Li–Yau inequalities for the Helfrich functional

joint work with Christian Scharrer (MPIM Bonn)

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Outline

Geometric background

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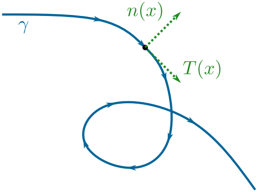

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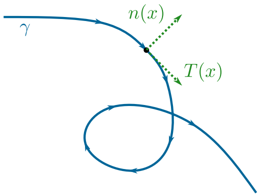

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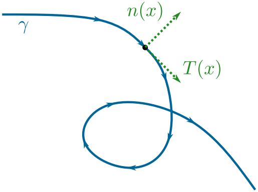

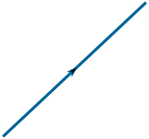

Figure: A straight line: $\kappa \equiv 0$.

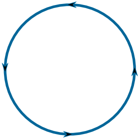

Figure: A circle: $\kappa \equiv \frac{1}{r}$.

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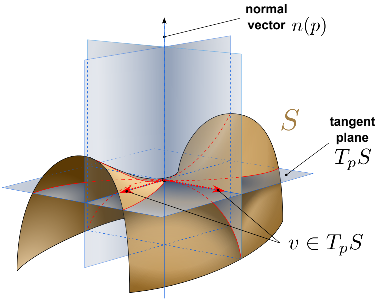

Figure: [a]

[a] Modified illustration of Eric Gaba at commons.wikimedia.org

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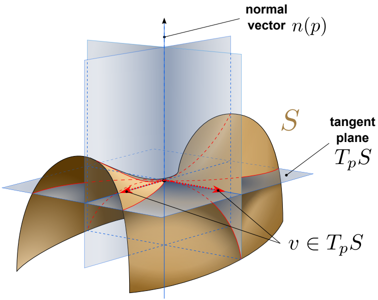

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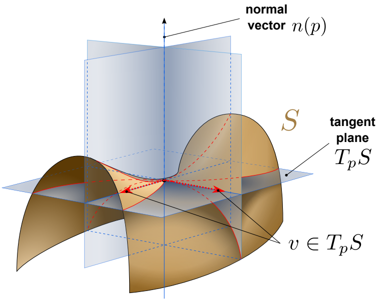

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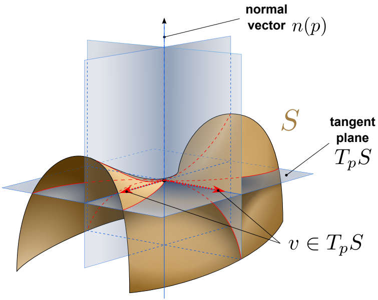

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- mean curvature $H := \kappa_1 + \kappa_2$, Gauss curvature $K := \kappa_1\kappa_2$

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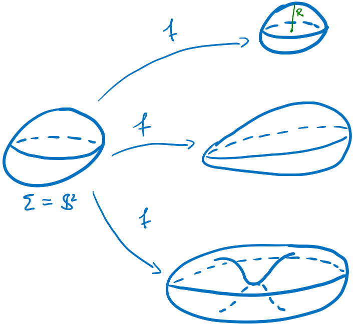

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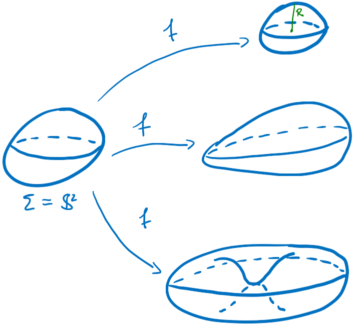

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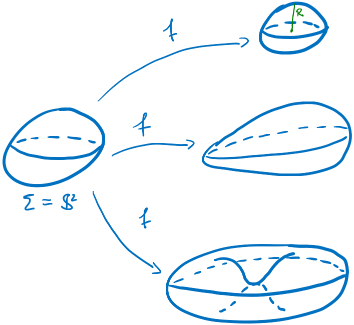

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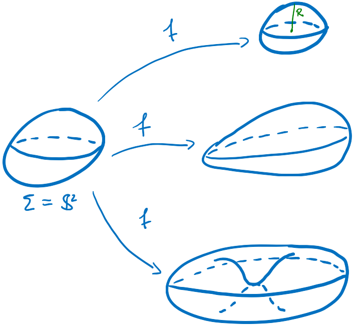

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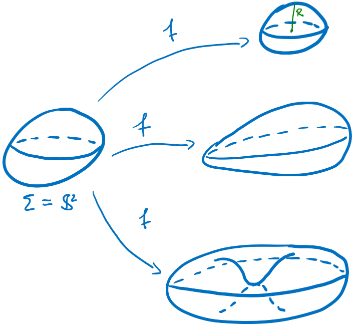

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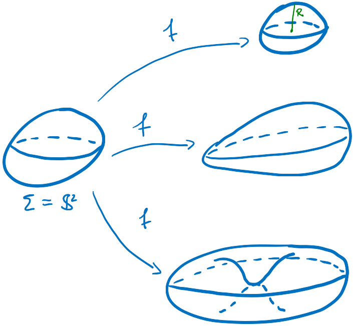

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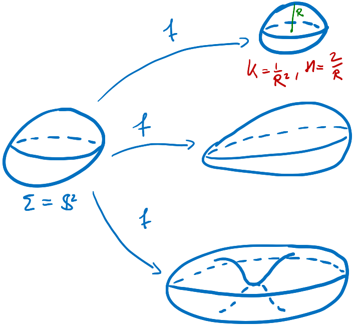

The Helfrich and Willmore functionals

The shape of biomembranes

- Canham–Helfrich model [Canham '70], [Helfrich '73]:
  Lipid bilayers are critical for the energy

$$\mathcal{H}_{c_0,\bar{k}_c}(f) := \int_\Sigma \left(\frac{1}{4}(H - c_0)^2 + \bar{k}_c K \right) \mathrm{d}\mu$$

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$$\mathcal{H}_{c_0}(f) = \frac{1}{4}\int_\Sigma (H - c_0)^2 \, \mathrm{d}\mu.$$

- If $c_0 = 0$, this is the Willmore energy

$$\mathcal{W}(f) := \frac{1}{4}\int_\Sigma H^2 \, \mathrm{d}\mu.$$

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Variational Canham–Helfrich problem

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where $g \in \mathbb{N}_0$ and $A_0, V_0 > 0$ satisfy the isoperimetric inequality $36\pi V_0^2 \leq A_0^3$.

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- $c_0 \neq 0$: Existence of varifold minimizers [Brazda–Lussardi–Stefanelli '19], immersed bubble trees [Mondino—Scharrer '20].

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Motivating question

Does the Helfrich energy $\mathcal{H}_{c_0}$ allow for an inequality like (1)?

A Li–Yau inequality and applications

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Question

Can we find $C > 0$ such that for immersions $f\colon \Sigma \to \mathbb{R}^3$ we have

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- For $|c_0|$ small, one may use the Li–Yau inequality for $\mathcal{W}$ and $\lim_{r\searrow 0}\mathcal{H}_{c_0}(rf) = \mathcal{W}(f)$.

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Let $f : \Sigma \to \mathbb{R}^3$ be a smooth immersion of an oriented closed surface Σ. Let $c_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^3$ and define the concentrated volume

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Let $f: \Sigma \to \mathbb{R}^3$ be a smooth immersion of an oriented closed surface Σ. Let $c_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^3$ and define the concentrated volume

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- Asymptotically sharp for spheres

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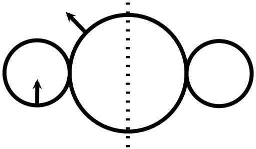

Surfaces with a divergence theorem

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Let $f\colon \Sigma \to \mathbb{R}^3$ be an Alexandrov immersion, i.e. there exists a compact 3-manifold M with $\partial M = \Sigma$, the inner unit normal field ν along Σ and an immersion $F\colon M \to \mathbb{R}^3$ with $f = F|_\Sigma$ and $n = \mathrm{d}F(\nu)$.

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If $c_0 < 0$ then $\nexists \min \mathcal{H}_{c_0}^{\lambda,p}(f)$ among embeddings $f \in C^\infty(\mathbb{S}^2;\mathbb{R}^3)$.

(Non-)examples of Alexandrov immersions

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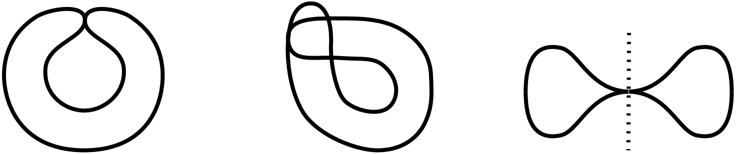
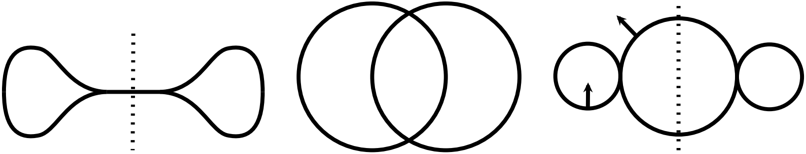


(a) (b) (c)



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An application to the variational model

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Theorem (R.–Scharrer '22)

Let $c_0 \in \mathbb{R}$ *and suppose* $A_0, V_0 > 0$ *satisfy the isoperimetric inequality* $36\pi V_0^2 \leq A_0^3$. *Let*

$$\eta(c_0, A_0, V_0) := \inf \left\{ \mathcal{H}_{c_0}(f) \mid f \in C^\infty(\mathbb{S}^2; \mathbb{R}^3) \text{ embedding}, \mathcal{A}(f) = A_0, \mathcal{V}(f) = V_0 \right\}. \quad (3)$$

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Then there exists $\Gamma(c_0, A_0, V_0) > 0$ *such that if*

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- For all $c_0 \leq 0$ there exist $A_0, V_0 > 0$ such that $\eta(c_0, A_0, V_0) < 8\pi$.
- Existence of smoothly embedded minimizers for the Canham–Helfrich model for $c_0 \leq 0$ and $\Sigma = \mathbb{S}^2$ if (4) holds.

Sketch of the proof of the Li–Yau inequality

Review of the result

Theorem (R.–Scharrer, '22)

Let $f\colon \Sigma \to \mathbb{R}^3$ be a smooth immersion of an oriented closed surface Σ, let $c_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^3$. Then

$$\#f^{-1}(x_0) \leq \frac{1}{4\pi}\mathcal{H}_{c_0}(f) + \frac{c_0}{2\pi}\mathcal{V}_c(f, x_0).$$

where $\mathcal{V}_c(f, x_0) := -\int_\Sigma \frac{\langle f - x_0, n\rangle}{|f - x_0|^2}\,\mathrm{d}\mu$ denotes the concentrated volume.

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Idea of the proof:

$$\#f^{-1}(x_0) = \lim_{\rho\to 0}\frac{\mu(f^{-1}(B_\rho(x_0)))}{\pi\rho^2}$$

and use first variation of area [Simon '93].

The first variation formula

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Let $0 < \sigma < \rho$, $x_0 = 0 \in \mathbb{R}^3$. Consider

$$\varphi(t) := \left(\max\{t, \sigma\}^{-2} - \rho^{-2}\right)_+,$$

define $X(x) = \varphi(|x|)x$.

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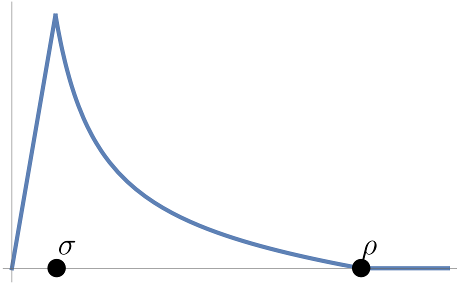

Figure: Plot of $|X(x)|$.

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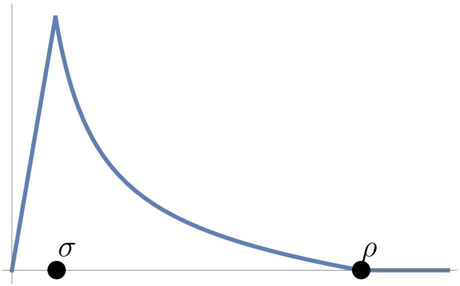
$$\int_\Sigma \operatorname{div}_\top X \circ f \,\mathrm{d}\mu = -\int_\Sigma \langle X \circ f, \vec{H}\rangle \,\mathrm{d}\mu.$$



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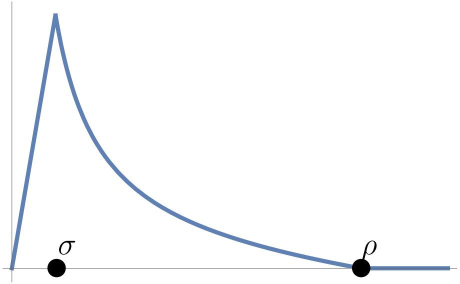

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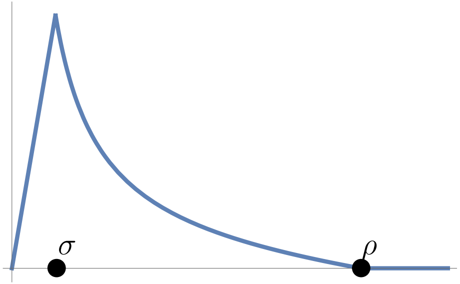

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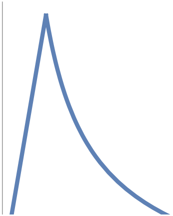

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$$\frac{\langle f, \vec{H} - c_0 n \rangle}{|f|^2} + \frac{2\langle f, n \rangle^2}{|f|^4} = 2\left|\frac{1}{4}(\vec{H} - c_0 n) + \frac{\langle f, n \rangle n}{|f|^2}\right|^2 - \frac{1}{8}|\vec{H} - c_0 n|^2$$

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$$\gamma(\rho) := \frac{\mu(\hat{B}_\rho)}{\rho^2} + \frac{1}{16}\int_{\hat{B}_\rho}|H-c_0 n|^2\,\mathrm{d}\mu - \frac{c_0}{2}\int_{\hat{B}_\rho}\frac{\langle f,n\rangle}{|f|^2}\,\mathrm{d}\mu + \frac{1}{2\rho^2}\int_{\hat{B}_\rho}\langle f,H\rangle\,\mathrm{d}\mu.$$

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- $\lim_{\rho\to\infty}\gamma(\rho) = \frac{\mathcal{H}_{c_0}(f)}{4} + \frac{c_0}{2}\mathcal{V}_c(f, x_0)$.

A monotonicity argument

Hence, we find

$$\frac{2\mu(\hat{B}_\sigma)}{\sigma^2} + \int_{\hat{B}_\rho\setminus\hat{B}_\sigma} 2\left|\frac{1}{4}(\vec{H} - c_0 n) + \frac{\langle f, n\rangle n}{|f|^2}\right|^2 \mathrm{d}\mu = \frac{2\mu(\hat{B}_\rho)}{\rho^2} - \sigma^{-2}\int_{\hat{B}_\sigma}\langle f, \vec{H}\rangle\,\mathrm{d}\mu$$
$$+\rho^{-2}\int_{\hat{B}_\rho}\langle f, \vec{H}\rangle\,\mathrm{d}\mu - c_0\int_{\hat{B}_\rho\setminus\hat{B}_\sigma}|f|^{-2}\langle f, n\rangle\,\mathrm{d}\mu + \frac{1}{8}\int_{\hat{B}_\rho\setminus\hat{B}_\sigma}\left|\vec{H} - c_0 n\right|^2\mathrm{d}\mu.$$

In particular, the following function is monotonically nondecreasing

$$\gamma(\rho) := \frac{\mu(\hat{B}_\rho)}{\rho^2} + \frac{1}{16}\int_{\hat{B}_\rho}|H - c_0 n|^2\,\mathrm{d}\mu - \frac{c_0}{2}\int_{\hat{B}_\rho}\frac{\langle f, n\rangle}{|f|^2}\,\mathrm{d}\mu + \frac{1}{2\rho^2}\int_{\hat{B}_\rho}\langle f, H\rangle\,\mathrm{d}\mu.$$

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$$\Rightarrow \#f^{-1}(x_0) \leq \frac{\mathcal{H}_{c_0}(f)}{4\pi} + \frac{c_0}{2\pi}\mathcal{V}_c(f, x_0).$$

□

Thank you for your attention!

If time allows. . .

Corollary (scale-invariant version)

Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion of a closed oriented surface. Then for all $x_0 \in \mathbb{R}^3$

$$\#f^{-1}(x_0) \leq \frac{1}{4\pi}\bar{\mathcal{H}}(f) + \frac{\bar{H}\mathcal{V}_c(f,x_0)}{2\pi} - \frac{\mathcal{V}_c(f,x_0)^2}{\pi\mathcal{A}(f)}.$$

Here $\bar{H} := \int_\Sigma H \,\mathrm{d}\mu$ and $\bar{\mathcal{H}}(f) := \inf_{c_0\in\mathbb{R}} \mathcal{H}_{c_0}(f) = \frac{1}{4}\int_\Sigma (H-\bar{H})^2 \,\mathrm{d}\mu$.

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- Minkowski inequality: If $\Omega \subset \mathbb{R}^3$ is bounded, open, convex with C^2-boundary, then

$$\frac{1}{2}\int_{\partial\Omega} H \,\mathrm{d}\mathcal{H}^2 \geq \sqrt{4\pi \mathcal{H}^2(\partial\Omega)}.$$

If time allows...

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- If for some $x_0 \in \mathbb{R}^3$ we have $\mathcal{V}_c(f,x_0) > 0$ and $\bar{\mathcal{H}}(f) \leq 4\pi\#f^{-1}(x_0)$, then

$$\frac{1}{2}\int_\Sigma H \,\mathrm{d}\mu \geq \sqrt{\Big(4\pi\#f^{-1}(x_0) - \bar{\mathcal{H}}(f)\Big)\mathcal{A}(f)} \geq 0.$$