proof. We only prove the statement for $N = 2$, then the general case follows by induction on N.

Suppose that the union graph G is connected. Since constant functions are always in the kernel of a discrete Laplace operator, it always holds

$$\ker \mathcal{L}(G_1) \cap \ker \mathcal{L}(G_2) \supseteq \langle \mathbb{1} \rangle$$

and then we just need to prove the opposite inclusion. Let $f \in \ker \mathcal{L}(G_1) \cap \ker \mathcal{L}(G_2)$, hence f is constant on the connected components of G_1 and G_2: we claim that f is actually constant on each vertex of V. Let $u, v \in V$. Since G is connected, there exists a path $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{m-1}, v_m)\}$ of edges in $E_1 \cup E_2$ such that

$$u \sim v_1 \sim \ldots \sim v_m \sim v,$$

where $\sim$ can be $\sim_{G_1}$ or $\sim_{G_2}$. We now apply f to u and v_1: since this couple of vertices are adjacent in G_1 or in G_2 and f is constant on the connected components of both graphs, then we have $f(u) = f(v_1)$. Applying the same procedure for every vertex in the path, we get

$$f(u) = f(v_1) = \ldots = f(v_m) = f(v),$$

thus f is constant and $\ker \mathcal{L}(1) \cap \ker \mathcal{L}(2) \subseteq \langle \mathbb{1} \rangle$.

As for the opposite implication, suppose G is not connected: then we can find a function in the intersection of the two kernels which is not constant on V. Being G not connected, so do G_1 and G_2 and we write

$$G = \bigsqcup_{h=1}^{l} G^{(h)}, \qquad G_1 = \bigsqcup_{j=1}^{m} G_1^{(j)}, \qquad G_2 = \bigsqcup_{k=1}^{n} G_2^{(k)},$$

as disjoint unions of connected components. In particular, each component of the union graph can be written as follows:

$$G^{(h)} = \bigsqcup_{j \in A_1^{(h)}} G_1^{(j)} \quad \text{or} \quad G^{(h)} = \bigsqcup_{k \in A_2^{(h)}} G_2^{(k)}, \qquad h = 1, \ldots, l,$$

where

$$\begin{aligned} A_1^{(h)} &:= \left\{ j \in \{1, \ldots, m\} \ : \ G_1^{(j)} \subseteq G^{(h)} \right\}, \\ A_2^{(h)} &:= \left\{ k \in \{1, \ldots, n\} \ : \ G_2^{(k)} \subseteq G^{(h)} \right\}. \end{aligned}$$

Due to the disjointness, we have

$$A_1^{(1)} \sqcup A_1^{(2)} \sqcup \ldots \sqcup A_1^{(l)} = \{1, \ldots, m\},$$
$$A_2^{(1)} \sqcup A_2^{(2)} \sqcup \ldots \sqcup A_2^{(l)} = \{1, \ldots, n\}.$$

At this point, we just define f as

$$f(v) = c_h, \quad \text{if } v \in V(G^{(h)}), \ \ i = 1, \ldots, l,$$

where c_h is constant $\forall h = 1, \ldots, l$, and $c_{h_1} \neq c_{h_2}$ for $h_1 \neq h_2$. By definition, for all $h = 1, \ldots, l$, f is constant on each $G^{(h)}$, therefore it is also constant on $G_1^{(j)}$, $j \in A_1^{(h)}$ and on $G_2^{(k)}$, $k \in A_2^{(h)}$. This means that

$$f \in \ker \mathcal{L}(G_1) \cap \ker \mathcal{L}(G_2),$$

but by construction f is not constant and the proof is complete. □

Lemma 2.12. *Let G be a combinatorial graph and denote by $\mathcal{L}$ its discrete Laplace operator. Then*

$$\|e^{-t\mathcal{L}} f\| < \|f\|, \qquad \forall f \notin \ker \mathcal{L}, \ \forall t > 0.$$

Proof. The proof is simple and comes from the spectral decomposition of the exponential matrix of $\mathcal{L}$. In fact, let $t > 0$ and $f \notin \ker \mathcal{L}$, then we can write

$$e^{-t\mathcal{L}} f = \sum_{k=1}^{d} e^{-t\lambda_k} (f, v_k) \ v_k,$$

where $\{v_k\}_{k=1}^d$ is an orthonormal basis of eigenvectors of $\mathcal{L}$, associated with the eigenvalues λ_ks. Taking the square norm, we get

$$\|e^{-t\mathcal{L}} f\|^2 = \sum_{k=1}^{d} e^{-2t\lambda_k} |(f, v_k)|^2 = \sum_{\{k:\ \lambda_k = 0\}} |(f, v_k)|^2 + \sum_{\{k:\ \lambda_k \neq 0\}} e^{-2t\lambda_k} |(f, v_k)|^2, \tag{2.22}$$

where the last addendum is different from 0 since $f \notin \ker \mathcal{L}$. Recalling that the nonzero λ_k are strictly positive and then $e^{-2t\lambda_k} < 1$, one can estimate the norm in (2.22) as follows:

$$\begin{aligned} \|e^{-t\mathcal{L}} f\|^2 &= \sum_{\{k:\ \lambda_k = 0\}} |(f, v_k)|^2 + \sum_{\{k:\ \lambda_k \neq 0\}} e^{-2t\lambda_k} |(f, v_k)|^2 \\ &< \sum_{\{k:\ \lambda_k = 0\}} |(f, v_k)|^2 + \sum_{\{k:\ \lambda_k \neq 0\}} |(f, v_k)|^2 \\ &= \|f\|^2 \end{aligned}$$

and the thesis follows. $\square$

Lemma 2.13. *Assume that the union graph $G = (V, E_1 \cup \ldots \cup E_N)$ is connected. Given any sequence of indexes $\alpha_1, \ldots, \alpha_L$ covering the full set $\{\alpha_1, \ldots, \alpha_L\} = \{1, \ldots, N\}$, then*

$$||Qe^{-\mathcal{L}(G_{\alpha_L})} \cdots e^{-\mathcal{L}(G_{\alpha_1})}|| < 1. \tag{2.23}$$

Proof. We argue by contradiction. Let us assume now that (2.23) does not hold: since the projection Q and all the exponential matrices are contractions, it follows that

$$||Qe^{-\mathcal{L}(G_{\alpha_L})} \cdots e^{-\mathcal{L}(G_{\alpha_1})}|| = 1.$$

Hence, there exists $f \in \mathbb{C}^{\mathrm{d}}$, $f \neq 0$ such that

$$Qe^{-\mathcal{L}(G_{\alpha_L})} \cdots e^{-\mathcal{L}(G_{\alpha_1})} f = f,$$

which means

$$f \in Fix\left(Qe^{-\mathcal{L}(G_{\alpha_L})} \cdots e^{-\mathcal{L}(G_{\alpha_1})}\right) \subseteq \operatorname{rg} Q.$$

Following the same procedure as in the proof of Proposition 2.7, we get

$$f \in \operatorname{Fix}\left(e^{-\mathcal{L}(G_{\alpha_L})} \cdots e^{-\mathcal{L}(G_{\alpha_1})}\right).$$

Let us assume that

$$\operatorname{Fix}\left(e^{-\mathcal{L}(G_{\alpha_L})} \cdots e^{-\mathcal{L}(G_{\alpha_1})}\right) = \bigcap_{h=1}^{L} \operatorname{Fix}\left(e^{-\mathcal{L}(G_{\alpha_h})}\right) \tag{2.24}$$

holds. As a consequence, we can apply the result of the previous lemma and we obtain

$$f \in \bigcap_{h=1}^{L} \operatorname{Fix}\left(e^{-\mathcal{L}(G_{\alpha_h})}\right) = \bigcap_{i=1}^{N} \operatorname{Fix}\left(e^{-\mathcal{L}(G_i)}\right) = \bigcap_{i=1}^{N} \ker\ e^{-\mathcal{L}(G_i)} = \langle \mathbb{1} \rangle.$$

Hence, f is constant and, since $f \in \operatorname{rg} Q$, the only possibility is $f = 0$, then we reach a contradiction.

To complete the proof, we only have to verify (2.24). Clearly,

$$\bigcap_{h=1}^{L} \operatorname{Fix}\left(e^{-\mathcal{L}(G_{\alpha_h})}\right) \subseteq \operatorname{Fix}\left(e^{-\mathcal{L}(G_{\alpha_L})} \cdots e^{-\mathcal{L}(G_{\alpha_1})}\right),$$

so we only have to prove the opposite inclusion. Given $f \in \operatorname{Fix}\left(e^{-\mathcal{L}(G_{\alpha_L})} \cdots e^{-\mathcal{L}(G_{\alpha_1})}\right)$, we start to show that f is in the kernel of $e^{-\mathcal{L}(G_{\alpha_1})}$. Recalling that we are dealing with contractive operators, we get

$$\begin{aligned} ||f|| &= ||e^{-\mathcal{L}(G_{\alpha_L})} \cdots e^{-\mathcal{L}(G_{\alpha_1})} f|| \\ &\leq ||e^{-\mathcal{L}(G_{\alpha_L})} \cdots e^{-\mathcal{L}(G_{\alpha_2})}|| \, ||e^{-\mathcal{L}(G_{\alpha_1})} f|| \\ &\leq ||e^{-\mathcal{L}(G_{\alpha_1})} f|| \\ &\leq ||f||, \end{aligned}$$

then

$$||e^{-\mathcal{L}(G_{\alpha_1})} f|| = ||f||. \tag{2.25}$$

Thanks to Lemma 2.12, we have that equality (2.25) implies

$$e^{-\mathcal{L}(G_{\alpha_1})} f = f$$

and so $f \in \mathrm{Fix}\left(e^{-\mathcal{L}(G_{\alpha_1})}\right)$. It follows that

$$e^{-\mathcal{L}(G_{\alpha_L})} \cdots e^{-\mathcal{L}(G_{\alpha_2})} f = f \tag{2.26}$$

and now one can recursively follow the same procedure starting from (2.26) until showing that

$$f \in \bigcap_{h=1}^{L} Fix\left(e^{-\mathcal{L}(G_{\alpha_h})}\right).$$

□

Thanks to the previous lemmas, we are finally able to prove the main result of this chapter.

Theorem 2.14. *Let us consider model (DRE) and let $\{j_k\}_{k\geq 0}$ be an irreducible Markov chain. Moreover, let us assume that the graphs in $\mathcal{C}$ have a connected union graph $G = (V, E_1 \cup \ldots \cup E_N)$.*

Then the evolution operator $S(t)$ converges uniformly and almost surely towards the projection P_0.

Proof. We fix $\omega \in \Omega$ and, due to the irreducibility of $\{j_k\}_{k\geq 0}$, with probability 1 we can find a sequence of indexes $\alpha_1, \ldots, \alpha_L$ such that

– they cover the full set of indexes:

$$\{\alpha_1, \ldots, \alpha_L\} = \{1, \ldots, N\};$$

– the Markov chain follows this path with a strictly positive probability:

$$p_{\alpha_1,\alpha_2}\, p_{\alpha_2,\alpha_3} \cdots p_{\alpha_{L-1},\alpha_L} > 0.$$

Consider now the Markov chain $\{j'_k\}_{k\geq 0}$ on the L-tuple of states $\{1,\ldots,N\}^L$ induced by $\{j_k\}_{k\geq 0}$ as follows:

$$\begin{aligned} j'_0 &:= (j_0,\ldots,j_{L-1})\,, \\ j'_1 &:= (j_L,\ldots,j_{2L-1})\,, \\ j'_k &:= \big(j_{kL},\ldots,j_{(k+1)L-1}\big)\,, \qquad k \geq 2. \end{aligned}$$

Clearly, the number of times the chain $\{j_k\}_{k\geq 0}$ follows the path $\alpha_1,\ldots,\alpha_L$ is greater than or equal to the number of times $\{j'_k\}_{k\geq 0}$ visits the state $\underline{\alpha} = (\alpha_1,\ldots,\alpha_L)$ and the latter is infinite $\mathbb{P}$-almost surely, due to the results proved in Appendix A. This means that the evolution of (DRE) follows the sequence of graphs $G_{\alpha_1},\ldots,G_{\alpha_L}$ infinitely many times with probability 1 as $t \to +\infty$.

From Lemma 2.13 we are able to estimate

$$\|Qe^{-\mathcal{L}(\alpha_L)}\cdots e^{-\mathcal{L}(\alpha_1)}\| < 1$$

and then for t large there exists $k(t)$ such that it diverges as $t \to +\infty$ and, almost surely, we get

$$\|QS(t)\| \leq \|Qe^{-\mathcal{L}(\alpha_L)}\cdots e^{-\mathcal{L}(\alpha_1)}\|^{k(t)} \xrightarrow{t\to+\infty} 0.$$

Now, the thesis

$$\lim_{t\to+\infty} \|S(t) - P_0\| = 0 \qquad \mathbb{P} - \text{a.s.}$$

follows analogously to Proposition 2.7. □

The result we have just proved ensures that $(\star)$ holds for a large class of families $\mathcal{C}$ of graph. At this point, one can wonder again if there exist cases which are further general. We are now going to show that this is not possible: hence, condition (2) not only is sufficient, but also necessary. To prove this, we are going to adapt some results in [**1**, **12**] about convergence of random products of contractions in Hilbert spaces.

Let $\mathcal{C} = \{G_1,\ldots,G_N\}$ a finite family of graphs such that their union

$$G = (V, E_1 \cup \ldots \cup E_N)$$

is a disconnected graph. As a consequence of Lemma 2.11, this implies that

$$\bigcap_{i=1}^{N} \ker \mathcal{L}(G_i) \neq \langle \mathbb{1} \rangle.$$

In particular, we can prove a more precise result, which also contains the statement of Lemma 2.11.

Lemma 2.15. *Let $G_1, \ldots, G_N$ be combinatorial graphs and let $G = (V, E_1 \cup \ldots \cup E_N)$ be their union. Then*

$$\bigcap_{i=1}^{N} \ker \mathcal{L}(G_i) = \ker \mathcal{L}(G). \tag{2.27}$$

Clearly, this result makes Lemma 2.11 a trivial consequence of (2.27).

Proof. For simplicity we only prove the case with two graphs. We still need the notation used in the proof of Lemma 2.11. In general, suppose that G, G_1 and G_2 are written as disjoint unions of connected components:

$$G = \bigsqcup_{h=1}^{l} G^{(h)}, \qquad G_1 = \bigsqcup_{j=1}^{m} G_1^{(j)}, \qquad G_2 = \bigsqcup_{k=1}^{n} G_2^{(k)}.$$

As we have already seen, each connected component of the union G can be expressed as

$$G^{(h)} = \bigsqcup_{j \in A_1^{(h)}} G_1^{(j)} \quad \text{or} \quad G^{(h)} = \bigsqcup_{k \in A_2^{(h)}} G_2^{(k)}, \qquad h = 1, \ldots, l,$$

with the same definition of $A_1^{(h)}$ and $A_2^{(h)}$ as in the previous proof.

In Chapter 1, we have seen that $\mathcal{B}_G = \{\mathbb{1}_h,\ h = 1, \ldots, l\}$ such that

$$\forall\, v \in V : \qquad \mathbb{1}_h(v) := \begin{cases} 1 & \text{if } v \in G^{(h)} \\ 0 & \text{otherwise} \end{cases}$$

is a basis for $\ker \mathcal{L}(G)$. Similarly, $\mathcal{B}_1 = \{\mathbb{1}_{1,j},\ j = 1, \ldots, m\}$ and $\mathcal{B}_2 = \{\mathbb{1}_{2,k},\ k = 1, \ldots, n\}$ are basis of $\ker \mathcal{L}(G_1)$ and $\ker \mathcal{L}(G_2)$, respectively. Hence, we only need to prove that for all $h = 1, \ldots, l$, $\mathbb{1}_h$ is in the intersection of the kernels in $\mathcal{C}$ and then extend the result to $\ker \mathcal{L}(G)$ by linearity. In particular, by construction the function $\mathbb{1}_h$ will be

$$\forall v \in V : \qquad \mathbb{1}_h(v) := \begin{cases} 1 & \text{if } v \in G_1^{(j)},\ j \in A_1^{(h)} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\forall v \in V: \qquad \mathbb{1}_h(v) := \begin{cases} 1 & \text{if } v \in G_2^{(k)},\ k \in A_2^{(h)} \\ 0 & \text{otherwise,} \end{cases}$$

thus $\mathbb{1}_h \in \ker \mathcal{L}(G_1) \cap \ker \mathcal{L}(G_2)$, in fact it can be written as linear combination of both basis $\mathcal{B}_1$ and $\mathcal{B}_2$ as

$$\mathbb{1}_h = \sum_{j \in A_1^{(h)}} \mathbb{1}_{1,j} \quad \text{or} \quad \mathbb{1}_h = \sum_{k \in A_2^{(h)}} \mathbb{1}_{2,k}.$$

On the other hand, given $f \in \ker \mathcal{L}(G_1) \cap \ker \mathcal{L}(G_2)$, we have

$$f = \alpha_1 \mathbb{1}_{1,1} + \cdots + \alpha_m \mathbb{1}_{1,m} \tag{2.28}$$

and

$$f = \beta_1 \mathbb{1}_{2,1} + \cdots + \beta_n \mathbb{1}_{2,n}, \tag{2.29}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ as above. We recall that by construction every function $\mathbb{1}_h$ of the basis $\mathcal{B}_G$ can be written for every $h = 1, \ldots, l$ as

$$\mathbb{1}_h = \sum_{j \in A_1^{(h)}} \mathbb{1}_{1,j} \quad \text{or} \quad \mathbb{1}_h = \sum_{k \in A_2^{(h)}} \mathbb{1}_{2,k}.$$

Then, comparing the expressions (2.28) and (2.29), we get that

$$\begin{aligned} \alpha_j = \beta_k = c_1, &\qquad \forall j \in A_1^{(1)},\ \forall k \in A_2^{(1)}, \\ &\vdots \\ \alpha_j = \beta_k = c_l, &\qquad \forall j \in A_1^{(l)},\ \forall k \in A_2^{(l)} \end{aligned}$$

and f can also be expressed in terms of $\mathcal{B}_G$ as

$$f = \sum_{h=1}^{l} c_h \mathbb{1}_h,$$

thus $f \in \ker \mathcal{L}(G)$. □

Let now denote by $\overline{P}$ the orthogonal projection onto the kernel of $\mathcal{L}(G)$:

$$\begin{aligned} &\overline{P} \colon \mathbb{C}^{\mathrm{d}} \to \ker \mathcal{L}(G), \\ &\overline{P}(f) := \sum_{h=1}^{l} \frac{1}{|V(G^{(h)})|} \left(f,\ \mathbb{1}_h\right) \mathbb{1}_h. \end{aligned}$$

Before proving the main result of this chapter, we introduce two lemmas.

Lemma 2.16. *The orthogonal projection $\overline{P}$ commutes with every exponential matrix $e^{-\mathcal{L}(G)}$, $\forall i = 1, \ldots, N$.*

Proof. We follow the proof of the theorem in Section 3 of [**1**].

Denote $T_i := e^{-\mathcal{L}(G_i)}$ for all $i = 1, \ldots, N$. Since $\overline{P}$ has range $\bigcap_{i=1}^{N} \operatorname{Fix} T_i$ it follows that

$$\overline{P}\, T_i = \overline{P}, \qquad i = 1, \ldots, N.$$

This implies (see [**23**]) that it also holds

$$\overline{P}\, T_i^* = \overline{P}, \qquad i = 1, \ldots, N.$$

Hence, since the projection $\overline{P}$ is selfadjoint, for all $h = 1, \ldots, N$ and $x,\ y \in \mathbb{C}^{\mathrm{d}}$:

$$\overline{P} = \overline{P}^* = (\overline{P}\, T_i^*)^* = T_i\, \overline{P}^* = T_i\, \overline{P},$$

thus $\overline{P}\, T_i = T_i\, \overline{P}$. □

Lemma 2.17. *Given any sequence of indexes $\alpha_1, \ldots, \alpha_L$ covering the full set of indexes $\{\alpha_1, \ldots, \alpha_L\} = \{1, \ldots, N\}$, then*

$$||(I - \overline{P})\ e^{-\mathcal{L}(G_{\alpha_L})} \cdots e^{-\mathcal{L}(G_{\alpha_1})}|| < 1. \tag{2.30}$$

Proof. The proof is exactly the same as the one for Lemma 2.13, just by substituting Q with $(I - \overline{P})$ and by taking into account Lemma 2.16. By contradiction, suppose that there exists a nonzero function f in the fixed points set of the operator in (2.30). In the end, $f \in \operatorname{rg} \overline{P} \cap \operatorname{rg}(I - \overline{P})$, so that $f = 0$, a contradiction. □

We are going to prove the necessity of condition (2) through the following

Theorem 2.18. *Let us consider model (DRE) and let $\{j_k\}_{k\geq 0}$ be an irreducible Markov chain. Moreover, let $\mathcal{C}$ be a finite family of graphs with union $G = (V, E_1 \cup \ldots \cup E_N)$.*

Then the evolution operator $S(t)$ converges uniformly and almost surely towards the projection $\overline{P}$.

We notice that this theorem also contains the result proved in Theorem 2.14. In that case, we have that $\overline{P} \equiv P_0$, indeed.

Proof. As one can guess, thanks to Lemma 2.17, the proof is very similar to that of Theorem 2.14.

Thanks to Lemma 2.16, we can write the evolution operator as

$$S(t) = \overline{P}\ S(t) + (I - \overline{P})\ S(t) = \overline{P} + (I - \overline{P})\ S(t).$$

Hence, we just need to prove that

$$\lim \|(I - \overline{P})\ S(t)\| = 0, \qquad \mathbb{P} - \text{a.s.} \tag{2.31}$$

and this follows as we in the proof of Theorem 2.14. □

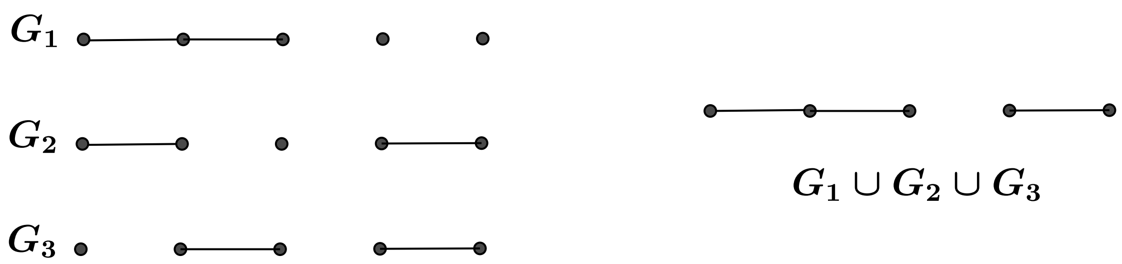


Figure 2.4: G_1, G_2 and G_3 have a non connected union graph: the heat distribution will converge towards the average of the initial datum u_0 on each connected component of $G_1 \cup G_2 \cup G_3$.

Chapter 3

The heat equation on a metric graph

In the previous chapter we have obtained and then analyzed a random evolution system on combinatorial graphs. Now the aim is to generalize that model in the framework of metric graphs.

As anticipated in the Introduction, in a metric graph the edges are interpreted as one-dimensional intervals. We will see that we can consider a metric graph as a topological space and we will equip it with a metric. Furthermore, we will see that it makes sense to define a Lebesgue measure on it and consequently several spaces of measurable functions acting on each edge.

At this point, the next natural step is to consider differential operators on these function spaces. In this work, we will only focus on one simple parabolic PDE studied on a metric graph, namely the heat diffusion equation. Before analyzing it in a random evolution's framework, it is reasonable to illustrate the model on one fixed metric graph. Hence, we first need to present the main properties about metric graphs, in order to better understand in which sense we can talk albout PDEs on graphs.

All the results reported in this chapter are well-known: we will only remark those ones which are necessary to generalize the random evolution in this new setting. In particular, we will especially refer to the books [**5**, **22**] and to the articles [**20**] and [**3**]. In addition, it will be necessary to apply several results about the classical theory of semigroups: in this thesis, we assume all the basic notions on evolution semigroups and their generators to be known. A complete presentation of these topics can be found for instance in [**2**, **6**, **13**, **24**]: in particular, we will mainly refer to Section II.6 of [**13**] and to Chapter 1 of [**24**].

3.1 Introduction to metric graphs

The aim of this section is to introduce metric graphs in a rigorous way, following the guidelines of [**5**]. As the combinatorial ones, metric graphs have a structure composed by a set of vertices and a set of edges. However, what really makes these two classes of graphs so different from each other is explained in an intuitive way by G. Berkolaiko and P. Kuchment in [**5**]: "*now one will need to imagine the edges not as abstract relations between vertices, but rather as physical "wires" connecting them.*".

Therefore, a *metric graph* is defined by a set V of vertices and a set E of edges, where in particular the latters are interpreted as one-dimensional intervals, each one with a certain length $\ell_e \in (0, +\infty]$. We will never discuss about graphs with infinite length and, additionally, we will only deal with finite graphs also in this framework. Furthermore, we will exclude those vertices which are not incident to any edge.

Metric graphs have a topological structure of one-dimensional CW-complexes: roughly speaking, they can be represented as a certain number of 1-cells (the edges) and 0-cells (the vertices), glued to each other in a suitable way. In this structure, vertices in common between more edges are identified.

We will often refer to a metric graph $\mathcal{G}$ as the couple (G, ℓ), where G is the support of $\mathcal{G}$, i.e. the combinatorial graph with vertices, edges and orientation induced by $\mathcal{G}$. Additionally G is equipped with the weight ℓ_e for each edge $e \in E$: such a graph is called weighted graph.

At this point, one can naturally introduce all the topological properties: in particular, also in this case, we are interested in the notion of connectedness, which is equivalent to the path-connectedness in this finite setting.

Definition 3.1. A metric graph $\mathcal{G}$ is said to be connected when for any two points of $\mathcal{G}$ there exists a path joining them.

If $\mathcal{G}$ is disconnected, then as in the combinatorial setting it is given by a disjoint union of connected components.

Since each $e \in E$ is like an interval, we can introduce a parametrization on every edge. In fact, taken $e \in E$ with incident vertices u and v, then the surjection $\{0, \ell_e\} \to \{u, v\}$ ensures that each edge can be identified as follows:

$$e \cong [0, \ell_e], \qquad \forall e \in E.$$

In particular, the image of 0 and the image of ℓ_e will be respectively called initial and terminal vertex. Now we can assign an increasing coordinate $x_e \in [0, \ell_e]$ to each edge of the graph.

Notice that in this way we have imposed an orientation on $\mathcal{G}$, given by the natural orientation from 0 to ℓ_e of every edge. As one can expect, if we would like to reverse the orientation of some $e \in E$, the relation between the old and the new coordinate $\tilde{x}_e$ will be $\tilde{x}_e = \ell_e - x_e$. It is also possible to define a notion of distance between two points in the same connected component of the metric graph $\mathcal{G}$. Obviously, if x and y belong to the same edge, we choose the Euclidean metric $d(x, y) = |x - y|$. Otherwise, if x and y are placed in two different intervals (or, in particular, they are vertices), then $d(x, y)$ is given by the total length of the minimal path connecting x and y.

Thanks to this system of coordinates which identify every point of $\mathcal{G}$, it seems reasonable to define functions on a metric graph: the natural idea is to consider functions acting edgewise. Therefore, ordering in a certain way the edges $E = \{e_1, \dots, e_m\}$, a function f defined on a metric graph $\mathcal{G}$ is identified by the m-tuple $(f_1, \dots, f_m)$, where for all $j = 1, \dots, m$, f_j has $e_j \cong [0, \ell_j]$ as corresponding domain.

We are especially interested in a particular class of functions. Based on how we have described metric graphs, we can naturally define the Lebesgue measure dx on them. Hence, a function $f = (f_e)_{e \in E}$ is *measurable* if every $f_e \colon (0, \ell_e) \to \mathbb{C}$ is measurable. In addition, f is *integrable* if so does f_e for all $e \in E$ and we define

$$\int_{\mathcal{G}} f(x) dx = \sum_{e \in E} \int_0^{\ell_e} f_e(x_e) dx_e.$$

We are now able to well define the Lebesgue space

$$L^2(\mathcal{G}) := \bigoplus_{e \in E} L^2(0, \ell_e)$$

with scalar product given by

$$(f, g)_{L^2(\mathcal{G})} := \int_{\mathcal{G}} f(x) \overline{g(x)} dx = \sum_{e \in E} (f_e, g_e)_{L^2(0, \ell_e)}, \qquad f, g \in L^2(\mathcal{G})$$

and induced norm

$$\|f\|_{L^2(\mathcal{G})} := \left(\sum_{e \in E} \|f_e\|^2_{L^2(0, \ell_e)} \right)^{\frac{1}{2}}, \qquad f \in L^2(\mathcal{G}).$$

If we denote by $f^{(k)} = (f_e^{(k)})$ the $|E|$-tuple of the weak derivatives of order $k = 1, 2$ of f_e

for all $e \in E$, we can also introduce the Sobolev spaces

$$\tilde{H}^k(\mathcal{G}) := \bigoplus_{e\in E} H^k(0,\ell_e), \qquad k = 1,2,$$

whose scalar product can be expressed in terms of those in $L^2(\mathcal{G})$ and $H^k(0,\ell_e)$

$$\begin{aligned}(f,g)_{\tilde{H}^k(\mathcal{G})} &:= \sum_{n\leq k}(f^{(n)}, g^{(n)})_{L^2(\mathcal{G})} \\ &= \sum_{e\in E}(f_e, g_e)_{H^k(0,\ell_e)}, \qquad f,g \in \tilde{H}^k(\mathcal{G})\end{aligned}$$

and induced norm

$$\begin{aligned}\|f\|_{\tilde{H}^k(\mathcal{G})} &:= \left(\sum_{n\leq k}\|f^{(n)}\|^2_{L^2(\mathcal{G})}\right)^{\frac{1}{2}} \\ &= \left(\sum_{e\in E}\|f_e\|^2_{H^k(0,\ell_e)}\right)^{\frac{1}{2}}, \qquad f \in \tilde{H}^k(\mathcal{G}).\end{aligned}$$

Notice that these definitions do not depend on the orientation of the metric graph.

3.2 Diffusion on a metric graph

In the previous section, we have presented the main properties of metric graphs and how to suitably define some function spaces on them. Additionally, we pointed out that once introduced Lebesgue and Sobolev spaces, it is now natural to consider differential operators acting on metric graphs.

In particular, the aim of this section is to illustrate one of the first simple PDE problems which can be studied on a graph: the heat diffusion equation. As in the classical setting, the differential operator governing the evolution is still the (continuous) Laplacian. Therefore, we first are going to introduce how a Laplace operator on a metric graph can be defined. Then, we proceed the discussion describing in detail the diffusion problem, its well-posedness and its asymptotic behaviour.

All the results of this chapter are already well-known and in particular we will follow the guidelines of [**5**, **20**, **22**].

3.2.1 Introduction to the Kirchhoff-Laplace operator

Throughout this section we will consider a metric graph $\mathcal{G} = (G,\ell)$ over a finite oriented combinatorial graph $G = (V,E)$ with vertices $V = \{v_1,\dots,v_n\}$ and edges

$E = \{e_1, \dots, e_m\}$. In the metric framework, we admit also graphs with multiple edges and loops.

In order to introduce the Laplace operator on $\mathcal{G}$, the natural idea is to define a second derivative operator on each edge $e_j \in E$:

$$u_j \mapsto \frac{d^2}{dx_j^2} u_j, \qquad u_j : [0, \ell_j] \to \mathbb{C}, \; e_j \in E.$$

Notice that this definition does not depend on the choice of orientation of the edge e_j for all $j = 1, \dots, m$.

Therefore, we will call *Laplace operator on* $\mathcal{G}$ and we will denote it by Δ the operator which acts on functions $u = (u_1, \dots, u_m)$ as follows:

$$\Delta\colon D(\Delta) \subset L^2(\mathcal{G}) \to L^2(\mathcal{G}),$$

$$\Delta := \begin{pmatrix} \frac{d^2}{dx_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{d^2}{dx_m^2} \end{pmatrix}, \qquad \Delta u := \left(\frac{d^2}{dx_1^2} u_1, \dots, \frac{d^2}{dx_m^2} u_m \right). \tag{L}$$

The next step is to understand how the domain $D(\Delta)$ is defined. Clearly, since we are working with a differential operator in a network setting, we need to impose suitable boundary conditions at each junction and node. Since these constraints will involve values of functions at the vertices of $\mathcal{G}$, they are also referred as *vertex conditions*. In order to find their nature, first of all we suppose that the Laplacian acts on $\tilde{H}^2(\mathcal{G})$. Thanks to the Sobolev embedding and trace theorems in the 1D framework, we deduce that each u_j and their first derivatives are continuous, thus they admit well-defined values at the endpoints of each edge $e_j \in E$. On the other hand, this does not hold for second derivatives of functions in $H^2(0, \ell_j)$, which just belong to $L^2(0, \ell_j)$ and are not well-defined at the vertices. Then, the vertex conditions we are going to prescribe will involve only values of u_j and $\frac{du_j}{dx_j}$ at each vertex.

The most natural – and also simple – choice is to consider the so-called *standard vertex conditions*, which require that a function $u = (u_e)_{e \in E}$ on $\mathcal{G}$ is in $D(\Delta)$ if and only if

- u is *continuous* on $\mathcal{G}$, i.e. it does not admit discontinuity jumps at each vertex:

$$\forall v \in V: \qquad u_j(v) = u_h(v), \qquad \forall\, e_j,\, e_h \in E_v; \tag{1}$$

- u satisfies the *Kirchhoff condition* at each vertex, namely

$$\forall v \in V : \qquad \sum_{e_j \in E_v^-} \frac{du_j}{dx_j}(v) - \sum_{e_j \in E_v^+} \frac{du_j}{dx_j}(v) = 0, \tag{2}$$

where

$$E_v := \{e \in E : e \text{ is incident to } v\},$$

$$E_v^+ := \{e \in E : v \text{ is the terminal vertex of } e\},$$

$$E_v^- := \{e \in E : v \text{ is the initial vertex of } e\}.$$

In order to rewrite these conditions in a more compact form, we introduce the following $n \times m$ matrices:

$$\phi^+ = \left(\varphi_{ij}^+\right), \qquad \varphi_{ij}^+ := \begin{cases} 1 & \text{if } e_j \in E_{v_i}^+ \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi^- = \left(\varphi_{ij}^-\right), \qquad \varphi_{ij}^- := \begin{cases} 1 & \text{if } e_j \in E_{v_i}^- \\ 0 & \text{otherwise.} \end{cases}$$

Then, we define

$$\phi := \phi^+ - \phi^-,$$

which is the incidence matrix of the graph G, already introduced in Chapter 1. We are now able to express (1) and (2) in terms of these matrices. In fact, one easily notices that the continuity in (1) is equivalent to require that

$$\begin{gathered} \exists\, d \in \mathbb{C}^{\mathrm{n}} \ \text{ s.t. } (\phi^-)^\top d = \begin{pmatrix} u_1(0) \\ \vdots \\ u_m(0) \end{pmatrix} =: u(0) \\ \text{and} \\ (\phi^+)^\top d = \begin{pmatrix} u_1(\ell_1) \\ \vdots \\ u_m(\ell_m) \end{pmatrix} =: u(\ell) \end{gathered} \tag{Cc}$$

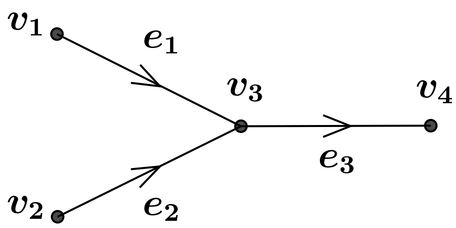
Additionally, we get a more efficient way of writing Kirchhoff condition, namely

$$\phi^+ \frac{d}{dx} u(\ell) - \phi^- \frac{d}{dx} u(0) = 0, \tag{Kc}$$

where with $\frac{d}{dx}u$ we mean that function defined on $\mathcal{G}$ whose components are given by the derivatives of u_js with respect to the spatial coordinate x_j.

Remark 3.2. Since these conditions are expressed in terms of matrices ϕ^+ and ϕ^- which clearly depend on the orientation of G, one might suspect that (Cc) and (Kc) do the same. However, this is not true: in fact, they remain unchanged if we choose a different orientation. To better understand it and how (Cc) and (Kc) work, we propose the following

Example 3.3. Let us consider a metric graph $\mathcal{G} = (G, \ell)$ where G is the finite and oriented graph represented by



with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3\}$. Looking at the picture, we can easily construct ϕ^+ and ϕ^- as

$$\phi^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \phi^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

thus

$$\phi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, applying condition (Cc) one obtains the expected formula

$$\exists\, d = (d_1, d_2, d_3, d_4)^\top \in \mathbb{C}^4 : \begin{cases} d_1 = u_1(0), \\ d_2 = u_2(0), \\ d_3 = u_3(0) \end{cases} \quad \text{and} \quad \begin{cases} d_3 = u_1(\ell_1), \\ d_3 = u_2(\ell_2), \\ d_4 = u_3(\ell_3), \end{cases} \tag{3.1}$$

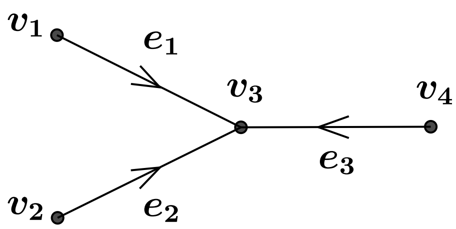
where expression (3.1) imposes in particular that the function's value d_3 at v_3 is uniquely determined.

After some easy computations, condition (Kc) gives

$$\begin{cases} \frac{d}{dx_1}u_1(0) = 0, \\ \frac{d}{dx_2}u_2(0) = 0, \\ -\frac{d}{dx_3}u_3(0) + \frac{d}{dx_1}u_1(\ell_1) + \frac{d}{dx_2}u_2(\ell_2) = 0, \\ \frac{d}{dx_3}u_3(\ell_3) = 0, \end{cases} \tag{3.2}$$

which is more intuitive than the compact form (Kc). First of all, this example let us remark that prescribing (Kc) at a vertex with degree 1 (as v_1, v_2, v_4) is equivalent to impose a Neumann condition. Secondly, looking at the image above and at (3.2), we can also understand the reason of the name *Kirchhoff*. In fact, if we interpret G as an electric circuit, then (3.2) is similar to the well-known *Kirchhoff's law* of Electrodynamics: at each node v, the total corrent density flowing into v has to be equal to the total current density flowing out of v.

If we now reverse the orientation of e_3, just considering the new coordinate $\tilde{x}_3 = \ell_3 - x_3$, we obtain a new oriented graph:



Obviously, ϕ^+ and ϕ^- are now different, but applying again the definitions of (Cc) and (Kc) one has

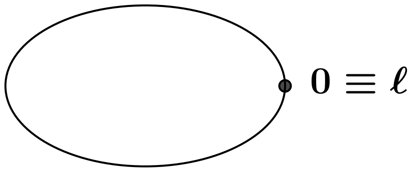
$$(Cc): \begin{cases} d_1 = u_1(0), \\ d_2 = u_2(0), \\ d_3 = u_1(\ell_1) = u_2(\ell_2) = u_3(\ell_3), \\ d_4 = u_3(0) \end{cases} \tag{3.3}$$

and

$$(Kc): \begin{cases} \frac{d}{dx_1}u_1(0) = 0, \\ \frac{d}{dx_2}u_2(0) = 0, \\ \frac{d}{dx_3}u_3(\ell_3) + \frac{d}{dx_1}u_1(\ell_1) + \frac{d}{dx_2}u_2(\ell_2) = 0, \\ \frac{d}{dx_3}u_3(0) = 0, \end{cases} \tag{3.4}$$

which basically represent the same relations with another parametrization. For instance, the value d_i at each vertex v_i remains unchanged and both third equations of (3.2) and (3.4) represent the sum of the exterior derivatives computed at v_3.

Remark 3.4. One can also wonder how these conditions are interpreted when we deal with loops. In order to explain it, take a graph composed by a single loop:



In this case, the initial and the terminal ends coincide, therefore if we apply (Cc) and (Kc), we get the periodic conditions

$$\begin{cases} u(0) = u(\ell), \\ \frac{d}{dx}u(0) = \frac{d}{dx}u(\ell). \end{cases}$$

Finally, once introduced the standard vertex conditions, the operator Δ given by (L) is more precisely called *Kirchhoff-Laplace operator* on $\mathcal{G}$ and its domain is defined as

$$D(\Delta) = \left\{ u \in \tilde{H}^2(\mathcal{G}) \mid u \text{ satisfies (Cc) and (Kc)} \right\}. \tag{3.5}$$

This is the first example of *quantum graph*, namely a metric graph, equipped with a differential operator and with suitable vertex conditions.

3.2.2 The continuous heat equation on a metric graph

As we have already said, we are interested in the operator Δ because it is the leading operator of the abstract Cauchy problem (ACP) describing a diffusion phenomenon on the network $\mathcal{G}$.

Suppose we want to study the continuous heat equation (CHE) on a fixed metric graph $\mathcal{G}$ with $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_m\}$. If we set all the conductance coefficients equal to 1 and we denote

$$\dot{u} := \frac{d}{dt} \quad \text{and} \quad u' := \frac{d}{dx},$$

where from now on x is the spatial coordinate for each edge, we get the following parabolic problem:

$$\begin{cases} \dot{u}_j(t,x) = u_j''(t,x), & t \geq 0, \quad x \in (0,\ell_j), \quad j = 1,\dots,m, \\ u_j(t,v_i) = u_h(t,v_i), & t \geq 0, \quad j,\, h \in E_{v_i}, \quad i = 1,\dots,n, \\ \sum_{e\in E_{v_i}^+} u_e'(v_i) - \sum_{e\in E_{v_i}^-} u_e'(v_i) = 0, & t \geq 0, \quad i = 1,\dots,n, \\ u_j(0,x) = f_j(x), & x \in (0,\ell_j), \quad j = 1,\dots,m. \end{cases} \tag{CHE}$$

In system (CHE), $u_j(t,\cdot)$ represents the heat distribution along $e_j \in E$ at time t; in addition, we consider $f := (f_1,\dots,f_m)$ as initial heat distribution on $\mathcal{G}$, while as boundary conditions we impose (Cc) and (Kc). As in the combinatorial case, we can hide the space dependence and rewrite (CHE) as an abstract Cauchy problem:

$$\begin{cases} \dot{u}(t) = \Delta u(t), & t \geq 0 \\ u(0) = f, \end{cases} \tag{ACP}$$

where the leading operator is exactly the Kirchhoff-Laplace operator of the metric graph $\mathcal{G}$ and the boundary conditions in (CHE) are contained in $D(\Delta)$. Now our goal is to study in detail (ACP), proving that it is well-posed in $L^2(\mathcal{G})$ and to analyze the asympotic behaviour of its solution.

We start introducing the following sesquilinear form on $L^2(\mathcal{G})$:

$$\begin{aligned} &a\colon V \times V \to \mathbb{C} \\ &a(f,g) = \sum_{j=1}^{m} \int_0^{\ell_j} f_j'(x)\overline{g_j'(x)}dx, \qquad f,g \in V, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} V = D(a) &:= \left\{ f \in \tilde{H}^1(\mathcal{G}) \mid \exists\, d \in \mathbb{C}^{\mathrm{n}} \text{ s.t. } (\phi^-)^\top d = f(0) \text{ and } (\phi^+)^\top d = f(\ell) \right\} \\ &= \left\{ f \in \tilde{H}^1(\mathcal{G}) \mid f \text{ satisfies (Cc)} \right\}. \end{aligned} \tag{3.7}$$

Since for $j = 1,\dots,m$, f_j and $\overline{g_j}$ are in $H^1(0,\ell_j)$, then their derivatives belong to $L^2(0,\ell_j)$, thus the product $f_j'\overline{g_j'} \in L^1(0,\ell_j)$ itself and each integral in (3.6) is finite. Hence, a is well defined and satisfies additionally several properties, which we are going to show following the techniques of [**20**].

Lemma 3.5. *The sesquilinear form a is densely defined in $L^2(\mathcal{G})$ and its domain V is closed in $\tilde{H}^1(\mathcal{G})$, in particular V is a Hilbert space with scalar product induced by $\tilde{H}^1(\mathcal{G})$.*

Proof. First of all, since $\tilde{H}^1(\mathcal{G})$ is a subspace of $L^2(\mathcal{G})$, it follows that V is a subspace of $L^2(\mathcal{G})$. We notice that $\bigoplus_{j=1}^m C_c^\infty(0,\ell_j)$ is contained in V and then, since it is also dense in $L^2(\mathcal{G})$, the density of V follows.

Regarding its closedness, we just recall that for every $j = 1, \dots, m$ and $x \in (0,\ell_j)$ the evaluation function from $H^1(0,\ell_j)$ to $\mathbb{C}$ defined as $\{f_j \mapsto f_j(x)\}$ is linear and bounded. Hence, one has just to take a sequence in V convergent in $\tilde{H}^1(\mathcal{G})$: then, due to the continuity of the evaluation functions at the involved vertices, it is easy to show that the limit still fulfills (Cc). $\square$

The previous lemma allows us to study this evolution problem in terms of the theory on sesquilinear forms, based on results in Chapter 1 of [**24**]. In this case, we denote

$$(H, \|\cdot\|_H) = \big(L^2(\mathcal{G}), \|\cdot\|_{L^2(\mathcal{G})}\big) \quad \text{ and } \quad (V, \|\cdot\|_V) = (V, \|\cdot\|_{\tilde{H}^1(\mathcal{G})}),$$

which are both separable Hilbert spaces with V continuously and densely embedded in H. At this point, in the following lemma we are going to show some properties statisfied by the sesquilinear form a.

Lemma 3.6. *Let a be the sesquilinear form defined in* (3.6)-(3.7)*. Then a is*

(i) symmetric, i.e. $a(f,g) = \overline{a(g,f)}$, $\forall f, g \in V$;

(ii) accretive, in particular positive, i.e. Re $a(f,f) = a(f,f) \geq 0$, $\forall f \in V$;

(iii) continuous, i.e. $\exists M > 0 : \ |a(f,g)| \leq M\|f\|_V\|g\|_V$, $\forall f, g \in V$;

(iv) closed, i.e. V is complete with respect to $\|f\|_a := \sqrt{a(f,f) + \|f\|_V^2}$;

(v) of Lions type, i.e. $\exists M > 0 : \ |Im\ a(f,f) \leq M\|f\|_V\|f\|_H$, $\forall f \in V$;

(vi) H-elliptic, i.e. $\exists \mu > 0$ *and* $\exists \omega \in \mathbb{R} :$ *Re* $a(f,f) + \omega\|f\|_H^2 \geq \mu\|f\|_V^2$, $\forall f \in V$.

Proof. (i) Due to the symmetry of the L^2-scalar product, for all $f,\ g \in V$ we have

$$\overline{a(g,f)} = \sum_{j=1}^m \int_0^{\ell_j} \overline{g_j'(x)} f_j'(x)dx = \sum_{j=1}^m \int_0^{\ell_j} f_j'(x)\overline{g_j'(x)}dx = a(f,g).$$

(ii) Thanks to the symmetry of a, it follows that $a(f,f) \in \mathbb{R}$ for every $f \in V$ and accretivity and positivity are, thus, the same property. One then easily obtains that

$$a(f,f) = \sum_{j=1}^m \int_0^{\ell_j} |f_j'(x)|^2 dx \geq 0, \qquad \forall f \in V.$$

(iii) Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|a(f,g)| &= \left| \sum_{j=1}^{m} \int_0^{\ell_j} f_j'(x)\overline{g_j'(x)}dx \right| \\
&\le \sum_{j=1}^{m} \left| \int_0^{\ell_j} f_j'(x)\overline{g_j'(x)}dx \right| \\
&\le \sum_{j=1}^{m} \|f_j'\|_{L^2(0,\ell_j)} \ \|g_j'\|_{L^2(0,\ell_j)} \\
&\le \sum_{j=1}^{m} \|f_j\|_{H^1(0,\ell_j)} \ \|g_j\|_{H^1(0,\ell_j)} \\
&\le \left(\sum_{j=1}^{m} \|f_j\|^2_{H^1(0,\ell_j)} \right)^{\frac{1}{2}} \left(\sum_{j=1}^{m} \|g_j\|^2_{H^1(0,\ell_j)} \right)^{\frac{1}{2}} \\
&= \|f\|_V \|g\|_V, \qquad \forall f, g \in V.
\end{aligned}$$

(iv) Due to the fact that $\|\cdot\|_a = \|\cdot\|_V$, the closedness of a follows as a trivial consequence.

(v) Since $a(f,f) \in \mathbb{R}$, the property is trivially satisfied due to the posivity of norms:

$$\forall M > 0 : \qquad 0 \le M\|f\|_V\|f\|_H, \qquad f \in V.$$

(vi) Taking into account that

- $\|f\|_V^2 = \|f\|_H^2 + a(f,f)$,
- Re $a(f,f) = a(f,f)$,

H-ellipticity's inequality

$$a(f,f) + \omega\|f\|_H^2 \ge \mu\|f\|_V^2 = \mu \ a(f,f) + \mu\|f\|_H^2$$

holds for $1 \ge \mu > 0$ and for $\omega \ge \mu > 0$. For instance, taking $\mu = \omega = 1$ we get an equality.

□

Definition 3.7. Take a general sesquilinear form $a : V \times V \to \mathbb{C}$. We define the *operator* $(A, D(A))$ *associated with* a as follows:

$$\begin{aligned}
&D(A) = \{f \in V \mid \exists h \in H \text{ such that } a(f,g) = (h,g)_H, \ \forall g \in V\}, \\
&Af = -h.
\end{aligned}$$

Notice that due to the density of V in H, the operator associated with a is univocally defined.

Lemma 3.8. *The operator associated with a is the Laplacian Δ with standard vertex conditions.*

Proof. To prove this lemma we follow the proof of Lemma 2.3 in [**20**] in the simpler case $\mu_j = c_j \equiv 1$ for all $j = 1, \dots, m$.

Based on Definition 3.7, we have to show that $D(A) = D(\Delta) = D$ and $Af = \Delta f$ $\forall f \in D$. We start showing that $\Delta \subset A$. Let $f \in D(\Delta)$, then, integrating by parts, for all $g \in V$ we have

$$\begin{aligned} a(f,g) &= \sum_{j=1}^{m} \int_0^{\ell_j} f_j'(x)\overline{g_j'(x)}dx \\ &= \sum_{j=1}^{m} \Big[\, f_j'(x)\overline{g_j(x)}\,\Big]_0^{\ell_j} - \sum_{j=1}^{m} \int_0^{\ell_j} f_j''(x)\overline{g_j(x)}dx. \end{aligned} \tag{3.8}$$

Observing the first term of (3.8), we notice that to evaluate each f_j' and $\overline{g_j}$ at 0 and ℓ_j means to compute their value at the endpoints of each edge $e_j \in E$. Then, we can rewrite the first part in terms of ϕ^+ and ϕ^- as

$$\Big[\, f_j'(x)\overline{g_j(x)}\,\Big]_0^{\ell_j} = \sum_{j=1}^{m}\sum_{i=1}^{n}(\varphi_{ij}^+ - \varphi_{ij}^-)f_j'(v_i)\overline{g_j(v_i)}.$$

In this way, now it is more convenient to use (Cc) satisfied by $g \in V$. In fact, since the continuity at the vertices holds, we can reformulate

$$\exists\ d \in \mathbb{C}^{\mathrm{n}} \text{ such that } (\phi^-)^\top d = g(0) \text{ and } (\phi^+)^\top d = g(\ell)$$

and say that there exist $d_1, \dots, d_n \in \mathbb{C}$ such that $g_j(v_i) = d_i$, $j \in E_i$, $i = 1, \dots, n$. Then, $\forall g \in V$ we get

$$\begin{aligned} a(f,g) &= \sum_{i=1}^{n} \overline{d_i} \underbrace{\sum_{j=1}^{m}(\varphi_{ij}^+ - \varphi_{ij}^-)f_j'(v_i)}_{=0} - \sum_{j=1}^{m} \int_0^{\ell_j} f_j''(x)\overline{g_j(x)}dx \\ &= -(\Delta f, g)_H, \end{aligned} \tag{3.9}$$

where the first term vanishes since f fulfills (Kc). Notice that the equality (3.9) makes sense because $\Delta f \in H$ and proves that $\Delta \subset A$.

Now, we start from a function $f \in D(A)$; then by definition there exists $h \in H$ such that

$$\sum_{j=1}^{m} \int_0^{\ell_j} f_j'(x)\overline{g_j'(x)}dx = a(f,g) = (h,g)_H = \sum_{j=1}^{m} \int_0^{\ell_j} h_j(x)\overline{g_j(x)}dx, \tag{3.10}$$

for all $g \in V$. Let us consider the following function

$$g^j := \begin{pmatrix} 0 \\ \vdots \\ g_j \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{th row}, \qquad g_j \in H_0^1(0,\ell_j),\ j = 1,\dots,m.$$

We notice that $h^j \in V$ for all $j = 1,\dots,m$ and hence we can use it as a test function in (3.10)

$$\int_0^{\ell_j} f_j'(x)\overline{g_j'(x)}dx = \int_0^{\ell_j} h_j(x)\overline{g_j(x)}dx, \qquad g_j \in H_0^1(0,\ell_j),\ j = 1,\dots,m. \tag{3.11}$$

In particular, (3.11) holds for all $g_j \in C_c^\infty(0,\ell_j)$; then we deduce that $-h_j \in L^2(0,\ell_j)$ is the weak derivative of $f_j' \in L^2(0,\ell_j)$. This implies that $f_j' \in H^1(0,\ell_j)$ for all $j = 1,\dots,m$, thus $f \in \tilde{H}^2(\mathcal{G})$. Now, we just have to integrate by parts (3.10) as done for (3.8) and for all test functions we still obtain

$$\sum_{i=1}^{n} \overline{d_i} \sum_{j=1}^{m} (\varphi_{ij}^+ - \varphi_{ij}^-) f_j'(v_i) = 0, \tag{3.12}$$

where the right-hand side of (3.12) is 0 thanks to the definition of weak derivative of f_j' and d_i is the joint value attained by g_j at the vertex v_i, for all $e_j \in E_i$. Due to the arbitrariness of $g \in V$, it follows that

$$\sum_{j=1}^{m} (\varphi_{ij}^+ - \varphi_{ij}^-) f_j'(v_i) = 0$$

and so f satisfies (Kc), too. In this way, we have proved that $f \in D(\Delta)$ and for all $g \in V$:

$$-(\Delta f, g)_H = -\sum_{j=1}^{m} \int_0^{\ell_j} f_j''(x)\overline{g_j(x)}dx = \sum_{j=1}^{m} \int_0^{\ell_j} h(x)\overline{g_j(x)}dx.$$

This implies that $\Delta f = -h = Af$ and so the proof is complete. □

The lemma we have just proved is fundamental, because it let us associate the operator Δ with the sesquilinear form a. In this way, we are now able to apply the whole theory of operators associated with sesquilinear forms reported in [**24**], where in particular Proposition 1.22 states the following result.

Proposition 3.9. *If A is the operator associated with a densely defined, accretive, continuous and closed sesquilinear form a, then A is densely defined and for every $\lambda > 0$, the operator $\lambda I - A$ is invertible (from $D(A)$ into H).*

As a consequence, we deduce that

Corollary 3.10. *The Kirchhoff-Laplace operator Δ is densely defined in H and $\lambda I - \Delta$ is invertible for every $\lambda > 0$.*

At this point, we define the *adjoint* a^* of a sesquilinear form a as

$$a^*(f, g) := \overline{a(g, f)}, \qquad f, g \in V = D(a) = D(a^*).$$

By some easy computations, one can check that a^* is densely defined, accretive, continuous and closed, provided that so is a. In order to find the operator associated with a^*, we prove the following

Lemma 3.11. *The operator associated with a^* is A^*, i.e. the adjoint of the operator associated with a.*

Proof. We follow the argument of Proposition 1.24 in [**24**].

First of all, we recall that the adjoint A^* of an operator A is defined as follows

$$\begin{aligned} &D(A^*) = \{f \in H \mid \exists h \in H \text{ such that } (Ag, f)_H = (g, h)_h, \ \forall g \in D(A)\} \\ &A^* f = h. \end{aligned}$$

Now, denoting by B the operator associated with a^*, our goal is to prove that $D(B) = D(A^*)$ and $Bf = A^*f$ for all $f \in D(A^*)$. We start considering $f \in D(B)$, then by definition for all $g \in V$ we obtain

$$(-Bf, g)_H = a^*(f, g) = \overline{a(g, f)} = \overline{(-Ag, f)_H} = (f, -Ag)_H = (-A^*f, g).$$

Thus, $f \in D(A^*)$, namely $D(B) \subset D(A^*)$ and $Bf = A^*f$ for all $f \in D(B)$.

On the other hand, take $f \in D(A^*)$ and consider $(I - A^*)f \in H$. By Proposition 3.9, $(I - B)$ is surjective, thus $\exists g \in D(B)$ such that

$$(I - A^*)f = (I - B)g.$$

Since $A^* = B$ in $D(B)$, we obtain

$$(I - A^*)f = (I - A^*)g$$

and, hence, for all $h \in D(A)$:

$$(f - g, (I - A)h)_H = ((I_A^*)(f - g), h)_H = 0.$$

Due to the invertibility of $(I - A)$ and the density of $D(A)$ in H, we get

$$(f - g, h)_H = 0 \qquad \forall h \in H,$$

thus $f = g$ and the thesis follows. $\square$

Corollary 3.12. *The Kirchhoff-Laplace operator Δ on $\mathcal{G}$ is self-adjoint.*

Proof. The proof is a simple consequence of symmetry of a. In fact, for all $f, g \in V$, one gets

$$a^*(f, g) = \overline{a(g, f)} = a(f, g).$$

Since $a = a^*$, then their associated operators agree with each other too:

$$(\Delta, D(\Delta)) = (\Delta^*, D(\Delta^*)).$$

$\square$

From this corollary, it follows that

Corollary 3.13. *The Kirchhoff-Laplace operator Δ is closed, i.e. its graph $G(\Delta) = \{(f, \Delta f) \in D(\Delta) \times H\}$ is closed in $H \times H$.*

Proof. This follows from the fact that a linear densely defined operator A on H has its adjoint A^* closed. In particular, every self-adjoint operator is closed.

In fact, we only need to consider the operator matrix

$$U := \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix},$$

which is unitary on $H \times H$ (i.e. U is invertible and $U^{-1} = U^*$). Then, denoting by $\big(\cdot \mid \cdot\big)$ and $(\cdot, \cdot)_H$ the scalar products on $H \times H$ and on H respectively, for all $f \in D(A)$ and

$g \in D(A^*)$ we get

$$\begin{aligned}\big((g, A^*g) \mid U(f, Af)\big) &= \big((g, A^*g) \mid (Af, -f)\big) \\ &= (g, Af)_H - (A^*g, f)_H \\ &= (g, Af)_H - (g, Af)_H = 0,\end{aligned}$$

hence, this implies that the $G(A^*) \subset U(G(A))^{\perp}$.

On the other hand, take $(g, h) \in U(G(A))^{\perp}$, then for all $f \in D(A)$ the vectors (g, h) and $U(f, Af)$ are orthogonal, thus

$$0 = \big((g, h) \mid (Af, -f)\big) = (g, Af)_H - (g, h)_H.$$

Therefore, we obtain $(Af, y)_H = (f, h)_H$, and, due to the density of $D(A)$ in H, $h = A^*g$, $(g, h) = (g, A^*g) \in G(A^*)$ and $G(A^*) \supset U(G(A))^{\perp}$. Since $U(G(A))^{\perp}$ is an orthogonal subspace, it follows that $G(A^*)$ is closed. $\square$

In order to finally prove the well-posedness of (ACP), we need to establish one last property fulfilled by Δ, namely

Corollary 3.14. *The Kirchhoff-Laplace operator Δ on $\mathcal{G}$ is dissipative, i.e.*

$$Re(\Delta f, f) \leq 0, \ \forall f \in D(\Delta)$$

.

Proof. We just have to use the accretivity (which is equivalent to the positivity in this case) of the sesquilinear form a. In fact

$$0 \leq a(f, f) = \mathrm{Re}\ a(f, f) = -\mathrm{Re}(\Delta f, f)_H,$$

for every $f \in D(\Delta)$. Thus,

$$\mathrm{Re}(\Delta f, f)_H \leq 0$$

follows. $\square$

We are now able to apply the Lumer-Phillips Theorem, namely[1]

Theorem 3.15 (Lumer-Phillips)**.** *For a densely defined, closed, dissipative operator $(A, D(A))$ on a Banach space X the following statements are equivalent.*

(i) A generates a contraction semigroup.

[1]We follow the text book [**13**].

(ii) rg $(\lambda I - A)$ *is dense in* X *for some (hence all)* $\lambda > 0$.

In this context, $X = H$ and we are dealing with the operator $(\Delta, D(\Delta))$ which is:

– densely defined,

– dissipative,

– closed

and additionally

– for every $\lambda > 0$ the operator $(\lambda I - \Delta)$ is surjective, hence its range is clearly dense in H.

Thus, by the Lumer-Phillips Theorem, it follows that

Theorem 3.16. *The Kirchhoff-Laplace operator generates a contractive strongly continuous semigroup, denoted by* $\left\{e^{t\Delta}\right\}_{t\geq 0}$.

Based on the good properties satisfied by Δ, we expect that its associated (ACP) is well-posed. The relation between the well-posedness of an abstract Cauchy problem associated with an operator A and the generation of a semigroup by A is well-known. For completeness, we recall the fundamental Theorem 6.7 in [**13**].

Theorem 3.17. *Let* X *be a Banach space and let* $A\colon D(A) \subset X \to X$ *be a closed operator. For the associated abstract Cauchy problem*

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = f \end{cases} \tag{ACP}$$

we consider the following existence and uniqueness condition:

$$\begin{gathered} \textit{for every } f \in D(A), \textit{ there exists a unique} \\ \textit{classical solution } u(\cdot, x) \textit{ of (ACP),} \end{gathered} \tag{EU}$$

namely a continuously differentiable function $u\colon \mathbb{R}^+ \to X$*, such that* $u(t) \in D(A)$ *for all* $t \geq 0$ *and (ACP) holds.*

Then the following are equivalent.

(i) *A generates a strongly continuous semigroup.*

(ii) (ACP) is well-posed, i.e. A satisfies (EU) and continous dependence on the data holds, i.e. for every sequence $\{f_n\}_{n\geq 0} \subset D(A)$ *satisfying* $\lim_{n\to\infty} f_n = 0$ *one has* $\lim_{n\to\infty} u(t, f_n) = 0$ *uniformly in compact intervals* $[0, t_0]$*. Additionally, A has dense domain.*

Thus, we finally obtain

Theorem 3.18 (Well-posedness of (ACP)). *The abstract Cauchy problem*

$$\begin{cases} \dot{u}(t) = \Delta u(t), \quad t \geq 0 \\ u(0) = f \end{cases} \tag{ACP}$$

is well-posed, i.e. for every $f \in D(\Delta)$ *there exists a unique (classical) solution*

$$u(t) := e^{t\Delta} f, \qquad \forall t \geq 0.$$

Moreover, continuous dependence on the data holds.

Once found the solution of (ACP), the next goal is to analyze its asymptotic behaviour as $t \to +\infty$ as we have done in the finite-dimensional settings. In fact, the idea is still to interpret the convergence results in terms of the structure of the metric graph. The positive fact is that we will find a similar situation as in the case of the discrete heat equation studied on a combinatorial graph.

3.2.3 On the spectrum of the Kirchhoff-Laplace operator

Before talking about long-time behaviour of the studied evolution model, we first need to briefly present how the spectrum of the Kirchhoff-Laplace operator is composed. In fact, as in the combinatorial framework we frequently applied the spectral decomposition of the semigroup generated by the leading operator (the minus discrete Laplacian), we will see also here that the knowledge of some main spectral properties of Δ will be helpful.

In order to recall those information about the spectrum $\sigma(\Delta)$ which are necessary for our research, we will mainly refer to [**20**].

Lemma 3.19. *The spectrum of* $(\Delta, D(\Delta))$ *is purely discrete, i.e. consists of real eigenvalues only. Moreover, its eigenvalues are negative and* $s(\Delta) = 0 \in \sigma(\Delta)$*, where*

$$s(\Delta) = \sup\{|\lambda| : \lambda \in \sigma(\Delta)\}$$

is the spectral radius.

Proof. We follow the proof of Lemma 4.1 in [**20**].

First of all, the function $\mathbb{1} = (1, \dots, 1)$ which is identically equal to 1 on each edge of $\mathcal{G}$ is in $D(\Delta)$ and $\Delta \mathbb{1} = 0$. This means that Δ is not invertible and thus $0 \in \sigma(\Delta)$. Moreover, let us suppose that Δ is a dissipative operator:

$$\mathrm{Re}(\Delta f, f)_H \leq 0 \qquad \forall f \in D(\Delta).$$

We take λ eigenvalue of Δ with eigenfunction f, then by dissipativity we get

$$\mathrm{Re}(\Delta f, f)_H = (\Delta f, f)_H = \lambda (f, f)_H \leq 0$$

and due to the positivity of the scalar product $(\cdot, \cdot)_H$ it follows that $\lambda \leq 0$. Hence any eigenvalue of Δ is negative.

Furthermore, one can prove (see [**20**] and [**13**]) that the resolvent of Δ is compact and this implies that the Kirchhoff-Laplace operator has only point spectrum. Additionally, being Δ also self-adjoint and given f eigenfunction associated with some eigenvalue λ, by sesquilinearity we obtain

$$\lambda (f, f)_H = (\Delta f, f)_H = (f, \Delta f)_H = \overline{\lambda} (f, f)_H,$$

thus $\lambda = \overline{\lambda}$ and we conclude that all the eigenvalues of Δ are real. $\square$

The study of the Laplacian's spectrum is still an open problem: right now, only in few cases it is fully determined and in general just some upper and lower bounds of eigenvalues are known. In this work, we are going to remark only one property of $\sigma(\Delta)$, which is also true in the combinatorial framework and for this reason frequently used in Chapter 2. However, before that we want to introduce an important class of functions. Let us consider a metric graph with the combinatorial graph G as support. In general, G is given by the disjoint union of $G^{(h)} = (V^{(h)}, E^{(h)})$, $h = 1, \dots, l$ connected components. Then we define

$$\mathbb{1}_h = \begin{cases} 1 & \text{if } x \in e_j,\ e_j \in E^{(h)}, \\ 0 & \text{otherwise,} \end{cases}$$

for all $h = 1, \dots, l$. Notice that these functions defined on each edge $e_j,\ j = 1, \dots, m$ are in $\tilde{H}^2(\mathcal{G})$ and satisfy (Cc) and (Kc): thus, they all belong to $D(\Delta)$.

Proposition 3.20. *Let $\mathcal{G} = (G, \ell)$ be a finite metric graph with support $G = (V, E)$. Assume that G is given by the disjoint union of $G^{(h)} = (V^{(h)}, E^{(h)})$, $h = 1, \dots, l$ connected components.*

Then, the multiplicity of 0 *as eigenvalue of the Kirchhoff-Laplace operator agrees with* *the number of connected components of the metric graph $\mathcal{G}$. In particular, a possible basis of* $\ker\Delta$ *is given by the piecewise constant functions* $\{\mathbb{1}_h\}_{h=1}^{l}$.

Proof. We have expanded Case 2 of the proof of Theorem 4.3 in [**20**].

We want to solve the equation

$$\Delta f = 0, \qquad f \in D(\Delta).$$

In this simple case, it is easy to see that the eigenfunctions corresponding to the eigenvalue 0 has the form

$$\begin{aligned} f(x) &= \begin{pmatrix} f_1(\ell_1) \\ \vdots \\ f_m(\ell_m) \end{pmatrix} = \begin{pmatrix} a_1 x_1 + b_1 \\ \vdots \\ a_m x_m + b_m \end{pmatrix} = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \\ &= Ax + b, \end{aligned}$$

where $a_j, b_j \in \mathbb{C},\ j = 1, \dots, m$. We can now use (Cc) and then, evaluating f in each vertex, for some $d \in \mathbb{C}^{\mathrm{n}}$ we get

$$\begin{aligned} f(0) &= b = (\phi^-)^\top d \\ f(\ell) &= \begin{pmatrix} f(\ell_1) \\ \vdots \\ f(\ell_m) \end{pmatrix} = A \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_m \end{pmatrix} + (\phi^-)^\top d = (\phi^+)^\top d, \end{aligned}$$

then

$$\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = L\phi^\top d,$$

where $L := \operatorname{diag}(\frac{1}{\ell_1}, \dots, \frac{1}{\ell_m})$. Hence, the eigenfunction f has to be of the form

$$f(x) = (\phi^-)^\top d + x^\top L \phi^\top d.$$

Applying (Kc), it follows that

$$\phi^+ L \phi^\top d = \phi^- L \phi^\top d \iff \phi L \phi^\top d = 0,$$

i.e. $d \in \ker\left(\phi L \phi^\top\right)$. The $n \times n$ matrix

$$\phi L \phi^\top$$

is known as the discrete Laplacian of the weighted oriented graph G: its kernel is l-dimensional (where l is the number of connected component of G) and it is spanned by functions $\varphi_h \in \mathbb{C}^{\mathrm{n}},\ h = 1, \ldots, l$ defined on each vertex v_i as follows

$$(\varphi_h)_i := \begin{cases} 1 & \text{if } v_i \in V^{(h)} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{\varphi_h\}_{h=1}^{l}$ is a basis of $\ker\left(\phi L \phi^\top\right)$, then $\{f_h\}_{h=1}^{l}$ forms a basis for $\ker \Delta$, where

$$f_h(x) = (\phi^-)^\top \varphi_h + x^\top L \phi^\top \varphi_h.$$

This proves that the multiplicity of 0 as eigenvalue of Δ is actually equal to l. Moreover, after some easy computations, one can find that

$$f_h \equiv \mathbb{1}_h, \qquad \forall h = 1, \ldots, l.$$

Conversely, by linearity we see that for all $h = 1, \ldots, l$, $\mathbb{1}_h$ also exhausts all possible eigenfunctions of $\lambda = 0$ and the thesis is complete. □

3.2.4 Asympotics

Having introduced some properties related to the spectrum of the Kirchhoff-Laplace operator, the idea is now to apply the Spectral Theorem for (possibly unbounded) self-adjoint operators (see, for instance, Theorem 7.1 in [**22**]) and its corollaries to our framework.

First of all, we check that V is not only continuously and densely, but also compactly embedded in H. To show it, we just need to apply Rellich's Theorem, which in particular provides that for all $j = 1, \ldots, m$, $H^1(0, \ell_j)$ is compactly embedded in $L^2(0, \ell_j)$. From this result, one can easily prove the same for their cartesian products and deduce that $\tilde{H}^1(\mathcal{G})$ is also compactly embedded in $L^2(\mathcal{G}) = H$. As a consequence, being V a subspace of $\tilde{H}^1(\mathcal{G})$, we conclude that so does V.

At this point, we are in position to apply Corollary 7.3 of [**22**], namely

Corollary 3.21. *The following statements hold.*

(i) There exists an orthonormal basis $\{e_k,\ k \geq 1\}$ of H, composed by eigenfunctions of Δ;

(ii) for all $k \geq 1$ the eigenspaces associated with the eingenvalue λ_k of Δ is finite-dimensional;

(iii) denoting the associated eigenvalues $\lambda_k \in \mathbb{R}^-$ by

$$0 = \lambda_1 \leq |\lambda_2| \leq \ldots \leq |\lambda_k| \leq \ldots,$$

then they can only accumulate at infinity:

$$\lim_{k \to +\infty} |\lambda_k| = +\infty.$$

(iv) Δ has the following spectral representation

$$D(\Delta) = \left\{ f \in H : \sum_{k=1}^{+\infty} \lambda_k^2 \ (f, e_k)_H^2 < +\infty \right\},$$

$$\Delta f = \sum_{k=1}^{+\infty} \lambda_k \ P_k f, \qquad f \in D(\Delta),$$

where P_k is the orthogonal (with respect $(\cdot,\cdot)_H$) projection onto the subspace spanned by the eigenvector e_k

$$P_k f = (f, e_k)_H \ e_k, \qquad f \in H.$$

As remarked in [**3**], the orthonormal basis $\{e_k,\ k \geq 1\}$ is clearly not unique. However, given any eigenfunction $g \in D(\Delta)$, i.e. $\Delta g = \lambda g$ for some eigenvalue λ, then g belongs to $\operatorname{span}\{e_k,\ \lambda_k = \lambda\} = \ker(\Delta - \lambda)$. This also means that λ_k is actually the kth eigenvalue of Δ counting the multiplicity. Regarding the eigenvalue $\lambda = 0$, we can construct one possible orthonormal basis of $\ker \Delta$ just by normalizing the basis $\{\mathbb{1}_h,\ h = 1, \ldots, l\}$, provided that the graph is composed by l connected components according to the hypotheses in Proposition 3.20. After easy computations, it follows that

$$e_h := \left(\sum_{j : e_j \in E^{(h)}} \ell_j \right)^{-\frac{1}{2}} \mathbb{1}_h, \qquad h = 1, \ldots, l. \tag{3.13}$$

In the next theorem, we are going to report a result of Corollary 7.5 in [**22**] and in particular to generalize Corollary 5.2 of [**20**] (where the authors propose only the case of a connected metric graph). As a consequence of the results in Corollary 3.21, one can write

$$e^{t\Delta} f = \sum_{k=1}^{+\infty} e^{t\lambda_k} \ (f, e_k)_H \ e_k, \qquad f \in H$$

and finally prove the following

Theorem 3.22 (**Asymptotic behaviour of (ACP)**). *Let P be the orthogonal projection onto the eigenspace of Δ associated with $\lambda = 0$, namely the orthogonal projection onto the kernel of the Kirchhoff-Laplace operator. Then the following hold:*

1. *The limit $Pf := \lim_{t\to+\infty} e^{t\Delta} f$ exists for every $f \in H$;*
2. *For every $\epsilon > 0$, there exists a constant $M > 0$ such that*
$$\|e^{t\Delta} - P\|_{\mathcal{L}(H)} \leq M e^{(\lambda_2+\epsilon)t}, \qquad \forall t \geq 0,$$
where λ_2 is the largest nonzero eigenvalue of the generator Δ.

In other words, the previous theorem states that the semigroup generated by the Kirchhoff-Laplace operator converges strongly and also exponentially uniformly to the eigenprojection P. As we expected, we have found the same result as in the combinatorial case, even if the framework is now more complicated.

Theorem 3.22 determines also the convergence rate of the semigroup towards P, which agrees with the absolute value of the second largest eigenvalue λ_2 of Δ. As pointed out in [**20**], it is important to remark that the convergence to equilibrium holds regardless of the intrisic structure of the graph on which we are studying the heat diffusion. On the other hand, since there exist estimates of the largest nonzero eigenvalue (for instance, see [**21**]) which depend on some properties of combinatorial graphs, the speed of convergence is directly related to the network.

Anyway, despite the convergence towards equilibrium always holds in this evolution model, the projection P obviously varies depending on the connectivity of the graph. For this reason, we want to conclude this chapter by analyzing in detail what P represents both in the cases of a connected and a disconnected graph.

Suppose we are studying the heat diffusion on a finite metric graph $\mathcal{G} = (G, \ell)$. Then, we can consider two different cases.

1. When G is connected, then 0 is a single eingenvalue for Δ. As a consequence, the kernel of the Kirchhoff-Laplacian is one-dimensional and composed by all the possible functions constant on the edges and, since (Cc) is satisfied, on the whole graph too. In particular $\ker \Delta$ can be spanned by the single constant eigenfunction
$$e_1 = \frac{1}{\left(\sum_{j=1}^{m} \ell_j\right)^{\frac{1}{2}}}\,(1, \ldots, 1).$$

When we apply P to an initial datum $f \in D(\Delta)$, we obtain

$$Pf = (f, e_1)_H \; e_1 = \frac{1}{\sum_{j=1}^m \ell_j} \sum_{j=1}^m \int_0^{\ell_j} f_j(x)dx \; (1, \dots, 1).$$

We notice that the constant function Pf can be interpreted as an average of the initial datum f computed on the graph $\mathcal{G}$. Then, also in the metric case we can denote

$$\fint_{\mathcal{G}} f := \frac{1}{\sum_{j=1}^m \ell_j} \sum_{j=1}^m \int_0^{\ell_j} f_j(x)dx$$

and it turns out that P agrees with the orthogonal projection onto the constant functions on all the edges of the graph. For resemblance with the combinatorial setting, we decide to denote this projection by P_0.

As one can expect knowing the classical results about the heat equation on finite one-dimensional domains with Neumann boundary conditions, also in this case for long times the heat spreading will tend to the constant distribution

$$P_0 f = \left(\fint_{\mathcal{G}} f, \dots, \fint_{\mathcal{G}} f \right).$$

2. The situation is different if G is non connected. As usual, we can suppose that it is given by the disjoint union of connected components $G^{(1)}, \dots, G^{(l)}$. According to Proposition 3.20, 0 is an eigenvalue with multiplicity $\operatorname{mult}(0) = l$, thus an orthonormal basis for $\ker \Delta$ is given by the functions defined in 3.13. So, the kernel contains not only the constants on the whole graph, but also all those piecewise constant functions, which – due to (Cc) – have to attain the same value on each connected component of $\mathcal{G}$.

   In this case, the limit distribution agrees with

$$\begin{aligned} Pf &= \sum_{h=1}^{l} (f, e_h)_H \; e_h \\ &= \frac{1}{\sum_{j:j\in E^{(1)}} \ell_j} \sum_{j:j\in E^{(1)}} \int_0^{\ell_j} f(x)dx \; \mathbb{1}_1 + \ldots + \frac{1}{\sum_{j:j\in E^{(l)}} \ell_j} \sum_{j:j\in E^{(l)}} \int_0^{\ell_j} f(x)dx \; \mathbb{1}_l. \end{aligned} \tag{3.14}$$

   Expression (3.14) seems complicated to read, but each coefficient is actually the average of the initial datum f on the corresponding connected component

$$\fint_{\mathcal{G}^{(h)}} f := \frac{1}{\sum_{j:j\in E^{(h)}} \ell_j} \sum_{j:j\in E^{(h)}} \int_0^{\ell_j} f(x)dx, \qquad h = 1, \dots, l.$$

Then, in the case of a disconnected graph, the limit distribution of (ACP)'s solution is just the average of f on the connected components of $\mathcal{G}$

$$Pf(x) = \fint_{\mathcal{G}^{(h)}} f, \qquad x \in \mathcal{G}^{(h)}.$$

This result is reasonable: in fact, in such a model each connected component consists of one indipendent system where we can study a different heat evolution, starting from the distribution f restricted on $\mathcal{G}^{(h)}$, $h = 1, \dots, l$. Therefore, recalling the previous case of a connected graph, we expect that for long times each heat distribution on a different component $\mathcal{G}^{(h)}$ will converge towards the average of the intial datum on the same one.

Those illustrated in this chapter are the main results about one simple example of diffusion on a fixed metric graph. The next goal, as we have done in the combinatorial setting, is to study the continuous heat equation on a network which is no longer fixed in time.

Chapter 4

Random evolution on metric graphs

We are finally able to generalize the random evolution model to the framework of metric graphs. Studying the diffusion on a fixed metric graph $\mathcal{G}$ in Chapter 3, we have seen that its dynamics behaves for long times in a way similar to the corresponding combinatorial model. The heat spreads along the edges tending towards a homogeneous distribution, which depends on the connectivity of $\mathcal{G}$.

For this reason, we expect that all the results obtained in Chapter 2 still hold in an infinite-dimensional context. In fact, as long as some proofs are extended from the discrete to the continuous case, all the known results we have used (such as the techniques of [**1**] and [**12**]) are still valid.

In the end, we will obtain again a necessary and sufficient condition for the convergence of the evolution operator to P_0. However, while almost every definition and property can be naturally generalized, there is a concept which we will focus on, as it is not obvious to be treated in the metric setting: the *union* of two (or more) metric graphs. In fact, in the literature a general concept of union graph which we can refer to is not available. Therefore, we are going to introduce a reasonable definition of union when all the metric graphs share the same number of edges. In this way, we shall obtain the metric version of the characterization of Theorem 2.14.

4.1 Evolution problems with variable coefficients. Generalization to infinite dimension

As we have done in Chapter 2, we first propose a general non autonomous model, where the leading operator varies at every integer time.

Consider a finite family $\mathcal{K} = \{A_0, \ldots, A_{N-1}\}$ such that for all $i = 0, \ldots, N-1$

$$A_i \colon D(A_i) \subset X \to X$$

is a linear operator on a Banach space X. We also assume that every A_i generates a strongly continuous semigroup $\{e^{tA_i}\}_{t\geq 0}$. All the results about the well-posedness of the abstract Cauchy problem

$$\begin{cases} \dfrac{du}{dt}(t) = A_i u(t), & t \geq 0, \\ u(0) = u_0 \in X \end{cases} \tag{ACP$_i$}$$

are assumed to be known (see Theorem 3.17 and Chapter II.6 of [**13**]).

We are interested in analyzing the following non autonomous problem:

$$\begin{cases} \dfrac{d^+u}{dt}(t) = A_{j_k} u(t), & t \in [k, k+1),\ \ k \in \mathbb{N}, \\ u(0) = u_0 \in X, \end{cases} \tag{4.1}$$

where $\{j_k\}_{k\geq 0}$ is a sequence taking values in $\{0, \dots, N-1\}$. Looking at the system (4.1), it is apparent that a classical solution is in general mot expected. In fact, unless we require that $u(k) \in D(A_{j_k})$, for all $k \geq 0$, there is no hope that a possible solution is differentiable even at each integer time k. Hence, as we have done in the finite-dimensional case, we can only weaken the concept of solution with the following

Definition 4.1. A continuous function $u\colon [0, +\infty) \to X$ is called *mild solution* of (4.1) if $u(0) = u_0 \in X$ and

$$u(t) = u_0 + \sum_{h=0}^{k-1} A_{j_h} \int_h^{h+1} u(s)ds + A_{j_k} \int_k^t u(s)ds, \qquad t \in [k, k+1],\ \ k \in \mathbb{N}. \tag{4.2}$$

Notice that (4.2) is well defined: in fact, since every $(A_{j_k}, D(A_{j_k}))$ generates a C_0-semigroup, it follows that

$$\int_h^{h+1} u(s)ds \in D(A_{j_h}),\ \forall h = 0, \dots, k-1 \qquad \text{and} \qquad \int_k^t u(s)ds \in D(A_{j_k}).$$

In addition, one can naturally generalize the proof of Theorem 2.3 to the current framework and, hence, prove that if problem (4.1) admits a mild solution, then this is also unique.

Following the guidelines of Chapter 2, we now introduce the evolution operator in the space of bounded operators $\mathcal{B}(X)$

$$S(t) := \begin{cases} e^{tA_{j_0}} & t \in [0,1], \\ e^{(t-k)A_{j_k}} S(k) & t \in [k, k+1], \end{cases} \qquad \forall t \geq 0, \tag{4.3}$$

which is still strongly continuous since so are $e^{tA_{j_k}}$ for all $k \geq 0$ and, also in this case, $\{S(t)\}_{t\geq 0}$ is no longer a semigroup. We are going to prove that

Theorem 4.2. *The continuous function given by*

$$t \mapsto S(t)u_0, \qquad \forall t \geq 0 \tag{4.4}$$

is the unique mild solution of (4.1).

Proof. Since we have already remarked that the uniqueness holds, we only need to verify that (4.4) satisfies (4.2) and the initial condition. The latter is clearly verified due to the property of the semigroup generated by A_{j_0}.

Then, for every $t > 0$ let $k \in \mathbb{N}$ such that $t \in (k, k+1]$ and consider the following abstract Cauchy problem

$$\begin{cases} \dfrac{du}{dt}(t) = A_{j_k}u(t), & t \in [k, k+1) \\ u(k) = e^{A_{j_{k-1}}}e^{A_{j_{k-2}}} \cdots e^{A_{j_0}}u_0 \in X. \end{cases} \tag{4.5}$$

It is well known that one can express the unique mild solution of (4.5) in terms of the semigroup generated by A_{j_k} is given by

$$e^{(t-k)A_{j_k}}u(k) = u(k) + A_{j_k}\int_k^t e^{(s-k)A_{j_k}}u(k)ds \tag{4.6}$$

and since $s \in [k, k+1]$, (4.6) can be rewritten as

$$u(t) = u(k) + A_{j_k}\int_k^t u(s)ds. \tag{4.7}$$

Now we consider the problem

$$\begin{cases} \dfrac{du}{dt}(t) = A_{j_{k-1}}u(t), & t \in [k-1, k), \\ u(k) = e^{A_{j_{k-2}}}e^{A_{j_{k-3}}} \cdots e^{A_{j_0}}u_0 \in X. \end{cases} \tag{4.8}$$

This admits a unique mild solution $u(\cdot)$ which can be continuously extended to $t = k$ by computing

$$u(k) := \lim_{t\to k^-} u(t).$$

In this way, $u(\cdot)$ uniquely solves (4.8) also on $[k, k+1]$ and verifies

$$\begin{aligned} u(k) &= e^{A_{j_{k-1}}} u(k-1) \\ &= u(k-1) + A_{j_{k-1}} \int_{k-1}^{k} e^{(s-(k-1))A_{j_{k-1}}} u(k-1) ds \\ &= u(k-1) + A_{j_{k-1}} \int_{k-1}^{k} u(s) ds. \end{aligned} \tag{4.9}$$

Hence, by substituing (4.9) in expression (4.7), we get

$$u(t) = u(k-1) + A_{j_{k-1}} \int_{k-1}^{k} u(s) ds + A_{j_k} \int_{k}^{t} u(s) ds.$$

Therefore, it is clear that, proceeding recursively, $u(t)$ satisfies (4.2) for all $t > 0$. $\square$

We will still be interested in studying the asymptotic behaviour of models like (4.1). In particular, we want to know whether $S(t)$ converges uniformly towards some bounded operator M, namely

$$\lim_{t \to +\infty} \|S(t) - M\|_{\mathcal{L}(X)} = 0.$$

Similarly to the finite-dimensional setting, when $\{j_k\}_{k \geq 0}$ is a Markov chain with state space $\{0, \ldots, N-1\}$, the notion of almost sure convergence

$$\mathbb{P}\left(\lim_{t \to +\infty} \|S(t) - M\|_{\mathcal{L}(X)} = 0 \right) = 1$$

remains the same.

Passing to the limit for $t \to +\infty$, we will obtain an infinite (random) product of linear operators, written in terms of semigroups generated by $A_0, \ldots, A_{N-1}$. In addition, when we study the random diffusion on metric graphs, we will see that every factor of the infinite product will be a contraction. Hence, in this case it is still possible to apply results in [**1**] and [**12**].

4.2 The continuous heat equation on evolving graphs

4.2.1 An introductive example

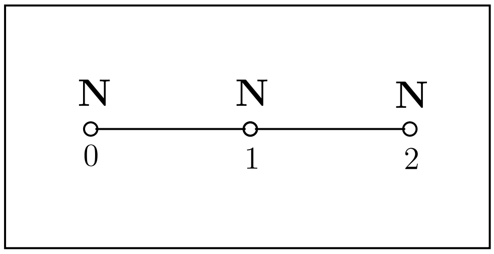
In order to introduce the reader to the dynamics of a random evolution model in the continuous framework, we first propose a simple example.

Consider one interval of length $\ell = 2$, along which we intend to study the heat equation

$$
\begin{cases}
\dot{u}(t,x) = u''(t,x), & x \in [0,2], \\
u(0,x) = u_0(x),
\end{cases}
$$

where $u_0 \in L^2(0,2)$ and as usual we adopt the notations $\dot{u} = \frac{d}{dt}$ and $u' = \frac{d}{dx}$. In particular, we are going to analyze two different and well-known boundary value problems: in one case, we decide to impose two Neumann conditions at $x = 0$ and $x = 2$, whereas the second setting keeps the same constraints at the boundaries, plus one additional Neumann condition at the middle point $x = 1$.

Model A



Model B

Model A can be interpreted as an example of diffusion problems studied in Chapter 3. Here the graph is trivial, since it is just one edge of length $\ell = 2$, and the leading operator Δ_1 is the Kirchhoff-Laplacian

$$
\Delta_1 u(x) = u''(x), \qquad x \in [0,2],
$$

with domain

$$
D(\Delta_1) = \left\{ u \in H^2(0,2) : \; u'(0) = 0 = u'(2) \right\}.
$$

Based on what we have shown in Chapter 3, we deduce that Δ_1 is densely defined, closed, self-adjoint and generates a contractive C^0-semigroup $\left\{ e^{t\Delta_1} \right\}_{t \geq 0}$. In this simple example, we are also able to fully determine the spectrum of Δ_1, which is given by

$$
\sigma(\Delta_1) = \left\{ \lambda_k = -\frac{k^2\pi^2}{4}, \; k = 0,1,2,\ldots \right\},
$$

with associated eigenfunctions

$$
\begin{aligned}
&e_0(x) = \frac{1}{\sqrt{2}}, \; x \in [0,2], \\
&e_k(x) = \cos\left(\frac{k\pi}{2} x \right), \; x \in [0,2], \qquad k \geq 1.
\end{aligned}
$$

In this way, for every initial condition $u_0 \in L^2(0,2)$, we can explicitly write the mild solution in terms of the spectral representation

$$u(t) = e^{t\Delta_1}u_0 = \sum_{k=0}^{+\infty} e^{t\lambda_k}(u_0, e_k)_{L^2(0,2)}\ e_k$$

and, as we expect, the limit distribution for long times agrees with the average of u_0 computed on the interval $[0,2]$

$$\lim_{t\longrightarrow+\infty} u(t) = P_0u_0 = \frac{1}{2}\int_0^2 u_0(x)dx = \fint_{[0,2]} u_0.$$

Regarding Model B, the leading operator is still the second derivative

$$\Delta_2 u(x) = u''(x), \qquad x \in [0,2],$$

but with different domain

$$\begin{aligned} D(\Delta_2) = \{u \in L^2(0,2) : u_1 := u\big|_{(0,1)} \in H^2(0,1), u_2 := u\big|_{(1,2)} \in H^2(1,2),\\ u_1'(0) = u_1'(1) = 0 = u_2'(1) = u_2'(2)\}. \end{aligned} \tag{4.10}$$

Here the dynamics is completely different from the previous one: in fact, the Neumann condition placed in $x = 1$ acts like an insulating "wall" through which heat exchanges are not allowed. Hence, we are studying two parallel and isolated diffusion systems: one on $[0,1]$, one on $[1,2]$.

This suggests that, also in this case, an efficient and equivalent way of analazying Model B is based on those diffusion models presented in Chapter 3. In fact, one can easily see that

$$L^2(0,2) \ \text{ and } \ L^2(0,1) \oplus L^2(0,1)$$

are identified by the isometric isomorphism

$$\begin{aligned} \Phi\colon L^2(0,2) &\to L^2(0,1) \oplus L^2(0,1)\\ u \quad &\mapsto (u_1, u_2), \end{aligned}$$

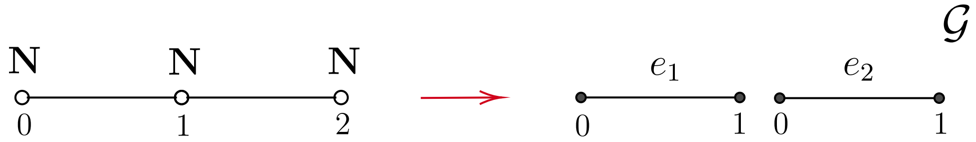
where

$$\Phi(u(\tilde{x})) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} u(\tilde{x}) \\ u(\tilde{x}-1) \end{pmatrix}, \qquad x \in [0,1],\ \tilde{x} \in [0,2].$$

Through Φ, we notice that Model B is equivalent to a new diffusion problem, where

$$\overline{\Delta}_2 := \Phi\Delta_2,$$
$$\overline{\Delta}_2 \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} u_1''(x) \\ u_2''(x) \end{pmatrix}$$

corresponds to the Laplace operator of a graph $\mathcal{G}$ with two disconnected edges $\{e_1, e_2\}$ of equal length $\ell_1 = \ell_2 = 1$.



In this new framework, one can show that the heavy notation of expression (4.10) gets simple and the domain of $\overline{\Delta}_2$ is actually

$$D(\overline{\Delta}_2) = \left\{u \in H^2(\mathcal{G}) : \ u \text{ satisfies (Cc) and (Kc)}\right\}.$$

Hence, $\overline{\Delta}_2$ has all the nice properties of Δ_1 and also in this case we are able to find the spectrum

$$\sigma(\overline{\Delta}_2) = \left\{\overline{\mu}_k = -k^2\pi^2, \ k = 0, 1, 2, \dots\right\},$$

where every eigenvalue has now double multiplicity. It is convenient to change notation and count all the eigenvalues twice as follows

$$0 = \mu_0 = \mu_1 < |\mu_2| = |\mu_3| < \ldots < |\mu_{2k}| = |\mu_{2k+1}| < \ldots$$

One orthonormal basis of eigenfunctions is then given by

$$f_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad f_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$f_{2k} = \begin{pmatrix} \sqrt{2}\cos(k\pi\cdot) \\ 0 \end{pmatrix}, \qquad f_{2k+1} = \begin{pmatrix} 0 \\ \sqrt{2}\cos(k\pi\cdot) \end{pmatrix}, \qquad \forall k \geq 1.$$

Applying again the Spectral Theorem, for every intial condition $g \in L^2(\mathcal{G})$, one gets

$$u(t) = e^{t\overline{\Delta}_2} f = \sum_{k=0}^{+\infty} e^{t\mu_k} (g, f_k)_{L^2(\mathcal{G})} \ f_k$$

and then for long times

$$\lim_{t \longrightarrow +\infty} u(t) = (g, f_0)_{L^2(\mathcal{G})} \; f_0 + (g, f_1)_{L^2(\mathcal{G})} \; f_1 = \begin{pmatrix} \int_0^1 g(x)dx \\ \\ \int_0^1 g(x)dx \end{pmatrix} = \begin{pmatrix} ⨍_{\mathcal{G}^{(1)}} g \\ \\ ⨍_{\mathcal{G}^{(2)}} g \end{pmatrix},$$

namely the heat distribution converges towards the average on each connected component. Going back to the initial framework and starting from $\Phi^{-1}g =: u_0 \in L^2(0,2)$, the limit distribution will agree with

$$\begin{pmatrix} \int_0^1 u_0(x)dx \\ \\ \int_1^2 u_0(x)dx \end{pmatrix} = \begin{pmatrix} ⨍_{[0,1]} u_0 \\ \\ ⨍_{[1,2]} u_0 \end{pmatrix},$$

which is consistent with what one expects from the classical results about diffusion problems. In particular, as represented in Fig. 4.1, notice that the averages of u_0 on $[0,1]$ and on $[1,2]$ are generally different and clearly

$$⨍_{[0,1]} u_0 + ⨍_{[1,2]} u_0 \neq ⨍_{[0,2]} u_0.$$

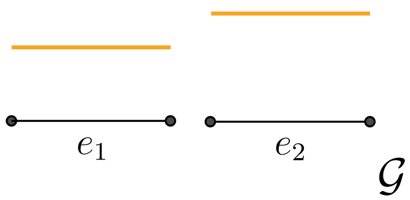
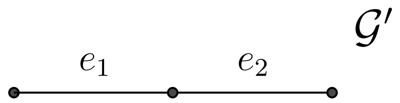


Figure 4.1: Possible configuration of the limit distribution on graph $\mathcal{G}$, given by the average of the initial datum on each edge.

Remark 4.3. It is convenient to point out that also Model A can be equivalently studied as a diffusion problem on a new graph $\mathcal{G}'$ with two edges. In fact, adding an internal vertex at the middle and requiring continuity of functions and their derivatives do not change the dynamics of the model. Through isomorphism Φ, one can still consider the Kirchhoff-Laplace operator $\overline{\Delta}_1 := \Phi\Delta_1$ on the new connected graph:



The spectrum $\sigma(\overline{\Delta}_1)$ remains unchanged, but now the eigenfunctions in $D(\overline{\Delta}_1) \subset L^2(0,1) \oplus L^2(0,1)$ are

$$e_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad e_k = \begin{pmatrix} \cos\left(\frac{k\pi}{2}\,\cdot\right) \\ \cos\left(\frac{k\pi}{2}\,\cdot\right) \end{pmatrix}, \ k \geq 1.$$

Clearly, given an initial datum $g \in L^2(0,1) \oplus L^2(0,1)$, the long time behaviour does not vary and we obtain again the convergence of the solution towards a constant function on the whole graph $\mathcal{G}'$:

$$\begin{aligned} \lim_{t \longrightarrow +\infty} u(t) &= (g, e_0)_{L^2(\mathcal{G}')}\ e_0 \\ &= \frac{1}{2}\left(\int_0^1 g_1(x)dx + \int_0^1 g_2(x)dx\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{2}\left(\int_0^1 u_0(x)dx + \int_1^2 u_0(x)dx\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \fint_{[0,2]} u_0 \\ \fint_{[0,2]} u_0 \end{pmatrix} \end{aligned}$$

which is actually the same constant as in the old framework based on $(\Delta_1, D(\Delta_1))$.

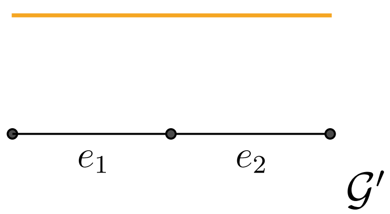
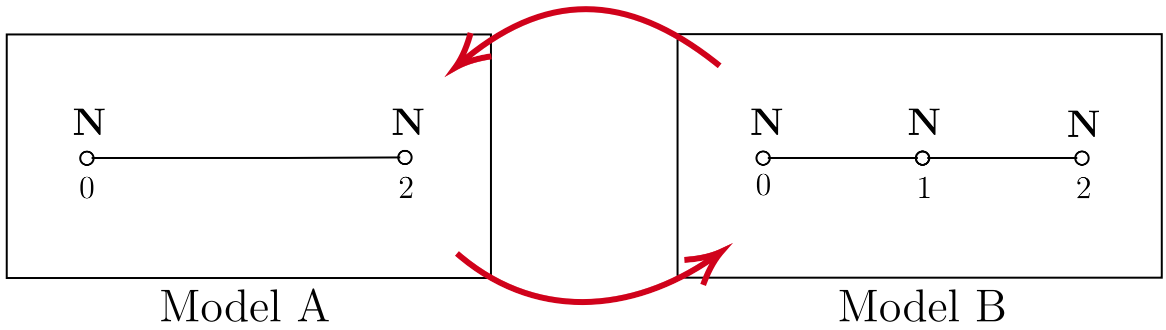


Figure 4.2: The heat distribution converges to the average $\fint_{[0,2]} u_0$ (in orange) for some initial condition $u_0 \in L^2(0,2)$.

Starting from these two models, we now introduce the following scenario: imagine that we are going to study the heat diffusion along the interval $[0,2]$ with Neumann boundary conditions. However, problem's features are not the same during the evolution in time: in fact, at each integer time we can decide to add or remove one third Neumann condition at $x = 1$. In particular, the choice of considering three or two constraints is determined by some suitable random law (for instance, a Markov chain).

This means that the system may continuously switch from Model A to Model B: hence, it is like we were analyzing a problem composed of two subsystems, which are linked in some intervals and isolated in others, according to the chosen random law. As one may guess, a problem like this can be easily described through one of the models presented in Section 4.1. In particular, at every integer time the Markov chain governing the random evolution can jump from the connected graph $\mathcal{G}'$ to the disconnected graph $\mathcal{G}$ and vice versa.



At this point, we wonder how this system behaves for long times: we would like to understand which configuration will more influence the limit heat distribution. Will the heat exchanges through $x = 1$ help to reach an uniform distribution along the whole interval like in Fig. 4.2? Or, on the contrary, will we get an asymptotic behaviour similar to the one in Fig. 4.1, due to the presence of the barrier during some time intervals? Obviously, this will depend on the particular mechanism driving the jumps from one to the other graph.

In the next section, we will discuss on this and other general models, where the system can also assume more than two configurations described by various metric graphs. Thanks to the results we will prove, which are similar to those presented in the combinatorial case, we will see what happens for long times in this first example. Provided that we require the irreducibility of the Markov chain, the solution will converge to the average of the initial datum over the entire interval $[0, 2]$.

4.2.2 The general model

In this section we give a generalization of the analysis made in Chapter 2, studying diffusion problems on metric graphs which are no longer autonomous.

As we have seen in Chapter 3, fixed a metric graph $\mathcal{G}$ constant in time, the long-time behaviour of the C^0-semigroup generated by the Kirchhoff-Laplace operator depends on the topological structure of the graph. In fact, the connectivity of the network plays a fundamental role, as the example introduced above confirms. When $\mathcal{G}$ is connected, the limit heat distribution is an average of the initial datum on the whole graph, otherwise the average is computed on each connected component.

As a consequence, according to the idea proposed at the end of the previous section, we would like to understand what happens when we study a diffusion on an environment which does not remain the same during time. In addition, what influence these changes may have on the dynamics of the system? The previous example gives us an intuitive idea: we can imagine a scenario where the non autonomous element is given by the presence or the absence of a central Neumann condition. The latter randomly joins and disjoins the network on which the heat spreads, making the dynamics change.

The same approach can be repeated for more general cases, as we have done in the finite-dimensional framework. However, we first need to point out a substantial difference between the heat equation in metric and combinatorial settings. In the latter, the heat is only concentrated at the vertices: in fact, its distribution is given by a vector with $|V|$ complex components, each of them indicating the concentration at every node. In this model, edges only represent the possible communication between their endpoints, namely they give an idea about the general connectivity of the combinatorial graph.

The situation is reversed in the continuous framework, which is certainly closer to the physical problem: in fact, as the adjective "continuous" suggests, here the heat no longer involves only the vertices of the graph. The heat distribution is always a vector, but now it has $|E|$ components, where each one is an L^2-function on an edge. Therefore, we can understand that values at points are not even taken into account singularly: what is really important is to study the heat distribution along the edges. Here vertices have to join the edges, they now represent the connectivity of the metric graph and, through the standard node conditions, they govern the heat transfer at the junctions.

Heuristically, edges and vertices exchange their roles in discrete and continuous settings. Based on this "duality", we can now better understand what we mean when we talk about random evolution of metric graphs. As in the combinatorial case, we are interested in studying a network environment whose connectivity randomly changes in time. Hence, if in Chapter 2 this is modeled by the evolution of edges, here we will focus on the changes of the vertices during time.

Consider a family of metric graphs $\mathcal{C} = \{\mathcal{G}_1, \ldots, \mathcal{G}_N\}$ which are finite, but not necessarily simple. These represent the possible stages reached by the network environment during the time, based on the same idea as for the discrete random evolution. While in Chapter 2 we require that graphs in $\mathcal{C}$ have the same set V, here it is not necessary. What is really fundamental is that $\mathcal{G}_1, \dots, \mathcal{G}_N$ share the same set of edges

$$E = \{e_1 \equiv [0, \ell_1], \dots, e_m \equiv [0, \ell_m]\},$$

whereas vertices can have different configurations graph by graph.

Also in this case, we introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stationary Markov chain $\{j_k\}_{k \geq 0}$ which still takes values in the set of graphs $\mathcal{C}$. Based on the results proved in Chapter 2, it seems reasonable to request that $\{j_k\}_{k \geq 0}$ is still irreducible.

At this point, we can consider the Kirchhoff-Laplace operator associated with each graph in $\mathcal{C}$:

$$(\Delta_1, D(\Delta_1)), \dots, (\Delta_N, D(\Delta_N)).$$

As we have seen in the previous chapter, they satisfy several nice properties, in particular they all generate a contractive strongly continuous semigroup $\left\{e^{t\Delta_i}\right\}_{t \geq 0}$, for $i = 1, \dots, N$.

Hence, we now can focus on the unknown function of the model we are about to present, namely

$$u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{pmatrix} \in \bigoplus_{j=1}^{m} L^2(0, \ell_j),$$

which represents the heat distribution at time t along every edge. Hence, based on systems in Section 4.1, the continuous random evolution (CRE) on metric graphs is given by

$$\begin{cases} \dfrac{d^+u}{dt}(t) = \Delta(j_k)u(t), \quad t \in [k, k+1), \ \ k \in \mathbb{N}, \\ u(0) = f \in \bigoplus_{j=1}^{m} L^2(0, \ell_j). \end{cases} \tag{CRE}$$

We immediately notice that (CRE) and the discrete random evolution model (DRE) has the same structure: they both are described by an evolution problem whose leading operator is non autonomous and stochastic, according to the evolution of $\{j_k\}_{k \geq 0}$. Therefore, we are still interested in analyzing the dynamics of (CRE) trajectory by trajectory, fixing an arbitrary realization $\omega \in \Omega$.

Based on the evolution of (DRE), the reader can already imagine how the dynamics works. In fact, during every unit time interval, also in this case the system describes the heat diffusion on the graph determined by the chain. However, in order to better understand how the passages from one to another graph occur, let us see a simple example. Graphs in Fig. 4.3 are taken all equilateral, namely all the edges have the same length, which we set equal to 1. They have different structure from each other, so it may be

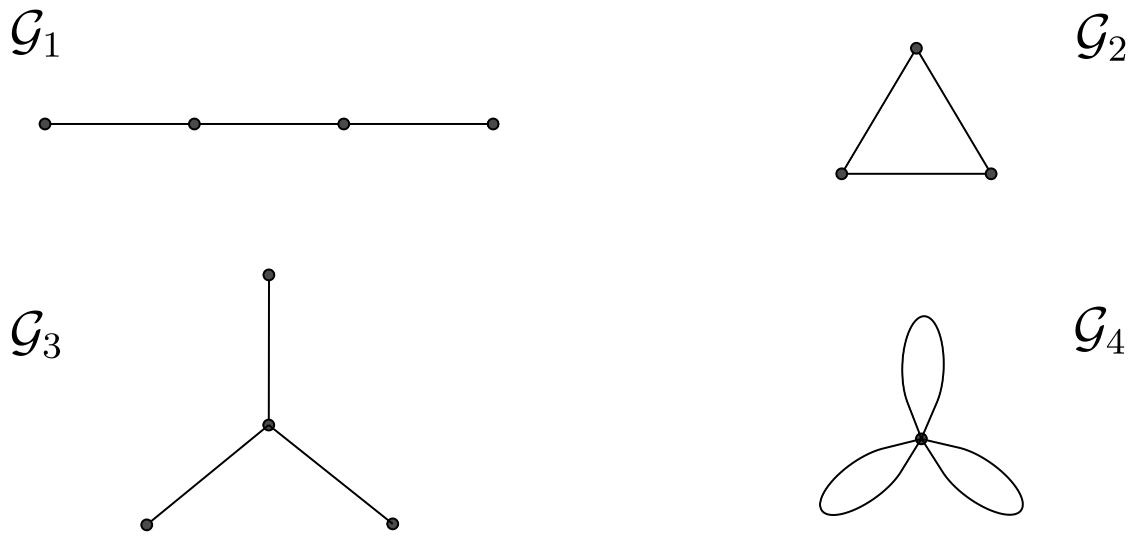


Figure 4.3: Possible graphs contained in family $\mathcal{C}$: the *path graph* $\mathcal{G}_1 = \mathsf{I}(L)$ of total length $L = 3$; the *complete graph* $\mathcal{G}_2 = \mathsf{K}_V$ with $V = 3$ vertices; the *star graph* $\mathcal{G}_3 = \mathsf{S}(L, E)$ of total length $L = 3$ and $E = 3$ edges; the *flower graph* $\mathcal{G}_4 = \mathsf{F}(L, E)$ of total length $L = 3$ and $E = 3$ edges, also called petals.

difficult to realize how (CRE) model is meant in this case.

Looking at the $\mathcal{G}_i$s, one can guess that the diffusion on every network does not follow the same dynamics. In fact, in every configuration the edges are differently linked to each other by the nodes and consequently the standard vertex conditions change every time. For instance, in the path graph $\mathcal{G}_1$, the external edges are not directly connected and heat exchanges between them necessarily occurs through the central one. On the other hand, the structure of the triangle K_3 makes every edge joined to the others.

Furthermore, one can explicitly compute the nonzero largest eigenvalue $\lambda_1(\mathcal{G}_i)$ (also called *spectral gap*) of the Kirchhoff-Laplacian of these graphs. It is well-known (see, for instance, [**19**]) that

$$\lambda_1(\mathcal{G}_1) = -\frac{\pi^2}{9}, \qquad \lambda_1(\mathcal{G}_2) = -\frac{4\pi^2}{9}, \qquad \lambda_1(\mathcal{G}_3) = -\frac{\pi^2}{4}, \qquad \lambda_1(\mathcal{G}_1) = -\pi^2.$$

Hence, despite the solution always converges towards

$$P_0 f = \frac{1}{3}\sum_{j=1}^{3}\int_0^1 f_j(x)dx, \qquad f \in \bigoplus_{j=1}^{3} L^2(0,1),$$

this limit distribution is reached at a different convergence rate $|\lambda_1(\mathcal{G}_i)|$, depending on the specific case.

Let us going back to (CRE) model: fixing a realization $\omega \in \Omega$, a sequence of graphs $\{j_k(\omega)\}_{k\geq 0}$ is determined. Roughly speaking, at every integer time we freeze the system on which we are working and we pass to a new graph, where we study the diffusion for the next unit time interval. In this way, for all $t = k,\ k \in \mathbb{N}$, each edge e_j of the new network – now differently organized – inherits as initial condition the final heat distribution on the same e_j of the old graph.

Referring to the example in Fig. 4.3, this means that the evolution can switch, for instance, from the complete graph (where the heat has two passageways on every edge) to the star graph (where there is only one vertex joining the three edges and their second endpoints have a Neumann boundary condition respectively). And again, after one unit time interval, we may reach the flowergraph, whose edges are all linked by one single common vertex, and so on.

This is how (CRE) behaves when $\mathcal{C}$ contains only connected graph. Obviously, when the Markov chain visits graphs which can also be disconnected, this further influences the dynamics' evolution.

Coming back to mathematical aspects, applying results of Section 4.1, the unique mild solution of (CRE) is given by

$$u(t) = S(t)u_0 = e^{(t-k)\Delta(j_k)}e^{\Delta(j_{k-1})}\cdots e^{\Delta(j_0)}u_0, \qquad t \in [k, k+1]$$

and the stochastic evolution operator $\{S(t)\}_{t\geq 0}$ – despite it is no longer a semigroup – inherits both contractivity and strong continuity from the semigruops of the $\Delta(j_k)$s.

Asympotics and union of metric graphs

Now we focus on the long-time behaviour of $\{S(t)\}_{t\geq 0}$. As we have remarked in Section 4.1, studying the limit of this operator for $t \to +\infty$ means dealing with an infinite random product, where each term is given by a contraction semigroup of the form $e^{\Delta(j_k)}$. As long as we easily extend some proofs to the infinite-dimensional framework, we prove the same results proposed in Chapter 2.

Given the identically 1-function $\mathbb{1} = (1, \ldots, 1)^\top$, we still denote by

$$P_0 \colon \bigoplus_{j=1}^{m} L^2(0, \ell_j) \to \langle \mathbb{1} \rangle$$

the orthogonal projection onto the constants on the whole graph. Therefore, we focus again on the following

Problem (P): Determine whether and under which conditions the evolution operator $\{S(t)\}_{t\geq 0}$ converges almost surely to the projection P_0.

At this point, we introduce the orthogonal projection

$$\overline{P}\colon \bigoplus_{j=1}^{m} L^2(0,\ell_j) \to \bigcap_{i=1}^{N} ker\Delta_i, \tag{4.11}$$

whose range is the intersection of all the kernels of the initial Kirchhoff-Laplace operators. Then, we are able to extend

Theorem 4.4. *Let us consider model (CRE) and let $\{j_k\}_{k\geq 0}$ be an irreducible Markov chain. Then the evolution operator $S(t)$ converges uniformly and almost surely towards the projection $\overline{P}$, namely*

$$\mathbb{P}\left(\lim_{t\longrightarrow+\infty} S(t) = \overline{P}\right) = 1.$$

Proof. We can still follow the guidelines in [**1**] and [**12**], where the authors also deal with an infinite-dimensional framework.

We only need to check that

$$\ker\Delta_i = \text{Fix }\ e^{t\Delta_i}, \qquad \forall t > 0,\ i = 1,\dots,N \tag{1}$$

and

$$\|e^{t\Delta_i}f\| < \|f\|, \qquad \forall f \notin ker\Delta_i,\ \forall t > 0,\ i = 1,\dots,N. \tag{2}$$

Since each semigroup $e^{t\Delta_i}$ can be still spectral represented, one can easily extend to the infinite-dimensional case the computations shown in Lemma 1.3 and Lemma 2.12. Thus, (1) and (2) follow and, arguing as in Lemma 2.16, Lemma 2.17 and Theorem 2.18, the proof is achieved. $\square$

As we can easily guess, Theorem 4.4 directly implies some corollaries.

Corollary 4.5. *Consider family $\mathcal{C} = \{\mathcal{G}_1,\dots,\mathcal{G}_N\}$ containing the initial metric graphs of model (CRE). Then*

(i) If $\mathcal{C}$ contains only connected graphs, then the evolution operator $\{S(t)\}_{t\geq 0}$ converges uniformly to the orthogonal projection P_0:

$$\lim_{t\to+\infty} S(t) = P_0 \qquad \textit{uniformly.}$$

(ii) If $\mathcal{C}$ has at least one connected graph, then then the evolution operator $\{S(t)\}_{t\geq 0}$ converges uniformly to the orthogonal projection P_0:

$$\lim_{t\to+\infty} S(t) = P_0 \qquad \mathbb{P}\text{-}almost\ surely.$$

Proof. One immediately notice that with at least one connected graph in $\mathcal{C}$, then its kernel exactly agrees with the subspace of constant functions on each edge. Thus

$$\operatorname{rg} \overline{P} = \bigcap_{i=1}^{N} \ker \Delta_i = \langle \mathbb{1} \rangle = \operatorname{rg} P_0$$

and, by uniqueness of the orthogonal projection onto a closed subspace of an Hilbert space, it follows that $\overline{P} \equiv P_0$.

Taking into account the proof of Proposition 2.7, the thesis follows.

□

Therefore, when we study the diffusion on a network which is infinitely often connected, then the heat spreads until converging to the average on the whole set of edges. This is also the situation which occurs in the introductive example described at the very beginning, where family $\mathcal{C}$ contains two graphs and one of them is connected.

Remark 4.6. Theorem 4.4 confirms that the connectedness of at least one graph in $\mathcal{C}$ is sufficient, but also not a necessary condition. In fact, we recall that (P) holds if and only if the orthogonal projections $\overline{P}$ and P_0 coincide, namely when their ranges agree:

$$\bigcap_{i=1}^{N} \ker \Delta_i = \langle \mathbb{1} \rangle. \tag{4.12}$$

We can easily show that (4.12) is satisfied even if no one $\ker \Delta_i$ is the subspace of constant functions (i.e. no one graph is connected). For instance, suppose $N = 2$ and $E = \{e_1, e_2, e_3\}$; additionally assume that the kernels of Δ_1 and Δ_2 are respectively spanned by

$$\ker \Delta_1 = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle,$$

$$\ker \Delta_2 = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \rangle.$$

Then, one can prove that given $f \in \ker \Delta_1 \cap \ker \Delta_2$, f is necessarily constant and (4.12) follows.

The next goal to achieve is to find a necessary and sufficient condition in order that (P) is solved. Based on the results proved in the discrete setting, one may suppose that the correct hypothesis to be imposed is linked to the connectedness of the union of graphs in $\mathcal{C}$. This is reasonable, but not obvious: in fact, a formal and rigorous definition of *union* of metric graphs is not present in literature.

Given two combinatorial graphs $G = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we can then define the notion of their union, as long as we enumerate both sets of vertices. For instance, assuming that $|V_1| < |V_2|$, let us denote by

$$V_1 = \{v_1, \dots, v_k\}, \qquad \text{for some } k = |V_1| \geq 1$$

and

$$V_2 = \{v_1, \dots, v_k, \dots, v_h\}, \qquad \text{for some } h = |V_2| \geq 1.$$

In this way, we are able to identify each vertex in V_1 with one in V_2 and, thus, the definition

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2) \tag{U}$$

is well-posed. Notice that (U) is not invariant under automorphism of V_i, $i = 1, 2$. In fact, if we change the enumeration of one or both set of vertices, we will obtain a different union graph.

Coming back to the continuous framework, let us consider now two metric graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. In particular, we are only interested in the case when they share the same set of edges $E = \{e_1, \dots, e_m\}$. In order to introduce a corresponding notion of (U), the first intuitive idea one may have is to consider the union graph of the combinatorial supports of the $\mathcal{G}_i$s. In fact, let

$$\mathcal{G}_1 = (G_1, \ell) \ \text{ and } \ \mathcal{G}_2 = (G_2, \ell),$$

where $G_1 = (V_1, E)$, $G_2 = (V_2, E)$ and ℓ is the edge weight function

$$\begin{aligned} \ell\colon \quad E &\to (0, +\infty) \\ e &\mapsto \ell_e. \end{aligned} \tag{4.13}$$

Hence, we may introduce the union $\mathcal{G}_1 \cup \mathcal{G}_2$ as a metric graph with the same edge weight function (4.13) and with combinatorial support

$$G_1 \cup G_2 = (V_1 \cup V_2, E), \tag{U'}$$

provided that V_1 and V_2 are enumerated in a suitable way and then comparable.

Though, this definition is not well-posed, as the following example shows.

Example 4.7. We consider again graphs $\mathcal{G}$ and $\mathcal{G}'$ introduced in the introductive example and we enumerate their vertices as shown in Fig. 4.4. Looking at this example, one

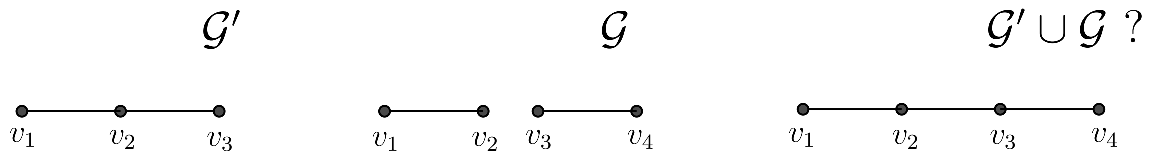


Figure 4.4

intuitively expects that the union gives again the graph $\mathcal{G}'$. Instead, applying (U'), we obtain the graph in Fig. 4.4, which is not $\mathcal{G}'$ and it does not even have the same set E of the two initial graphs.

In view of the above example, we give a more reasonable and consistent definition, for which the union graph contains all the junctions of both $\mathcal{G}_1$ and $\mathcal{G}_2$.

We start by considering again the notion of metric graph: we are going to give a more formal version of the definition introduced in Chapter 3. Let us consider an initial family of intervals

$$E = \{e_1, \dots, e_m\},$$

where each of them has two endpoints, which we denote by e_j^- and e_j^+. Let

$$E^{\pm} = \{e_1^-, e_1^+, \dots, e_m^-, e_m^+\}$$

be the set of the $2m$ points of the intervals in E. Hence, every equivalence relation R defined on $E^{\pm}$ induces an oriented metric graph with set of edges E and set of vertices given by the quotient

$$V := \frac{E^{\pm}}{R}.$$

In addition, let $S \subset E^{\pm} \times E^{\pm}$ and consider the reflexive, symmetric and transitive closure $\overline{S}$ of S, obtained just by adding:

- every (x, x) for all $(a, b) \in S$, where $x = a$ or $x = b$ (reflexive closure);

– every (y, x) for all $(x, y) \in S$ (symmetric closure);

– every (x, z) for all $(x, y), (y, z) \in S$ (transitive closure).

By construction, $\overline{S}$ is the smallest equivalence relation containing S. Hence, it follows that the reflexive, symmetric and transitive closure of any subset of $E^{\pm} \times E^{\pm}$ defines a metric graph.

We can now consider some intuitive examples, starting from the family

$$E = \{e_1, e_2, e_3\},$$

where, from now on, the orientation of each e_j is represented by an arrow with tail e_j^- and head e_j^+.

1. Let
$$R_1 = E^{\pm} \times E^{\pm},$$
then we obtain the flower graph with three petals, where every endpoint is identified with each other.

2. Consider the following subset of $E^{\pm} \times E^{\pm}$:
$$S = \left\{(e_1^-, e_2^-),\ (e_2^-, e_3^-),\ (e_1^-, e_3^-),\ (e_1^+, e_2^+),\ (e_2^+, e_3^+),\ (e_1^+, e_3^+)\right\}.$$
If we take its reflexive and the symmetric closure denoted by R_2, then the transitivity follows, too. Hence, the equivalence relation R_2 induces the pumpkin graph with three slices.

3. Take
$$T = \left\{(e_1^+, e_2^-),\ (e_2^+, e_3^-), (e_3^+, e_1^-)\right\}$$
and we denote by R_3 its reflexive, symmetric closure, which is automatically transitive again. In this case, the equivalence relation R_3 induces the complete graph with three vertices, namely a clockwise-oriented triangle.

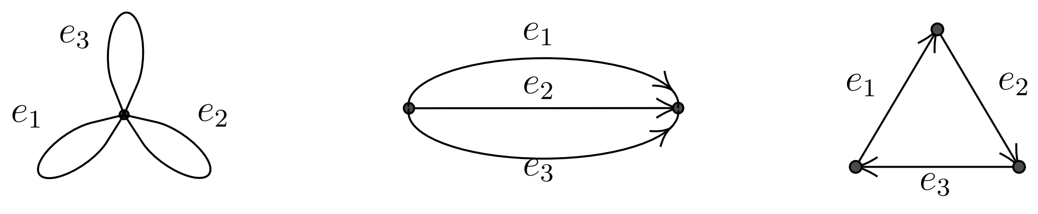


Figure 4.5: A flower graph, a pumpkin graph and a complete graph induced by the equivalence relations R_1, R_2, R_3, respectively.

We are now able to give a definition of union of $N = 2$ oriented metric graphs, which can be easily generalized for every $N \geq 1$.

Definition 4.8 (**Union of metric graphs**). Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two oriented metric graphs which share the same set of edges E. Moreover, let $R_1, R_2 \subset E^{\pm} \times E^{\pm}$ be the equivalence relations on $E^{\pm}$ which induce $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Consider the symmetric and transitive closure of $R_1 \cup R_2 \subset E^{\pm} \times E^{\pm}$ and denote it by $R_{\cup}$.

Then, we call *union graph* $\mathcal{G}_1 \cup \mathcal{G}_2$ the oriented metric graph induced by $R_{\cup}$.

Remark 4.9. The union of two equivalence relations is not necessarily an equivalence relation, because only the reflexivity is invariant under this operation. For this reason, in the above definition we need to consider the symmetric and transitive closure of $R_1 \cup R_2$.

In Fig. 4.6 we can consider some examples of union graphs.

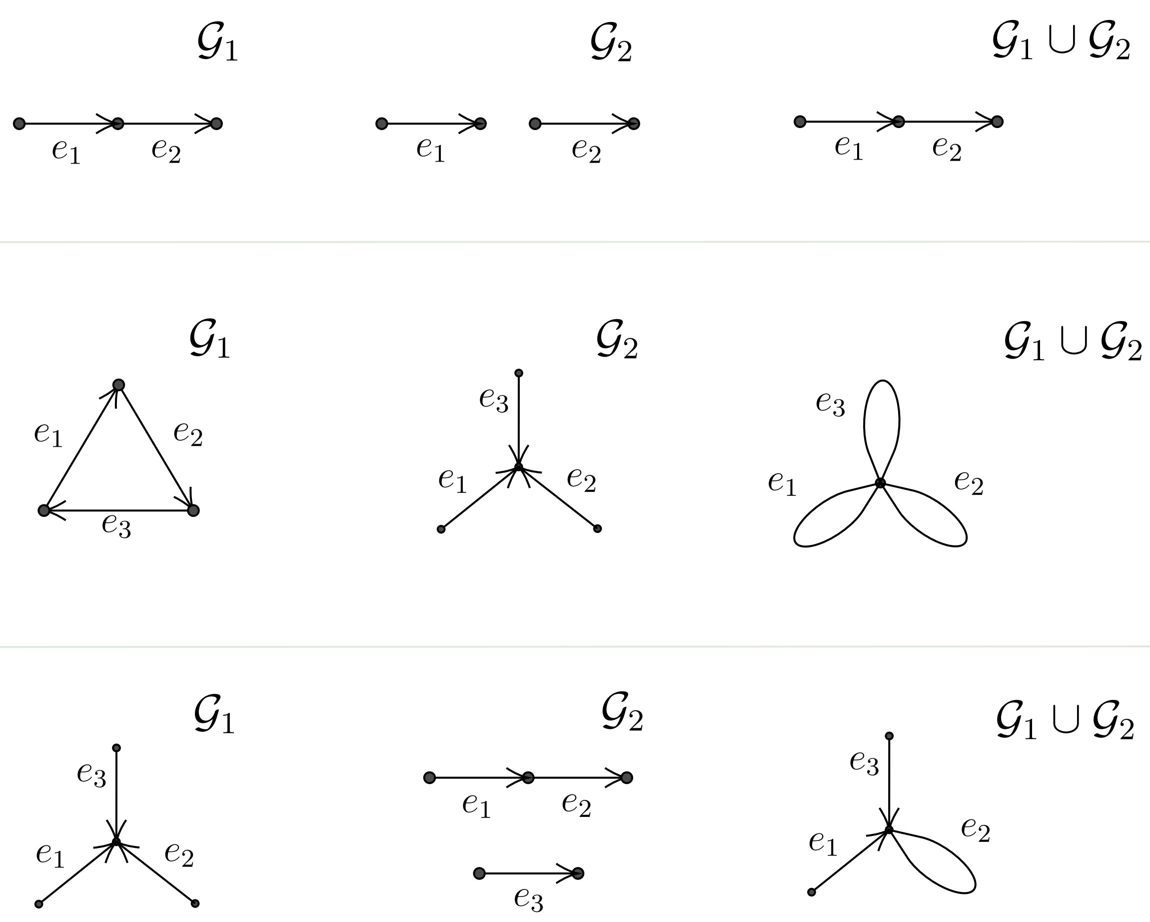


Figure 4.6: Some examples of union graph.

Remark 4.10. We want to stress that the notion of union just introduced is not invariant under the orientation imposed on each initial graph. For instance, we can take the same graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in the third example in Fig. 4.6 and just reverse the orientation of one edge as shown in Fig. 4.7.

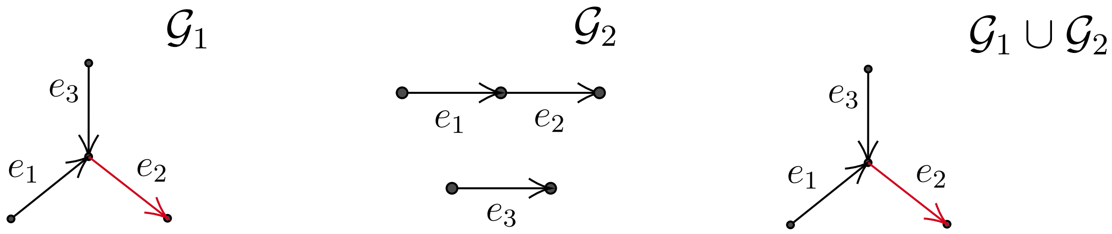


Figure 4.7: If we reverse the orientation of just one edge, the resulting union is different.

We notice that the resulting union gives the star graph again, which is clearly different from the previous $\mathcal{G}_1 \cup \mathcal{G}_2$. This is reasonable, in fact changing the orientation in $\mathcal{G}_1$ is related to considering a different equivalence relation $\overline{R}_1$ on $E^{\pm}$, which induces a different oriented metric graph, even though the topological structure is the same as the previous one. Therefore, since Definition 4.8 directly depends on R_1 and R_2, we can easily deduce why the union $\mathcal{G}_1 \cup \mathcal{G}_2$ is different from the previous case. On the other hand, it turns out that the union is invariant under the simultaneous reversion of every edge of both graphs.

We recall that the goal is to find a necessary and sufficient condition for (P), in particular we would like to obtain again

$$\text{(P) is solved} \iff \text{the union of graphs in } \mathcal{C} \text{ is connected.} \qquad (\star\star)$$

In view of this, the dependence on the orientation highlighted in Remark 4.10 may be a problem. In fact, since the diffusion problem is invariant under orientation, the latter is never fixed on graphs in $\mathcal{C}$. Hence, depending on a particular choice of orientation of the single graphs, we could obtain diffent unions $\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_N$, among which some of them is not connected, and this would make $(\star\star)$ unconsistent.

We show that this cannot happen thanks to the following

Lemma 4.11. *Let $\mathcal{G}_1, \ldots, \mathcal{G}_N$ be metric graphs, with associated Kirchhoff-Laplace operators $\Delta_1, \ldots, \Delta_N$. Then*

$$\mathcal{G} := \mathcal{G}_1 \cup \ldots \cup \mathcal{G}_N \text{ is connected} \iff \bigcap_{i=1}^{N} \ker \Delta_i = \langle \mathbb{1} \rangle. \qquad (4.14)$$

Notice that this lemma is exactly the metric version of Lemma 2.11 and the proof will be similar, indeed. One has only to extend the result, taking into account the new definition of union.

Proof. We show the proof for $N = 2$, then one can easily extend the result for an arbitrary N by induction. In general, both $\mathcal{G}_1$ and $\mathcal{G}_2$ have a certain number of disjoint connected components:

$$\mathcal{G}_1^{(1)}, \dots, \mathcal{G}_1^{(m)} \quad \text{s.t.} \quad \mathcal{G}_1 = \bigsqcup_{j=1}^{m} \mathcal{G}_1^{(j)}, \qquad \text{for some } m \in \mathbb{N}$$

and

$$\mathcal{G}_2^{(1)}, \dots, \mathcal{G}_2^{(n)} \quad \text{s.t.} \quad \mathcal{G}_2 = \bigsqcup_{k=1}^{n} \mathcal{G}_1^{(k)}, \qquad \text{for some } n \in \mathbb{N}.$$

Since the connectedness is just a topological property, notice that they do not change and depend on the particular choice of orientation.

Now assume that $\mathcal{G}$ is connected: we need to show that every function $\ker \Delta_1 \cap \ker \Delta_2 \subseteq \langle \mathbb{1} \rangle$. Thus, we take $f \in \ker \Delta_1 \cap \ker \Delta_2$, in particular from the theory presented in Chapter 3 it is well-known that f is constant on each connected component of both $\mathcal{G}_1$ and $\mathcal{G}_2$. Take $x, y \in \mathcal{G}$ and without loss of generality we can assume that $x \in e_h$ and $y \in e_l$ with $h \neq l$. Hence, by connectedness of $\mathcal{G}$, there exists a path of incident edges $\Gamma_{xy} = \{e_h, e_{i_1}, \dots, e_{i_M}, e_l\}$ linking x and y:

$$x \in e_h \sim e_{i_1} \sim \cdots \sim e_{i_M} \sim e_l \ni y,$$

where here $\sim$ means the incidence relation. In particular, edges in Γ_{xy} can be incident in $\mathcal{G}_1$ and/or in $\mathcal{G}_2$. Thus, taking into account that f is constant on the connected components of both graphs, we deduce that f is constant along Γ_{xy} and in particular

$$f(x) = f(y).$$

Byr the arbitrariness of x, y, we conclude that f is constant.

In order to prove the opposite implication, we are going to show that if $\mathcal{G}$ is disconnected, then we can find a non constant function such that $f \in \ker \Delta_1 \cap \ker \Delta_2$. For simplicity, assume that $\mathcal{G}$ has only two connected components: $\mathcal{G}^{(A)}$ and $\mathcal{G}^{(B)}$. Then, both contain a certain number of connected components of $\mathcal{G}_1$ and $\mathcal{G}_2$. In particular, we set

$$J_A = \left\{ j \in \{1, \dots, m\} : \mathcal{G}_1^{(j)} \subseteq \mathcal{G}^{(A)} \right\}, \qquad J_B = \left\{ j \in \{1, \dots, m\} : \mathcal{G}_1^{(j)} \subseteq \mathcal{G}^{(B)} \right\}$$

and

$$K_A = \left\{ k \in \{1, \dots, n\} : \mathcal{G}_2^{(k)} \subseteq \mathcal{G}^{(A)} \right\}, \qquad K_B = \left\{ k \in \{1, \dots, n\} : \mathcal{G}_2^{(k)} \subseteq \mathcal{G}^{(B)} \right\}.$$

Due to the fact that $\mathcal{G}$ is disconnected, it follows that

$$J_A \cap J_B = \emptyset, \qquad K_A \cap K_B = \emptyset,$$

in fact there cannot exist some connected components of $\mathcal{G}_1$ or $\mathcal{G}_2$ shared by $\mathcal{G}^{(A)}$ and $\mathcal{G}^{(B)}$. This is always true, even if, roughly speaking, we reverse the endpoints of some edge in one of the initial graphs. Since the connected components of $\mathcal{G}_1$ and $\mathcal{G}_2$ are invariant under orientation, we will never find an orientation for which some index j is in $J_A \cap J_B$ or some k is in $K_A \cap K_B$. Hence, taking the characteristic functions on each connected component such that

$$\ker \Delta_1 = \langle \{\mathbb{1}_{1,j}\}_{j=1}^{m} \rangle \quad \text{and} \quad \ker \Delta_2 = \langle \{\mathbb{1}_{2,k}\}_{k=1}^{n} \rangle,$$

it is always true that

$$\sum_{j \in J_A} \mathbb{1}_{1,j} = \mathbb{1}_A = \sum_{k \in K_A} \mathbb{1}_{2,k} \tag{4.15}$$

and

$$\sum_{j \in J_B} \mathbb{1}_{1,j} = \mathbb{1}_B = \sum_{k \in K_B} \mathbb{1}_{2,k}. \tag{4.16}$$

At this point, we only need to take some function of the form

$$f = \alpha \mathbb{1}_A + \beta \mathbb{1}_B, \qquad \alpha, \beta \in \mathbb{C}$$

and from (4.15) and (4.16) one gets that f can be written as a linear combination of both basis of $\ker \Delta_i$, $i = 1, 2$:

$$f = \alpha \sum_{j \in J_A} \mathbb{1}_j + \beta \sum_{j \in J_B} \mathbb{1}_j \qquad \Longrightarrow \qquad f \in \ker \Delta_1$$

and

$$f = \alpha \sum_{k \in K_A} \mathbb{1}_k + \beta \sum_{k \in K_B} \mathbb{1}_k \qquad \Longrightarrow \qquad f \in \ker \Delta_2.$$

Thus, the proof is complete. $\square$

This lemma is fundamental, since we have just characterized the connectedness of the union graphs in terms of the intersection of all the kernels $\ker \Delta_i$. Hence, from the fact that every Kirchhoff-Laplace operator does not depend on the orientation of graphs in $\mathcal{C}$, we can conclude that so does the connectedness of $\mathcal{G}$.

At this point, in order to verify that $(\star\star)$ holds, we follow the same argumentation made in Chapter 2 and we extend the following

Lemma 4.12. *Let $\mathcal{G}_1, \dots, \mathcal{G}_N$ be metric graphs, with associated Kirchhoff-Laplace operators $\Delta_1, \dots, \Delta_N$. Let $\mathcal{G} := \mathcal{G}_1 \cup \dots \cup \mathcal{G}_N$ be their union with associated Kirchhoff-Laplace operator Δ. Then*

$$\ker \Delta = \bigcap_{i=1}^{N} \ker \Delta_i. \tag{4.17}$$

Remark 4.13. Before proving this lemma, we underline that also $\ker \Delta$ does not depend on the orientation of the graphs in $\mathcal{C}$. It only refers to the connected components of $\mathcal{G}$, which do not vary. In fact, one has just to focus on each component and consider it, in turn, as an union of subgraphs of the $\mathcal{G}_i$s. Therefore, applying Lemma 4.11, it turns out that even the connectedness of these is invariant under orientation.

Proof. Once proved Lemma 4.11, this proof is similar to the one of Lemma 2.15, as long as we modify the basis of $\Delta_i,\ i = 1, \dots, N$. We show the case when $N = 2$ and, to avoid a further heavy notation, assume that the union graph has only two connected components. After proving the thesis under these hypotheses, it will be easy to obtain the general case.

Then, if the union graph has $\mathcal{G}^{(A)}$ and $\mathcal{G}^{(B)}$ as connected components, then as we know the kernel of Δ is spanned by the characteristic functions

$$\{\mathbb{1}_A, \mathbb{1}_B\}.$$

In the previous proof, we have obtained that

$$\sum_{j \in J_A} \mathbb{1}_{1,j} = \mathbb{1}_A = \sum_{k \in K_A} \mathbb{1}_{2,k} \tag{4.18}$$

and

$$\sum_{j \in J_B} \mathbb{1}_{1,j} = \mathbb{1}_B = \sum_{k \in K_B} \mathbb{1}_{2k}, \tag{4.19}$$

where J_A, J_B, K_A, K_B have the same definition as before.

Then, if $f \in \ker \Delta$, we have shown in the proof of Lemma 4.11 that this implies that $f \in \ker \Delta_i$, for all $i = 1, 2$.

On the other hand, we now take $f \in \ker \Delta_1 \cap \ker \Delta_2$, and so f can be written as

$$f = \alpha_1 \mathbb{1}_{1,1} + \dots + \alpha_m \mathbb{1}_{1,m} \tag{4.20}$$

and

$$f = \beta_1 \mathbb{1}_{2,1} + \dots + \beta_n \mathbb{1}_{2,n}. \tag{4.21}$$

Then, as we have done in the discrete case, we compare the expressions (4.20) and (4.21), so that by construction we obtain that equality holds if and only if

$$\alpha_j = \beta_k = c_A, \qquad \forall\, j \in J_A,\ \forall\, k \in K_A,$$
$$\alpha_j = \beta_k = c_B, \qquad \forall\, j \in J_B,\ \forall\, k \in K_B.$$

Hence, recalling (4.18)-(4.19), we deduce that f can also be expressed in terms of the characteristic functions associated with $\mathcal{G}^{(A)}$ and $\mathcal{G}^{(B)}$, thus

$$f = c_A \mathbb{1}_A + c_B \mathbb{1}_B,$$

and $f \in \ker \Delta$.

□

Once proved Lemma 4.12, we can finally state that ($\star\star$) still holds in the continuous framework. In fact, as we have found in Chapter 2, now projection $\overline{P}$ introduced in (4.11) exactly coincide with

$$\overline{P} \colon \bigoplus_{j=1}^{m} L^2(0, \ell_j) \to \ker \Delta$$

and then, by Theorem 4.4 and Lemma 4.11, we get the following

Theorem 4.14. *Let us consider model (CRE) and let $\{j_k\}_{k\geq 0}$ be an irreducible Markov chain. Moreover, let $\mathcal{C}$ be a finite family of graphs with union $\mathcal{G} = \mathcal{G}_1 \cup \ldots \cup \mathcal{G}_N$.*

Then the evolution operator $S(t)$ converges uniformly and almost surely towards the projection $\overline{P}$. In particular,

(i) When $\mathcal{G}$ is connected, then $\overline{P}$ coincides with the orthogonal projection onto the constant functions on the edges $E = \{e_1, \ldots, e_m\}$. In addition, for any initial datum $f \in \bigoplus_{j=1}^{m} L^2(0, \ell_j)$, the solution of (CRE) converges strongly and almost surely towards the average of f computed on E:

$$\lim_{t \longrightarrow +\infty} u(t) = P_0 f = \frac{1}{\sum_{j=1}^{m} \ell_j} \sum_{j=1}^{m} \int_0^{\ell_j} f_j(x)dx =: \fint_{\mathcal{G}} f$$

(ii) When $\mathcal{G}$ is not connected and given by a disjoint union of $\mathcal{G}^{(1)}, \ldots, \mathcal{G}^{(L)}$, then $\overline{P}$ is the orthogonal projection onto the constant functions on each connected component of the union. In addition, for any initial datum $f \in \bigoplus_{j=1}^{m} L^2(0, \ell_j)$, the solution of (CRE) converges strongly and almost surely towards the average of f computed on $\mathcal{G}^{(1)}, \ldots, \mathcal{G}^{(L)}$:

$$\lim_{t \longrightarrow +\infty} u(t) = \overline{P} f,$$

where

$$Pf(x) = \frac{1}{\sum_{j:e_j\in\mathcal{G}^{(h)}} \ell_j} \sum_{j:e_j\in\mathcal{G}^{(h)}} \int_0^{\ell_j} f_j(x)dx =: \fint_{\mathcal{G}^{(h)}} f, \qquad x \in \mathcal{G}^{(h)}.$$

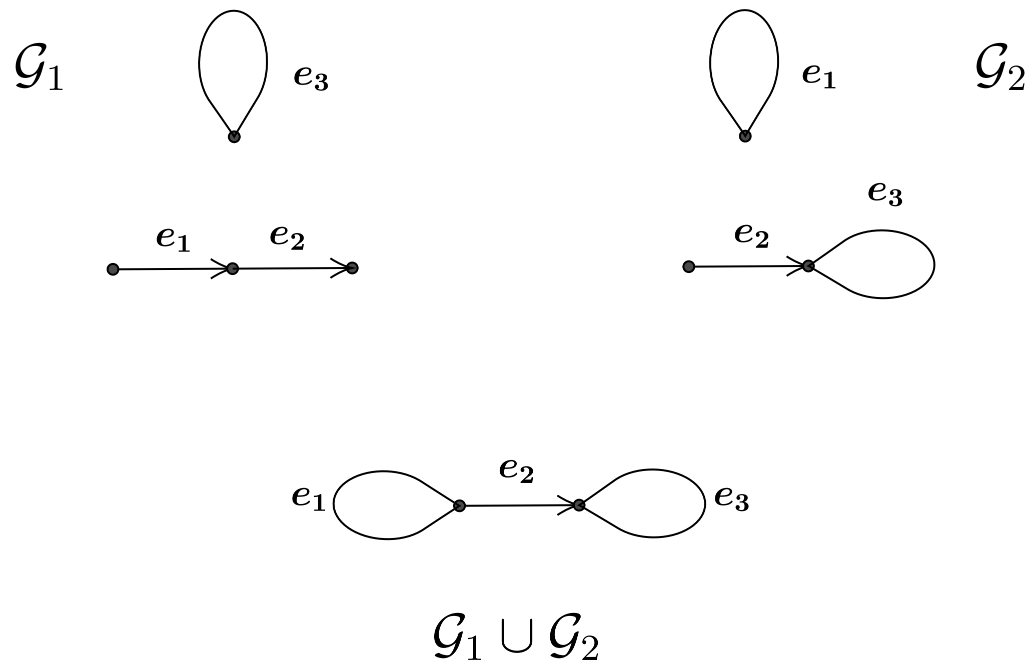


Figure 4.8: Example of two disconnected graphs with connected union: in this case the limit heat distribution assumes the same constant value on e_1, e_2 and e_3.

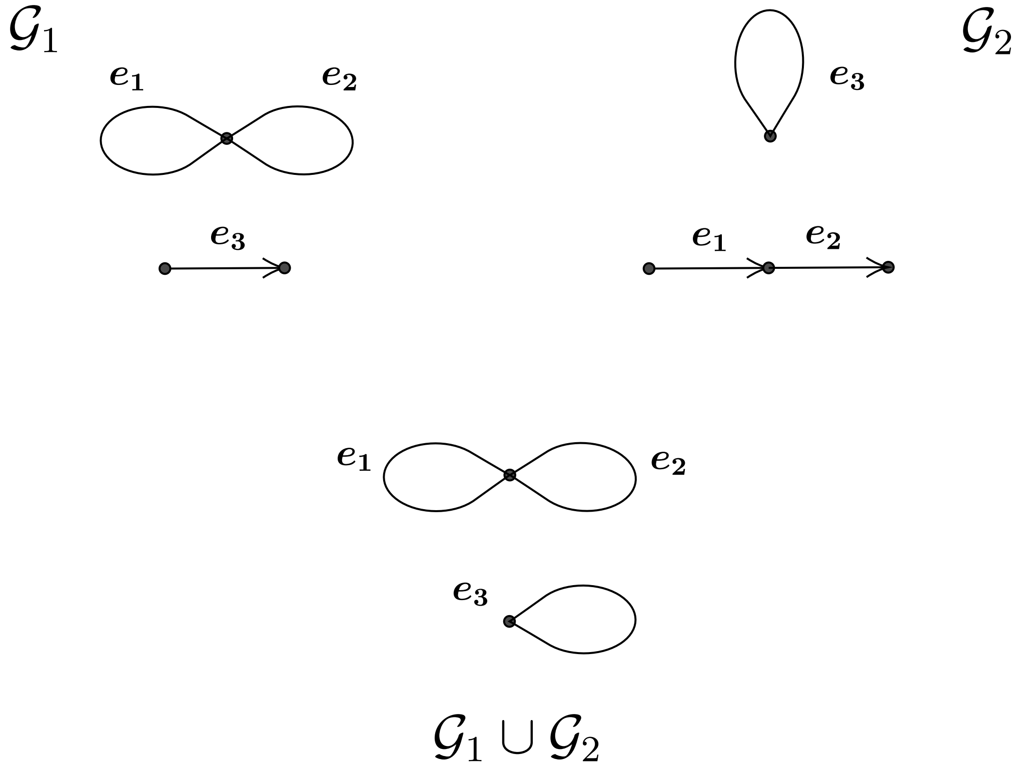


Figure 4.9: Example of two disconnected graphs with disconnected union: in this case the limit heat distribution assumes two different constant values on e_1, e_2 and on e_3.

Chapter 5

Final remarks and conclusions

As we have seen, the asympotic behaviour of the proposed evolution problems can be described in terms of the convergence of infinite products of matrices and, more generally, contractive operators on Hilbert spaces. We have often mentioned and directly applied results in [**1**, **12**] about random products of contractive operators, however there is a wide theory about the convergence of infinite and deterministic products of matrices, where the fundamental papers [**9**, **10**] by I. Daubechies and J. C. Lagarias are the most representative. They have introduced the following

Definition 5.1. A set $\Sigma = \{M_1, \ldots, M_N\}$ of complex square matrices is called *LCP set* (left-convergent product) if all the infinite products whose factors are in Σ are left-convergent (with respect to the matrix norm).

In these articles, they provide a set of necessary and sufficient conditions in order that Σ is *LCP* by means of the eigenvalues, the eigenspaces and the spectral radii of the involved matrices. In particular, we recall Theorem 4.1 in [**9**]

Theorem 5.2. *Let $\Sigma = \{M_1, \ldots, M_N\}$ be a finite set of square matrices. Then the following are equivalent.*

(i) Σ is LCP set whose limit function is identically 0 for every possible infinite product.

(ii) The joint spectral radius satisfies

$$\hat{\rho}(\Sigma) := \limsup_{k\to+\infty} \left(\hat{\rho}_k(\Sigma)\right)^{\frac{1}{k}} < 1,$$

where[1]

$$\hat{\rho}_k(\Sigma) := \sup\left\{ \left\| \prod_{i=1}^{k} M_i \right\| \ : \ M_i \in \Sigma,\ 1 \le i \le k \right\}.$$

[1]Being the supremum of a finite set, $\hat{\rho}_k(\Sigma)$ is actually a maximum.

The problems studied by Daubechies and Lagarias are exactly the starting point and the general framework of the models introduced in this thesis. In fact, concerning the finite-dimensional setting, let

$$\Sigma = \left\{ M_1 = Qe^{-\mathcal{L}(G_1)}, \dots, M_N = Qe^{-\mathcal{L}(G_N)} \right\},$$

where each one of these matrices is related to one graph on which we study the random evolution. Hence, we recall that problem (P), namely proving that the evolution operator $S(t)$ converges uniformly towards the orthogonal projection P_0 onto the subspace of constant functions, is equivalent to showing that

$$\lim_{t \to +\infty} \|QS(t)\| = 0.$$

Since $QS(t)$ is expressed by means of a product of matrices in Σ, then it immediately follows that (P) can be solved referring to the *LCP* property of Σ.

However, if we only consider the results in [**9**, **10**] and in particular Theorem 5.2, we do not obtain satisfying results, because it turns out that Σ is *LCP* under restrective hypotheses. More precisely, we recall that for Proposition 2.7 the evolution operator converges uniformly to P_0 for every possible sequence given by $\{j_k\}_{k \geq 0}$ (thus, Σ is *LCP*) provided that every graph in $\mathcal{C}$ is connected. This can also be verified considering the joint spectral radius of Σ. In fact, according to Lemma 2.6, we know that

$$\|M_i\| < 1, \qquad \forall M_i \in \Sigma,$$

hence

$$\left\| \prod_{i=1}^{k} M_i \right\| \leq \prod_{i=1}^{k} \|M_i\| < 1, \qquad \forall M_i \in \Sigma,\ 1 \leq i \leq k,\ \forall k$$

and taking the supremum

$$\hat{\rho}_k(\Sigma) < 1.$$

This implies that

$$\hat{\rho}(\Sigma) = \limsup_{k \to +\infty} \left(\hat{\rho}_k(\Sigma) \right)^{\frac{1}{k}} < 1,$$

then Theorem 5.2 confirms the statement of Proposition 2.7.

The *LCP* property no longer holds when at least one graph in the family $\mathcal{C}$ is not connected. In fact, assuming that the only disconnected graph is G_1 and denoting by $\mathcal{L}_1$ its Laplacian, one gets that

$$M_1 = Qe^{-\mathcal{L}_1}$$

has norm 1. This is easily shown by taking $f \in \operatorname{rg} Q \cap \ker \mathcal{L}_1$ (which is non empty if G is non connected), in fact

$$M_1 f = f \implies \|M_1\| = 1.$$

Taking the same f as before, we can prove that every kth power of M_1 has norm 1 and then

$$\hat{\rho}_k(\Sigma) \geq \|M_1^k\| = 1.$$

Additionally, since we are dealing with contractions, it also holds that

$$\hat{\rho}_k(\Sigma) \leq 1,$$

hence

$$\hat{\rho}_k(\Sigma) = 1$$

and condition (ii) is not satisfied: we conclude that Σ is not *LCP* in this case.

In order to accept more general families $\mathcal{C}$, where graphs are not supposed to be all connected, one has to leave the framework of *LCP* sets. In fact, we have decided to weaken the notion of convergence: instead of analyzing deterministic non autonomous diffusion models, from the beginning we requested that a Markov chain leads the evolution. Consequently, we have studied the almost sure convergence of the stochastic operator $S(t)$, excluding all those products which happen with probability 0.

This suggests to introduce the following definition, as a stochastic version of the notion according to Definition 5.1 of *LCP* set proposed by Daubechies and Lagarias.

Definition 5.3. Let $\Sigma = \{M_1, \ldots, M_N\}$ be a set of square matrices and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Moreover, let $j = \{j_n\}_{n\geq 0}$ be a stochastic process with values in the set of indexes $\sigma = \{1, \ldots, N\}$.

We say that Σ is *stochastically LCP* if the limit product

$$M(j) = \prod_{n=0}^{\infty} M_{j_n} := \lim_{n\to\infty} M_{j_n} M_{j_{n-1}} \cdots M_{j_0}$$

exists $\mathbb{P}$-almost surely. We say that Σ is *stochastically* LCP_0 set if it is *LCP* and additionally $M(j) = 0$ $\mathbb{P}$-almost surely.

This is clearly weaker than the *LCP* property, because it only requires the convergence of the sequences with positive probability. However, it is the most suitable notion to consider in the framework of the random evolution models. In this way, imposing certain conditions on the stochastic process $\{j_n\}_{n\geq 0}$, we are able to present the convergence results proved in Chapter 2 by means of the *stochastic LCP* property.

Theorem 5.4. *Let $\Sigma = \{e^{-\mathcal{L}(G_1)}, \ldots, e^{-\mathcal{L}(G_N)}\}$ and consider the framework of the random evolution model (DRE) on combinatorial graphs in $\mathcal{C} = \{G_1, \ldots, G_N\}$, governed by an irreducible Markov chain $\{j_k\}_{k\geq 0}$.*

Then the following are equivalent.

(i) Σ is stochastically LCP with limit product $M(j) = P_0$.

(ii) The union graph $G = (V, E_1 \cup \ldots \cup E_N)$ is connected.

Equivalently, given $Q\Sigma = \{Qe^{-\mathcal{L}(G_1)}, \ldots Qe^{-\mathcal{L}(G_N)}\}$, the following are equivalent.

(i) $Q\Sigma$ is stochastically LCP_0.

(ii) The union graph $G = (V, E_1 \cup \ldots \cup E_N)$ is connected.

In a similar way, one can extend the definition of the *stochastic LCP* property to contractive operators on infinite-dimensional Hilbert spaces and, hence, obtain the version of Theorem 5.4 for the random evolution model on metric graphs.

Since the stochastic LCP property provides an elegant way to express all the results of our work, this suggests that many other evolution problems, even not related to graphs, could be solved in the same terms. In fact, it would be challenging to find necessary and/or sufficient conditions to ensure the stochastic LCP notion for the set Σ of all the matrices (or bounded operators) composing a certain evolution operator $S(t)$.

Concerning the possible modifications and improvements starting from the random evolution models proposed in this thesis, there are several open problems to be discussed in the next future. First of all, we have proved the convergence of $S(t)$ towards an equilibrium in both the combinatorial and the metric settings, but we have not yet investigated the rate of convergence. Hence, the first goal to achieve is to estimate it, based on the knowledge from the diffusion models on a fixed graph, where the speed of convergence is well-known. Intuitively, this rate will still depend on the spectral gap of every Laplace operator, but we guess that it will be lower than in the problem on one fixed network. In fact, we imagine that the continuous jumps from one to another graph can slow down the diffusion and then the convergence process towards P_0.

Another challenging work is to make the stochasticity of the models more complicated, replacing the simple Markov chain with a different Markov process. The first idea is to choose a continuous time pure jump Markov process: all the good properties of the Markov chain would be preserved, but the waiting times between two consecutive jumps would be random variables, too. We guess that everything will still work, even if the framework is now more complicated. In fact, as a consequence, every factor in the product $S(t)$

would depend on two types of stochasticity: one provided by the chain governing the jumps, the other one represented by the times. This would mean to deal with a set of matrices/operators Σ which is no longer finite, hence more sophisticated arguments would be needed to prove the stochastic LCP property.

Regarding the random evolution on metric graphs, it would be interesting to consider more general operators instead of the Kirchhoff-Laplace one. We could consider different elliptic operators: based on the well-known results about PDEs on fixed networks, it should not be difficult to generalize our results. Alternatively, we could still deal with the Laplacian and consider different vertex conditions which randomly vary in time. We intend to find which additional hypotheses we need to impose in order to have analogous results.

In this way, we could obtain evolution models capable of describing more complicated scenarios. For instance, one could apply them in the area of epidemiology: in our opinion, it would be interesting to study the diffusion of an epidemic disease among individuals whose interactions – represented by graphs – changing randomly in time. This is just an example of application: in fact, graphs are studied in several areas within physics and chemistry, but also in sociology, ecology, neurobiology, etc., because they can depict several types of real networks and, as it happens in most cases, the latters are not constant during the time.

For this reason, having dynamical systems capable of dealing with the (random) time evolution of newtworks would be challenging and useful. In this context, the results presented in our work can be considered as one simple and first step towards more complicated and sophisticated models.

Appendix A

Some notions of Markov chains

In this appendix, we are going to show some results about Markov chains we have used in the proof of Theorem 2.14. In order to give a clear presentation, we are going to breafly recall some important properties, too. We assume the basic notions about Markov chains to be known: the reader can refer for instance to [**4**, **18**] for a detailed presentation of the topic.

A.1 Preliminaries

Let $\{X_n\}$ be a Markov chain with state space E and transition matrix T. We assume that X is irreducible, i.e. for any initial state x there exists a positive probability of hitting any state y in finite time,

$$\mathbb{P}(T_y < \infty \mid X_0 = x) = H(x,y) > 0, \qquad (x,y) \in E \times E.$$

Alternatively, we say that X is irreducible if for any couple of states (x,y) there exist times $n, m \in \mathbb{N}$ such that $T^n(x,y)T^m(y,x) > 0$. We say that a state is *recurrent* if $H(x,x) = 1$. In a finite chain, there always exists a recurrent state. Further, if the chain is irreducible, then all the states are recurrent.

Let us introduce the *mean return time* $\mu(y) = \mathbb{E}[T_y \mid X_0 = y]$. Intuitively, its inverse is the frequency of returns: the following result makes precise this intuition.

Theorem A.1. *Denote $N_{y;n}$ the number of passages in the state y up to time n:*

$$N_{y;n} = \#\{k \le n \,:\, X_k = y\}.$$

Then

$$\mathbb{P}\left(\frac{N_{y;n}}{n} \to \frac{1}{\mu(y)} \ \textit{as}\ n \to \infty \mid X_0 = x\right) = H(x,y).$$

In particular, if $H(x,y) = 1$ *then* $\mathbb{P}(N_y = \infty \mid X_0 = x) = 1$.

A.1.1 Period

The period of a state is the gcd (greatest common divisor) of the set of times when a return is possible:

$$d(x) = \gcd\left\{n > 1 \,:\, T^n(x,x) > 0\right\}.$$

Thus, a state has period d if and only if the chain can return to x only at multiples of the period d, and d is the largest such integer. A state x is *aperiodic* if $d(x) = 1$ and *periodic* otherwise. If the Markov chain is irreducible then all the states has the same period; we can speak then about the period of the chain (or, the chain being periodic or aperiodic).

If a Markov chain is periodic, then we can partition the state space in classes, according to the period. Let x_0 be an arbitrary state in E. Then for any $k = 0, 1, \ldots, d-1$ we let

$$A_k = \{x \in E \,:\, \exists\, n \geq 0 \text{ s.t. } T^{nd+k}(x_0, x) > 0\}.$$

Notice that for $x \in A_k$ and $y \in A_j$ then $T^n(x,y) > 0$ implies that $n \equiv j - k (\mathrm{mod}\ d)$.

If we consider the d-step Markov chain $\{Y_n = X_{nd},\ n \geq 0\}$ on E with transition matrix $T' = T^d$, then we see that it is reducible, and its final classes are exactly the cyclic classes $A_0, \ldots, A_{d-1}$. In particular, the reduced chain on every class is irreducible and aperiodic.

A.1.2 Invariant distribution

Given an irreducible Markov chain X, positively recurrent (in particular, this is the case if the state space is finite) then there exists a positive distribution π on E that is left invariant for the transition matrix T: $\pi T = \pi$. Moreover, π is explicitly given in terms of the mean return times:

$$\pi(x) = \frac{1}{\mu(x)}, \qquad x \in E.$$

In the above assumptions, we can prove the following almost sure convergence:

$$T^n(x, y) \to \frac{1}{\mu(x)} \qquad \text{as } n \to \infty.$$

Let us consider what this means for the case of periodic chains. We first recall that the d-step chain Y is reducible, which means that we shall consider it separately on the various cyclic classes. Then $Y\big|_{A_k}$ is irreducible, aperiodic with transition matrix $(T^d)\big|_{A_k}$. It follows that

$$\big((T^d)\big|_{A_k} \big)^n (x, y) \to \frac{1}{\mu_d(y)}, \qquad x, y \in A_k,$$

where the mean return time $\mu_d(y)$ is computed with respect to the reduced matrix $(T^d)\big|_{A_k}$. But then the mean return time for the original Markov chain X is d-times the previous one:

$$T^{nd}(x, y) \to \frac{d}{\mu(y)}, \qquad x, y \in A_k. \tag{A.1}$$

Notice that, being x and y in the same cyclic class, it also holds

$$T^{nd+k}(x, y) = 0, \qquad k = 1, \dots, d-1.$$

Moreover, if the states are in different cyclic classes, it holds

$$T^{nd+(j-k)}(x, y) \to \frac{d}{\mu(y)}, \qquad x \in A_k,\ y \in A_j. \tag{A.2}$$

A.2 Multistep chains

In this section, we consider a Markov chain X on a finite state space E. For simplicity we assume that X is irreducible and, in a first moment, that it is aperiodic. This Markov chain induces another process Y by setting

$$Y_0 = (X_0, X_1), \quad Y_1 = (X_2, X_3), \quad \dots \quad Y_k = (X_{2k}, X_{2k+1}).$$

The state space is the subset $E_2 \subset E \times E$ of admissible pairs:

$$E_2 = \{(x_1, x_2) \,:\, T(x_1, x_2) > 0\}.$$

The transition probability matrix associated to the Markov chain Y is

$$P(\underline{x}, \underline{y}) = T(x_2, y_1)T(y_1, y_2).$$

Moreover,

$$P^n(\underline{x}, \underline{y}) = T^{2n-1}(x_2, y_1)T(y_1, y_2).$$

Since the original chain is irreducible and aperiodic, definitely $T^{2n-1}(x_2, y_1) > 0$, hence Y is irreducible and aperiodic as well. In particular, given any admissible pair $\underline{a} = (a_1, a_2)$, the chain Y visits this state infinite times, a.s..

Previous construction can be extended to the case of vectors of length k without difficulty. Instead, we are now interested to discuss the case of periodic chains.

Let d be the period of the irreducible chain X. We consider sequences of length k where $d \mid k$ and introduce the space E_k of admissible sequences

$$E_k = \{\underline{x} = (x_0, x_1, \dots, x_{k-1}), \colon\ p(\underline{x}) = \prod T(x_j, x_{j+1}) > 0\}.$$

If $\underline{x}$ is an admissible sequence then there exists ℓ such that

$$x_0 \in A_\ell, x_1 \in A_{\ell+1}, \dots, x_j \in A_{\ell+j}, \dots, x_{k-1} \in A_{\ell-1},$$

where we assume the indexes are computed modulo d. The transition probability matrix associated to the Markov chain Y is

$$P(\underline{x}, \underline{y}) = T(x_{k-1}, y_0)p(\underline{y}).$$

Notice that by formula (A.2) it follows that

$$P^n(\underline{x}, \underline{y}) = T^{(n-1)k+1}(x_{k-1}, y_0)p(\underline{y})$$

is positive for n large enough; therefore, the Markov chain Y is irreducible and aperiodic, and every admissible sequence is visited infinite times, a.s..

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Acknowledgements

This thesis owes a lot to so many people who on several occasions and in different ways have accompanied and supported me and who I now would like to thank.

First of all, I wish to express my deep gratitude to my supervisors, Professor Stefano Bonaccorsi and Professor Delio Mugnolo. The first encouraged and convinced me to live the experience of research abroad; the second welcomed me generously in his department at the FernUniversität in Hagen (Germany). I am grateful to both of them for involving me in the project they were working on: thanks to their competence, their patient guidance and their time, I was able to give my best and learn a lot. I thank them for their trust in me, for giving me challenges that even a year ago I would have never imagined I could face. Thanks to them I have revived my enthusiasm for mathematics: they have been and will always be a great example, which will inspire me along my academic path.

I am also grateful to my Bachelor's supervisor, Professor Alfredo Marzocchi. Although I am no longer officially a student at his University and even though our fields of research are different, he has always been available to help and advice me: he is a precious guide.

In addition, I would like to thank all the professors, the researchers and the staff I met during my stay at the FernUniversität in Hagen. From the beginning I was involved in their community, where I found a culturally lively atmosphere. A special thank you goes to Marvin Plümer, who during my work on this thesis has always been available to help me and to clarify my doubts, both during the months in Germany and after my return to Italy.

My gratitude goes out to all my friends who have stayed by my side during these years full of changes. They are spread around Italy and Europe and busy with the most diverse projects, yet they are there for me – together we share the joy of our achievements and support each other during moments of despair.

Finally, thanks to my family, especially to my father, my mother and my brother Alessandro, for their patience, for their support and for being always interested in my work. Although they are not mathematicians, now they too, know what a graph is.

Francesca