Three Dimensional Elastic Frames: Rigid Joint Conditions In Variational And Differential Formulation

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Overview

1. **Introduction and Motivation**
   - Beam Structures in Practice
   - Planar Frames and Matching Vertex Conditions
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2. **General Three Dimensional Graphs**
   - Parameterization of Beam Deformation
   - Full Description of Euler–Bernoulli Energy Functional
3. **Energy Form and Differential Operator**
   - Quadratic Form and Vertex Conditions
   - Hamiltonian on Graph and Vertex Conditions
   - Decoupling of Fields for Planar Graph
4. **Symmetry and Irreducible Representations**
   - Numerical Results and Discussion
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Section 1

Beam Structures in Practice

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Beam Structures in Practice

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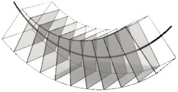
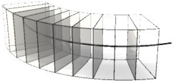
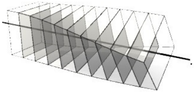
2. Each beam is described by (Euler-Bernoulli) energy functional considering energy of:



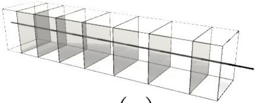
$v(x)$



$w(x)$



$\eta(x)$



$u(x)$

Matching Vertex Conditions in Planar Graph

1. Energy functional corresponding to network of beams $\Gamma = (E, V)$ is given by

$$\Pi = \frac{1}{2} \sum_{e \in E} \int_e a_e(x) \left| v_e''(x) \right|^2 dx.$$

- Domain of Π consists of functions $v \in \bigoplus_{e \in E} H^2(e)$ that satisfy *certain* vertex conditions.
- One way is to assign analogue of *standard* vertex conditions introduced for the Laplacian (e.g. see the work by B. Dekoninck and S. Nicaise, 2000)

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- In order to study the spectral gap, the following vertex conditions are assumed (see the work by P. Kurasov and J. Muller, 2020):

$$v_i(c) = v_j(c) \quad \text{when } e_i, e_j \text{ adjacent to } c$$
$$v_i'(c) = 0$$

2. The corresponding Beam operators, mapping $v_e \mapsto a_e v_e''''$, is defined from $v \in \bigoplus_{e \in E} H^4(e)$ satisfying at each vertex

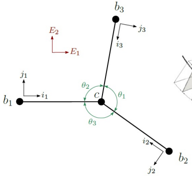
$$v_i(c) = v_j(c) \quad \text{when } e_i, e_j \text{ adjacent to } c$$
$$v_i'(c) = 0$$
$$\sum_{e_i \sim c} v_i'''(c) = 0$$

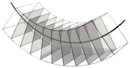
Matching Vertex Conditions in Planar Graph

Energy functional and corresponding vertex conditions

$$\Pi = \frac{1}{2}\sum_{e\in E}\int_e a_e(x)\left|v_e''(x)\right|^2 dx.$$

1. Vertex conditions on the quadratic form (see the work by J.C. Kiik, P. Kurasov, and M. Usman, 2015)
   - $v_1(c) = v_2(c) = v_3(c)$
   - $\sin(\theta_1)v_1'(c) + \sin(\theta_2)v_2'(c) + \sin(\theta_3)v_3'(c) = 0$





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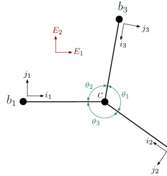

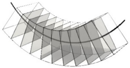

2. Corresponding self-adjoint Beam operator, mapping $v_e \mapsto a_e v_e''''$, on every edge $e \in E$ satisfying (in addition to above conditions)
   - $\frac{v_1''(c)}{\sin(\theta_1)} = \frac{v_2''(c)}{\sin(\theta_2)} = \frac{v_3''(c)}{\sin(\theta_3)}$
   - $v_1'''(c) + v_2'''(c) + v_3'''(c) = 0$

Questions Regarding Generalization?

1. Role of Degrees of Freedom

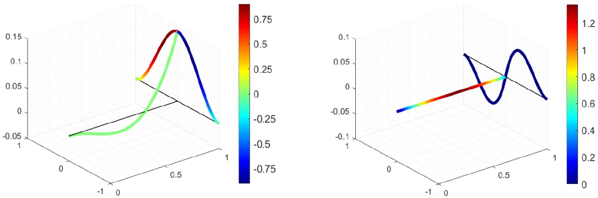

Figure: Eigenfunctions corresponding first and second eigenvalues.

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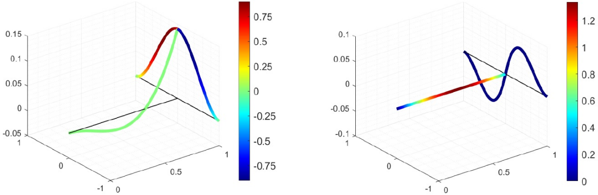
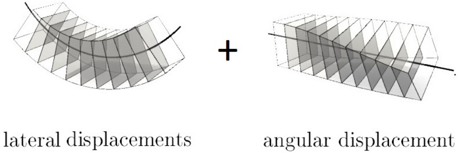


Figure: Eigenfunctions corresponding first and second eigenvalues.

From Scalar to Vector Quantities



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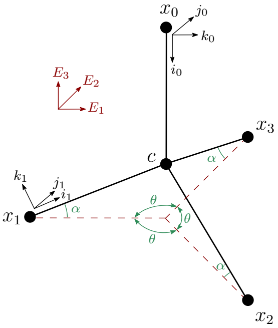

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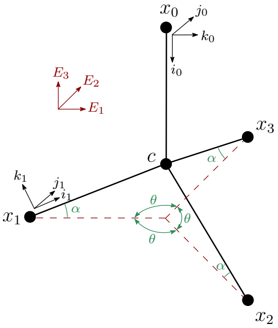
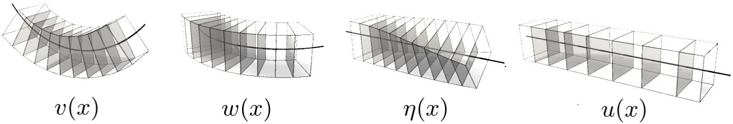


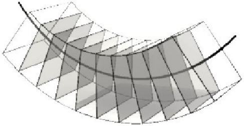
2. Including all Degrees of Freedom



Section 2

Parameterization of Beam Deformation

1. Euler-Bernoulli hypothesis:
   - Plane sections remain plane,
   - Geometry of the spatial beam is described by the **centroid line** and a family of the corresponding **cross-sections**.



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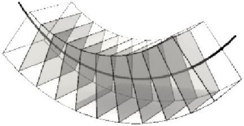

2. Description of problem in basis:
   - Orthonormal basis $\{\vec{E}_1, \vec{E}_2, \vec{E}_3\}$ span the physical space in which the beam is embedded,
   - Orthonormal basis $\{\vec{i}, \vec{j}, \vec{k}\}$ describes orientation of the cross section of beam
3. Deformed configuration fully described by:
   - Position vector $\vec{g}(x)$ with x representing the arc-length coordinate,
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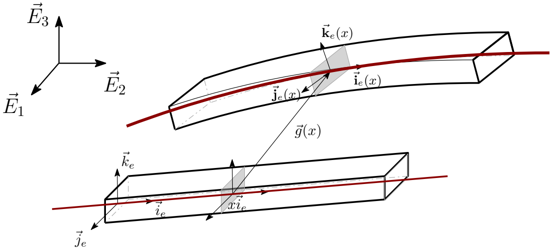

Rigid Vertex Condition

1. The relationship between the cross-section basis in the initial un-deformed and the deformed configurations can be expressed through $\mathcal{R}(x) \in \mathrm{SO}(3)$

$$\vec{\boldsymbol{i}}(x) = \mathcal{R}(x)\vec{i}, \qquad \vec{\boldsymbol{j}}(x) = \mathcal{R}(x)\vec{j}, \qquad \vec{\boldsymbol{k}}(x) = \mathcal{R}(x)\vec{k}$$

2. Introduce (linearized)-rotation vector $\vec{\omega}(x) := \alpha\vec{\vartheta}(x)$

$$\vec{\boldsymbol{i}}(x) = \vec{i} + \vec{\omega}(x) \times \vec{i}, \qquad \vec{\boldsymbol{j}}(x) = \vec{j} + \vec{\omega}(x) \times \vec{j}, \qquad \vec{\boldsymbol{k}}(x) = \vec{k} + \vec{\omega}(x) \times \vec{k}$$

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Definition

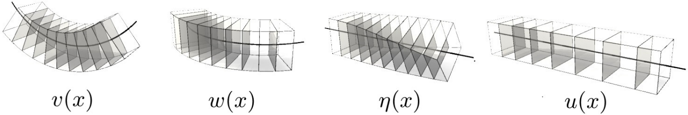
A joint v with n incident beans $\{e_i\}_{i=1}^{n}$ is called **rigid**, if the displacement and rotation vectors on beams e_i satisfy

$$\vec{g}_1(v) = \cdots = \vec{g}_n(v), \quad \text{and} \quad \vec{\omega}_1(v) = \cdots = \vec{\omega}_n(v)$$

Towards General Graph $\Gamma = (V, E)$

1. Kinematic Bernoulli assumptions for beam frame:
   - no vertex energy, pre-sress, or external force

$$\mathcal{U}^{(\Gamma)} = \frac{1}{2} \sum_{e \in E} \int_e \Big(a_e(x) \big| v_e''(x) \big|^2 + b_e(x) \big| w_e''(x) \big|^2 + c_e(x) \big| u_e'(x) \big|^2 + d_e(x) \big| \eta_e'(x) \big|^2 \Big) dx.$$



2. Associated to each edge $e \in E$ is a local orthonormal basis $\{\vec{i}_e, \vec{j}_e, \vec{k}_e\}$
   - $\vec{g}_e(x)$ is **displacement vector** of edge e in global coordinate system at $x \in e$.

$$(u_e, w_e, v_e)(x) := \big(\vec{g}_e \cdot \vec{i}_e, \vec{g}_e \cdot \vec{j}_e, \vec{g}_e \cdot \vec{k}_e\big)(x)$$

   - $\vec{\omega}_e(x)$ is (linearized) **rotation vector** of edge e in global coordinate system at $x \in e$

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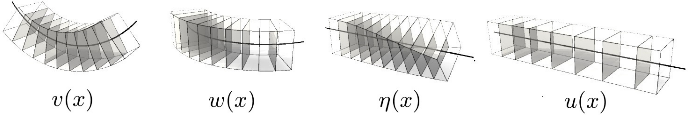

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3. A **key ingredient** of generalization to elastic frame is the **property of vertex** at which the incident edges are met.

Section 3

Quadratic Form and Vertex Conditions

Theorem

*Energy functional Π of the beam frame with free rigid joints is the quadratic form corresponding to the positive closed **sesquilinear form***

$$h\left[\widetilde{\Psi}, \Psi\right] := \sum_{e\in E} \int_e \left(a_e \overline{\widetilde{v}''_e} v''_e + b_e \overline{\widetilde{w}''_e} w''_e + c_e \overline{\widetilde{u}'_e} u'_e + d_e \overline{\widetilde{\eta}'_e} \eta'_e\right) dx,$$

densely defined on

$$\mathcal{H} = \bigoplus_{e\in E} L_2(e) \times \bigoplus_{e\in E} L_2(e) \times \bigoplus_{e\in E} L_2(e) \times \bigoplus_{e\in E} L_2(e),$$

*with the **domain** of h consisting of the vectors*

$$\Psi := \begin{pmatrix} v, & w, & u, & \eta \end{pmatrix}^T \in \bigoplus_{e\in E} H^2(e) \times \bigoplus_{e\in E} H^2(e) \times \bigoplus_{e\in E} H^1(e) \times \bigoplus_{e\in E} H^1(e)$$

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*that satisfy, at every vertex $v\in V$, the "****free rigid joint****" conditions*

- ***continuity*** *of displacement,*

$$u_1\vec{i}_1 + w_1\vec{j}_1 + v_1\vec{k}_1 = \cdots = u_n\vec{i}_n + w_n\vec{j}_n + v_n\vec{k}_n$$

- ***continuity*** *of rotation,*

$$\eta_1\vec{i}_1 - v'_1\vec{j}_1 + w'_1\vec{k}_1 = \cdots = \eta_n\vec{i}_n - v'_n\vec{j}_n + w'_n\vec{k}_n$$

where n is the degree of v.

Hamiltonian on Graph and Vertex Conditions

Theorem

Energy form Π on a beam frame with free rigid joints corresponds to the ***self-adjoint*** *operator $H : \mathcal{H} \to \mathcal{H}$ acting as*

$$\Psi_e := \begin{pmatrix} v_e \\ w_e \\ u_e \\ \eta_e \end{pmatrix} \mapsto \begin{pmatrix} a_e v_e'''' \\ b_e w_e'''' \\ -c_e u_e'' \\ -d_e \eta_e'' \end{pmatrix}$$

on every edge $e \in E$ of the graph. The ***domain*** *of the operator H consists of the functions*

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- ***continuity*** *of* ***displacement*** *and* ***rotation*** *conditions respectively*

$$u_1\vec{i}_1 + w_1\vec{j}_1 + v_1\vec{k}_1 = \cdots = u_{n_v}\vec{i}_{n_v} + w_{n_v}\vec{j}_{n_v} + v_{n_v}\vec{k}_{n_v},$$
$$\eta_1\vec{i}_1 - v_1'\vec{j}_1 + w_1'\vec{k}_1 = \cdots = \eta_{n_v}\vec{i}_{n_v} - v_{n_v}'\vec{j}_{n_v} + w_{n_v}'\vec{k}_{n_v},$$

- ***equilibrium*** *of* ***forces*** *and* ***moments****, respectively*

$$\sum_{e\sim v}\left(c_e u_e'\vec{i}_e - b_e w_e'''\vec{j}_e - a_e v_e'''\vec{k}_e\right) = \vec{0},$$
$$\sum_{e\sim v}\left(d_e \eta_e'\vec{i}_e - a_e v_e''\vec{j}_e + b_e w_e''\vec{k}_e\right) = \vec{0}.$$

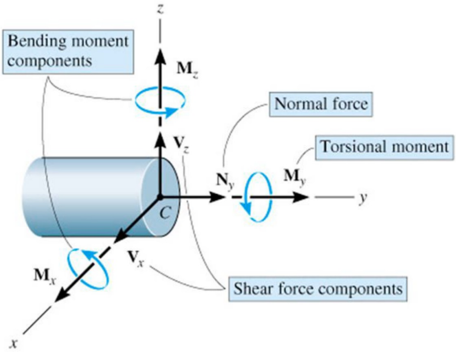
Self-adjointness and Physics Behind Vertex Conditions

1. Equilibrium of forces at vertex

$$\sum_{e\sim v}\Big(\underbrace{c_e u_e'}_{N_y}\vec{i}_e - \underbrace{b_e w_e'''}_{V_x}\vec{j}_e - \underbrace{a_e v_e'''}_{V_z}\vec{k}_e\Big) = \vec{0}$$

2. Equilibrium of moments at vertex

$$\sum_{e\sim v}\Big(\underbrace{d_e \eta_e'}_{M_y}\vec{i}_e - \underbrace{a_e v_e''}_{M_x}\vec{j}_e + \underbrace{b_e w_e''}_{M_z}\vec{k}_e\Big) = \vec{0}.$$



Decoupling of Fields for Planar Graph

Corollary

*Free **planar** network of beams is described by **Hamiltonian***

$$\mathcal{H}^{(\Gamma)}(v,\eta,w,u) = \left(\mathcal{H}_1^{(\Gamma)}(v,\eta)\right) \oplus \left(\mathcal{H}_2^{(\Gamma)}(w,u)\right)$$

where $\mathcal{H}_1^{(\Gamma)}$ *and* $\mathcal{H}_2^{(\Gamma)}$ *are differential operators with action*

$$\mathcal{H}_1^{(\Gamma)}(v,\eta)\big|_e = \begin{pmatrix} a_e \frac{d^4}{dx^4} & 0 \\ 0 & -d_e \frac{d^2}{dx^2} \end{pmatrix}, \quad \text{and} \quad \mathcal{H}_2^{(\Gamma)}(w,u)\big|_e = \begin{pmatrix} b_e \frac{d^4}{dx^4} & 0 \\ 0 & -c_e \frac{d^2}{dx^2} \end{pmatrix}$$

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- $\mathcal{H}_1^{(\Gamma)}$ ***satisfying*** *at each vertex* v

$$v_1 = \cdots = v_{n_v}, \quad \text{and} \quad \eta_1 \vec{i}_1 - v_1' \vec{j}_1 = \cdots = \eta_{n_v} \vec{i}_{n_v} - v_{n_v}' \vec{j}_{n_v},$$

$$\sum_{e\sim v} d_e \eta_e' \vec{i}_e - a_e v_e'' \vec{j}_e = \vec{0}, \quad \text{and} \quad \sum_{e\sim v} a_e v_e''' = 0.$$

- $\mathcal{H}_2^{(\Gamma)}$ ***satisfying*** *at each vertex* v

$$w_1' = \cdots = w_{n_v}', \quad \text{and} \quad u_1 \vec{i}_1 + w_1 \vec{j}_1 = \cdots = u_{n_v} \vec{i}_{n_v} + w_{n_v} \vec{j}_{n_v},$$

$$\sum_{e\sim v} c_e u_e' \vec{i}_e - b_e w_e''' \vec{j}_e = \vec{0}, \quad \text{and} \quad \sum_{e\sim v} b_e w_e'' = \vec{0}.$$

Example: Planar Graph

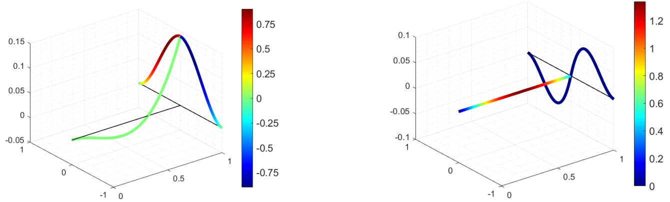


Figure: Eigenfuctions corresponding to first and second eigenvalues for parameters $a = d = d_0 = 1$. Color bar shows value of in-axis torsion of edges.

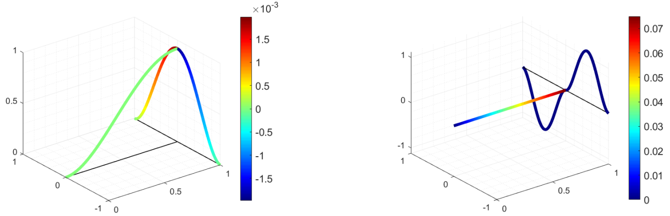
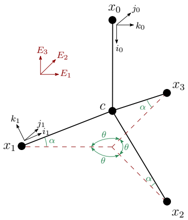


Figure: Eigenfuctions corresponding to first and second eigenvalues for parameters $b = 1$ and $d = d_0 = 10^3$. Color bar shows value of in-axis torsion of edges.

Section 4

Irreducible Representations



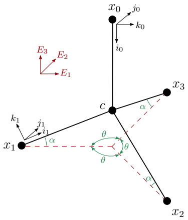
1. The graph Γ_{AT} is **invariant** under the symmetry group $G = D_3$, the dihedral group of degree 3.
   - R : **rotation** of Γ by $\theta = 2\pi/3$ with axis of rotation along E_3
   - F : **reflection** with respect to the planes passing through vertices x_1, x_0 and c.
2. The group G then can be described as

$$G = \langle R, F \mid R^3 = I,\ F^2 = I,\ FRFR = I \rangle$$

with I to be the identity element. This implies that G contains the elements

$$G = \{I, R, R^2, F, FR, FR^2\}$$

Decomposition of $\mathcal{H}$



Theorem

*The **Hamiltonian** operator H of the beam frame Γ_{AT} is **reduced** by the **decomposition***

$$\mathcal{H} = \mathcal{H}_{\mathrm{id}} \oplus \mathcal{H}_{\mathrm{alt}} \oplus \mathcal{H}_{\omega} \oplus \mathcal{H}_{\overline{\omega}},$$

where

$$\mathcal{H}_{\mathrm{id}} := \{\Psi \in \mathcal{H} \colon v_0 = w_0 = \eta_0 = 0,\ w_s = \eta_s = 0,\ v_1 = v_2 = v_3,\ u_1 = u_2 = u_3\},$$
$$\mathcal{H}_{\mathrm{alt}} := \{\Psi \in \mathcal{H} \colon v_0 = w_0 = u_0 = 0,\ u_s = v_s = 0,\ w_1 = w_2 = w_3,\ \eta_1 = \eta_2 = \eta_3\},$$
$$\mathcal{H}_{\omega} := \left\{\Psi \in \mathcal{H} \colon u_0 = \eta_0 = 0,\ w_0 = \mathrm{i}v_0,\ \Psi_3 = \omega\Psi_2 = \omega^2\Psi_1\right\} = \overline{\mathcal{H}_{\overline{\omega}}},$$

where $s \in \{1, 2, 3\}$ labels the legs, $\Psi_s := (v_s, w_s, u_s, \eta_s)^T$, and $\omega = e^{2\pi\mathrm{i}/3}$.

Decomposition of $\mathcal{H}$

Remark

A **decomposition** $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$ is **reducing** for an operator H if

- H is **invariant** on each of the subspaces, and
- the operator domain $\mathrm{Dom}(H)$ is **aligned** with respect to the decomposition, namely

$$\mathrm{Dom}(H) = \bigoplus_\alpha \big(\mathcal{H}_\alpha \cap \mathrm{Dom}(H)\big).$$

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- This means that we can **restrict** H to each subspace in turn and every aspect of the spectral data of the operator H is the **sum** (or union) of the spectral data of the restricted parts.
- In particular, since $\mathcal{H}_\omega = \overline{\mathcal{H}_{\overline{\omega}}}$, the **eigenvalues** of the corresponding restrictions are **equal** and thus each eigenvalue of the restriction $H_\omega = H\big|_{\mathcal{H}_\omega}$ enters the spectrum of H with **multiplicity two**.
- Kinematically, these eigenvalues correspond to the **rotational wobbles** of the antenna beam.

Decomposition of $\mathcal{H}$

Theorem

*The **Hamiltonian** operator H of the beam frame Γ_{AT} is **reduced** by the **decomposition***

$$\mathcal{H} = \mathcal{H}_{\rm id} \oplus \mathcal{H}_{\rm alt} \oplus \mathcal{H}_{\omega} \oplus \mathcal{H}_{\overline{\omega}},$$

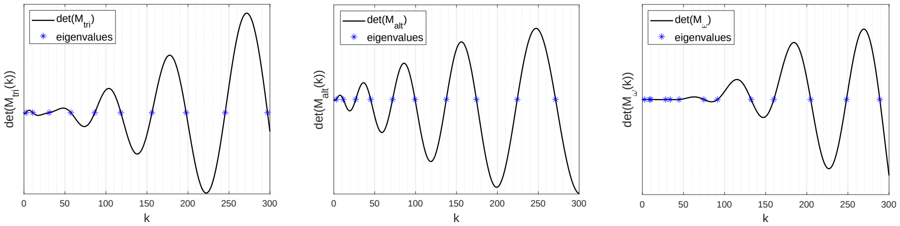

Figure: Variation of determinant of matrices corresponding irreducible representations and their corresponding eigenvalues: (left) trivial $M_{\rm tri}$, (middle) alternative $M_{\rm alt}$, and (right) standard M_{ω}. All the results are based on unit materials parameters and beams lengths.

$\mathcal{H} = \mathcal{H}_{\mathrm{id}} \oplus \mathcal{H}_{\mathrm{alt}} \oplus \mathcal{H}_{\omega} \oplus \mathcal{H}_{\overline{\omega}}$

$$\mathcal{H}_{\mathrm{id}} := \{\Psi \in \mathcal{H} \colon v_0 = w_0 = \eta_0 = 0,\ w_s = \eta_s = 0,\ v_1 = v_2 = v_3,\ u_1 = u_2 = u_3\},$$
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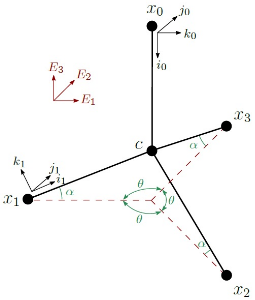
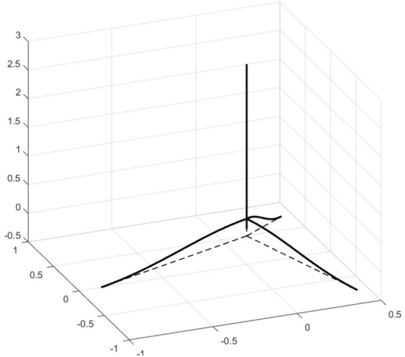




$\mathcal{H} = \mathcal{H}_{\mathrm{id}} \oplus \mathcal{H}_{\mathrm{alt}} \oplus \mathcal{H}_{\omega} \oplus \mathcal{H}_{\varpi}$

$$\mathcal{H}_{\mathrm{id}} := \{\Psi \in \mathcal{H} \colon v_0 = w_0 = \eta_0 = 0,\ w_s = \eta_s = 0,\ v_1 = v_2 = v_3,\ u_1 = u_2 = u_3\},$$

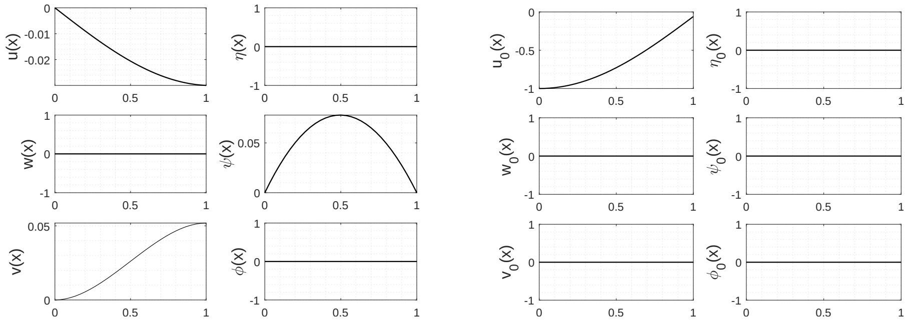


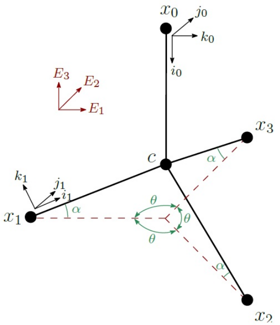
Figure: Plot of the components of the first eigenfuction from $\mathcal{H}_{\mathrm{id}}$. Plots are obtained from a finite elements numerical computation and are displayed in the local coordinate system of the corresponding edge. All the results are based on unit materials parameters and beams lengths.

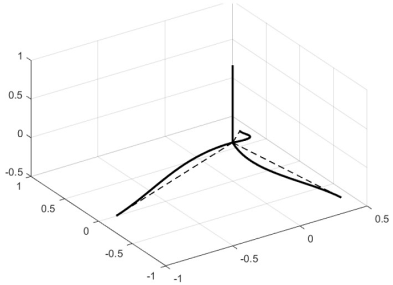
$\mathcal{H} = \mathcal{H}_{\text{id}} \oplus \mathcal{H}_{\text{alt}} \oplus \mathcal{H}_{\omega} \oplus \mathcal{H}_{\overline{\omega}}$

$$\mathcal{H}_{\text{id}} := \{\Psi \in \mathcal{H} \colon v_0 = w_0 = \eta_0 = 0,\ w_s = \eta_s = 0,\ v_1 = v_2 = v_3,\ u_1 = u_2 = u_3\},$$

$$\mathcal{H}_{\text{alt}} := \{\Psi \in \mathcal{H} \colon v_0 = w_0 = u_0 = 0,\ u_s = v_s = 0,\ w_1 = w_2 = w_3,\ \eta_1 = \eta_2 = \eta_3\},$$

$$\mathcal{H}_{\omega} := \left\{\Psi \in \mathcal{H} \colon u_0 = \eta_0 = 0,\ w_0 = \mathrm{i}v_0,\ \Psi_3 = \omega\Psi_2 = \omega^2\Psi_1\right\} = \overline{\mathcal{H}_{\overline{\omega}}},$$





$\mathcal{H} = \mathcal{H}_{\mathrm{id}} \oplus \mathcal{H}_{\mathrm{alt}} \oplus \mathcal{H}_{\omega} \oplus \mathcal{H}_{\overline{\omega}}$

$$\mathcal{H}_{\mathrm{alt}} := \{\Psi \in \mathcal{H} \colon v_0 = w_0 = u_0 = 0,\ u_s = v_s = 0,\ w_1 = w_2 = w_3,\ \eta_1 = \eta_2 = \eta_3\},$$

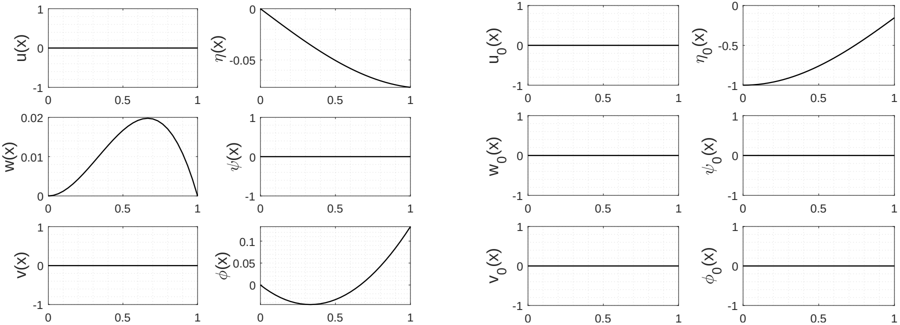


Figure: Plot of the components of the first eigenfuction from $\mathcal{H}_{\mathrm{alt}}$. Plots are obtained from a finite elements numerical computation and are displayed in the local coordinate system of the corresponding edge. All the results are based on unit materials parameters and beams lengths.

$\mathcal{H} = \mathcal{H}_{\text{id}} \oplus \mathcal{H}_{\text{alt}} \oplus \mathcal{H}_{\omega} \oplus \mathcal{H}_{\overline{\omega}}$

$$\mathcal{H}_{\text{id}} := \{\Psi \in \mathcal{H} \colon v_0 = w_0 = \eta_0 = 0,\ w_s = \eta_s = 0,\ v_1 = v_2 = v_3,\ u_1 = u_2 = u_3\},$$

$$\mathcal{H}_{\text{alt}} := \{\Psi \in \mathcal{H} \colon v_0 = w_0 = u_0 = 0,\ u_s = v_s = 0,\ w_1 = w_2 = w_3,\ \eta_1 = \eta_2 = \eta_3\},$$

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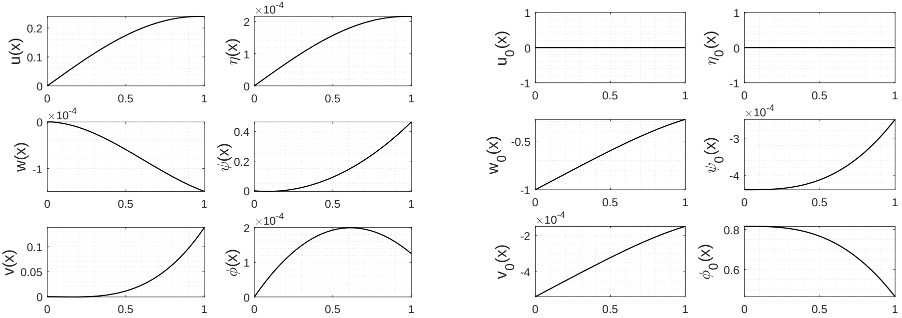


Figure: Plot of (first) eigenfunction fields corresponding to the eigenvalue of multiplicity two in edge's local coordinate system by finite element approximation.

$\mathcal{H} = \mathcal{H}_{\text{id}} \oplus \mathcal{H}_{\text{alt}} \oplus \mathcal{H}_{\omega} \oplus \mathcal{H}_{\overline{\omega}}$

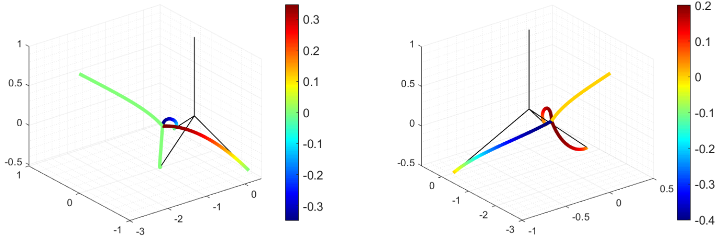
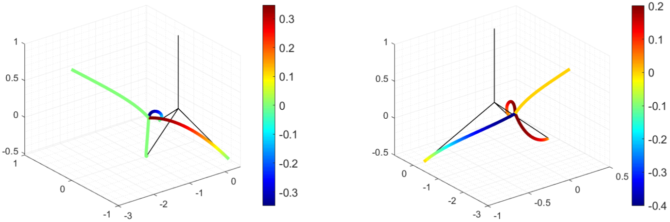


Figure: Plot of (left) **first**, and (right) **second** displacement eigenfunction corresponding to eigenvalue of **standard representation** in global coordinate system by finite element approximation. Color bar shows value of in-axis rotation of edges.

$\mathcal{H} = \mathcal{H}_{\text{id}} \oplus \mathcal{H}_{\text{alt}} \oplus \mathcal{H}_{\omega} \oplus \mathcal{H}_{\overline{\omega}}$



Questions?

Thanks for your attention !