Nonlinear Peridynamic Models

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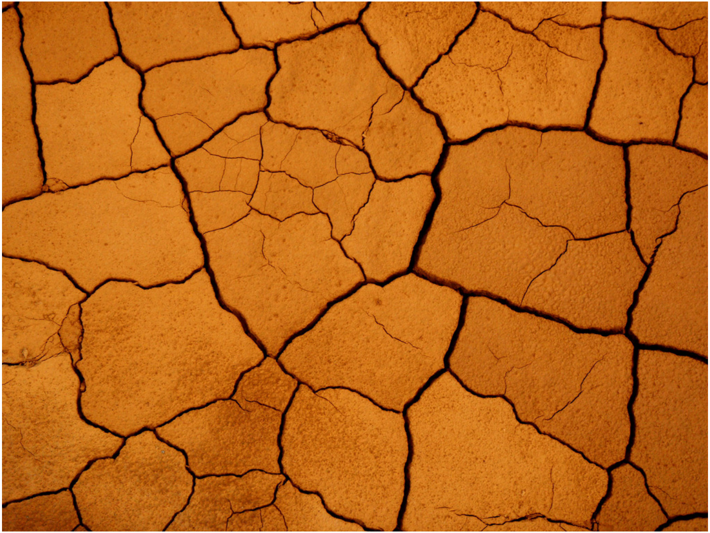
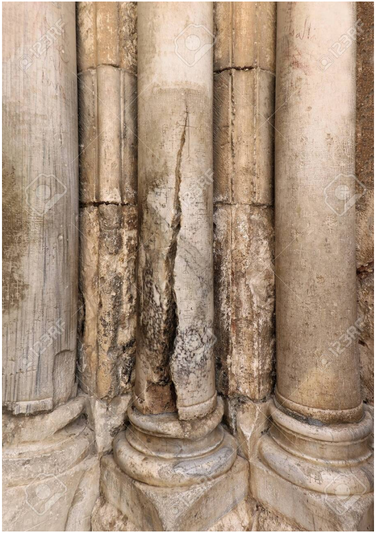
Motivation of the Model

- Key problem in solid mechanics
  - spontaneous formation of singularity
    - spontaneous $\equiv$ a singularity forms where one was not present initially
    - crack in a homogeneous solid
    - folding
    - ripples
    - damage mechanics
    - evolution of phase boundaries in phase transformations
    - defects
    - dislocations
    - nonlocal effects

Clay (crack)



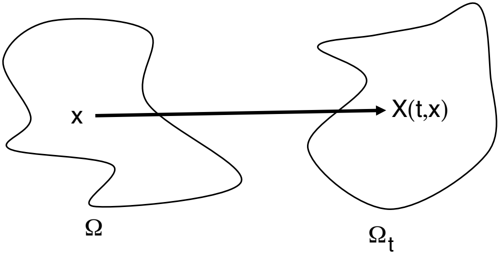
Marble (crack)



Morandi Bridge Genova (14/08/2018) (damage)

Aluminium Tin (folding)

Classical (local) Elasticity



- $\Omega \subset \mathbb{R}^N$ rest configuration of a material body
- $\rho : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}_+$ mass density
  - $\rho \equiv 1$
- $X(t,x)$ deformation map
  - $X(t,x)$ position at time t of the particle in x at $t = 0$
- $u(t,x) = X(t,x) - x$ displacement
  - $u : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}^N$

Energy

$$E[u](t) = \int_\Omega \Bigg(\underbrace{\frac{|\partial_t u|^2}{2}}_{\text{kinetic energy}} + \underbrace{W(\nabla u)}_{\text{potential energy}} \Bigg) dx$$

- $u(t,x) \in \mathbb{R}^N$, $\nabla u(t,x) \in \mathbb{R}^{N\times N}$
- W depends on the material

Compulsory assumption on W

- invariance under rigid rotations

Equation of motion

$$\partial_{tt}^2 u = \text{div}\left(W'(\nabla u)\right) + \underbrace{b(t,x)}_{\text{external force}}$$

- Newton Law $F = ma$
- Conservation of momentum

Example (Linear elasticity)

$$W(\nabla u) = \mu |E|^2 + \frac{\lambda}{2}(\text{tr}(E))^2, \qquad E = \frac{\nabla u + (\nabla u)^T}{2}$$

- λ, μ Lamé coefficients
- E symmetric part of ∇u

Approaches to deal with discontinuities

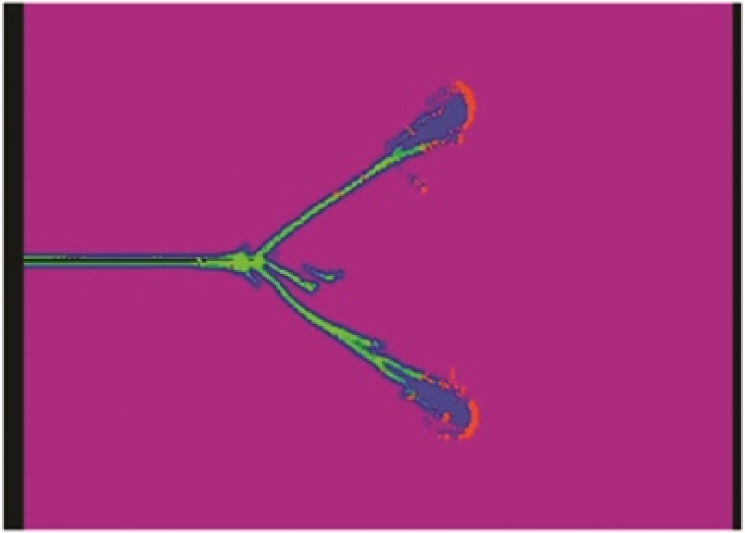
Distributional solutions of the equation of motion

- Knowles & Sternberg - 1978
- weak derivatives
- It fails in case of severe discontinuities
  - too much regularity
  - cracks
  - discontinuous displacement field

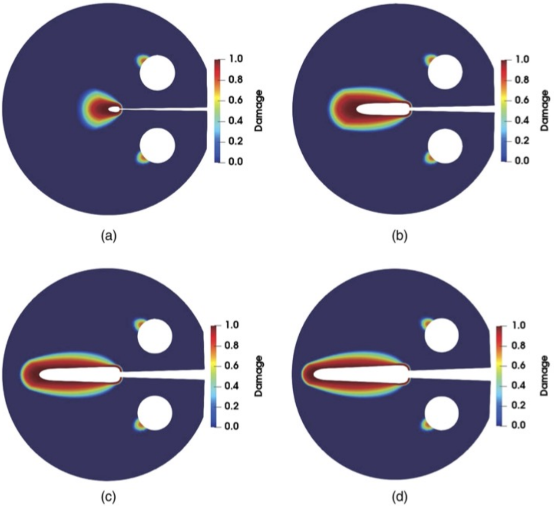
See cracks as free boundaries

- Hellan - 1984
- redefine the body so that the crack lies on the boundary
  - one need to know where the discontinuity is located

Nonlocal Effects



Du & Lipton - 2014

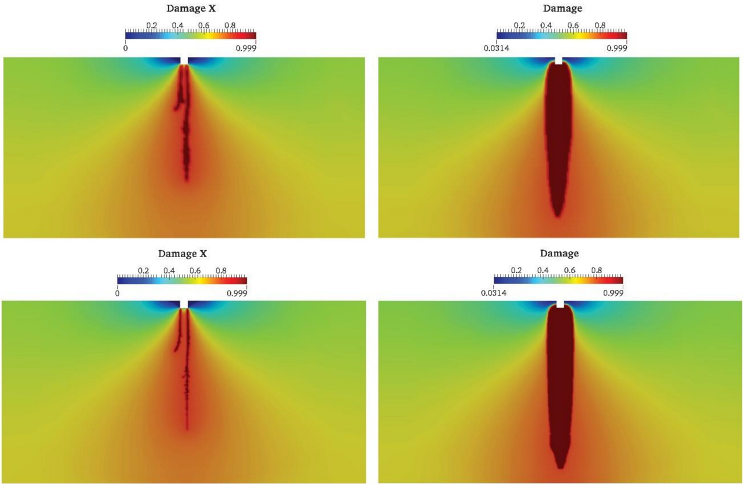


Wagnoner & Buttlar & Paulino - 2005

Local Damage model — Non-local Damage model

Coarse Mesh:

Fine Mesh:



Waisman & Bassis & Duddu - 2021

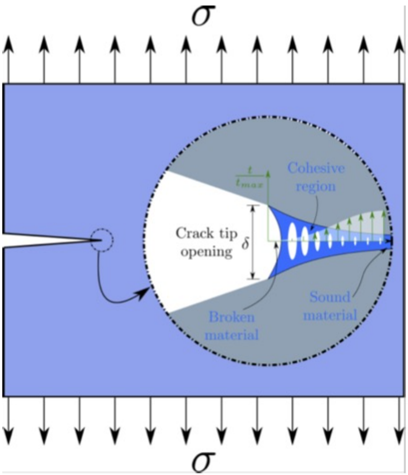


Fig. 5 Schematic representation of the cohesive zone: transition from sound material to broken material. The green arrows represent the distribution of tractions over the cohesive region.

Shakoo & Trejo-Navas & Muñoz & Bernacki & Bouchard - 2018

Nonlocal Elasticity

Equation of motion

$$\partial_{tt}^2 u = \operatorname{div}\left(\int_\Omega K(x,y) f(\nabla u(t,y)) dy\right) + b(t,x)$$

- Kröner - 1967
- Eringen - 1972
- Eringen & Edelen - 1972
- K convolution kernel
- still too much regularity

Peridynamic (Silling - 2000)

Greek Etymology

$$\text{peridynamic} = \text{near} + \text{force}$$

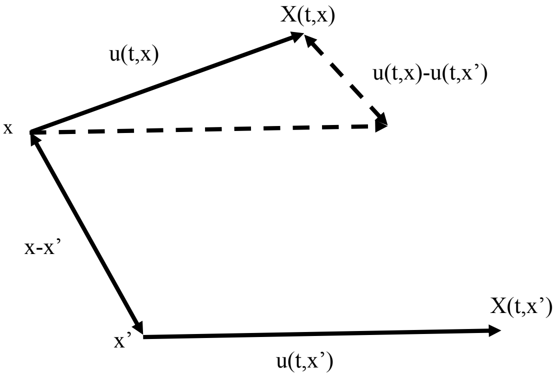
Nonlocal Equation of motion

$$\underbrace{\partial^2_{tt} u(t,x)}_{\text{accelleration}} = \underbrace{\int_{H(x)} f(x - x', u(t,x) - u(t,x'))dx' + \underbrace{b(t,x)}_{\text{external forces}}}_{\text{forces acting on } x}$$

- Newton Law: $F = ma$
- $H(x) \equiv$ neighborhood of x
  - $x' \in H(x) \iff x'$ interacts with x

Pairwise force function f

$$f\Big(\underbrace{x - x'}_{x' \text{ interacts with } x}, \underbrace{u(t,x) - u(t,x')}_{\text{relative displacement}}\Big)$$



- f depends on the material

Compulsory assumption on f

- Newton law of actio et reactio

$$f\left(x - x', u(t,x) - u(t,x')\right) = -f\left(x' - x, u(t,x') - u(t,x)\right)$$

Example (Linear Elastic Material)

$$f(x - x, u - u') = f_0(x - x') + \lambda(|x - x'|)(x - x') \otimes (x - x')(u - u')$$

Micropotential & Energy

Existence of a micropotential

$$\exists\Phi \text{ s.t. } f(y,u) = \nabla_u \Phi(y,u)$$

- semi-compulsory

Energy

$$E[u](t) = \frac{1}{2}\int_\Omega |\partial_t u|^2 dx + \frac{1}{2}\int_\Omega \int_{H(x)} \Phi\Big(x - x', u(t,x) - u(t,x')\Big) dx dx'$$

Essential Mathematical Literature

Classical solutions

- Erbay & Erkip & Muslu - 2012
  - $N = 1$
  - $f(x - x, u - u') = a(x - x')g(u - u')$
- Emmrich & Puhst - 2013
  - $N > 1$
  - $|f(x - x, u - u')| \leq a(x - x')|u - u'|$

Weak and Measure Valued solutions

- Emmrich & Puhst - 2015
  - $N \geq 1$
  - $f(x - x, u - u') \cdot (u - u')$ quadratic positive definite form
  - Polyconvexity assumptions

Our Problem

Cauchy Problem

$$\begin{cases} \partial^2_{tt} u = (Ku(t,\cdot))(x), & t > 0, \quad x \in \mathbb{R}^N \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N \\ \partial_t u(0,x) = v_0(x), & x \in \mathbb{R}^N \end{cases}$$

Nonlocal Operator

$$(Ku)(x) = \int_{B_\delta(x)} f(x - x', u(x) - u(x'))dx'$$

$$\Omega = \mathbb{R}^N \qquad H(x) = B_\delta(x) \qquad \delta \not\longrightarrow 0$$

Assumptions on f

Regularity

- "$f \in C^1$"
  - $f(u, 0) = \infty$

Symmetry

- $f(-y, -u) = -f(y, u)$
  - Newton law of actio et reactio
  - Alternative writing

$$(Ku)(x) = -\int_{B_\delta(0)} f(y, u(x) - u(x-y))dy$$

Existence of a micropotential

- $f = \nabla_u \Phi$
- $\Phi(y,u) = k\dfrac{|u|^p}{|y|^{N+\alpha p}} + \Psi(y,u)$, $2 \leq p < \infty$, $0 < \alpha < 1$
- $\Psi(y,0) = 0 \leq \Psi(y,u)$
- $|\nabla_u \Psi(y,u)|$, $|D_u^2 \Psi(y,u)| \leq g(y) \in L^2_{loc}$
  - Anisotropic material

$$\Phi(y,u) = k u^T K u \frac{|u|^{p-2}}{|y|^{N+\alpha p}} + \Psi(y,u), \qquad K \in \mathbb{R}^{N\times N}$$

  - No convexity assumptions on Φ

Functional spaces & Definition of Solutions

Fractional Sobolev Spaces

$$\|u\|_{W^{\alpha,p}} = \left(\int_{\mathbb{R}^N} |u|^p dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(x-y)|^p}{|y|^{N+\alpha p}} dxdy \right)^{1/p}$$

- $W^{\alpha,p} \hookrightarrow\hookrightarrow L^q_{loc}$, $1 \leq q < p$
- Di Nezza & Palatucci & Valdinoci - 2012

Our Functional Space $\mathcal{W}$

$$\|u\|_{\mathcal{W}} = \|u\|_{L^p} + \left(\int_{\mathbb{R}^N} \int_{B_\delta(0)} \frac{|u(x) - u(x-y)|^p}{|y|^{N+\alpha p}} dxdy \right)^{1/p}$$

- $\mathcal{W} \hookrightarrow\hookrightarrow L^2_{loc}$

Lemma

$$\forall u \in \mathcal{W} : Ku \in \mathcal{W}' \left(\Longleftrightarrow \forall u,\, v \in \mathcal{W} : \int_{\mathbb{R}^N} (Ku) v dx < \infty \right)$$

Lemma

$$\left.\begin{array}{r} \{u_n\}_n \subset \mathcal{W} \ \ bounded \\ u \in \mathcal{W} \\ u_n \longrightarrow u \ \ in \ L^2_{loc} \end{array}\right\} \Longrightarrow Ku_n \longrightarrow Ku \ \ in \ \mathcal{D}'$$

Definition (Dissipative Weak Solutions)

$u : [0, \infty) \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a dissipative weak solution if

- $u \in L^\infty(0, T; \mathcal{W}), \; T > 0$
- $\partial_t u \in L^\infty(0, \infty; L^2)$
- for all $\varphi \in C_c^\infty$

$$\int_0^\infty \int_{\mathbb{R}^N} \Big(u \partial_{tt}^2 \varphi - (Ku)\varphi \Big) dtdx - \int_{\mathbb{R}^N} v_0(x)\varphi(0, x)dx + \int_{\mathbb{R}^N} u_0(x)\partial_t \varphi(0, x)dx = 0$$

- $E[u](t) \leq E[u](0), \; t \geq 0$

Existence of Solutions

Theorem (Existence)

$$u_0,\ v_0 \in L^2$$

$$\underbrace{\int_{\mathbb{R}^N}\int_{B_\delta(0)} \Phi(y, u_0(x) - u_0(x-y))dxdy < \infty}$$

$$\Downarrow$$

$\exists u$ *dissipative weak solution*

Higher order approximation

$$\varepsilon > 0 \qquad \begin{cases} \partial^2_{tt} u_\varepsilon = (K u_\varepsilon(t,\cdot))(x) - \varepsilon\Delta^2 u_\varepsilon \\ u_\varepsilon(0,x) = u_0(x) \\ \partial_t u_\varepsilon(0,x) = v_0(x) \end{cases}$$

Lemma (Energy Estimate)

$$\|\partial_t u_\varepsilon(t,\cdot)\|_{L^2}^2 + \varepsilon \|\Delta u_\varepsilon(t,\cdot)\|_{L^2}^2 + \int_{\mathbb{R}^N}\int_{B_\delta(0)} \Phi(y, u_\varepsilon(t,x) - u_\varepsilon(t,x-y))dxdy \le C$$

Proof. Multiply by $\partial_t u_\varepsilon$. **Q.E.D.**

Lemma

$$\|u_\varepsilon(t,\cdot)\|_{L^2} \le C(1+t)$$

Proof.

$$|u_\varepsilon(t,x)| \le |u_0(x)| + \int_0^t |\partial_t u_\varepsilon(s,x)|\, ds$$

Square both sides. Use the Hölder inequality and the energy estimate.

Q.E.D.

Compactness.

$$\{u_\varepsilon\}_\varepsilon \text{ bounded in } L^\infty(0,T;\mathcal{W}),\ T>0$$
$$\{\partial_t u_\varepsilon\}_\varepsilon \text{ bounded in } L^\infty(0,\infty;L^2)$$
$$\Downarrow$$
$$\exists u \text{ s.t. } \begin{cases} u \in L^\infty(0,T;\mathcal{W}),\ T>0 \\ \partial_t u \in L^\infty(0,\infty;L^2) \\ u_\varepsilon \longrightarrow u \text{ a.e. and in } L^2_{loc} \\ u \text{ is a distributional solution} \end{cases}$$
$$\Downarrow$$

existence of a distributional solution

Energy Dissipation.

$$
\begin{aligned}
E[u](0) \geq & \frac{\|\partial_t u_\varepsilon(t,\cdot)\|_{L^2}^2}{2} + \frac{1}{2}\int_{\mathbb{R}^N}\int_{B_\delta(0)} \Phi(y, u_\varepsilon(t,x) - u_\varepsilon(t,x-y))dxdy \\
= & \frac{\|\partial_t u_\varepsilon(t,\cdot)\|_{L^2}^2}{2} + \frac{k}{2}\int_{\mathbb{R}^N}\int_{B_\delta(0)} \frac{|u_\varepsilon(t,x) - u_\varepsilon(t,x-y)|^p}{|y|^{N+\alpha p}}dxdy \\
& + \frac{1}{2}\int_{\mathbb{R}^N}\int_{B_\delta(0)} \Psi(y, u_\varepsilon(t,x) - u_\varepsilon(t,x-y))dxdy \\
\geq & \frac{\|\partial_t u_\varepsilon(t,\cdot)\|_{L^2}^2}{2} + \frac{k}{2}\int_{\mathbb{R}^N}\int_{B_\delta(0)} \frac{|u_\varepsilon(t,x) - u_\varepsilon(t,x-y)|^p}{|y|^{N+\alpha p}}dxdy \\
& + \frac{1}{2}\int_{B_R(0)}\int_{B_\delta(0)} \Psi(y, u_\varepsilon(t,x) - u_\varepsilon(t,x-y))dxdy
\end{aligned}
$$

- use convexity and the Dominated Convergence Theorem
- $\varepsilon \longrightarrow 0$

$$E[u](0) \geq \frac{\|\partial_t u(t,\cdot)\|_{L^2}^2}{2} + \frac{k}{2}\int_{\mathbb{R}^N}\int_{B_\delta(0)} \frac{|u(t,x)-u(t,x-y)|^p}{|y|^{N+\alpha p}}dxdy + \frac{1}{2}\int_{B_R(0)}\int_{B_\delta(0)} \Psi(y, u(t,x)-u(t,x-y))dxdy$$

- $R \longrightarrow \infty$

$$E[u](0) \geq E[u](t)$$

$\Downarrow$

existence of a dissipative weak solution

The "linear" case

$$p = 2$$

- $\Phi(y,u) = k\dfrac{|u|^2}{|y|^{N+2\alpha}} + \Psi(y,u)$
- $f(y,u) = \nabla_u \Phi(y,u) = 2k\dfrac{u}{|y|^{N+2\alpha}} + \nabla_u \Psi(y,u)$

$$\partial_{tt}^2 u(t,x) = \underbrace{-2k\int_{B_\delta(0)} \frac{u(t,x)-u(t,x-y)}{|y|^{N+2\alpha}}dy}_{\text{linear part}}$$

$$-\int_{B_\delta(0)}\int_{B_\delta(0)} \nabla_u \Psi(y, u(t,x)-u(t,x-y))dxdy$$

Uniqueness and Stability

u, $\widetilde{u}$ dissipative weak solutions

$$\begin{cases} \partial_{tt}^2 u = (Ku(t,\cdot))(x) \\ u(0,x) = u_0(x) \\ \partial_t u(0,x) = v_0(x) \end{cases} \qquad \begin{cases} \partial_{tt}^2 \widetilde{u} = (K\widetilde{u}(t,\cdot))(x) \\ \widetilde{u}(0,x) = \widetilde{u}_0(x) \\ \partial_t \widetilde{u}(0,x) = \widetilde{v}_0(x) \end{cases}$$

Stability Estimate

$$
\begin{aligned}
&\|\partial_t u(t,\cdot) - \partial_t \widetilde{u}(t,\cdot)\|_{L^2}^2 \\
&\quad + \frac{k}{2}\int_{\mathbb{R}^N}\int_{B_\delta(0)} \frac{|(u(t,x)-u(t,x-y)) - (\widetilde{u}(t,x)-\widetilde{u}(t,x-y))|^2}{|y|^{N+2\alpha}}dxdy \\
&\leq e^{kt}\Bigg(\|v_0 - \widetilde{v}_0\|_{L^2}^2 \\
&\quad + \frac{k}{2}\int_{\mathbb{R}^N}\int_{B_\delta(0)} \frac{|(u_0(x)-u_0(x-y)) - (\widetilde{u}_0(x)-\widetilde{u}_0(x-y))|^2}{|y|^{N+2\alpha}}dxdy\Bigg)
\end{aligned}
$$

$\Downarrow$

uniqueness and stability of dissipative weak solutions

Spontaneous development of singularities

- α can be very small
  - our functional setting allows severe discontinuities

"Local" elasticity (Evans & Gariepy book - 1992)

- $u \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^N) \Longrightarrow \dim_{\mathcal{H}}(\{\text{Discontinuity set of } u\}) \leq N - p$
  - $u \in H^1(\mathbb{R}^3;\mathbb{R}^3) \Longrightarrow \dim_{\mathcal{H}}(\{\text{Discontinuity set of } u\}) \leq 1$

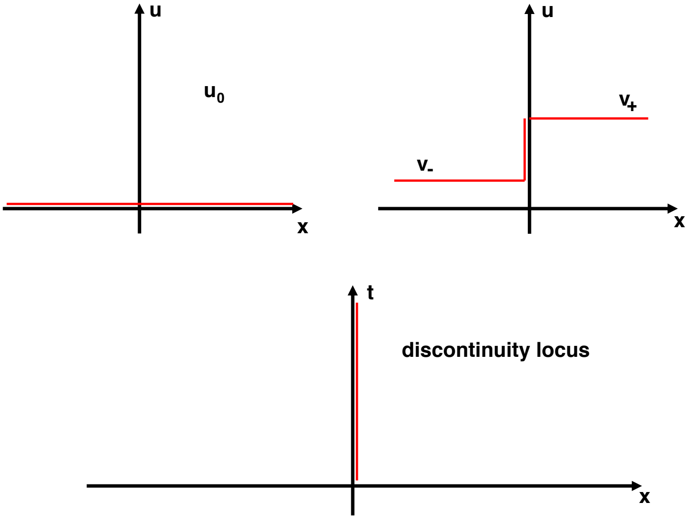
Peridinamic (Mingione - 2003)

- $u \in W^{\alpha,p}(\mathbb{R}^N;\mathbb{R}^N) \Longrightarrow \dim_{\mathcal{H}}(\{\text{Discontinuity set of } u\}) \leq N - \alpha p$
  - $u \in H^\alpha(\mathbb{R}^3;\mathbb{R}^3) \Longrightarrow \dim_{\mathcal{H}}(\{\text{Discontinuity set of } u\}) \leq 3 - 2\alpha$

Riemann Problem ($p = 2,\ N = 1,\ \Psi \equiv 0$)

$$\begin{cases} \partial^2_{tt} u(t,x) = -2k \displaystyle\int_{-\delta}^{\delta} \frac{u(t,x) - u(t,x-y)}{|y|^{1+2\alpha}} dy, & t > 0,\ x \in \mathbb{R} \\ u(0,x) = 0, & x \in \mathbb{R} \\ u_t(0,x) = \begin{cases} v_+, & \text{if } x \geq 0 \\ v_-, & \text{if } x < 0 \end{cases} \end{cases}$$

- $u(0,\cdot)$ continuous
- $u(t,0^+) - u(t,0^-) = \pi\,(v_+ - v_-)\,t \neq 0$
- infinite energy


u
u0
x
u
v+
v-
x
t
discontinuity locus
x

Dispersion relation & Representation formula ($p = 2,\ N = 1,\ \Psi \equiv 0$)

$$\begin{cases} \partial^2_{tt} u(t,x) = -2k \displaystyle\int_{-\delta}^{\delta} \frac{u(t,x) - u(t,x-y)}{|y|^{1+2\alpha}} dy, & t > 0,\ x \in \mathbb{R} \\ u(0,x) = u_0(x), & x \in \mathbb{R} \\ u_t(0,x) = v_0(x), & x \in \mathbb{R} \end{cases}$$

Representation formula

$$u(t,x) = \int_{\mathbb{R}} e^{-i\xi x} \left[\widehat{u_0}(\xi) \cos\left(\omega(\xi)\, t\right) + \frac{\widehat{v_0}(\xi)}{\omega(\xi)} \sin\left(\omega(\xi)\, t\right) \right] d\xi$$

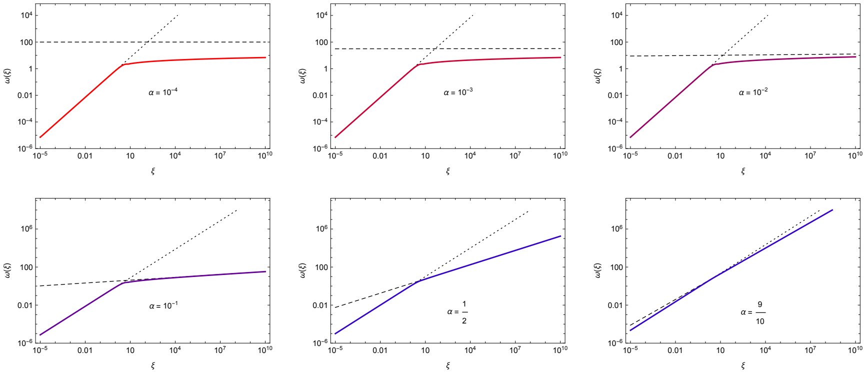
Dispersion relation

$$\omega(\xi) = \sqrt{\frac{2\kappa}{\delta^{2\alpha}} \int_{-1}^{1} \frac{1-\cos(\xi\delta z)}{|z|^{1+2\alpha}} dz}$$

$$\omega(\xi) = \begin{cases} \begin{cases} |\xi|, & \xi \to 0 \\ |\xi|^{\alpha}, & \xi \to \pm\infty \end{cases} & \text{peridynamic} \\ \\ |\xi| & \text{wave equation} \end{cases}$$

Scale effects

- $\alpha < 1$
- different behavior for $|\xi| \ll \frac{1}{\delta}$ and $|\xi| \gg \frac{1}{\delta}$



- Logarithmic scale of $\omega(\xi)$
  - ξ and ξ^{α} lines in logarithmic scale
  - change of slope
  - $\delta = 1$

Improved L^p estimates

$$\frac{\widehat{u_0}}{\omega'} \in W^{1,1}(\mathbb{R}), \qquad \frac{\widehat{v_0}}{(\omega^2)'} \in W^{1,1}(\mathbb{R})$$

$$\Downarrow$$

$$\|u(t,\cdot)\|_{L^2(\mathbb{R})} \le C \left\| \widehat{u_0} + \frac{\widehat{v_0}}{\omega} \right\|_{L^2(\mathbb{R})}$$

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \le C$$

$$|u(t,x)| \le C \frac{1+|x|}{t}$$

$$u_0, v_0 \in \mathcal{S}(\mathbb{R}) \implies \frac{\widehat{u_0}}{\omega'} \in W^{1,1}(\mathbb{R}), \ \frac{\widehat{v_0}}{(\omega^2)'} \in W^{1,1}(\mathbb{R})$$

Strichartz estimates

$$1 < q \leq 2$$

$$f(\xi) = \frac{\widehat{u_0}(\xi)}{\omega'(\xi)} + \frac{\widehat{v_0}(\xi)}{(\omega^2(\xi))'}, \qquad f \in W^{1,q}(\mathbb{R})$$

$$g(\xi) = \left(\frac{\widehat{u_0}(\xi)}{\omega'(\xi)}\right)' + \left(\frac{\widehat{v_0}(\xi)}{\omega^2(\xi)}\right)', \qquad g \in L^q(\mathbb{R})$$

$$\Downarrow$$

$$\|u\|_{L^r(1,\infty;L^p(\mathbb{R}))} \leq C\left(\|f'\|_{L^q(\mathbb{R})} + \|g\|_{L^q(\mathbb{R})}\right)$$

$$2 \leq p < \infty, \qquad 1 \leq r < \infty$$

$$u_0, v_0 \in \mathcal{S}(\mathbb{R}) \implies f \in W^{1,q}(\mathbb{R}),\ g \in L^q(\mathbb{R})$$