Nonlinear Spectral Theory

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Contents

Introduction

The term *Spectral Theory* was coined by David Hilbert in his studies of quadratic forms in infinitely-many variables. This theory evolved into a beautiful blend of Linear Algebra and Analysis, with striking applications in different fields of science. For example, the formulation of the calculus of Quantum Mechanics (e.g., POVM) would not have been possible without such a (Linear) Spectral Theory at hand. With such a successful beginning, the obvious question is, if it is possible to extend Spectral Theory to the study of nonlinear operators. One reason is that the macroscopic world is definitely not linear. A famous example is the dynamics of fluids, modeled by the nonlinear Navier-Stokes equations. This work is meant as an introduction to such a *Nonlinear* Spectral Theory.

In contrast to the linear theory, there is not a single spectrum, which does the job in all situations. A plethora of spectra have been defined in the last decades, tailored to the solution of specific (nonlinear) problems. We present a biased selection with the Rhodius, Neuberger, Kačurovskiĭ, Dörfner, Furi-Martelli-Vignoli (FMV), and the Feng spectrum, respectively.

The linear spectrum enjoys a couple of favorable properties, like nonemptiness, closedness, boundedness, and hence compactness. It is also upper semicontinuous, meaning that it cannot expand suddenly, when the parameterized underlying operator changes continuously. In Chapter 3, we study such properties for the spectra mentioned above. The content has been mainly taken from [**ADPV04**], but is presented differently. As main tools for analysis, deep results from Fixed-Point Theory have been used. We present the necessary material in Chapter 2, with the proof of the Theorem of Schauder-Tychonoff for locally-convex spaces as a highlight.

Chapter 4 is devoted to applications, where we apply Nonlinear Spectral Theory to the p-Laplace operator, a nonlinear generalization of the ordinary Laplacian. We derive discreteness results for its spectrum and a nonlinear Fredholm Alternative. As a prerequisite for the proofs of these results, we need to develop the Theory of Monotonic Operators, presented in Chapter 2. This material is generalized to locally-convex spaces. Another object of study in Chapter 4 are the (stationary) Navier-Stokes equations. We present the existence and smoothness of strong solutions in bounded domains. The main insight for this proof is the above-mentioned generalization of the Theory of Monotonic Operators beyond Banach spaces. By construction, the theory traditionally only yields weak solutions, but with the right function space plugged into this generalized theory, it is possible to obtain a strong solution from a weak one.

As locally-convex spaces come with a lot of abstract and arguably difficult overhead, both in definitions and in insights, when compared to Banach spaces, we devote whole Chapter 1 to recap important results used in the sequel.

Coming back to Nonlinear Spectral Theory, one could argue that it is still an infant theory, at the very beginning of its existence. It borrows heavily from other theories, in particular Fixed-Point Theory, Selection Theory, Theory of Monotonic Operators, and one could argue that everything could also be proven without the terminology of Nonlinear Spectral Theory. But we think that – like with Category Theory – this different angle in viewing on the subject matter is very fruitful.

Still, one of the major drawbacks of the current state of affairs is that the theory has been developed mostly in the setting of Banach spaces, and not been extended properly to arbitrary locally-convex spaces. Banach spaces are often too narrow for applications, especially for partial differential equations. We think that going in this more general direction of locally-convex spaces would be very promising. Needless to say, it is even difficult to find textbooks presenting Linear Spectral Theory in such a general setting.

Foremost due to limited time and space, and also due to our arguably biased selection of topics, we do not cover important topics like

- Spectral aspects of Distribution Theory and, more general, of nuclear spaces and operators, up to Schwartz's kernel theorem for linear, nuclear operators.[1] This result is the most general form of a spectral theorem possible.
- Leray-Schauder-Degree Theory for locally-convex spaces as a quantitative extension of fixed-point theorems.[2] There are similarities between the properties of some of the solvability measures and such degrees.[3]
- Other important spectra defined in the literature, like the Väth phantoms and their associated theory, the Weber, the Singhof-Weyer, or the Infante-Webb spectrum, respectively.
- The study of numerical ranges for nonlinear problems.[4] In the linear case, they provide a powerful instrument to locate the spectrum of the operator under consideration.

The mentioned topics easily fill, and in our opinion deserve, whole books on their own to do them justice.

Nevertheless, despite all the shortcomings, we hope that you, the reader, enjoy reading this work as much as we had pleasure in writing it!

Please note that this version of the thesis differs from the submitted one in the correction of an error in Theorem 47, and in corresponding adapations of all results based on this. In particular, this applies to Chapter 4, Section 3. We mark these changes in more detail at the respective places. We also corrected few typographical errors.

[1] [**SW99**, Chapter III, Section 7.1]
[2] [**GD03**, §8–§17]
[3] [**ADPV04**, Chapter 7, Properties 7.1–7.5]
[4] [**ADPV04**, Chapter 11]

CHAPTER 1

Spaces

To set the stage for the next chapters, we systematically recap well-known basic definitions and statements, all circling around the intuitive notion of *space*. Here, we use notation, definitions, and results from the excellent textbooks of Schaefer [**SW99**], Querenburg [**vQ01**], Shirali [**SV06**], and Bourbaki [**Bou98a, Bou98b**], respectively. Everything presented is known, except three highlights, interwoven in this chapter, which are due to the author of this thesis: (i) A characterization of two historically-relevant classes of barreled spaces, defined in the 1960's by Pták and connected with questions on the limits of Functional Analysis. (ii) A separation of these classes, revisiting and solving a long-standing open problem in this part of Functional Analysis. (iii) The definition of a new class of locally-convex spaces (W spaces), which will play an important role in the generalization of the Theory of Monotonic Operators, presented in Chapter 2. We also derive that relatives of the Schwartz spaces are contained in this class.

1. Topological Spaces

1.1. Open Sets, Closed Sets, and Filters. A *topological space* is a set X, together with a *topology* defined over X. The elements of X are called *points*. The topology can be defined in three equivalent ways, via a system of open sets, a system of closed sets, and a neighborhood system.

A *system of open sets* is defined as a set of subsets of X, closed under arbitrary unions and finite intersections.

A *system of closed sets* is defined as a set of subsets of X, closed under finite unions and arbitrary intersections.

Clearly, given a set of open sets, the complements of these sets form a system of closed sets, and vice versa. Given a set system of open or closed sets, a subset of X is called *open* or *closed* set, if it is contained in the respective set system. Furthermore, by definition of union and intersection, sets $\emptyset$ and X are contained in every system of open or closed sets, and thus are both open and closed.

A *filter* $\mathcal{F}$ is a set of subsets of X such that $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$, $\mathcal{F}$ is closed under finite intersections, and $\mathcal{F}$ is closed under supersets, i.e., every $F' \supseteq F$ is in $\mathcal{F}$ for an F in $\mathcal{F}$. We say that $\mathcal{F}$ is a filter *on point* x, if $x \in \bigcap \mathcal{F}$.

A *neighborhood system* is defined as a map $\mathcal{N}\colon X \to \mathcal{P}(X)$, $x \mapsto \mathcal{N}_x = \mathcal{N}(x)$, such that each $\mathcal{N}_x$ is a filter on x, and for each $N \in \mathcal{N}_x$, there exists an $M \in \mathcal{N}_x$ such that for all $y \in M$, we have $N \in \mathcal{N}_y$.

For each x, the filter $\mathcal{N}_x$ called the *neighborhood filter of* x, and its sets are called *neighborhoods* of x.

On the one hand, a given neighborhood system defines a set of open sets.[1] Here, a set is open iff it is a neighborhood of each of its points. On the other hand, a given set of open sets defines a neighborhood system.[2] Here, each neighborhood filter of a point x is defined as the set of all supersets of open sets containing x.

[1] [**vQ01**, 2.9]
[2] [**vQ01**, 2.8]

A *filter base* $\mathcal{B}$ is a set of nonempty subsets of X such that with every $B_1, B_2 \in \mathcal{B}$, there exists $B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2$. We say that $\mathcal{B}$ is a filter base *on point* x, if $x \in \bigcap \mathcal{B}$.

Clearly, a filter base defines a filter. Here, the filter contains all supersets of sets in the filter base.

A *neighborhood-system base* is defined as a map $\mathcal{B}\colon X \to \mathcal{P}(X)$, $x \mapsto \mathcal{B}_x = \mathcal{B}(x)$, such that each $\mathcal{B}_x$ is a filter base on x, and for each $D \in \mathcal{B}_x$, there exists $B \in \mathcal{B}_x$ such that for all $y \in B$, there exists $C \in \mathcal{B}_y$ with $C \subseteq D$.

A given neighborhood-system base defines a neighborhood system. Here, the neighborhood filters are defined via the filter bases.

Let X be a topological space, $V \subseteq X$ a subset. Set $N \subseteq X$ is called a *neighborhood* of V, if N is a neighborhood for each point of V. Equivalently, there is an open set $O \subseteq X$ such that $V \subseteq O \subseteq N$.

1.2. Interior, Exterior, and Boundary. Fix a subset $V \subseteq X$. Point x is called *inner point* of V, if V is a neighborhood of x. It is called *outer point* of V, if $X \backslash V$ is a neighborhood of x. It is a *boundary point* of V, if it is neither an inner nor an outer point of V. A *contact point (adherent point)* of V is an inner or a boundary point of V.

Let the *interior* of V be defined as the set V° of inner points of V, let the *closure* of V be the set $\overline{V}$ of contact points of V, and let the *boundary* of V be the set ∂V of boundary points of V, respectively.

Equivalently, V° is the largest open set contained in V, and $\overline{V}$ is the smallest closed set containing V.[3]

For the interior, we have $\emptyset^\circ = \emptyset$, $A^\circ \subseteq A$, $(A^\circ)^\circ = A^\circ$, $(A \cap B)^\circ = A^\circ \cap B^\circ$.

For the closure, we have $\overline{\emptyset} = \emptyset$, $A \subseteq \overline{A}$, $\overline{\overline{A}} = \overline{A}$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

For the boundary, we have $\partial\emptyset = \emptyset$, $\partial\partial A = \emptyset$.

Let X be a topological space. A subset $V \subseteq X$ is *dense* in X, if $\overline{V} = X$.

1.3. Baire Spaces. Let X be a topological space. A subset A of X is called *nowhere dense (rare)*, if the interior of its closed hull is empty, i.e., $\left(\overline{A}\right)^\circ = \emptyset$. Otherwise, it is called *somewhere dense*. A subset B of X is called *meager (of first category)*, if B is a countable union of nowhere-dense sets, i.e., $B \subseteq \bigcup_n A_n$, with $B \cap A_n$ nowhere dense. Otherwise, it is called *non-meager (of second category)*.

A topological space is called a *Baire* space, if every nonempty and open subset is non-meager.

1.4. Open and Closed Maps. A map $f\colon X \to Y$ between topological spaces X and Y is called *open*, if for every open set $O \subseteq X$, the image $f(O)$ is open in Y. It is called *closed*, if for every closed set $A \subseteq X$, the image $f(A)$ is closed in Y.

Clearly, the identity $\mathrm{id}_X\colon X \to X$, $x \mapsto x$, is open / closed, and if $f\colon X \to Y$ and $g\colon Y \to Z$ are open / closed, their composition $f \circ g$ is open / closed.

1.5. Convergence and Continuity. A filter $\mathcal{F}$ of space X *converges to* point x, written $\mathcal{F} \to x$, if $\mathcal{F}$ contains the neighborhood filter $\mathcal{N}_x$ of x.

Map $f\colon X \to Y$ produces an *image filter* $f(\mathcal{F})$ with filter base $\{f(M) \mid M \in \mathcal{F}\}$. Map f is *(locally) continuous* at point x, if for every filter $\mathcal{F}$ converging to x, the image filter $f(\mathcal{F})$ converges to $f(x)$. This is equivalent to the property that for every neighborhood V of $f(x)$, set $f^{-1}(V)$ is a neighborhood of x.[4]

Map f is *(globally) continuous*, if it is continuous at every point.[5] This is equivalent to the property that for every open set V of Y, set $f^{-1}(V)$ is open in X.

[3] [**vQ01**, 2.15]

[4] [**vQ01**, 2.24(b), 5.17(b)]

[5] [**vQ01**, 2.24(a)]

Clearly, the composition $g \circ f\colon X \to Z$ of two continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$ is continuous at point $x \in X$, if f is continuous at x and g is continuous at $f(x)$.

In addition, the identity id_X is (globally) continuous, and if $f\colon X \to Y$ and $g\colon Y \to Z$ are (globally) continuous, their composition $g \circ f$ is (globally) continuous.[6]

Let X and Y be topological spaces, let $f\colon X \to Y$ be a map, and let $A, B \subseteq X$ be two closed subsets of X with $A \cup B = X$. Then f is continuous iff its restrictions $f|A$ and $f|B$ are continuous.[7]

A map $f\colon X \to Z$ is a *homeomorphism*, if f is bijective, and if both f and f^{-1} are continuous. Two topological spaces X and Y are *homeomorphic*, if there exists a homeomorphism between them.

The identity map id_X is a homeomorphism. Being homeomorphic is an equivalence relation on the class of topological spaces.

A family $\mathcal{F}$ of maps between spaces X and Y is *equicontinuous*, if for all neighborhoods V of F, the set $\bigcap_{u \in \mathcal{F}} u^{-1}(V)$ is a neighborhood in X.

A map $i\colon X \to Y$ is called *embedding*, if $i\colon X \to i(X)$ is a homeomorphism. This is exactly the case if i is injective, continuous, and image-open.[8]

A sequence $(x_n)_n$ defines an associated filter $\mathcal{F}$ via filter base $\mathcal{B} := \{B_m\}$, with $B_m := \{x_n \mid n \geq m\}$.[9] Sequence $(x_n)_n$ *converges* to a point x, written $x_n \to x$, if for all neighborhoods U of x, there exists an n_0 such that $x_n \in U$ for all $n \geq n_0$. Clearly, $x_n \to x$ iff $\mathcal{F} \to x$.

1.6. Initial and Final Topologies. Let $\mathcal{S}$ and $\mathcal{T}$ be two topologies over set X. We call $\mathcal{S}$ *coarser* than $\mathcal{T}$ and $\mathcal{T}$ *finer* than $\mathcal{S}$, if $\mathcal{S} \subseteq \mathcal{T}$.

The set of topologies on a fixed set X is partially ordered by inclusion. The coarsest topology on X is the *indiscrete topology*, $\{\emptyset, X\}$, the finest is the *discrete topology*, 2^X.

Given a fixed set X, for every family $(\mathcal{T}_\iota)_{\iota \in I}$, of topologies on X, there exists a uniquely-determined coarsest topology $\inf_{\iota \in I} \mathcal{T}_\iota$, which is coarser then every $\mathcal{T}_\iota$. Analogously, there exists a uniquely-determined finest topology $\sup_{\iota \in I} \mathcal{T}_\iota$, which is finer then every $\mathcal{T}_\iota$. Hence, the set of topologies over set X is a complete lattice.

Given a set X and a family of maps $(f_\alpha\colon X \to X_\alpha)_{\alpha \in A}$, the *initial topology* is defined as the unique coarsest topology on X such that each map f_α is continuous.[10]

As special cases of initial topologies, we mention subsets and products of topological spaces. A subset X of a topological space Y is given a topology, the *subset topology*, by the initial topology via the *inclusion map* $i\colon X \to Y$. A cartesian product $\prod_{\alpha \in A} X_\alpha$ of topological spaces X_α is given a topology, the *product topology*, by the initial topology via the *projection maps* $p_\beta\colon \prod_{\alpha \in A} X_\alpha \to X_\beta$, $(x_\alpha)_\alpha \mapsto x_\beta$.

Dually[11], given a set X and a family of maps $(f_\alpha\colon X_\alpha \to X)_{\alpha \in A}$, the *final topology* is defined as the unique finest topology on X such that each map f_α is continuous.[12]

In addition, as special cases of final topologies, we mention quotients and sums. A quotient of a topological space X is a set Y together with a *quotient map* $q\colon X \to Y$ such that Y is given a topology, the *quotient topology*, by the final topology via

[6][**vQ01**, 2.20]
[7][**vQ01**, 3.4]
[8][**vQ01**, 3.5]
[9][**vQ01**, 5.11(c)]
[10][**vQ01**, 3.12, 3.13]
[11]We do not introduce Category Theory, and we do not discuss categorical aspects of these topological constructions, because this would digress too much from the main topic of this thesis.
[12][**vQ01**, 3.15, 3.16]

map q. A (disjoint) sum $\sum_{\alpha\in A} X_\alpha$ is given a topology via the final topology determined by the inclusion maps $i_\beta\colon X_\beta \to \sum_{\alpha\in A} X_\alpha$.

Map $f\colon X \to Y$ between topological spaces X and Y is called *image-open*, if for every open set $O \subseteq X$ the image $f(O)$ is open in $f(X)$, the latter with the subset topology.[13]

Clearly, the identity id_X is image-open.

1.7. Countability and Separation Axioms. A topological space X is *1st-countable (satisfies the first countability axiom)*, if there exists a neighborhood-system base $\mathcal{B}$ for X such that $\mathcal{B}(x)$ is countable for each point $x \in X$. Space X is called *2nd-countable (satisfies the second countability axiom)*, if there exists a neighborhood-system base $\mathcal{B}$ for X such that $\bigcup_{x\in X} \mathcal{B}(x)$ is countable. Space X is *separable*, if it contains a countable and dense subset.

A topological space X is T_0, if for each pair of points of X, one of them possesses a neighborhood, not containing the other point. It is T_1, if for each pair of points of X, each point possesses a neighborhood, not containing the other point. It is T_2 *(Hausdorff)*, if every two points possess disjoint neighborhoods. It is T_3, if every nonempty and closed subset $A \subset X$ and point $x \notin A$ possess disjoint neighborhoods. It is T_{3a}, if for every closed subset $A \subset X$ and point $x \notin A$, there exists a continuous function $f\colon X \to [0,1]$ such that $f(A) = \{0\}$ and $f(x) = 1$. It is T_4, if every two disjoint and closed subsets possess disjoint neighborhoods. It is T_{4a}, if for every two disjoint, nonempty, and closed subsets $A, B \subset X$, there exists a continuous function $f\colon X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Space X is called *regular*, if it is T_1 and T_3. It is *fully regular*, if it is T_1 and T_{3a}. Finally, it is *normal*, if it is T_1 and T_4.

A space X is T_1 iff every one-point set is closed.[14] Hence, every T_1 and T_{4a} space is T_{3a} and thus fully regular.

It is T_2 iff one of the following statements is true:[15]

(i) Every one-point set is the closure of its neighborhoods.
(ii) Every convergent filter has exactly one limit point.
(iii) The diagonal $\Delta := \{(x,x) \mid X\}$ is closed in $X \times X$.

LEMMA 1 (Urysohn). *Every T_4 space is T_{4a}.*[16]

Every normal space is fully regular, every fully-regular space is regular, every regular space is T_2, every T_2 space is T_1, and finally, every T_1 space is T_0. In addition, every T_{4a} space is T_4, and every T_{3a} space is T_3.

THEOREM 2 (Tietze). *Let X be a T_4 space. Then for every closed subset $A \subseteq X$ and continuous function $f\colon A \to \mathbb{R}$, there exists a continuous extension $g\colon X \to \mathbb{R}$ of f, i.e., $g(x) = f(x)$ for all $x \in A$.*[17]

We will prove a generalization of this theorem in Chapter 2.

1.8. Permanence Properties. The 1st-countable spaces are closed under initial and final topologies. In particular, arbitrary products and subsets of 1st-countable spaces are 1st-countable. The 2nd-countable spaces are closed under final topologies. In general, they are not closed under initial topologies. At least, they are closed under countable products and arbitrary subsets.

[13] In [**SW99**] image-open maps are called open. This may lead to confusion and wrong results. Hence, we do not follow this deviation from standard terminology [**Bou98a**, I.§5].

[14] [**vQ01**, 6.3]

[15] [**vQ01**, 6.4]

[16] [**vQ01**, 7.1, 7.2]

[17] [**vQ01**, 7.7]

Spaces, which are T_i, $i \in \{0,1,2,3,3a\}$, are closed under initial topologies.[18] T_4 spaces are not closed under subsets or arbitrary products.[19] But T_4 (normal) spaces are closed under closed subsets.[20]

In general, T_i spaces, $i \in \{1,2,3,3a,4\}$, are not closed under final topologies.[21]

2. Uniform Spaces

2.1. Uniformities. For subsets $A, B \subseteq X \times X$, let $A^{-1} := \{(y,x) \mid (x,y) \in A\}$. Let $BA := \{(x,z) \mid \exists y \in C\colon (y,z) \in B, (x,y) \in A\}$ In particular, $A^2 = AA$.

Given set X, a *uniformity* on X is a filter $\mathcal{U}$ on $X \times X$ with the following properties:

Reflexivity: $\Delta := \{(x,x) \mid x \in X\} \subseteq U$ for all $U \in \mathcal{U}$.
Symmetry: $U^{-1} \in \mathcal{U}$ for all $U \in \mathcal{U}$.
Triangle Inequality: There exists $V \in \mathcal{U}$ with $V^2 \subseteq U$ for all $U \in \mathcal{U}$.

The pair $(X,\mathcal{U})$ is called a *uniform space*. Each set $U \in \mathcal{U}$ is called a *uniform neighborhood*.

A *uniformity base* on X is a filter base $\mathcal{B}$ on $X \times X$ with the following properties:

Reflexivity: $\Delta := \{(x,x) \mid x \in X\} \subseteq B$ for all $B \in \mathcal{B}$.
Symmetry: There exists $C \in \mathcal{B}$ with $C \subseteq B^{-1}$ for all $B \in \mathcal{B}$.
Triangle Inequality: There exists $C \in \mathcal{B}$ with $C^2 \subseteq B$ for all $B \in \mathcal{B}$.

Then $\mathcal{U} := \{U \mid \exists B \in \mathcal{B}\colon B \subseteq U\}$ is the unique uniformity defined by the uniformity base $\mathcal{B}$.[22]

A uniformity uniquely induces a topology[23]: Uniformity $\mathcal{U}$ on X induces the neighborhood system $\mathcal{N}_x := \{U_x \mid U \in \mathcal{U}\}$, where $U_x := \{y \in X \mid (x,y) \in U\}$. A topological space is *uniformizable*, if there exists a uniformity on this space inducing its topology.

A topological space is uniformizable iff it is a T_{3a} space.[24] In particular, a T_1 space is uniformizable iff it is fully regular.

2.2. Uniform Continuity and Convergence. A map $f\colon X \to Y$ between uniform spaces X and Y is *uniformly continuous*, if for each uniform neighborhood W of Y, there exists a uniform neighborhood V of X such that $(f(x), f(y)) \in W$ for all $(x,y) \in V$.

The identity id_X is uniformly continuous, and the composition of uniformly-continuous maps is uniformly continuous.

2.3. Completeness. Let $(X,\mathcal{U})$ be a uniform space, let $A \subseteq X$, and let $U \in \mathcal{U}$ be a uniform neighborhood. Subset A is *small of order* U, if $A \times A \subseteq U$. A filter $\mathcal{F}$ on X is called *Cauchy filter*, if for every uniform neighborhood $U \in \mathcal{U}$ there exists a set $F \in \mathcal{F}$ small of order U.

In a uniform space, every convergent filter is a Cauchy filter.[25]

For a uniformly-continuous map, the image filter of a Cauchy filter is a Cauchy filter.[26]

A uniform space X is *complete*, if every Cauchy filter is a convergent filter in X.

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[18]See [**vQ01**, 6.11] for subsets, and [**vQ01**, 6.14] for products
[19]See [**vQ01**, 6.12] for subsets and [**vQ01**, 6.15] for products
[20][**vQ01**, 6.13]
[21][**vQ01**, 6.17]
[22][**vQ01**, 11.4]
[23][**vQ01**, 11.5]
[24][**vQ01**, 11.22, 11.30]
[25][**vQ01**, 12.4]
[26][**vQ01**, 12.6]

For every uniform space X, there exists a complete, uniform, and T_2 space $\hat{X}$ and a uniformly-continuous map $i\colon X \to \hat{X}$ such that the following universal property holds: for every complete, uniform, and T_2 space Y and every uniformly-continuous map $f\colon X \to Y$, there exists a uniquely-defined uniformly-continuous map $\hat{f}\colon \hat{X} \to Y$ with $\hat{f} \circ i = f$.[27]

In case X is a uniform T_2 space, X is isomorphic to a dense subset of $\hat{X}$.[28]

2.4. (Para-)Compactness. As the definition of a compact space does not make explicit reference to uniformities, the reader may wonder, why compact spaces are introduced here as part of uniform spaces, and not as part of general topological spaces. But this is correct, see e.g., [**Bou98a**, II.§4].

Let X be a topological space, A a subset of X, and $\mathcal{C}$ a set of subsets of X. Set $\mathcal{C}$ is called a *covering* of A, if its union contains A. Covering $\mathcal{C}$ is *open*, if it only contains open subsets of X. A *subcovering* of covering $\mathcal{C}$ is just a subset of $\mathcal{C}$. A *refinement* D of covering $\mathcal{C}$ is a covering of A such that for each $V \in \mathcal{D}$ there exists an $U \in \mathcal{C}$ with $V \subseteq U$.

A covering is called *finite* / *countable*, if it only contains a finite / countable number of sets.

A covering $\mathcal{C}$ of space X is called *locally finite*, if for every $x \in X$ there exists a neighborhood V of x such that only finitely-many sets $U \in \mathcal{C}$ intersect with V, i.e., $U \cap V \neq \emptyset$.

Given an open covering $\mathcal{C} = \{U_\alpha \mid \alpha \in A\}$, a family $\{f_\alpha \mid \alpha \in A\}$ of continuous functions $f_\alpha\colon X \to \mathbb{R}$ is called *partition of unity subordinate to* $\mathcal{C}$, if it has the following properties:

(i) $f_\alpha(x) \geq 0$ for all $\alpha \in A$ and $x \in X$.
(ii) $U_\alpha \subseteq \operatorname{supp}(f_\alpha)$ for all $\alpha \in A$.
(iii) Covering $\{\operatorname{supp}(f_\alpha) \mid \alpha \in A\}$ is locally finite.
(iv) $\sum_{\alpha \in A} f_\alpha(x) = 1$ for all $x \in X$.

We say that a topological space X *allows for a partition of unity*, if for every open and locally-finite covering of X there exists a partition of unity.

Every normal space allows for a partition of unity.[29]

A uniform T_2 space X is *compact*, if every open covering of X contains a finite subcovering of X. A uniform T_2 space is called *precompact*, if its completion is compact.

For a uniform T_2 space X, the following statements are equivalent.

(i) Space X is compact.
(ii) Each family of closed subsets of X has nonempty intersection, if every finite subfamily has nonempty intersection.
(iii) Every filter has a cluster point.

A compact space is normal.[30] Its topology is induced by a unique uniformity.[31]

Let $f\colon X \to Y$ be a continuous map between a uniform and compact space X and a uniform space Y. Then f is uniformly continuous.[32]

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[27] [**vQ01**, 12.15]
[28] [**vQ01**, 12.16]
[29] [**vQ01**, 7.16]
[30] [**vQ01**, 8.9]
[31] [**vQ01**, 11.A6]
[32] [**vQ01**, 11.14]

Let X be a compact space, and let $f\colon X \to Y$ be continuous. Then $f(X)$ is compact.[33] If Y is a T_2 space, then f is closed.[34] In particular, a continuous function $f\colon X \to \mathbb{R}$ attains its minimum and maximum on compact space X.[35]

Let X and Y be topological T_2 spaces. A map $f\colon X \to Y$ has *precompact image*, if $\overline{f(X)}$ is compact.[36]

A T_2 space is *locally compact*, if every point has a compact neighborhood. By definition, every compact space is locally compact.

Every locally compact space is fully regular.[37] Hence, it is uniformizable.

A continuous map $f\colon X \to Y$ between topological spaces X and Y is called *proper*, if for every compact subset $C \subseteq Y$, its preimage $f^{-1}(C)$ is compact.

Clearly, the identity id_X is proper, and the composition of proper maps is proper.

Let $f\colon X \to Y$ be a proper map between locally-compact spaces X and Y. Then f is closed and $f(X)$ is locally compact.[38]

A T_2 space X is *paracompact*, if every open covering of X possesses a locally-finite open refinement.

Every paracompact space is normal.[39] Hence, it allows for a partition of unity.

2.5. Compactification. Let X be a topological space, let Y be a compact space, and let $f\colon X \to Y$ be an embedding onto a dense subset of Y. Then pair (f, Y) is called a *compactification* of X. A *Stone-Čech compactification* is a compactification $(\beta, \beta X)$ such that the following universal property holds: for every T_2 space Y and every continuous map $f\colon X \to Y$, there exists a uniquely-defined continuous map $\beta f\colon \beta X \to Y$ such that $f = \beta f \circ \beta$.

THEOREM 3. *For every fully-regular space X, there exists a uniquely-determined Stone-Čech compactification $(\beta, \beta X)$.*[40]

Hence, $\beta X \backslash \beta(X)$ denotes all the "∞"-elements, added to X by the compactification.[41]

Let X and Y be fully-regular topological spaces, and let $f\colon X \to Y$ be continuous. Then there exists an extension $\beta f\colon \beta X \to \beta Y$ of f such that $\beta f \circ \beta = \beta \circ f$.[42]

2.6. Permanence Properties. Given a fixed set X, for every family $(\mathcal{U}_\iota)_{\iota\in I}$ of uniformities on X, then there exists a uniquely-determined coarsest uniformity $\inf_{\iota\in I} \mathcal{U}_\iota$, which is coarser then every $\mathcal{U}_\iota$. Analogously, there exists a uniquely-determined finest uniformity $\sup_{\iota\in I} \mathcal{U}_\iota$, which is finer then every $\mathcal{U}_\iota$. Hence, the set of uniformities over set X is a complete lattice.

Uniform spaces are closed under initial and final topologies. More precisely, given a family of uniform spaces, their topologies lead to an initial (respectively, final) topology, which is induced by a unique uniformity.[43]

Complete uniform spaces are closed under initial topologies: Given a family of uniform spaces, these spaces are complete iff the uniformity of the initial topology is

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[33] [**vQ01**, 8.10]

[34] [**vQ01**, 8.11]

[35] [**vQ01**, 8.A3]

[36] Such maps are also sometimes called compact, e.g., compare [**GD03**] and [**SW99**]. This may lead to serious confusion.

[37] [**vQ01**, 8.15, 8.A20]

[38] [**vQ01**, 8.21]

[39] [**vQ01**, 10.2]

[40] [**vQ01**, 12.18]

[41] This remark has been added after thesis submission.

[42] [**vQ01**, 12.A12]

[43] [**vQ01**, 11.17, 11.18]

complete.[44] In particular, closed subsets and arbitrary products of complete spaces are complete.[45]

Every compact subset of a T_2 space is closed.[46] Every closed subset of a compact set is compact.[47]

Precompact spaces are closed under subspaces and arbitary products.

THEOREM 4 (Tychonoff). *An arbitrary product of compact spaces is compact.*[48]

3. Metric Spaces

3.1. Definition. A pair (M, d) is called a *metric space*, if M is a set, and if for $d\colon M \times M \to \mathbb{R}$ the following statements hold:

Positive Definite: $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$ for all $x, y \in M$.
Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in M$.
Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

Function d is then called a *metric* on M.

For $x \in M$ and $r > 0$, set $B(x, r) := \{y \in M \mid d(x, y) < r\}$ is called *open ball* around x of radius r. Set $S(x, r) := \{y \in M \mid d(x, y) = r\}$ is called *sphere* around x of radius r.

A metric d on M induces a topology on M with the set $\mathcal{T}_{(M,d)} := \{B(x, r) \mid x \in M, r > 0\}$ of its open balls.[49]

It even induces a uniformity via its uniformity base $\mathcal{B}_{(M,d)} := \{U(r) \mid r > 0\}$, where $U(r) := \{(x, y) \mid d(x, y) < r\}$.[50]

A uniform space $(X, \mathcal{U})$ is *metrizable*, if there exists a metric d on M, inducing the uniformity $\mathcal{U}$.

A topological space X is *metrizable*, if there exists a metric on X, inducing the topology on X.

A topological space M is called *completely metrizable*, if there exists a metric d, inducing the topology of M, and if (M, d) is complete.

For every metric d we have the *reverse triangle inequality*, $|d(x, y) - d(x, z)| \leq d(y, z)$ for all $x, y, z \in M$. Hence, d is continuous as seen as a map between topological spaces $M \times M$ and $\mathbb{R}$.

Every metrizable space is first countable.[51]

A metrizable space is T_2 and T_4.[52] Hence, a metrizable space is normal.

THEOREM 5 (M.H. Stone). *Every metrizable space is paracompact.*

3.2. Isometries. Let (M, d) and (N, e) be two metric spaces. A map $f\colon M \to N$ is called *isometric*, if for all $x, y \in M$ we have $e(f(x), f(y)) = d(x, y)$. If f is bijective and isometric, f is called an *isometric isomorphism*. Then f^{-1} is also an isometric isomorphism.

The identity map id_M is an isometric isomorphism, and the composition of isometric isomorphisms is an isometric isomorphism.

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[44] [**vQ01**, 12.12]
[45] [**vQ01**, 12.13]
[46] [**vQ01**, 8.6(b)]
[47] [**vQ01**, 8.6(a)]
[48] [**vQ01**, 8.12]
[49] [**vQ01**, 1.4]
[50] [**vQ01**, 11.6(a)]
[51] [**vQ01**, 2.13(a)]
[52] [**vQ01**, 1.25]

3.3. Compactness. Let (M, d) be a metric space, let $\epsilon > 0$, and let $V \subseteq E$. A *finite ϵ-net* for V is a finite set $\{z_1, \ldots, z_n\} \subseteq M$ such that $V \subseteq \bigcup_{i\in[n]} z_i + B(0, \epsilon)$.

For a metrizable space X, the following statements are equivalent:

(i) X is compact.
(ii) X is sequentially compact.
(iii) For every $\epsilon > 0$, there exists a finite ϵ-net for X.

3.4. Permanence Properties. In general, metric spaces are not closed under initial and final topologies.

Countable products of metric spaces are metric spaces.[53] Every subset $A \subseteq M$ of a metric space (M, d) is a metric space with the induced metric $d|_A \colon A \times A \to \mathbb{R}$.

4. Vector Spaces

4.1. Definition. A *group* is a tuple $(G, \circ, e)$, consisting of a set G, the *neutral* element $e \in G$ and the group operation $\circ\colon G\times G \to G$, fulfilling the following axioms for all $x, y, z \in G$:

Associativity: $(x \circ y) \circ z = x \circ (y \circ z)$,
Neutrality: $e \circ x = x \circ e = x$,
Existence of Inverse: There exists $w \in G$ such that $x \circ w = w \circ x = e$.

Group G is *commutative*, if $x \circ y = y \circ y$ for all $x, y \in G$. In such a case, a group is often written additively, i.e., with notation $(G, +, 0)$.

As usual, let $\mathbb{K}$ denote the field $\mathbb{R}$ or $\mathbb{C}$, respectively. A *vector space* is a set E, together with an *addition* $+\colon E \times E \to E$ and a *scalar multiplication*$\cdot\colon \mathbb{K} \times E \to E$ such that $(E, +, 0)$ is a commutative group and the following axioms hold for all $x, y \in E$ and $\lambda, \mu \in \mathbb{K}$:

Distributivity: $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$,
Associativity: $\lambda \cdot (\mu \cdot x) = (\lambda \cdot \mu) \cdot x$,
Neutrality: $1 \cdot x = x$.

4.2. Basis. Let E be a vector space. A *linear combination* is an element $\lambda_1 \cdot x_1 + \cdots + \lambda_m \cdot x_m$, where $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ and $x_1, \ldots, x_m \in E$. A set $I \subseteq E$ and its elements are called *independent*, if for all linear combinations $\lambda_1 \cdot x_1 + \cdots + \lambda_m \cdot x_m$ with elements $\{x_1, \ldots, x_m\} \subseteq I$ we have that $\lambda_1 \cdot x_1 + \cdots + \lambda_m \cdot x_m = 0$ implies $\lambda_1, \ldots, \lambda_m = 0$ for all $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$. Otherwise, the set and its elements are called *dependent*.

For $I \subseteq E$, the *span* of I, $\operatorname{span}(I)$, is the set of all linear combinations of elements in I. Set I *generates* E, if $E = \operatorname{span}(I)$.

A *basis* of E is a generating and independent subset of E. By Zorn's lemma, every vector space has a basis. In addition, all bases of E have the same cardinality / number of elements. Hence, the *dimension* of E, defined as the cardinality of a basis of E, is well-defined.

4.3. Linear Operators. A map $u\colon E \to F$ between vector spaces E and F is called *linear (linear operator)*, if $u(x + y) = u(x) + u(y)$ and $u(\lambda \cdot x) = \lambda \cdot u(x)$ for all $x, y \in E$ and $\lambda \in \mathbb{K}$.

The identity id_E is linear. The composition of linear operators is linear.

A bijective linear operator is called a *linear isomorphism*. It has a linear inverse.

Given linear operator $u\colon E \to F$, the *kernel* of u, $\ker u$, is defined as the set $u^{-1}(0)$. The sets *graph* of u, $\operatorname{graph} u := \{(x, u(x)) \mid x \in E\}$ and *image* of u, $\operatorname{im} u := u(E)$, are defined as for general maps.

[53][**SV06**, 6.3.1]

Given basis B for vector space E and basis C for vector space F, a linear operator $h\colon E \to F$ has a representation via a *matrix* $M := (m_{b,c})_{b\in B, c\in C}$, where $m_{b,c}$ is defined via $u(b) = \sum_{c\in C} m_{b,c} \cdot c$.

In case E is finite-dimensional, a linear operator $u\colon E \to E$ is injective iff it is surjective.

The *space of linear operators*, $\mathcal{L}(E, F)$, contains all linear operators $u\colon E \to F$ between vector spaces E and F. It is itself a vector space, with addition and scalar multiplication defined pointwise.

4.4. Permanence Properties. Given a family $\mathcal{E} := (E_\iota)_{\iota\in I}$ of vector spaces and a vector space E, a family $\mathcal{P} := (p_\iota)_{\iota\in I}$ of linear operators $p_\iota\colon E \to E_\iota$ is called *projections* for E and $\mathcal{E}$, if for all vector spaces D and linear operators $f, g\colon D \to E$, we have that $f = g$ in case that $p_\iota \circ f = p_\iota \circ g$ for all $\iota \in I$.

Vector space E is called *projective* for $\mathcal{E}$, if there exists a family $\mathcal{P} := (p_\iota)_{\iota\in I}$ of projections $p_\iota\colon E \to E_\iota$ for E and $\mathcal{E}$ such that the following universal property holds: For every vector space F and family $\mathcal{Q} := (q_\iota)_{\iota\in I}$ of projections $q_\iota\colon F \to E_\iota$ for F and $\mathcal{E}$, there exists a linear operator $q\colon F \to E$ with $p_\iota \circ q = q_\iota$ for all $\iota \in I$.

For every family $\mathcal{E} := (E_\iota)_{\iota\in I}$ of vector spaces, there exists a projective vector space. To see this, take the *product* $E := \prod_{\iota\in I} E_\iota$, together with the projections $p_\alpha\colon E \to E_\alpha$, defined by $(x_\iota)_{\iota\in I} \mapsto x_\alpha$.

A projective vector space is uniquely-determined up to linear isomorphism.

Dually, Given a family $\mathcal{E} := (E_\iota)_{\iota\in I}$ of vector spaces and a vector space E, a family $\mathcal{J} := (j_\iota)_{\iota\in I}$ of linear operators $j_\iota\colon E_\iota \to E$ is called *inclusions* for E and $\mathcal{E}$, if for all vector spaces D and linear operators $f, g\colon E \to D$, we have that $f = g$ in case that $f \circ j_\iota = g \circ j_\iota$ for all $\iota \in I$.

Vector space E is called *inductive* for $\mathcal{E}$, if there exists a family $\mathcal{J} := (j_\iota)_{\iota\in I}$ of inclusions $j_\iota\colon E_\iota \to E$ for E and $\mathcal{E}$ such that the following universal property holds: For every vector space F and family $\mathcal{K} := (k_\iota)_{\iota\in I}$ of inclusions $k_\iota\colon E_\iota \to F$ for F and $\mathcal{E}$, there exists a linear operator $k\colon E \to F$ with $k \circ j_\iota = k_\iota$ for all $\iota \in I$.

For every family $\mathcal{E} := (E_\iota)_{\iota\in I}$ of vector spaces, there exists an inductive vector space. To see this, take the *coproduct (algebraic direct sum)* $E := \coprod_{\iota\in I} E_\iota$, consisting of all elements $(x_\iota)_{\iota\in I}$ in $\prod_{\iota\in I} E_\iota$ with only finitely-many nonzero x_ι. Take as inclusions $j_\iota\colon E_\iota \to E$, defined by $x \mapsto (x_\nu)_{\nu\in I}$ with $x_\nu := x$ for $\nu = \iota$ and $x_\nu := 0$ otherwise.

An inductive vector space is uniquely-determined up to linear isomorphism.

A subset $S \subseteq E$ of a vector space $(E, +, \cdot, 0)$ is called *subspace* of E, if addition $+$ and scalar multiplication $\cdot$ are closed under S, i.e., if we have $+\colon S \times S \to S$ and $\cdot\colon \mathbb{K} \times S \to S$, respectively. Then S is a vector space. The inclusion operator $i_S\colon S \to E$, $x \mapsto x$, is injective and linear.

Every subspace S of E can be complemented with a subspace T of E such that E is inductive for family $\{S, T\}$ with $\{i_S, i_T\}$ taken as inclusions.

Given a vector space E and a subspace S of E, one can define the *quotient* space of E and S, denoted by E/S. Here, $E/S := \{x + S \mid x \in E\}$, $(x + S) + (y + S) := (x + y) + S$, and $\lambda \cdot (x + S) := (\lambda \cdot x) + S$ for all $x, y \in E$ and $\lambda \in \mathbb{K}$. Then E/S is a vector space. The quotient operator $q_S\colon E \to E/S$, $x \mapsto x + S$, is surjective and linear.

For every subspace S of vector space E, the exists a surjective and linear operator $p_S\colon E \to S$ such that vector space E is projective for family $\{S, E/S\}$ with $\{p_S, q_S\}$ taken as projections.

Let $u\colon E \to F$ be a linear operator between vector spaces E and F. Then the kernel, $\ker u$, is a subspace of E, and the image, $u(E)$, is a subspace of F.

Operator u has a *canonical decomposition* into $u = j \circ u_0 \circ p$, with projection $p\colon E \to E/\ker u$, linear isomorphism $u_0\colon E/\ker u \to u(E)$, and inclusion $j\colon u(E) \to F$.

4.5. Algebraic Dual, Hyperplanes. A *linear form* is a linear operator $u\colon E \to \mathbb{K}$ between a vector space E and its field of scalars $\mathbb{K}$. The space of linear forms, $\mathcal{L}(E, \mathbb{K})$, is called *algebraic dual*, and denoted by E^*. It is itself a vector space, with addition and scalar multiplication defined pointwise.

An *affine* subspace of vector space E is a set $x + S$, where $x \in E$ and S is a subspace of E. A *hyperplane* is an affine subspace $x + S$ of E, where S is a maximally proper subspace of E.

Every linear form $u \in E^*$ defines a hyperplane $H := \ker u$, and for every hyperplane H of E, there exists a linear form $u \in E^*$ and $c \in \mathbb{K}$ such that $H = \{x \in E \mid u(x) = c\}$.

4.6. Circledness, Convexity. A subset A of vector space E is *circled*, if $\mu \cdot A \subseteq A$ for all $\mu \in \mathbb{K}$, $|\mu| \leq 1$.

Trivially, $\emptyset$ and E are circled. Kernel and image of a linear operator are circled. Circled sets are closed under arbitrary unions and intersections. If $A, B \subseteq E$ are circled, then $A + B$ and $\lambda \cdot A$ are circled, $\lambda \in \mathbb{K}$.

A *circled hull* of A is defined as a smallest circled set containing A. For every subset A, a circled hull exists and is uniquely-determined. It equals $\mathrm{ci}(A) := \bigcap_{\lambda \in \mathbb{K}, |\lambda| \geq 1} \lambda \cdot A$.

A subset A of a vector space E is *convex*, if for all $x, y \in A$ and real $\lambda \in [0, 1]$ we have $\lambda \cdot x + (1 - \lambda) \cdot y \in A$.

Trivially, $\emptyset$ and E are convex. Kernel and image of a linear operator are convex. In general, convex sets are not closed under unions. Convex sets are closed under arbitrary intersections. If $A, B \subseteq E$ are convex, then $A + B$ and $\lambda \cdot A$ are convex, $\lambda \in \mathbb{K}$.

A *convex hull* of A is defined as a smallest convex set containing A. For every subset A, a convex hull exists and is uniquely-determined. It is denoted by $\mathrm{co}(A)$. The convex hull of a circled set is circled. The circled hull of a convex set is convex.

A subset A of a vector space E is *absolutely convex*, if for all $x, y \in A$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ we have $\lambda \cdot x + (1 - \lambda) \cdot y \in A$. Then A is absolutely convex iff it is circled and convex.

Trivially, $\emptyset$ and E are absolutely convex. Kernel and image of a linear operator are absolutely convex. In general, absolutely-convex sets are not closed under unions. Absolutely-convex sets are closed under arbitrary intersections. If $A, B \subseteq E$ are absolutely convex, then $A + B$ and $\lambda \cdot A$ are absolutely convex, $\lambda \in \mathbb{K}$.

An *absolutely-convex hull* of A is defined as a smallest absolutely-convex set containing A. For every subset A, an absolutely-convex hull exists and is uniquely-determined. It is denoted by $\mathrm{aco}(A)$. We have $\mathrm{aco}(A) = \mathrm{co}(\mathrm{ci}(A)) = \mathrm{ci}(\mathrm{co}(A))$.

A function $f\colon E \to \mathbb{R}$ is called *convex*, if $f(\lambda \cdot x + (1-\lambda) \cdot y) \leq \lambda f(x) + (1-\lambda) \cdot f(y)$ for all $x, y \in E$ and $\lambda \in [0, 1]$.

By induction on n, one can prove *Jensen's inequality*. We have

$$f\left(\sum_{i \in [n]} \lambda_i \cdot x_i\right) \leq \sum_{i \in [n]} \lambda_i \cdot f(x_i) \tag{1}$$

for all $x_1, \ldots, x_n \in E$, and $\lambda_1, \ldots, \lambda_n \in [0, 1]$ with $\sum_{i \in [n]} \lambda_i = 1$.

5. Topological Vector Spaces

5.1. Definition. A *topological vector space (t.v.s.)* is a vector space $(E, +, \cdot, 0)$ over a field $\mathbb{K}$, together with a topology on E such that addition $+\colon E \times E \to E$ and scalar multiplication $\cdot\colon E \times \mathbb{F} \to E$ are continuous.

Let E be a linear space together with a topology, not necessarily a t.v.s.. A neighborhood-system base $\mathcal{B}$ is called *locally additive*, if for each neighborhood $N \in \mathcal{B}_x$, there exists $M \in \mathcal{B}_x$ such that $M + M \subseteq N$, $x \in E$.

Let E be a t.v.s., let $\lambda \in \mathbb{K}$, and let $A, B \subseteq E$. If A is open, then $A + B$ and $\lambda \cdot A$ are open. If A is closed, then $\lambda \cdot A$ is closed. Furthermore, if B is closed, then $A + B$ is closed.

If E is a t.v.s. that is not T_0, then one can use the quotient t.v.s. E/N instead, where $N = \bigcap \mathcal{N}_0$ consists of all points being in all neighborhoods of zero. Hence, in the sequel, we can assume E to be T_0.

Every t.v.s. is a uniform space.[54] And every T_0 t.v.s. is a T_1 t.v.s.. Hence, a T_0 t.v.s. is fully regular. In particular, it is T_2.

As every T_0 t.v.s. E is uniform, there exists a uniquely-defined completion $\tilde{E}$ of E, with E dense in $\tilde{E}$. The extensions $+\colon \tilde{E} \times \tilde{E} \to \tilde{E}$, $\cdot\colon \mathbb{K} \times \tilde{E} \to \tilde{E}$ of addition and scalar multiplication make $\tilde{E}$ a t.v.s..[55]

5.2. Linear and Continuous Operators. Addition and multiplication are homeomorphisms. More precisely, for every fixed $y \in E$, map $x \mapsto x + y$ is a homeomorphism, analogously for the second argument of $+$. For fixed $\lambda \neq 0$, $x \mapsto \lambda \cdot x$ is a homeomorphism.

Not every bijective, linear, and continuous operator has a continuous inverse.

Let $u\colon E \to Y$ be a linear operator between T_0 t.v.s.. If u is continuous, then $\ker f$ is closed. If E is finite-dimensional, then u is continuous.

Proposition 6 (Folklore). *If a linear operator between T_0 t.v.s. is continuous, then it is graph-closed.*

Proof. Let E and F be T_0 t.v.s.. Then both are T_2. Let $u\colon E \to F$ be a linear and continuous operator. Define linear operator $v\colon E \times F \to E \times F$ by $v(e, g) := (e, u(e))$. Then v is continuous and $v(E \times F) = \operatorname{graph} u$. Consider an arbitrary point (e, f) in the closure $\overline{\operatorname{graph} u}$. Then there exists a filter C containing $\operatorname{graph} u$ and converging to (e, f). By continuity of v, the image filter $v(C)$ converges to $v(e, f) = (e, u(e))$. As $E \times F$ is in C, we have $\operatorname{graph} u$ in $v(C)$. The set of intersections of sets from C and $v(C)$, i.e., $C \cap v(C) = \{A \cap B \mid A \in C, B \in v(C)\}$, constitutes a filter base for a finer filter $D \supseteq C, v(C)$. Filter D contains $\operatorname{graph} u$ and converges both to (e, f) and $(e, u(e))$, respectively. As $E \times F$ is T_2 as the product of two T_2 spaces, we have the uniqueness of the limit $(e, f) = (e, u(e))$. Hence, (e, f) is in $\operatorname{graph} u$, showing closedness of $\operatorname{graph} u$. $\square$

5.3. Circledness, Convexity. If A is circled, then $\overline{A}$ is circled. If $0 \in A^\circ$, then A° is circled, too.[56]

If A is convex, then A° and $\overline{A}$ are convex.[57] Hence, if A is absolutely convex, then A° and $\overline{A}$ are absolutely convex.[58]

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[54] [**SW99**, I.1.4]
[55] [**SW99**, I.1.5]
[56] [**SW99**, I.1.1]
[57] [**SW99**, II.1.2]
[58] Here, we do not need the assumption $0 \in A^\circ$. Either A° is empty and we are done, or there exists $x \in E$ and a circled neighborhood U of 0 such that $x + (\frac{1}{2} \cdot U + \frac{1}{2} \cdot U) \subseteq A$. Then $-x + U = -x + (-U) \subseteq A$, because A and U are circled. By convexity of A, we finally obtain $0 + U \subseteq (x + \frac{1}{2} \cdot U) + (-x + \frac{1}{2} \cdot U) \subseteq A$. Hence, $0 \in A^\circ$.

5.4. Topological Dual, Hyperplanes. The set of continuous linear forms $u\colon E \to \mathbb{K}$ is called *topological dual* of E, and is denoted with E'. It is a vector subspace of the algebraic dual E^*.

A linear form u defines a hyperplane H and vice versa.[59] Hyperplane H is closed iff u is continuous, and it is dense in E iff u is not continuous.[60]

The topological dual may be the trivial space. Take $E := \mathcal{L}^p[0,1]$, $0 < p < 1$, as an example.[61]

THEOREM 7 (Hahn-Banach, Geometrical Form). *Let E be a t.v.s., let M be a linear subspace of E, and let C be a nonempty, convex, and open subset of E, not intersecting M. Then there exists a closed hyperplane H, containing M and not intersecting C.*[62]

5.5. Projective and Inductive Topologies. Initial and final topologies are too general in the context of linear spaces. Their counterparts are projective and inductive topologies. These are just their restrictions to linear operators.

Given a vector space E and a family of linear operators $(u_\alpha\colon E \to E_\alpha)_{\alpha\in A}$ into t.v.s. E_α, the *projective topology* is defined as the initial topology of this family, i.e., the coarsest topology on E such that all linear operators u_α are continuous. The projective topology is a translation-invariant topology on E, and E becomes a t.v.s., equipped with the projective topology.

Analogously to initial topologies, as special cases of projective topologies, we mention subspaces and products of t.v.s.. A subspace L of a t.v.s. E is given a topology, the *subspace topology*, by the projective topology via the linear *inclusion operator* $i\colon L \to E$. A cartesian product $\prod_{\alpha\in A} E_\alpha$ of t.v.s. E_α is given a topology, the *product topology (t.v.s.)*, by the projective topology via the linear *projection operators* $p_\beta\colon \prod_{\alpha\in A} E_\alpha \to E_\beta$.

Dually, given a t.v.s. E and a family of linear operators $(u_\alpha\colon E_\alpha \to E)_{\alpha\in A}$, the *inductive topology* is defined as the final topology of this family, i.e., the finest topology such that all linear operators u_α are continuous. The inductive topology is a translation-invariant topology on E, and E becomes a t.v.s., equipped with the inductive topology.

Again, analogously to final topologies, as special cases of inductive topologies, we mention quotients and sums. A quotient of a t.v.s. E is a t.v.s. F together with a linear *quotient operator* $q\colon E \to F$ such that F is given a topology, the *quotient topology (t.v.s.)*, by the inductive topology via q.

A cartesian coproduct $\coprod_{\alpha\in A} E_\alpha$ of t.v.s. E_α is given a topology, the *coproduct topology (t.v.s.)*, by the inductive topology via the linear *inclusion operators* $j_\beta\colon E_\beta \to \coprod_{\alpha\in A} E_\alpha$. The coproduct, equipped with the coproduct topology, is also called *topological direct sum* and denoted with $\bigoplus_{\alpha\in A} E_\alpha$.

5.6. Projective and Inductive Limits. Projective and inductive limits are special cases of projective and inductive topologies, respectively. In the sequel, let A be an index set, directed under a partial order $\le$, and let $(E_\alpha)_{\alpha\in A}$ be a family of t.v.s..

Let linear and continuous operators $g_{\alpha,\beta}\colon E_\beta \to E_\alpha$ be given for all $\alpha \le \beta$. The *projective limit* of $(E_\alpha)_{\alpha\in A}$ and $(g_{\alpha,\beta})_{\alpha,\beta\in A,\alpha\le\beta}$, denoted with $\varprojlim g_{\alpha,\beta}E_\beta$, is defined as the subspace of $\prod_{\alpha\in A} E_\alpha$, whose elements (x_α) satisfy the relation $x_\alpha = g_{\alpha,\beta}(x_\beta)$, whenever $\alpha \le \beta$. By construction, the topology of the projective limit is the subspace topology of the projective topology.

59 [**SW99**, I.4.1]
60 [**SW99**, I.4.2]
61 [**SW99**, I.Ex.6]
62 [**SW99**, II.3.1]

Dually, let linear and continuous operators $h_{\beta,\alpha}\colon E_\alpha \to E_\beta$ be given for all $\alpha \leq \beta$. The *inductive limit* of $(E_\alpha)_{\alpha\in A}$ and $(h_{\beta,\alpha})_{\alpha,\beta\in A,\alpha\leq\beta}$, denoted with $\varinjlim h_{\beta,\alpha}E_\alpha$, is defined as the quotient space of $\coprod_{\alpha\in A} E_\alpha$ with the subspace, generated by the ranges of all linear and continuous operators $j_\alpha - j_\beta \circ h_{\beta,\alpha}$ for all $\alpha, \beta \in A$ with $\alpha \leq \beta$. By construction, the topology of the inductive limit is the quotient topology of the inductive topology.

5.7. Baire Vector Spaces. A subset A of a linear space E is called *absorbent (radial absorbing)*, if for every $x \in E$, there exists $r > 0$ such that $x \in \lambda \cdot A$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \geq r$.

Let $\lambda \in \mathbb{K}$. If sets A and B are absorbent, then $A + B$, $\lambda \cdot A$, $\overline{A}$, circled hull $\mathrm{ci}(A)$, and convex hull $\mathrm{co}(A)$ are also absorbent, respectively.

We call a closed and circled subset *disk-like*, and an absorbent and disk-like subset a *vessel*.[63]

THEOREM 8. *A vector space together with a topology is a t.v.s. exactly if it possesses a locally-additive neighborhood-system base of vessels.*[64]

This motivates the following definition. A t.v.s. is called *vesseled*, if every vessel is a neighborhood.

If a closed set is not rare, then it is a neighborhood of some point. Consequently, if a vessel is not a neighborhood, then it is rare.

If a set V is absorbent, then the whole t.v.s. E is a countable union of translates of V, i.e., $E = \bigcup_{n\geq 1} n \cdot V$. Hence, if a closed and absorbent set is rare, then the t.v.s. E is meager. Consequently, if a vessel is not a neighborhood, then the t.v.s. is meager.

The above considerations give one direction of the theorem below. For the other direction, see Kunzinger [**Kun93**, Thm. 4.1.5].

THEOREM 9. *Baire t.v.s. are exactly the vesseled t.v.s..*

5.8. Compactness. For a t.v.s., its compact subspaces are exactly the finite-dimensional ones.[65] A finite-dimensional t.v.s. is topologically isomorphic to a $\mathbb{K}^n$.[66]

Let $A, B \subseteq E$. If A and B are compact subsets, then $A + B$, $\lambda \cdot A$ for $\lambda \in \mathbb{K}$, closure $\overline{A} = A$, and circled hull $\mathrm{ci}(A)$ are compact sets, respectively.[67]

A t.v.s. E has the *Heine-Borel property*, if the compact subsets of E are exactly the closed and bounded subsets.

The convex hull of a compact set is not necessarily compact.[68]

5.9. Boundedness. Subset A *absorbs* B, if there exists a positive real number λ such that $B \subseteq \lambda \cdot A$. A is *bounded*, if it is absorbed by every neighborhood.

A set $A \subseteq E$ is *totally bounded*, if for every neighborhood U in E, there exists a finite subset $A_0 \subseteq A$ such that $A \subseteq A_0 + U$.

Let $A, B \subseteq E$. If A and B are (totally) bounded sets, then $A \cap B$, $A \cup B$, $A + B$, $\lambda \cdot A$ for $\lambda \in \mathbb{K}$, interior A°, closure $\overline{A}$, circled hull $\mathrm{ci}(A)$, and convex hull $\mathrm{co}(A)$ are (totally-)bounded sets, respectively. More generally, every subset of a (totally-)bounded set is (totally) bounded.[69]

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[63] The terms *disk-like* and *vessel* are not standard notions. We introduce them here to make more explicit the analogy between Baire and barreled spaces.

[64] [**SW99**, I.1.2, I.1.3]

[65] [**SW99**, I.3.6]

[66] [**SW99**, I.3.1, I.3.2]

[67] [**SW99**, I.5.2]

[68] [**SW99**, II.Ex.27]

[69] [**SW99**, I.5.1]

Every totally-bounded set is bounded. Every relatively-compact set is totally bounded.

A subset $A \subseteq E$ is bounded iff for every sequence $(x_n)_n$ in A and every zero sequence $(\lambda_n)_n$ in $\mathbb{K}$, sequence $(\lambda_n \cdot x_n)_n$ is a zero sequence in E.[70]

A map $f\colon E \to F$ is called *bounded*, if it maps bounded sets into bounded sets, i.e., $f(B)$ is bounded for every bounded set B.

The identity id_E is bounded, and the composition of bounded maps is bounded. Every linear and continuous operator is bounded.[71]

A map $f\colon E \to F$ is called *compact*, if it maps bounded sets into relatively-compact sets, i.e., $f(B)$ is relatively compact for every bounded set B.

The composition of compact maps is compact. The identity id_E is compact iff t.v.s. E is finite-dimensional.

A family $\mathcal{F}$ of linear maps between t.v.s. E and F is *equibounded*, if for all bounded sets B in E, the set $\bigcup_{u\in\mathcal{F}} u(B)$ is bounded in F. Every equicontinuous family is equibounded.[72]

5.10. Permanence Properties. A t.v.s. E, equipped with the inductive topology of family $(u_\alpha\colon E_\alpha \to E)_{\alpha\in A}$, is a T_0 t.v.s. iff all E_α are T_0 t.v.s.. Consequently, this holds for properties T_2 and being fully regular.

A t.v.s. E, equipped with the projective topology of family $(u_\alpha\colon E \to E_\alpha)_{\alpha\in A}$, is a T_0 t.v.s. iff all E_α are T_0 t.v.s.. Consequently, this holds for properties T_2 and being fully regular.

A subset B of a topological product $\prod_{\alpha\in A} E_\alpha$ is bounded iff $p_\alpha(B)$ is bounded for all projections p_α, $\alpha \in A$.[73]

Projective limits of complete T_0 t.v.s. are complete.[74]

6. Locally-Convex Spaces

6.1. Definition. A t.v.s. has a *locally-convex* topology, if it possess a neighborhood-system base of convex sets.

A T_0 t.v.s. with locally-convex topology is called a *locally-convex space (l.c.s.)*.

A *seminorm* is a map $p\colon E \to \mathbb{K}$, which is homogeneous, positive-semidefinite, and satisfies the triangle inequality. *Homogeneous* means that $p(\lambda\cdot x) = |\lambda|\cdot p(x)$ for all vectors x and scalars λ. Map p is *positive semidefinite*, if $p(x)$ is real and nonnegative for all vectors x. It satisfies the *triangle inequality*, if $p(x+y) \le p(x)+p(y)$ for all $x, y \in E$.

A *disk* is a closed and absolutely-convex subset of E. By definition, a disk is disk-like. A disk D defines a subspace E_D of E by $E_D := \mathrm{span}(D)$, together with its *gauge functional* p_D. The latter is defined by $p_D(x) := \inf\{\lambda > 0 \mid x \in \lambda \cdot D\}$. Here, D absorbs every point in E_D, and the gauge functional p_D is a seminorm on space E_D.

Hence, a topology on a t.v.s. is locally convex iff it is the initial topology of a family of seminorms.

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70 [**SW99**, I.5.3]

71 [**SW99**, I.5.4]

72 Let $\mathcal{F}$ be an equicontinuous family of linear maps between t.v.s. E and F. Let V be an arbitrary neighborhood in F, and let B be an arbitrary bounded set in E. Then $U = \bigcap_{u\in\mathcal{F}} u^{-1}(V)$ is a neighborhood in E. As U absorbs B, there exists a positive real number $\lambda > 0$ such that $B \subseteq \lambda \cdot U$, implying $u(B) \subseteq \lambda V$ for all $u \in \mathcal{F}$. Hence $\bigcup_{u\in\mathcal{F}} u(B)$ is absorbed by V and thus bounded, proving $\mathcal{F}$ to be equibounded.

73 [**SW99**, I.5.5]

74 See [**SW99**, II.5.3] for a proof formulated for l.c.s.. The local convexity of the spaces is not used in proof.

6.2. Topological Dual, Hyperplanes.

THEOREM 10 (Hahn-Banach, Analytical Form). *Let E be a t.v.s., let M be a subspace of E, and let $f\colon M \to \mathbb{K}$ be a linear form on M. If there is a seminorm $p\colon E \to \mathbb{R}$ with $|f(x)| \le p(x)$ for all $x \in M$, then there exists a linear form $g\colon E \to \mathbb{K}$, extending f to all of E with $|g(x)| \le p(x)$ for all $x \in E$.*[75]

THEOREM 11. *Let E be an l.c.s., let M be a subspace of E, and let $f\colon M \to \mathbb{K}$ be a continuous linear form on M. Then there exists a continuous linear form $g\colon E \to \mathbb{K}$, extending f to all of E.*[76]

Let $\langle\cdot,\cdot\rangle$ denote the duality between E and E', defined by $\langle f, x\rangle := f(x)$ for $x \in E$ and $f \in E'$.

A sequence $(x_n)_n$ *weakly converges* to x, denoted by $x_n \rightharpoonup x$ in E $(n \to \infty)$, if for all $f \in E'$ we have $\langle f, x_n\rangle \to \langle f, x\rangle$ $(n \to \infty)$. A sequence $(f_n)_n$ in E' $*$*-weakly converges* to f, denoted by $f_n \overset{*}{\rightharpoonup} f$, if it converges pointwise, i.e., if for all $x \in E$ we have $\langle f, x_n\rangle \to \langle f, x\rangle$ $(n \to \infty)$. Clearly, convergence $x_n \to x$ implies weak convergence $x_n \rightharpoonup x$, because each $f \in E'$ is continuous.

In addition, weak convergence $f_n \rightharpoonup f$ in E' $(n \to \infty)$ implies $*$-weak convergence $f_n \overset{*}{\rightharpoonup} f$ in E' $(n \to \infty)$. In case E is reflexive, the opposite is also true, i.e., weak and $*$-weak convergence are equivalent. This can be seen by applying the topological isomorphism $j\colon E \to (E'_\beta)'_\beta$, $\langle j(x), f_n\rangle = \langle f, x\rangle$.

6.3. Compactness. The convex hull of a compact set is compact.[77]

6.4. Permanence Properties. The class of l.c.s. is closed under arbitrary projective topologies.[78] In particular, it is closed under arbitrary products[79] and closed subspaces.[80] In addition, it is closed under arbitrary projective limits.[81]

The class of l.c.s. is closed under arbitrary inductive topologies.[82] In particular, it is closed under coproducts (topological direct sums)[83] and quotients under closed subspaces.[84]

It seems to be an open problem, if the class of l.c.s. is closed under arbitrary inductive limits.[85] For a special case, one can prove more: Given a family $(E_\alpha)_{\alpha\in A}$ of l.c.s., each E_α a subspace of a vector space $E = \bigcup_{\alpha\in A} A_\alpha$, and directed under inclusion, i.e., $E_\alpha \subseteq E_\beta$ for $\alpha \le \beta$. Then the inductive limit $\varinjlim E_\alpha$ exists and is locally convex.[86] It is called *strict*, if the topology of E_β induces the topology on E_α for all $\alpha \le \beta$.

In particular, let $(E_m)_m$ be an increasing sequence of l.c.s. with topologies $\mathcal{T}_m$. If each $(E_{m+1}, \mathcal{T}_{m+1})$ induces the topology $\mathcal{T}_m$ on E_m, then the inductive limit exists on $E := \bigcup_m E_m$ and induces the topology $\mathcal{T}_m$ on E_m.[87]

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75 [**SW99**, II.3.2]
76 [**SW99**, II.4.2]
77 [**SW99**, II.4.3]
78 [**SW99**, II.5]
79 [**SW99**, II.5.2]
80 [**SW99**, I.2.1, II.6.1]
81 [**SW99**, II.5.2]
82 [**SW99**, II.6]
83 [**SW99**, II.6.1]
84 [**SW99**, I.2.3, II.6.1]
85 [**SW99**, II.6.3]
86 [**SW99**, II.6.3]
87 [**SW99**, II.6.4]

A subset B of the coproduct $\coprod_{\alpha\in A} E_\alpha$ of a family of l.c.s. E_α is bounded iff there exists a finite subset $A_0 \subseteq A$ such that $p_\alpha(B) = 0$ for all $\alpha \notin A_0$ and $p_\alpha(B)$ is bounded for all $\alpha \in A_0$. Again, p_α denotes the projection into E_α.[88]

A subset B of a strict inductive limit $E := \varinjlim(E_m, \mathcal{T}_m)$ is bounded in E iff there exists a natural m such that $B \subseteq E_m$ is bounded in $(E_m, \mathcal{T}_m)$.[89]

The coproduct of a family of l.c.s. is complete iff each summand is complete.[90]

The strict inductive limit of a sequence of complete l.c.s. is complete.[91] In particular, LB and LF spaces are complete, respectively.

7. Bornological Spaces

7.1. Definition. An l.c.s. E is *bornological*, if every absolutely-convex subset of E, absorbing every bounded subset of E, is a 0-neighborhood.

7.2. Operators. Let E be bornological, let F be an l.c.s., and let $u\colon E \to F$ be a linear operator. Then the following statements are equivalent:[92]

(i) Operator u is continuous.
(ii) Operator u is bounded, i.e., $u(B)$ is bounded in F for every bounded set B of E.
(iii) Operator u maps zero sequences to zero sequences.

7.3. Permanence Properties. Bornological spaces are closed under arbitrary inductive topologies.[93] In particular, they are closed under T_0 quotients, topological direct sums, and inductive limits.[94]

In general, bornological spaces are not closed under arbitrary projective topologies. In particular, there exists a bornological space, which is not closed under closed subspaces.[95]

8. Barreled Spaces

In our opinion, the class of barreled spaces is *the* class of spaces suitable for Functional Analysis. They define the limits of certain constructions in Functional Analysis like Banach-Steinhaus, Closed-Graph, Open-Mapping, and Continuous-Inverse properties. In contrast to the narrow class of Banach spaces, they are also broad enough to cover all relevant function spaces, including distributions.

For more information on barreledness and related properties, see, e.g., [**Ada70, Val71a, Val71b, Val72a, Val72b, VD72, Val73, Val79, VC81**] and also [**Sax74, Hol77, PC87**].

8.1. Definition. A *barrel* is an absorbent disk.

THEOREM 12. *A vector space, together with a topology stemming from a family of seminorms, is an l.c.s. exactly if it possesses a locally-additive neighborhood-system base of barrels.*[96]

[88] [**SW99**, II.6.3]
[89] [**SW99**, II.6.5]
[90] [**SW99**, II.6.2]
[91] [**SW99**, II.6.6]
[92] [**SW99**, II.8.3]
[93] [**SW99**, II.8.2]
[94] [**SW99**, II.8.2, Cor.1]
[95] [**Kha82**, p.104]
[96] [**SW99**, II.4]

This motivates the following definition. An l.c.s. is called *barreled*, if every barrel is a neighborhood.

Every Baire l.c.s. is barreled.[97]

Important classes of l.c.s., analyzed in connection with the closed-graph property, were defined by Pták. Recall that an l.c.s. E is a *Pták space (B-complete)*, if every subspace $Q \subseteq E'$ is $\sigma(E', E)$-closed iff $Q \cap C$ is $\sigma(E', E)$-closed for all equicontinuous subsets $C \subseteq E'$. Furthermore, E is an *infra-Pták space (B_r-complete)*, if this holds for all dense subspaces Q. For the notions of Pták and infra-Pták space, see also [**SW99**, II.4, II.7, IV.8], respectively.

8.2. Reflexive Spaces. The *bidual* E'' of l.c.s. E is defined as the vector space (a priori without a topology) $(E'_\beta)'$. The *strong bidual* is defined as $(E'_\beta)'_\beta$, i.e., the bidual together with the strong topology.

Linear operator $j\colon E \to E''$, defined by $\langle jx, f\rangle := (jx)(f) := f(x)$, $x \in E$, $f \in E'$, is called the *(canonical) embedding* or *evaluation map*.

An l.c.s. E is *semireflexive* iff the (canonical) embedding j is a surjective linear operator onto the bidual E''. Space E is called *reflexive* iff j is a topological isomorphism onto the strong bidual.

An l.c.s. is reflexive iff it is semireflexive and barreled.[98]

The strong dual E'_β of a reflexive space E is reflexive.[99]

A *Montel space* is a T_0, complete, and reflexive l.c.s. with the Heine-Borel property. Hence, every bounded subset is relatively compact.

8.3. Banach-Steinhaus. We say that l.c.s. E has the *Banach-Steinhaus property*, if for all l.c.s. F and all families $\mathcal{F}$ of linear, continuous maps between E and F holds that if $\mathcal{F}$ is *pointwise bounded on* A, i.e., for all $x \in E$ the set $\mathcal{F}(x) = \{u(x) \mid u \in \mathcal{F}\}$ is bounded, family $\mathcal{F}$ is equicontinuous. We say that E has the *Banach-Steinhaus property for functionals*, if every family $\mathcal{F} \subseteq E'$ of continuous functionals pointwise-bounded on E is equicontinuous.

THEOREM 13. *An l.c.s. has the Banach-Steinhaus property iff it is barreled.*

Instead of Banach-Steinhaus property, one often uses the term *uniform boundedness*.

8.4. Permanence Properties. In general, barreled spaces are not closed under arbitrary projective topologies. In particular, a closed subspace of a barreled space does not need to be barreled.[100] At least, they are closed under finite products.

Barreled spaces are closed under arbitrary inductive topologies.[101] Hence, they are closed under arbitrary inductive limits, topological direct sums, and quotients with closed subspaces. In addition, LB and LF spaces are barreled.

8.5. Operators. Theorems on open mappings, continuous inverses, and closed graphs have a long history, with many applications in different branches of Functional Analysis [**Wer00, Mat98, Alt06, AV05**]. Initially only formulated for Banach spaces, one line of research was to extend these theorems to very general classes of spaces [**Ptá58, Ptá59, Ptá60, Ptá62, Ptá65, Ptá66, Ptá69, Ptá74, HM62, Hus62, Hus64a, Hus64b, Kri71, SW99, Val78, Ada83, Ada86, SR89, Rod91**]. While this research states such theorems for linear mappings

[97] [**SW99**, II.7.1]
[98] [**SW99**, IV.5.5]
[99] [**SW99**, IV.5.6]
[100] [**SW99**, IV.Ex.10]
[101] [**SW99**, I.7.2]

$u\colon E \to F$ with E taken from one class $\mathcal{A}$ of t.v.s. and F taken from a possibly different class $\mathcal{B}$, we approach the topic differently. We only allow E and F to come from the *very same* class of t.v.s. $\mathcal{C}$, and we ask, under which conditions on $\mathcal{C}$ the open-mapping theorem, the continuous-inverse theorem, and the closed-graph theorem are actually equivalent and hold. For the equivalence of these theorems for a class $\mathcal{C}$, the crucial insight is that $\mathcal{C}$ needs closure properties weaker than expected. Besides closure under quotients with closed subspaces, additionally, only closure under closed graphs is needed, not closure under closed finite products or closed subspaces. This insight leads to a characterization result, showing that the class of barreled Pták spaces is a natural habitat of these theorems, and that at least for locally-convex spaces, the barrier of being barreled and Pták cannot be overcome without losing important closure properties. As research in the 1960s considered Pták and barreled spaces already, this may explain, why research on these topics faded out in the 1970s.

Recall that a map $u\colon E \to F$ is called *graph-closed*, if the set $\operatorname{graph} u = \{(e, u(e)) \mid e \in E\}$ is a closed subset of $E \times F$.

We define three properties for a class $\mathcal{C}$ of t.v.s..

- ***(O) Open-Mapping Property*:** For every pair of t.v.s. E and F in $\mathcal{C}$, it holds that every surjective, linear, continuous map $u\colon E \to F$ is open.
- ***(C) Continuous-Inverse Property*:** For every pair of t.v.s. E and F in $\mathcal{C}$, it holds that every bijective, linear map $u\colon E \to F$ is continuous iff its inverse u^{-1} is continuous.
- ***(G) Closed-Graph Property*:** For every pair of t.v.s. E and F in $\mathcal{C}$, it holds that every linear map $u\colon E \to F$ is graph-closed iff it is continuous.

We say that a class $\mathcal{C}$ of t.v.s. is *closed under closed graphs*, if for every E and F in $\mathcal{C}$ and every linear, graph-closed map $u\colon E \to F$ its graph, $\operatorname{graph} u$, is in $\mathcal{C}$. A class $\mathcal{C}$ of t.v.s. is *closed under quotients with closed subspaces*, if for every E in $\mathcal{C}$ and S a closed subspace of E, the quotient space E/S is in $\mathcal{C}$. Furthermore, we say that a class $\mathcal{C}$ of t.v.s. has the *OCG-equivalence* property, if it is closed under quotients with closed subspaces, and if it is closed under closed graphs.

THEOREM 14. *Let $\mathcal{C}$ be a class of (T_0) t.v.s. satisfying the OCG-equivalence property. Then properties (O), (C), and (G) are equivalent for $\mathcal{C}$.*

The following arguments in the proof of the above theorem are well-known and thus not new. We present them for three reasons: (1) emphasis on where exactly the closure-properties of the class $\mathcal{C}$ are needed, (2) first-time crystal-clear presentation of these equivalences in this general setting, not found in textbooks in Functional Analysis, and (3) for the sake of completeness.

PROOF. *(O) implies (C):* Let E and F be t.v.s. in $\mathcal{C}$, and let $u\colon E \to F$ be bijective, linear, and continuous. By (O), u is open. Hence, u^{-1} is continuous. Analogously, argue for u^{-1}.

(C) implies (O): Let E and F be t.v.s. in $\mathcal{C}$, and let $u\colon E \to F$ be surjective, linear, and continuous. Subspace $N := \ker u$ is closed by continuity of u. As $\mathcal{C}$ is closed by quotients with closed subspaces, E/N is in $\mathcal{C}$. The induced map $u_0\colon E/N \to F$ is bijective and continuous. By (C), u_0^{-1} is continuous. Hence, u_0 is open. Then finally, the map $u = p \circ u_0$ is open as composition of open maps, where $p\colon E \to E/N$ denotes the linear, continuous, and open projection.

(C) implies (G): Let E and F be t.v.s. in $\mathcal{C}$, and let $u\colon E \to F$ be linear. Define the bijective, linear map $v\colon E \to \operatorname{graph} u$ by $v(e) := (e, u(e))$. Let p_E and p_F denote the linear, continuous projections from $E \times F$, respectively. If u is continuous, then by Proposition 6, $\operatorname{graph} u$ is closed. And if $\operatorname{graph} u$ is closed, then it is in $\mathcal{C}$ by

closure under closed graphs. As $v^{-1} = p_E\colon \operatorname{graph} u \to E$ is bijective, linear, and continuous, the map v is continuous by application of (C).

(G) implies (C): Let E and F be t.v.s. in $\mathcal{C}$. Define $s\colon E \times F \to F \times E$ by $s(x,y) := (y,x)$. Clearly, s is a topological isomorphism. Let $u\colon E \to F$ be bijective and linear. By (G), the map u is continuous iff $\operatorname{graph} u$ is closed. This holds iff $\operatorname{graph} u^{-1} = s(\operatorname{graph} u)$ is closed. Again by (G), the former holds iff u^{-1} is continuous. □

Note that a class $\mathcal{C}$ of t.v.s. is closed under closed graphs, if it is *closed under finite products*, i.e., with E and F in $\mathcal{C}$, we have $E \times F$ in $\mathcal{C}$, and *closed under closed subspaces*, i.e., with E in $\mathcal{C}$, every closed subspace of E is in $\mathcal{C}$. Main insight of above theorem is that the weaker property of closure under closed graphs suffices. Closure under finite products or closure under closed subspaces is not necessary.

It is well-known that the classes of complete T_0 l.c.s., Fréchet spaces, and Banach spaces all satisfy the OCG-equivalence property.

In contrast, it is unclear if subclasses of barreled spaces, Pták spaces, or Baire spaces satisfy the property of OCG-equivalence, because in general, barreled spaces and Baire spaces are not closed under closed subspaces, and Pták spaces are not closed under finite products. At least, barreled spaces are closed under finite products and quotients with closed subspaces, and Pták spaces are closed under closed subspaces and quotients with closed subspaces, respectively, see [**SW99**, IV.8.2, IV.8.3 Cor. 3].

8.6. Characterization. Recall that a linear map $u\colon E \to F$ is called *nearly-open*, if for each 0-neighborhood $U \subseteq E$, $u(U)$ is dense in some 0-neighborhood in $u(E)$.

We say that a class $\mathcal{C}$ of t.v.s. is *closed under continuous images*, if for every E in $\mathcal{C}$, every l.c.s. F, and every injective, linear, continuous, and nearly-open map $u\colon E \to F$, its image $u(E)$ is in $\mathcal{C}$.

Proposition 15. *The classes of Banach spaces, barreled Pták spaces, and barreled infra-Pták spaces are closed under continuous images.*

Proof. Let F be an arbitrary l.c.s., and let $u\colon E \to F$ be an arbitrary injective, linear, continuous, and nearly-open map. Space $u(E)$ is l.c.s. as a subspace of F.

If E is an (infra-)Pták space, then map u is a topological homomorphism by [**SW99**, IV.8.3, Thm.]. Hence, $u(E)$ is isomorphic to E and thus an (infra-)Pták space.

If E is a Banach space, then it is a Fréchet space, and thus a Pták space by the theorem of Krein-Šmulian, see [**SW99**, IV.6.4, Thm.]. By the above argument, $u(E)$ is isomorphic to E and thus a Banach space.

We show that $u(E)$ is barreled, if E is a barreled (infra)-Pták space. By [**SW99**, IV.8.3, Thm.], map $u\colon E \to u(E)$ is an isomorphism. Let B be an arbitrary Banach space, and let $v\colon u(E) \to B$ be an arbitrary linear and graph-closed map. Then the composition map $v \circ u\colon E \to B$ is linear and graph-closed, the latter because map $(u, \mathrm{id})\colon E \times B \to u(E) \times B$ is an isomorphism with $(u,\mathrm{id})\,(\operatorname{graph} v \circ u) = \operatorname{graph} v$. As E is barreled, B is infra-Pták, and $v\circ u$ is graph-closed, map $v\circ u$ is continuous by the Thm. of Robertson-Robertson, [**SW99**, IV.8.5, Thm.]. Hence, $v = (v \circ u) \circ u^{-1}$ is continuous. Finally, space $u(E)$ is barreled by the Thm. of Mahowald, [**SW99**, IV.8.6]. □

Proposition 16. *The classes of Banach spaces, barreled Pták spaces, and barreled infra-Pták spaces are closed under closed graphs.*

Proof. The statement holds for Banach spaces, because Banach spaces are closed under finite products and closed subspaces.

Let E and F be arbitrary barreled (infra-)Pták spaces, and let $u\colon E \to F$ be an arbitrary linear and graph-closed map. By the theorem of Robertson-Robertson, [**SW99**, IV.8.5, Thm.], u is continuous. Note that the space $\operatorname{graph} u$ is an l.c.s. as a closed subset of l.c.s. $E \times F$. Define the bijective and continuous map $v\colon E \to \operatorname{graph} u$ by $v(e) = (e, u(e))$. The map v is open and thus nearly-open, because its inverse $v^{-1} = p_E\colon \operatorname{graph} u \to E$ is continuous. Now, $\operatorname{graph} u$ is the continuous image of the barreled (infra-)Pták space E. The statement then follows from Prop. 15. □

Theorem 17 (Barreled Pták Characterization). *The class of barreled Pták spaces is exactly the largest class of (T_0) l.c.s., which contains all Banach spaces, is closed under quotients with closed subspaces, is closed under closed graphs, is closed under continuous images, and for which an open-mapping theorem (O), a continuous-inverse theorem (C), or a closed-graph theorem (G) holds (and thus all of them).*

Proof. The classes of Banach spaces and of barreled Pták spaces both have the mentioned closure properties: they contain all Banach spaces, are closed under quotients with closed subspaces, are closed under closed graphs (Prop. 16), and are closed under continuous images (Prop. 15). It is well-known that property (O) holds for Banach spaces, and it also holds for barreled Pták spaces by [**SW99**, IV.8.3, Cor. 1]. Consequently, for both of these classes, properties (O), (C) and (G) are equivalent (Thm. 14) and hold.

Let $\mathcal{C}$ be a maximal class of l.c.s. satisfying the assumed closure properties of the theorem. First of all, $\mathcal{C}$ satisfies all properties (O), (C), and (G), because it satisfies OCG-equivalence.

Let E be an arbitrary l.c.s. in $\mathcal{C}$. We want to show that E is barreled. Let B be an arbitrary Banach space. We have B in $\mathcal{C}$. Let $u\colon E \to B$ be an arbitrary linear, graph-closed map. By (G), u is continuous. Then by the theorem of Mahowald, [**SW99**, IV.8.6], E is barreled.

We want to show that E is a Pták space. Let F be an arbitrary l.c.s., and let $u\colon E \to F$ be an arbitrary linear, continuous, and nearly-open map. Subspace $N := \ker u$ is closed, because u is continuous. Hence, E/N is in $\mathcal{C}$ by closure under quotients with closed subspaces. The map $u_0\colon E/N \to F$, associated with u, is injective, linear, continuous, and nearly-open. Thus, image $u(E)$ is in $\mathcal{C}$ by closure under continuous images. Applying (C) to bijective and continuous map $u_0\colon E/N \to u(E)$ yields that u_0 is open. Hence, u_0 is an isomorphism and thus u a topological homomorphism by [**SW99**, III, 1.2]. By [**SW99**, IV.8.3, Thm.], E is a Pták space.

Consequently, every space in $\mathcal{C}$ is a barreled Pták space. Finally, $\mathcal{C}$ must equal the class of barreled Pták spaces by maximality. □

Valdivia [**Val77**] showed that the space of test functions $\mathcal{D}(\Omega)$ and the space of distributions $\mathcal{D}'(\Omega)$ are not even infra-Pták. Hence, they fall out of the above framework. Nevertheless, for these classes, a closed-graph theorem, open-mapping theorem, and continuous-inverse theorem hold. Maybe surprisingly, in sharp contrast, for the space of Schwartz functions $\mathcal{S}$ and the space of tempered distributions $\mathcal{S}'$ the story is different.

Proposition 18 (Maybe folklore). *The Schwartz space $\mathcal{S}$ and the space of tempered distributions $\mathcal{S}'$ are both barreled Pták spaces.*

Proof. As space $\mathcal{S}$ is a Montel space, [**SW99**, IV.5.8], the strong dual $(\mathcal{S}', \beta(\mathcal{S}', \mathcal{S}))$ is a Montel space, [**SW99**, IV.5.9]. As the strong topology $\beta(\mathcal{S}', \mathcal{S})$ coincides with the topology of compact convergence T_c, $\mathcal{S}'$ is a Montel space. Montel spaces are reflexive (by definition) and thus barreled, [**SW99**, IV.5.6, Thm.]. Hence, $\mathcal{S}$ and $\mathcal{S}'$ are barreled.

Space $\mathcal{S}$ is clearly a Fréchet space, [**SW99**, III.8]. Then by [**SW99**, IV.8, Examples], both $\mathcal{S}$ and $\mathcal{S}'$ are Pták spaces. □

In the same vein as above, we prove a characterization theorem for barreled infra-Pták spaces. For more information on infra-Pták spaces, see [**Val75**]. These spaces are more general then barreled Pták spaces. The missing closure under quotients with closed subspaces is exactly the differentiating property.

THEOREM 19 (Barreled infra-Pták Characterization). *The class of barreled infra-Pták spaces is exactly the largest class of (T_0) l.c.s., which contains all Banach spaces, is closed under closed graphs, is closed under continuous images, and for which an open-mapping theorem (O), a continuous-inverse theorem (C), or a closed-graph theorem (G) holds (and thus all of them).*

PROOF. The classes of Banach spaces and of barreled infra-Pták spaces both have the mentioned closure properties: they contain all Banach spaces, are closed under closed graphs (Prop. 16), and are closed under continuous images (Prop. 15). It is well-known that properties (O), (C), and (G) hold for Banach spaces. Property (G) also holds for barreled infra-Pták spaces by [**SW99**, IV.8.5, Thm.]. Property (G) implies (C) directly. We need to show (O). For this, let $u\colon E \to F$ be a surjective, linear, and continuous mapping between two barreled infra-Pták spaces E and F. As u is a surjective, linear map onto a barreled space, it is nearly open [**SW99**, IV.8.2]. As u is continuous and linear, its graph is closed. By Ptak's general open mapping theorem [**SW99**, IV.8.4], u is open. Hence, (O) holds. Consequently, for both of these classes, all properties (O), (C), and (G) hold (and thus are equivalent).

Let $\mathcal{C}$ be a maximal class of l.c.s. satisfying the assumed closure properties of the theorem. First of all, $\mathcal{C}$ always satisfies property (C), because (O) and (G) imply (C) directly. As $\mathcal{C}$ is closed under closed graphs, (G) always holds for $\mathcal{C}$, too.

Let E be an arbitrary l.c.s. in $\mathcal{C}$. We want to show that E is barreled. Let B be an arbitrary Banach space. We have B in $\mathcal{C}$. Let $u\colon E \to B$ be an arbitrary linear, graph-closed map. By (G), u is continuous. Then by the theorem of Mahowald, [**SW99**, IV.8.6], E is barreled.

We want to show that E is an infra-Pták space. Let F be an arbitrary l.c.s., and let $u\colon E \to F$ be an arbitrary injective, linear, continuous, and nearly-open map. Then image $u(E)$ is in $\mathcal{C}$ by closure under continuous images. Applying (C) to bijective and continuous map $u\colon E \to u(E)$ yields that u is a topological homomorphism. By [**SW99**, IV.8.3, Thm.], E is an infra-Pták space.

Consequently, every space in $\mathcal{C}$ is a barreled infra-Pták space. Finally, $\mathcal{C}$ must equal the class of barreled infra-Pták spaces by maximality. □

Valdivia [**Val84**] was apparently the first, who gave an example of a space, which is infra-Pták but not Pták. Separating these classes was a long-standing open problem in the theory of l.c.s.. Unfortunately, it is a priori unclear, if this example space is barreled or not. We give a much simpler example below, showing that the above class of barreled infra-Pták spaces is strictly larger than the class of barreled Pták spaces. Surprisingly, for this we make use of considerations by Husain [**Hus62**], published twenty years earlier than Valdivia's.

THEOREM 20. *The dual space $\left(\mathbb{R}^{\mathbb{N}}\right)'$ is barreled infra-Pták but not Pták.*

PROOF. For Pták space $E := \mathbb{R}^{\mathbb{N}}$, its dual (E', t_c) is reflexive and thus barreled. Here, strong topology β and topology of uniform convergence on compact, convex sets t_c coincide. It is not Pták [**Hus62**, Prop. 5]. As E is a complete and metrizable l.c.s. (i.e., Fréchet), it is an S-space with CP property [**Hus62**, Remark after Thm. 1 and remark after Def. 2]. Hence, by [**Hus62**, Thm. 10] its dual (E', t_c) is infra-Pták. □

9. Metric Vector Spaces

A *metric vector space* (E, d) is a vector space E and a metric space (E, d), equipped with a *translation-invariant* metric d, i.e., $d(x, y) = d(x + z, y + z)$ for all $x, y, z \in E$.

The translation-invariant metric d of a metric vector space (E, d) induces a uniform topology on E, which makes E a t.v.s.. A t.v.s. E is called *metrizable*, if there exists a translation-invariant metric d on E inducing the topology of E.

A T_0 t.v.s.is metrizable iff it is first countable.[102]

A complete and metrizable l.c.s. is called *Fréchet space*.

The differentiating property between complete and metrizable t.v.s. and l.c.s. is exactly the following.

Theorem 21 (Characterization Fréchet). *Let E be a complete and metrizable t.v.s.. Then E is a Fréchet space iff there exists a translation-invariant metric d on E such that for all $x, y \in E$ and $\lambda \in [0, 1]$ we have*

$$d(\lambda \cdot x, \lambda \cdot y) \leq \lambda \cdot d(x, y) \quad . \tag{2}$$

Proof. We modify the proof in [**SW99**, I.6.1]. There, a pseudonorm $|x|$ is constructed by a base of 0-neighborhoods V_n. The metric is then obtained via $d(x, y) = |y - x|$ and vice versa. As E is an l.c.s., we can assume that these V_n are not only circled but absolutely-convex, and that $2 \cdot V_{n+1} = V_n$. We prove $|2^{-k} \cdot x| \leq 2^{-k} \cdot |x|$ for arbitrary $x \in E$ and $k \geq 1$. Then by dyadic expansion and the triangle inequality, we obtain $|\lambda \cdot x| \leq \lambda \cdot |x|$ for all real $\lambda \in [0, 1]$. Set $V_H := \sum_{n \in H} V_n$ for finite $H \subseteq \mathbb{N}$. Then $V_{k+H} = \sum_{n \in H} V_{k+n} = \sum_{n \in H} 2^k \cdot V_n = 2^k \cdot (\sum_{n \in H} V_n) = 2^k \cdot V_H$. Hence, $2^{-k} \cdot x \in V_H$ iff $x \in 2^k \cdot V_H$ iff $x \in V_{k+H}$. For the numbers $p_H := \sum_{n \in H} 2^{-n}$, we get $p_{k+H} = 2^{-k} \cdot p_H$.

Given arbitrary $\epsilon > 0$, let H be such that $|x| \leq p_H - \epsilon$. Then $2^{-k} \cdot x \in V_H$ implies $x \in V_{k+H}$, and hence $|2^{-k} \cdot x| \leq p_{k+H} = 2^{-k} \cdot p_H \leq 2^{-k} \cdot (|x| - \epsilon)$. □

The $\mathcal{L}^p$ spaces give nice examples to show, when this stronger inequality (2) holds and when it does not. Let $\lambda \in [0, 1]$. For $1 \leq p \leq \infty$, space $\mathcal{L}^p$ is a normed and thus a Fréchet space, and we have $d(\lambda \cdot x, \lambda \cdot y) := \|\lambda \cdot (y - x)\|_p = \lambda \cdot d(x, y)$. In contrast, for $0 < p < 1$, space $\mathcal{L}^p$ is only a complete and metrizable t.v.s., and not an l.c.s.. Here, we have $d(\lambda \cdot x, \lambda \cdot y) = \int |\lambda \cdot (y - x)|^p = \lambda^p \cdot d(x, y) > \lambda \cdot d(x, y)$ for $\lambda \in]0, 1[$.

Define a *Limit-Fréchet space (LF space)* as the strict inductive limit of Fréchet spaces.

Every Fréchet space is barreled.[103] Hence, Baire's category theorem holds. Every LF space is barreled.[104]

Every Fréchet space is bornological.[105] In addition, every LF space is bornological.[106]

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[102] [**SW99**, I.6.1]
[103] [**SW99**, II.7.1]
[104] [**SW99**, II.7.2,Cor.2]
[105] [**SW99**, II.8.1]
[106] [**SW99**, II.8.2, Cor.2]

9.1. Permanence Properties. In general, Fréchet spaces are not closed under inductive and projective topologies. In particular, in general, they are not closed under uncountable products and topological sums.

Nevertheless, they are closed under finite and countable products[107], closed subspaces[108], finite topological sums[109], and quotients under closed subspaces[110], respectively.

10. Normed Vector Spaces

10.1. Definition. A *normed (vector) space* is a vector space E, equipped with a *norm*, $\|\cdot\|\colon E \to \mathbb{K}$, which is a positive-definite seminorm. *Positive definite* means that it is positive semidefinite and $\|x\| = 0$ iff $x = 0$.

A normed space $(E, \|\cdot\|)$ induces a metric space (E, d) by translation-invariant metric $d(x, y) := \|y - x\|$. Hence, a normed vector space is also an l.c.s.. A t.v.s. E is *normable*, if there exists a norm on E inducing its topology. A normable space is metrizable.

A T_0 t.v.s. is normable iff it possesses a convex and bounded zero neighborhood.[111]

If a metric vector space (E, d) has a *homogeneous* metric d, i.e., $d(\lambda\cdot x, \lambda\cdot y) = \lambda\cdot d(x, y)$ for all $x, y \in E$ and $\lambda \in \mathbb{K}$, then E is actually a normed vector space with norm $\|x\| := d(0, x)$.

A *Banach space* is a complete and normed vector space.

In addition, define a *Limit-Banach space (LB space)* as the strict inductive limit of Banach spaces.

A Banach space is also a Fréchet space. Hence, it is barreled. Every LB space is barreled.[112]

A disk D in an l.c.s. E is called a *Banach disk*, if $(E_D, |\cdot|_D)$ is a Banach space.

Every complete l.c.s. is topologically isomorphic to a projective limit of a family of Banach spaces.[113]

10.2. Operators. Let E and F be normed spaces. A *norm homomorphism* is a linear operator $u\colon E \to F$ such that $\|u(x)\|_F = \|x\|_E$ for all $x \in E$. A *norm isomorphism* is a bijective norm homomorphism. Its inverse is also a norm isomorphism.

The identity id_E is a norm isomomorphism. The composition of norm homomorphisms is a norm homomorphism.

A linear operator $u\colon E \to F$ is bounded iff there exists a constant $c > 0$ such that $\|u(x)\|_F \leq c \cdot \|x\|_E$ for all $x \in E$. For such a bounded operator, define

$$\|u\|_{E\to F} := \sup\left\{ \frac{\|u(x)\|_F}{\|x\|_E} \mid x \in E \right\} \quad .$$

Then the vector space of bounded operators $(\mathcal{B}(E, F), \|\cdot\|_{E\to F})$ is a normed space. It is a Banach space in case F is complete. In particular, the dual E' is a Banach space.

Of course, $\mathcal{B}(E, F) = \mathcal{C}(E, F)$, because every Banach space E is bornological.

10.3. Compactness. In a normed space E, a ball is compact iff E is finite-dimensional. Hence, a Banach space is a Montel space iff it is finite-dimensional.

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[107] [**SW99**, I.2, Ex.1(b)], [**Bou98a**, II, §3.5, §3.9]
[108] [**SW99**, I.2.1], [**Bou98a**, II, §3.4, §3.9]
[109] [**SW99**, I.Ex.10]
[110] [**SW99**, I.2.3, I.6.3]
[111] [**SW99**, II.2.1]
[112] [**SW99**, II.7.2,Cor.2]
[113] [**SW99**, II.5.4]

10.4. Permanence Properties. In general, Banach spaces are not closed under arbitrary projective topologies. They are closed under finite products.[114] In general, they are not closed under countably-infinite products. Banach spaces are closed under quotients with closed subspaces.[115]

In general, Banach spaces are not closed under arbitrary inductive topologies. They are closed under finite sums, which are equivalent to finite products. In general, they are not closed under countably-infinite sums. Banach spaces are closed under closed subspaces, by restricting the norm to the subspace.

11. Inner-Product Vector Spaces

An *inner-product (vector) space* is a linear space E, equipped with an *inner product*, $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{K}$, which is a conjugate-symmetric map, a linear map in its second argument, and positive definite, respectively. *Conjugate symmetry* means $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all vectors x and y. *Linear* in the second argument means $\langle z, \alpha x + \beta y \rangle = \alpha \langle z, x \rangle + \beta \langle z, y \rangle$ for all vectors x, y, and z, and scalars α and β. Map $\langle \cdot, \cdot \rangle$ is *positive semidefinite*, if $\langle x, x \rangle$ is real and non-negative for all vectors x. Finally, it is *positive definite*, if it is positive semidefinite and if $\langle x, x \rangle = 0$ iff $x = 0$.

An inner-product space $(E, \langle \cdot, \cdot \rangle)$ induces a normed space $(E, \| \cdot \|)$ with the *induced norm* $\|x\| = \sqrt{\langle x, x \rangle}$.

A *Hilbert* space is a complete inner-product space.

In addition, define a *Limit-Hilbert space (LH space)* as the strict inductive limit of Hilbert spaces.

A set $A \subseteq E$ of elements in E is *orthogonal*, if for all $x, y \in A$, $x \neq y$, we have $\langle x, y \rangle = 0$. We write $x \perp y$ in case $\{x, y\}$ is orthogonal. Set A is *orthonormal*, if we have $\langle x, y \rangle = [x = y]$ for all $x, y \in A$.

In an inner-product space, we have the famous *Cauchy-Schwarz inequality*,

$$\langle x, y \rangle \leq \|x\| \cdot \|y\|$$

for all $x, y \in E$. This implies the *Theorem of Pythagoras*: For $x \perp y$ we have

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 \quad .$$

By the Theorem of Jordan-Neumann[116], a normed space is an inner-product space iff the parallelogram law holds:

$$\|x + y\|^2 + \|x - y\|^2 = 2 \cdot (\|x\|^2 + \|y\|^2) \quad .$$

11.1. Permanence Properties. In general, Hilbert spaces are not closed under arbitrary projective topologies. Nevertheless, they are closed under finite products and under closed subspaces.

In general, Hilbert spaces are not closed under arbitrary inductive topologies. Nevertheless, they are closed under finite sums and under quotients with closed subspaces.

12. Examples

12.1. Sequence Spaces. In the examples in Chapter 3, we will make use of sequence spaces. While these spaces are all subspaces of $\mathbb{K}^{\mathbb{N}}$ algebraically, their topologies differ due to different norms used in their definitions. We define the *supremum norm* $\|(x_n)_n\|_\infty := \sup_n |x_n|$, and the *p-norm* $\|(x_n)_n\|_p := \sqrt[p]{\sum_n |x_n|^p}$, $1 \leq p < \infty$, respectively.

[114] [**SW99**, II.2.2], [**Bou98b**, IX, §3.4]
[115] [**SW99**, I.2.3, II.2.3], [**Bou98b**, IX, §3.4]
[116] [**Fur01**, Theorem J-N]

The *space of convergent sequences* $(c, \|\cdot\|_\infty)$ is defined as the set of sequences, converging to a limit in $\mathbb{K}$. Analogously, the *space of zero sequences* $(c_0, \|\cdot\|_\infty)$ is defined as the set of sequences, converging to zero.

The spaces ℓ^p, $1 \leq p \leq \infty$, are defined as the set of sequences bounded by $\|\cdot\|_p$.

The spaces $(c, \|\cdot\|_\infty)$, $(c_0, \|\cdot\|_\infty)$, and $(\ell^p, \|\cdot\|_p)$ are all Banach spaces.

12.2. Space of Radon Measures. For a nonempty and compact subset $G \subseteq \mathbb{R}^d$, $d \geq 1$, denote with $\mathcal{C}(G)$ the space of continuous, $\mathbb{K}$-valued functions $f\colon \mathbb{R}^d \to \mathbb{K}$ with support in G. It is a Banach space with supremum norm $\|f\|$ on $\mathcal{C}(G)$.

For a nonempty and open subset $\Omega \subseteq \mathbb{R}^d$, let G_m be a sequence of compact sets such that G_m is in the interior of G_{m+1} and $\Omega = \bigcup_m G_m$. Define $\mathcal{C}(\Omega)$ as the strict inductive limit of the spaces $\mathcal{C}(G_m)$.

The space of *Radon measures* is defined as the topological dual $\mathcal{C}'(\Omega)$.

As $\mathbb{R}^d$ is a countable union of compact subspaces, $\mathcal{C}'(\Omega)$ is an example of an LB space.[117] We will use Radon spaces as an example to justify an extension of a measure of noncompactness in Chapter 2.

12.3. Lebesgue Spaces. Lebesgue spaces are an important part of the foundations of (Functional) Analysis, see e.g., [**Wer00**]. Their study is closely related to (abstract) Measure Theory and Harmonic Analysis, see e.g., [**Rou05, Kat04**]. In this thesis, Lebesgue spaces will play an important role in Chapter 4, when we study Nemyckij operators. They are also fundamental to the definition and study of Sobolev space, see below.

Let (X, σ, μ) be a measure space, where measure μ is a countably-additive and nonnegative function, defined on the σ-algebra Σ of X. For $0 < p < \infty$, denote with $\mathcal{L}^p(X)$ the set of equivalence classes of Σ-measurable, $\mathbb{K}$-valued functions f such that $|f|^p$ is μ-integrable, modulo functions of μ-measure zero. By the Hölder inequality, see below, $\mathcal{L}^p(X)$ is a vector space with *p-norm* $\|f\|_p := \left(\int_X |f|^p \mathrm{d}\mu\right)^{1/p}$.

A Σ-measurable, $\mathbb{K}$-valued function f is called essentially μ-bounded, if there exists a μ-bounded function in the equivalence class of f. The set of equivalence classes of essentially μ-bounded functions is a vector space. Denote with $\mathcal{L}^\infty(X)$ the set of equivalence classes of Σ-measurable, $\mathbb{K}$-valued functions, and essentially μ-bounded functions. Space $\mathcal{L}^\infty(X)$ is a vector space with *(essential) supremum norm* $\|f\|_\infty$, defined by

$$\|f\|_\infty := \inf_{\substack{D \subseteq X \\ \mu\text{-measurable}}} \sup\{|f(x)| \mid x \in X \backslash D\} \quad . \tag{3}$$

The *Lebesgue spaces* $\mathcal{L}^p(X)$ are Banach spaces for $1 \leq p \leq \infty$. Spaces $\mathcal{L}^2(X)$ are even Hilbert spaces.

In the context of Lebesgue spaces, we note some integration-theory results, which can be virtually found in any introductory textbook on Analysis, and in the references given above.

THEOREM 22 (Majorized Convergence). *Let $(f_n)_n$ be a sequence of measurable functions on a measure space X with Lebesgue measure λ, converging to f pointwise almost everywhere, i.e., $f_n(x) \to f(x)$ for almost all x. Let $h \in \mathcal{L}^1(X)$ be a majorant, i.e., $|f_n(x)| \leq h(x)$ almost everywhere. Then $f \in \mathcal{L}^1(X)$ and*

$$\lim_{n\to\infty} \int f_n \mathrm{d}\lambda \to \int f \,\mathrm{d}\lambda \quad (n \to \infty) \quad .$$

A slight generalization is the following theorem.

[117] [**SW99**, II.6.3]

THEOREM 23. *Let $(f_n)_n$ and $(h_n)_n$ be sequences of measurable functions in $\mathcal{L}^1(X)$, defined on a measure space X with Lebesgue measure λ. Let $h \in \mathcal{L}^1(X)$, and let $f_n(x) \to f(x)$ and $h_n(x) \to h(x)$ almost everywhere ($n \to \infty$). Furthermore, $|f_n(x)| \leq h_n(x)$ almost everywhere, and*

$$\lim_{n\to\infty} \int h_n \,\mathrm{d}\lambda \to \int h \,\mathrm{d}\lambda \quad (n \to \infty) \quad .$$

Then

$$\lim_{n\to\infty} \int |f_n - f| \,\mathrm{d}\lambda \to 0 \quad (n \to \infty) \quad .$$

We will also need a certain reverse statement.

THEOREM 24. *Let $f_n, f \in \mathcal{L}^1(X)$ such that*

$$\lim_{n\to\infty} \int |f_n - f| \,\mathrm{d}\lambda \to 0 \quad (n \to \infty) \quad .$$

Then there exists a subsequence $(f_{n_k})_k$ converging to f almost everywhere.

For $1 \leq p \leq \infty$, denote with p' its dual parameter, defined by $p' := 1$, if $p = \infty$, $p' := \infty$, if $p = 1$, and $p' := p/(p-1)$, if $1 < p < \infty$. We note *Hölder's inequality*,

$$\int |f \cdot g| \,\mathrm{d}\lambda \leq \|f\|_p \cdot \|g\|_{p'} \quad , \tag{4}$$

as a generalization of the Cauchy-Schwartz inequality in $\mathcal{L}^p(X)$ spaces.

12.4. Sobolev Spaces. Sobolev spaces are vector spaces of functions, whose derivatives satisfy certain integrability conditions. They have been studied extensively since the 1930's. One reason is that they provide a technical foundation for spaces of weak solutions of partial differential equations. We can only give a glimpse of this theory and refer the reader to the now classical book of Adams [**Ada03**] for a thorough introduction to this topic.

Sobolev spaces will be needed in Chapter 4, when we study the nonlinear partial differential equation of the p-Laplacian.

Let $\Omega \subseteq \mathbb{R}^d$ be a domain, $d \geq 1$. Let $f, g\colon \Omega \to \mathbb{K}$ be locally integrable functions. Let $\alpha = (\alpha_1, \dots, \alpha_d)$ be a multiindex. We call g the *weak α-partial derivative* of f, denoted by $\partial^\alpha f := g$, if for all smooth functions $\phi \in \mathcal{C}_0^\infty(\Omega)$ with compact support we have

$$\int_\Omega f \cdot \partial^\alpha \phi \,\mathrm{d}\lambda = \int_\Omega g \cdot \phi \,\mathrm{d}\lambda \quad .$$

As usual, $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

The space of *Sobolev* functions $\mathcal{W}^{k,p}(\Omega)$ is defined as the set of all functions $f \in \mathcal{L}^p(\Omega)$ such that their weak α-derivatives $\partial^\alpha f$ exist up to $|\alpha| \leq k$ and belong to $\mathcal{L}^p(\Omega)$. For such f, we define

$$\|f\|_{k,p} := \|f\|_{\mathcal{W}^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \|\partial^\alpha f\|^p_{\mathcal{L}^p(\Omega)} \right)^{\frac{1}{p}} \quad .$$

Space $(\mathcal{W}^{k,p}(\Omega), \|\cdot\|_{k,p})$ is a Banach space.[118]

We denote with $\mathcal{W}_0^{k,p}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in $\mathcal{W}^{k,p}(\Omega)$.

Sobolev space $\mathcal{W}^{k,p}(\Omega)$ is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$.[119] The same holds for the closed subspace $\mathcal{W}_0^{k,p}(\Omega)$.

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[118] [**Ada03**, 3.3]
[119] [**Ada03**, 3.6]

If Ω is a bounded domain with Lipschitz-continuous boundary $\partial\Omega$, then the *Poincaré inequality*

$$\|f\|_{k,p} \leq c \cdot \|\nabla f\|_{\mathcal{L}^p(\Omega)}$$

holds for all functions $f \in \mathcal{W}_0^{k,p}(\Omega)$. Here, for the mentioned constant we have $c = c(p, d, \Omega) > 0$. This implies that norms $\|f\|_{k,p}$ and $\|\nabla f\|_{\mathcal{L}^p(\Omega)}$ are equivalent.[120]

With the same conditions (bounded, Lipschitz boundary) on Ω as above, one can also prove that the embedding

$$\mathcal{W}_0^{k,p}(\Omega) \hookrightarrow \mathcal{L}^p(\Omega)$$

is compact, to our knowledge this holds only for $p \leq d$.[121]

12.5. Distributions. Generalized functions are an enabler to define solutions to ordinary or partial differential equations, which in a classical sense would not be sufficiently differentiable or even not be defined. In some physical models, this may be necessary. For example, modelling point masses naturally leads to Dirac's *delta-distribution* δ_p, with properties like $\delta_p(x) = 0$ for $x \neq p$, and $\int \delta_p \phi = \phi(p)$, which cannot be fulfilled by ordinary functions. There exist several approaches to this topic. Arguably the most important one is Distribution Theory, systematically developed by Laurent Schwartz in the 1950's, see [**Sch57, Sch58**]. In addition, see Trèves [**Trè06**] and Friedman [**Fri63**] for the application of this theory in Partial Differential Equations. From the many flavors of possible distribution spaces, we only need $\mathcal{D}'$ and $\mathcal{S}'$ for our purposes.

For a nonempty and compact subset $G \subseteq \mathbb{R}^d$, denote with $\mathcal{D}(G)$ the space of *test functions*, consisting of all infinitely-differentiable, $\mathbb{K}$-valued functions $f\colon \mathbb{R}^d \to \mathbb{K}$ with support in G. For any multiindex $\alpha := (\alpha_1, \ldots, \alpha_d)$, define

$$D^\alpha f := \frac{\partial^{\alpha_1}}{\partial x_1} \circ \cdots \circ \frac{\partial^{\alpha_d}}{\partial x_d} f(x_1, \ldots, x_d) \quad .$$

Define a countable set of seminorms $p_\alpha(f) := \|D^\alpha f\|$ on $\mathcal{D}(G)$, making it into a Fréchet space.

For a nonempty and open subset $\Omega \subseteq \mathbb{R}^d$, let G_m be a sequence of compact sets such that G_m is in the interior of G_{m+1} and $\Omega = \bigcup_m G_m$. Define $\mathcal{D}(\Omega)$ as the strict inductive limit of the spaces $\mathcal{D}(G_m)$. It is an LF space.

The space of *distributions* is defined as the strong topological dual $\mathcal{D}'_\beta(\Omega)$.

The *Schwartz space* $\mathcal{S}$ is defined as the set of smooth and rapidly-decaying functions. More precisely, a function $f\colon \mathbb{R}^d \to \mathbb{R}$ is in the Schwartz space iff it is smooth, i.e., in $\mathcal{C}^\infty(\mathbb{R}^d)$, and if f and all its derivatives decay faster than any polynomial, i.e.,

$$p_{\alpha,\beta}(f) := \sup_{x \in \mathbb{R}^d} \left|(1 + |x|^2)^\alpha \cdot D^\beta f(x)\right| = \|(1 + |x|^2)^\alpha \cdot D^\beta f(x)\|_{\mathcal{L}^\infty(\mathbb{R}^d)} < \infty$$

for all numbers α and multiindices β. As the topology of $\mathcal{S}$ is generated by a countable number of seminorms, it is a Fréchet space.

We note that $\mathcal{S} = \mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{L}^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$.

The *tempered distributions* are defined as the strong topological dual $\mathcal{S}'_\beta$.

Spaces $\mathcal{S}$ and $\mathcal{S}'$ are both examples of Montel spaces.[122]

In Chapter 4 we study the Navier-Stokes equation. We need certain delicate properties at our disposal of distributions. We derive these properties in this section.

A *countably normed space* is a t.v.s. E such that there exists a (countable) sequence $(\|\cdot\|_n)_n$ of norms on E.[123]

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[120] [**Ada03**, 4.31]

[121] See inside of the proof of the Theorem of Rellich-Kondrachov in [**Ada03**, 6.3], part I.

[122] [**SW99**, III.8]

[123] [**Fri63**, Chapter I, Section 3]

For a complete countably normed space E, every weakly bounded set of E' is strongly bounded[124], the topological dual E' is complete with respect to the weak topology[125], and a set in E is (strongly) bounded iff it is weakly bounded.[126]

A *perfect space* is defined as a complete countably normed space E having the property that every bounded set of E is relatively sequentially compact.

Perfect spaces are separable.[127]

In a perfect space, weak convergence implies strong convergence.[128] Is E is a perfect space, then in E' weak convergence implies strong convergence.[129]

If E is a perfect space, then bounded sets in E' are relatively sequentially compact in both the weak and strong topologies.[130]

The Schwartz space $\mathcal{S}$ is a perfect space. First of all, it is countably normed. Secondly, every bounded set is relatively compact, because $\mathcal{S}$ is a Montel space. And a relatively-compact set is relatively sequentially compact, because $\mathcal{S}$ is a Fréchet space.

Finite products $\mathcal{S}^n$ of Schwartz spaces are also perfect spaces due to the permanence properties of countably normed, Fréchet, and Montel spaces.

We introduce the definition of a **W space**. A T_0, complete l.c.s. is a *(weak / strong) W space*, if it is reflexive, and if every bounded subset is relatively (weakly / strongly) sequentially compact. By definition, every strong W space is a weak W space. In our opinion, the W spaces draw a fine line along the border of those spaces, where we can apply the generalized Theory of Monotonic Operators, which we develop in Chapter 2.

For example, every reflexive Banach space is a weak W space by the Theorem of Eberlein and Šmuljan.

THEOREM 25. [131] *Every closed subspace of a finite product of Schwartz spaces, $E \subseteq (\mathcal{S}(\mathbb{R}^m))^n$, is a strong W space, and the same holds for its strong topological dual E'_β, the tempered distributions over E. In addition, E is separable.*

PROOF. Space $\mathcal{S}$ is separable. As separable spaces are closed under countable and thus finite products and under closed subspaces, space E is separable, too.

Space $\mathcal{S}$ is reflexive. As reflexive spaces are closed under finite products and under closed subspaces, space E is reflexive, too. Then its strong dual E'_β is reflexive.

Let $B \subseteq E$ be an abritrary bounded subset. Then $B = \prod_{i\in[n]} B_i \cap E$, where each B_i is bounded in $\mathcal{S}$. As $\mathcal{S}$ is a Montel space, each B_i is relatively compact. As $\mathcal{S}$ is a Fréchet space, then each B_i is relatively sequentially compact. Then the finite product $\prod_{i\in[n]} B_i$ is relatively sequentially compact. Finally, then B is relatively sequentially compact in E. Hence, E is a strong W space.

Space $\mathcal{S}$ is a countably normed space. As countable normed spaces are closed under finite products and under closed subspaces, space E is a countably normed space, too.

As E is a perfect space, every bounded subset of E'_β is relatively sequentially compact. Thus, E'_β is a strong W space. $\square$

124 [**Fri63**, Chapter 1, Theorem 15]
125 [**Fri63**, Chapter 1, Theorem 16]
126 [**Fri63**, Chapter 1, Theorem 19]
127 [**Fri63**, Chapter 1, Theorem 26]
128 [**Fri63**, Chapter 1, Theorem 21]
129 [**Fri63**, Chapter 1, Theorem 22]
130 [**Fri63**, Chapter 1, Theorem 27]
131 This theorem has been given a more detailed proof after thesis submission.

THEOREM 26. [132] *Let $\Omega \subseteq \mathbb{R}^m$ be a domain. Every closed subspace of a finite product of test-function spaces, $E \subseteq (\mathcal{D}(\Omega))^n$, is a strong W space, and the same holds for its strong topological dual E'_β, the distributions over E. In addition, E is separable.*

PROOF. Space $\mathcal{D}$ is the strict inductive limit $\varinjlim D_n$ of separable, countably normed spaces $D_n = \mathcal{D}(K_n)$, where $\cup K_n = \Omega$ are increasing compacta.

As separable spaces are closed under countable and thus finite products and under closed subspaces, space E is separable, too.

Space $\mathcal{S}$ is reflexive. As reflexive spaces are closed under finite products and under closed subspaces, space E is reflexive, too. Then its strong dual E'_β is reflexive.

Define $E_n := E \cap D_n$. Then either (*case 1*) there exists a strictly increasing sequence E_{n_k} of spaces, and E is the strict inductive limit $\varinjlim E_{n_k}$, or (*case 2*) space E is fully contained in a single E_m. Let B be an arbitrary bounded subset of E, and let B' be an arbitrary bounded subset of E'_β, respectively.

Case 1: There exists $D_{n_{k_0}}$ such that $B \subseteq D_{n_{k_0}}$. Then B is bounded in $E_{n_{k_0}}$ and is thus relatively sequentially compact. Furthermore, space E'_β is isomorphic to a closed subspace of $\prod_n (D_n)'_\beta$, see [**SW99**, 4.1, and p.173]. Then $B' = \prod_n B'_n \cap E'$ with bounded sets B'_n in $(D_n)'_\beta$. Consequently, as each D_n is perfect, they are all relatively sequentially compact and thus $\prod_n B'_n$ as a countable product. Finally, this holds for B'.

Case 2: Then B is bounded in E_m and is thus relatively sequentially compact. As D_m is perfect, bounded subset B' is relatively sequentially compact.

In both cases, every bounded subset B of E is relatively sequentially compact, same for B' and E'_β. Hence, both E and E'_β are strong W spaces. □

A common definition of the Schwartz space uses a family of norms, defined via a supremum. The same applies to our definition, where we used the norms $p_{\alpha,\beta}(f) := \|(1+|x|^2)^\alpha \cdot D^\beta f(x)\|_{\mathcal{L}^\infty(\mathbb{R}^d)}$. For applications, this is not always favorable. In Chapter 4, we rather need norms, based on the $\mathcal{L}^1$ norm, in order to prove the coercivity of an operator associated with the Navier-Stokes equations.

Recall that a norm p is *weaker* than a norm q, if there exists a constant $c > 0$ such that $p(x) \leq c \cdot q(x)$ for all points x. An *increasing* family $P = \{p_k\}$ of norms p_k has the property $p_0 \leq p_1 \leq p_2 \leq \ldots$. Given two countable and increasing families $P = \{p_k\}$ and $Q = \{q_k\}$ of norms, we say that P is *weaker* than Q, if for every norm p in P, there exists a norm q in Q such that p is weaker than q. We say that P and Q are equivalent iff P is weaker than Q and Q is weaker than P.

We define countable families $P := \{p_{\alpha,\beta}\}$, $\tilde{P} := \{p_k\}$, $Q := \{q_{\alpha,\beta}\}$, and $\tilde{Q} := \{q_k\}$ of norms

$$p_{\alpha,\beta}(f) := \|(1+|x|^2)^\alpha \cdot D^\beta f(x)\|_{\mathcal{L}^\infty(\mathbb{R}^d)} \quad , \tag{5}$$

$$\tilde{p}_k(f) := \sum_{\alpha,|\beta|\leq k} p_{\alpha,\beta}(f) \quad , \tag{6}$$

$$q_{\alpha,\beta}(f) := \|(1+|x|^2)^\alpha \cdot D^\beta f(x)\|_{\mathcal{L}^1(\mathbb{R}^d)} \quad , \tag{7}$$

$$\tilde{q}_k(f) := \sum_{\alpha,|\beta|\leq k} q_{\alpha,\beta}(f) \quad . \tag{8}$$

THEOREM 27. *The norms in Q generate the topology of the Schwartz space $\mathcal{S}$.*

[132] This theorem has been added after thesis submission.

PROOF. First of all, families P and $\tilde{P}$ generate the same topology, because the norms $p_{\alpha,\beta}$ and $\tilde{p}_k$ are sequentially continuous. Just consider an arbitrary sequence, converging to zero. The same argument applies to Q nd $\tilde{Q}$. We show that the countable and increasing families $\tilde{P}$ and $\tilde{Q}$ are equivalent. Thus, all families generate the same topology.

Family $\tilde{Q}$ is weaker than $\tilde{P}$: For arbitrary $q_{\alpha,\beta}$ we have

$$\begin{aligned} q_{\alpha,\beta}(f) &= \int_{\mathbb{R}^d} (1+|x|^2)^\alpha \cdot D^\beta f(x) \,\mathrm{d}\lambda \\ &= \int_{\mathbb{R}^d} (1+|x|^2)^\alpha \cdot (1+|x|^2)^1 \cdot D^\beta f(x) \cdot \frac{1}{1+|x|^2} \,\mathrm{d}\lambda \\ &\leq \|(1+|x|^2)^{\alpha+1} \cdot D^\beta f(x)\|_{\mathcal{L}^\infty(\mathbb{R}^d)} \cdot \int_{\mathbb{R}^d} \frac{1}{1+|x|^2} \,\mathrm{d}\lambda \leq c \cdot p_{\alpha+1,\beta}(f) \quad . \end{aligned}$$

Then we can bound

$$\tilde{q}_k(f) = \sum_{\alpha,|\beta|\leq k} q_{\alpha,\beta}(f) \leq \sum_{\alpha,|\beta|\leq k} c \cdot p_{\alpha+1,\beta}(f) \leq c \cdot \sum_{\alpha,|\beta|\leq k+1} p_{\alpha,\beta}(f) \leq c \cdot \tilde{p}_{k+1}(f) \ .$$

Family $\tilde{P}$ is weaker than $\tilde{Q}$: This is the arguably more difficult part. We use [**Esk**, Theorem 9.2].[133] For arbitrary $p_{\alpha,\beta}$ with $\alpha, |\beta| \leq k$, we have

$$\begin{aligned} p_{\alpha,\beta}(f) &= \sup_{x\in\mathbb{R}^d} \left|(1+|x|^2)^\alpha \cdot D^\beta f(x)\right| \leq \sup_{x\in\mathbb{R}^d} \left((1+|x|^2)^\alpha \cdot \sum_{0\leq|\eta|\leq k} |D^\eta f(x)| \right) \\ &\leq c \cdot \sum_{0\leq|\eta|\leq k+d} \int_{\mathbb{R}^d} (1+|x|^2)^\alpha \cdot |D^\eta f(x)| \,\mathrm{d}\lambda \leq c \cdot \sum_{0\leq|\eta|\leq k+d} q_{\alpha,\eta}(f) \quad . \end{aligned}$$

Then we can bound

$$\begin{aligned} \tilde{p}_k(f) &= \sum_{\alpha,|\beta|\leq k} p_{\alpha,\beta}(f) \leq \sum_{\alpha,|\beta|\leq k} \left(c \cdot \sum_{0\leq|\eta|\leq k+d} q_{\alpha,\eta}(f) \right) \\ &\leq \left(c \cdot \sum_{0\leq|\beta|\leq k} 1 \right) \cdot \left(\sum_{\alpha,|\eta|\leq k+d} q_{\alpha,\eta}(f) \right) \leq d \cdot \tilde{q}_{k+d}(f) \quad . \end{aligned}$$

□

[133] https://math.stackexchange.com/questions/485178/inequality-in-schwartz-space.

CHAPTER 2

Fixed Points

Fixed-point theorems are existence results of equations of the form $f(x) = x$ for certain classes of operators f. In such a case, point x is called a *fixed point* of f. Prominent examples of elementary fixed-point theorems are based on completeness (Banach), order (Knaster-Tarski), and convexity (Kakutani).

For example, the Banach fixed-point theorem, which can be found in every Analysis textbook, states the following.

THEOREM 28 (Banach). *Let (E, d) be a complete metric space, and let $f\colon E \to E$ be* contractive*, i.e., there exists a constant $c < 1$ such that $d(f(x), f(y)) \leq c \cdot d(x, y)$ for all $x, y \in E$. Then f has a unique fixed-point.*

Non-elementary ones are based on the topological KKM principle (Schauder-Tychonoff), topological transversality (Brouwer, Borsuk), or homology theories (Lefschetz-Hopf). Historically, these theorems have been formulated for subsets of Euclidean space. Often, overcoming the difficulties in proving these theorems in general, infinite-dimensional Banach spaces (compact $\neq$ bounded and closed!) has later led to versions extending to l.c.s., or even beyond. Fixed-point theorems play an important role in Analysis in general, and in Nonlinear Spectral Theory in particular. For our purposes it suffices to prove the fixed-point theorems of Fan-Browder, Schauder-Tychonoff, Brouwer, and Darbo. In addition, we introduce the Theory of Monotonic Operators as an application of Fixed-Point Theory.

1. Schauder-Tychonoff

Main result of this section is the proof of the Theorem of Schauder-Tychonoff, a fixed-point theorem in the setting of a general l.c.s.. Its proof is based on the geometric KKM principle for KKM maps, and the Theorem of Fan-Browder on set-valued maps. The Theorem of Schauder-Tychonoff is used in Nonlinear Spectral Theory on several occasions. First of all, it helps to show that a linear surjective operator is stably solvable. Secondly, it is a main tool in the proof of the closedness of the FMV and Feng spectra.

None of the results in this section are new. They can be found e.g., in the excellent textbooks of Appell and Väth [**AV05**], Fučík et al. [**FNSS73**], and of course in the opus magnum of Fixed-Point Theory [**GD03**], and the references therein.

Let X and Y be two subsets of a t.v.s., and let $s\colon X \to 2^Y$ be a set-valued map. The sets $s(x)$ are called its *values*.

Its *inverse* $s^{-1}\colon Y \to 2^X$ is defined by $s^{-1}(y) := \{x \in X \mid y \in s(x)\}$. Each set $s^{-1}(y)$ is called *fiber*.

Given s, its *dual* $s^*\colon Y \to 2^X$ is defined by $s^*(y) := X \backslash s^{-1}(y)$.

We say that s has a *fixed point*, if there exists an $x \in C$ such that $x \in s(x)$.

Let E be a vector space, and let $X \subseteq E$ be a subset. A set-valued map $s\colon X \to 2^E$ is called a *Knaster-Kuratowski-Mazurkiewicz map (KKM map)*, if for all finite subsets $A = \{x_1, \dots x_m\} \subseteq X$ we have $\operatorname{co}(A) \subseteq s(A) := \bigcup_{i \in [m]} s(x_i)$.

We can only give a glimpse of the theory based on such KKM maps without deviating too much from our topic of Nonlinear Spectral Theory. We refer the reader to [**GD03**] for more information.

PROPOSITION 29. *Let E be a vector space, $C \subseteq E$ a nonempty and convex set. Let $s\colon C \to 2^C$ be a set-valued map such that its dual s^* is not a KKM map. Then there exists a point $x \in C$ such that $x \in \operatorname{co}(s(x))$. In particular, if s has convex values, then s has a fixed point.*

PROOF. As s^* is not a KKM map, there exists a finite set $A = \{x_1, \dots x_m\} \subseteq C$ such that $\operatorname{co}(A) \nsubseteq s^*(A)$. Hence, there exists a point $x \in C$ with $x \in C \backslash s(A) = C \backslash \bigcup_{i\in[m]} (C\backslash s^{-1}(x_i)) = \bigcap_{i\in[m]} s^{-1}(x_i)$. Consequently, $x_i \in s(x)$ for each $i \in [m]$, implying $x \in \operatorname{co}(s(x))$. In case that s has convex values, we have $x \in \operatorname{co}(s(x)) = s(x)$, i.e., a fixed point. $\square$

Let E be a vector space. A *flat* of E is the translate of a subspace of E. A subset $A \subseteq E$ is called *finitely closed*, if its intersection with every finite-dimensional flat L of E is closed in the euclidean topology of L.

A family $\{A_\iota \mid \iota \in I\}$ of subsets of some set has the *finite-intersection property*, if the intersection $\bigcap_{\iota\in I_0} A_\iota$ of each finite subfamily $\{A_\iota \mid \iota \in I_0\}$, $I_0 \subseteq I$ finite, is nonempty. We say that a set-valued map $s\colon X \to 2^Y$ has the finite-intersection property, if the family $\{s(x) \mid x \in X\}$ of its values has this property.

LEMMA 30. *Let E be a vector space, $X \subseteq E$ a subset, and $s\colon X \to 2^E$ a KKM map with finitely-closed values. Then s has the finite-intersection property.*

PROOF. We show by induction on the number of elements m that for every finite subset $A = \{x_1, \dots, x_m\} \subseteq X$ we have

$$\operatorname{co}(A) \cap \bigcap_{i\in[m]} s(x_i) \neq \emptyset \quad . \tag{9}$$

For the induction base, we note that $x \in s(x)$ for each $x \in X$, because of $\{x\} = \operatorname{co}(\{x\}) \subseteq s(x)$ by the KKM property of s. Assume that the statement is true for m elements. For the induction step, choose $(m+1)$ elements y_i from

$$\operatorname{co}(A\backslash\{x_i\}) \cap \bigcap_{j\in[m+1], i\neq j} s(x_j) \quad . \tag{10}$$

Such elements exist, because these sets are nonempty by induction hypothesis. Define convex and compact set $Y := \operatorname{co}(\{y_1, \dots, y_{m+1}\}) \subseteq \operatorname{co}(A)$. To establish the statement, it suffices to show that $\bigcap_{i\in[m+1]} s(x_i) \cap Y \neq \emptyset$. For a contradiction, assume the opposite.

Let L be the finite-dimensional subspace spanned by the elements of A. Denote with d the Euclidean metric in L. For each $i \in [m+1]$, define distance functions $\ell_i\colon Y \to \mathbb{R}$ by $\ell_i(y) := d(y, Y \cap s(x_i))$. These functions are convex, because the metric d is induced by the euclidean norm. Each $Y \cap s(x_i)$ is closed, because the values of s are finitely-closed. Hence, $\ell_i(y) = 0$ iff $y \in Y \cap s(x_i)$. Furthermore, define $\ell\colon Y \to \mathbb{R}$ by $\ell(y) := \max\{\ell_i(y) \mid i \in [m+1]\}$.

Let $\check{z} \in Y$ be a point at which d attains its minimum. Such a point exists, because d is continuous and Y is compact. By assumption, $\bigcap_{i\in[m+1]} s(x_i) \cap Y = \emptyset$, implying $d(\check{z}) > 0$. As s is a KKM map, $Y \subseteq \operatorname{co}(A) \subseteq s(A) = \bigcup_{i\in[m+1]} s(x_i)$. Thus, point $\check{z}$ must belong to one of the sets $s(x_i)$. W.l.o.g. assume that $\check{z} \in s(x_{m+1})$. Then $\ell_{m+1}(\check{z}) = 0$.

Define points $z_t := t \cdot \check{z} + (1-t) \cdot y_{m+1}$, $t \in [0,1]$. First of all, we have

$$\ell_{m+1}(z_t) \leq t \cdot \ell_{m+1}(\check{z}) + (1-t) \cdot \ell_{m+1}(y_{m+1}) \leq (1-t) \cdot \ell_{m+1}(y_{m+1}) \quad . \tag{11}$$

Then $\ell_{m+1}(z_t) \to 0$ for $t \to 1$. We find a t_0 close to 1 such that $\ell_{m+1}(z_{t_0}) < \ell(\check{z})$. Secondly, for every $i \in [m]$, $\ell_i(y_{m+1}) = 0$ and thus

$$\ell_i(z_{t_0}) \leq t_0 \cdot \ell_i(\check{z}) + (1-t_0) \cdot \ell_i(y_{m+1}) \leq (1-t_0) \cdot \ell_i(y_{m+1}) < \ell(\check{z}) \quad . \tag{12}$$

Combining above estimates, we obtain $\ell(z_{t_0}) < \ell(\check{z})$, a contradiction to the property of $\check{z}$ being a minimum of ℓ. $\square$

The requirement in [**GD03**, I. §3. Thms. 1.4 and 1.5] that the values of s be convex is unnecessary, and it even creates a gap in the proof of the Fan-Browder Theorem [**GD03**, II. §7. Thm. 1.2].

Finally, as a consequence of the above theorem, we obtain

Theorem 31 (Geometric KKM Principle). *Let E be a t.v.s., $X \subseteq E$ a subset, and $s\colon X \to 2^E$ a KKM map with finitely-closed values such that $s(x)$ is compact for some $x \in X$. Then the intersection $\bigcap\{s(x) \mid x \in X\}$ is nonempty.*

Map s is called *Fan map*, if s has nonempty and convex values, and if it has open fibers.

Theorem 32 (Fan-Browder). *Let C be a nonempty, convex, and compact subset of a t.v.s., and let $s\colon C \to 2^C$ be a Fan map. Then s has a fixed point.*

Proof. Consider the dual map s^* of s. We show that s^* is not a KKM map. First of all, all its values $s^*(x) = C \backslash s^{-1}(x)$ are compact, because C is compact, each fiber $s^{-1}(x)$ is open, and closed subsets of compact sets are compact. Hence, the values of s^* are finitely-closed.

If s^* were a KKM map, then by Theorem 31 (Geometric KKM Principle), the intersection $\bigcap\{s^*(x) \mid x \in X\}$ would not be empty. We prove the opposite. First of all, note that the fibers of s^{-1} are all nonempty, because $(s^{-1})^{-1}(x) = s(x)$ and $s(x) \neq \emptyset$ for Fan map s. Secondly, all fibers being nonempty is equivalent to s^{-1} being surjective, i.e., $s^{-1}(C) = C$. Hence, we obtain

$$\begin{aligned}\bigcap\{s^*(x) \mid x \in C\} = \bigcap\{C \backslash s^{-1}(x) \mid x \in C\} &= C \backslash \bigcup\{s^{-1}(x) \mid x \in C\} \\ &= C \backslash \{y \in C \mid s(y) \neq \emptyset\} = C \backslash C = \emptyset \quad .\end{aligned}$$

As s has convex values and s^* is not a KKM map, map s has a fixed point by Proposition 29. $\square$

Given a set X, subset $U \subseteq X$, and map $f\colon X \to X$, we say that f has a *U-fixed point*, if there exists a point $x \in X$ with $f(x) \in x + U$.

Lemma 33. *Let E be a t.v.s., $C \subseteq E$ a nonempty and compact subset, U an open and absolutely-convex neighborhood of 0, and let map $f\colon C \to E$ be continuous such that $f(C) \subseteq C + U$. then f has a U-fixed point.*

Proof. Define set-valued map $s\colon C \to 2^C$ by $s(x) := \{y \in C \mid y \in f(x)+U\} = (f(x)+U) \cap C$, $x \in C$. Each value $s(x)$ is convex as the intersection of convex sets, and nonempty, because $f(C) \subseteq C + U$. In addition, each fiber $s^{-1}(y) = \{x \in C \mid f(x) \in y - U\} = f^{-1}(y-U)$ is open by continuity of f. Hence, s is a Fan map. By Theorem 32 (Fan-Browder), s has a fixed point, i.e., $x \in s(x) \subseteq f(x) + U$. Thus, f has a U-fixed point. $\square$

Lemma 34. *Let E be an l.c.s., $A \subseteq E$ be an arbitrary subset, and let $f\colon A \to A$ be compact. Let f have a U-fixed point for all absolutely-convex neighborhoods U of 0, then f has a fixed point.*

PROOF. Suppose for a contradiction that f does not have a fixed point. Then for each $x \in A$ we can find open and absolutely-convex neighborhoods V_x, W_x of 0 such that $(x+V_x) \cap (f(x)+W_x) = \emptyset$ and $f((x+V_x) \cap A) \subseteq f(x)+W_x$, respectively. As $\overline{f(A)}$ is compact, there exists a finite open covering $\{x_i + \frac{1}{2}V_{x_i}\}_{i \in [n]}$. Define set $U := \bigcap\{\frac{1}{2}V_{x_i} \mid i \in [n]\}$. By construction, U is open and absolutely-convex as a finite intersection of open and absolutely-convex sets. Now, for an arbitrary $x \in A$, there exists some index $i \in [n]$ such that $f(x) \subseteq x_i + \frac{1}{2}V_{x_i}$. If $x \in x_i + V_{x_i}$, then $f(x) \in f(x_i) + W_{x_i}$. By definition of V_{x_i}, W_{x_i} then $f(x) \notin x_i + V_{x_i}$, a contradiction. Thus, $x \notin x_i + V_{x_i}$. As $f(x) + \frac{1}{2}V_{x_i} = f(x) - x_i + x_i + \frac{1}{2}V_{x_i} \subseteq \frac{1}{2}V_{x_i} + x_i + \frac{1}{2}V_{x_i} \subseteq x_i + V_{x_i}$, we have $x \notin f(x) + \frac{1}{2}V_{x_i}$. This implies $f(x) \notin x + U$ for all $x \in A$ by definition of U. Hence, f has no U-fixed point in contradiction to the assumption. □

Now, we have everything prepared to prove the main result.

THEOREM 35 (Schauder-Tychonoff). *Let E be an l.c.s., $C \subseteq E$ be a nonempty and convex subset, and let map $f\colon C \to C$ be compact. Then f has a fixed point.*

PROOF. By Lemma 34, it suffices to show that f has a U-fixed point for every open and absolutely-convex neighborhood U of 0. As $\overline{f(C)}$ is compact by assumption on f, there exists a finite open covering $\{x_i + U\}_{i \in [n]}$ of $\overline{f(C)}$. Define set $K := \mathrm{co}(\{x_1, \dots, x_n\})$. Then $f(K) \subseteq K + U$. Hence, by Lemma 33, f has a U-fixed point. □

We derive a couple of theorems as consequences of the above result. The following generalizes the well-known Theorem of Schauder, who proved his fixed-point theorem for Banach spaces.

THEOREM 36 (Tychonoff). *Let E be an l.c.s., $C \subseteq E$ a nonempty, convex, and compact subset, and let map $f\colon C \to C$ be continuous. Then f has a fixed point.*

PROOF. As f is continuous, f maps compact sets to compact sets. In particular, image $f(C)$ is compact. Hence, f is compact. Then f has a fixed point by Theorem 35 (Schauder-Tychonoff). □

As every Banach space is an l.c.s., we have

THEOREM 37 (Schauder). *Let E be a Banach space, $C \subseteq E$ a nonempty, convex, and compact subset, and let map $f\colon C \to C$ be continuous. Then f has a fixed point.*

As a closed ball is compact in finite-dimensional spaces, we obtain

THEOREM 38 (Brouwer). *Every continuous map $f\colon B \to B$ from a closed ball $B \subseteq \mathbb{R}^n$ in itself has a fixed point.*

For a purely-analytical proof of Brouwer's theorem using Lagrange-zero functions, see [**Růž04**, Thm.2.7]. As an application of Brouwer's fixed-point theorem, we obtain

THEOREM 39. *Given a system of (nonlinear) equations*

$$g^i(x) = 0 \quad , \quad x \in \mathbb{R}^n \quad , \quad i \in [n] \quad , \tag{13}$$

for continuous functions $g^i\colon \mathbb{R}^n \to \mathbb{R}$. If there exists a radius $R > 0$ such that for all $x \in \mathbb{R}^n$ of length $|x| = R$ we have

$$\sum_{i \in [n]} g^i(x) x_i \geq 0 \quad , \tag{14}$$

then (13) has a solution $\hat{x} \in \mathbb{R}^n$ of length $|\hat{x}| \leq R$.

PROOF. Let $g := (g^1, \dots, g^n)$, and define $f^i \colon \in \mathbb{R}^n \to \mathbb{R}$ by

$$f^i(x) := -R\frac{g^i(x)}{|g(x)|} \quad .$$

Assume that (13) does not have a solution x with $|x| \le R$. Then $|g(x)| > 0$ for all such x, and $f := (f^1, \dots, f^n)$ maps closed ball $B_R(0)$ in itself. By Theorem 38 (Brouwer), map f has a fixed point $\hat{x}$ in $B_R(0)$. Then $|\hat{x}| = |f(\hat{x})| = \left|-R\frac{g(\hat{x})}{|g(\hat{x})|}\right| = R$. By assumption, we have (14), and thus

$$0 \le \sum_{i\in[n]} g^i(\hat{x})\hat{x}_i = -\frac{|g(\hat{x})|}{R}\sum_{i\in[n]} f^i(\hat{x})\hat{x}_i = -\frac{|g(\hat{x})|}{R}\sum_{i\in[n]} |\hat{x}_i|^2 < 0 \quad .$$

This contradiction shows that (13) must have a solution x with $|x| \le R$. $\square$

2. Monotonic Operators

The Theory of Monotonic Operators is an established theory, which provides very useful tools to obtain weak solutions to important classes of PDEs. Its development roughly begins in the 1960's, starting with contributions by Brezis, Browder, and Minty, to name a few of the pioneers. The theory has been mostly developed in the realm of Banach spaces. In this thesis, without claiming originality, we systematically lift some of these results to the more general setting of reflexive l.c.s. and subclasses. In the chapter on applications, the reader will see surprising results of this generalized theory. While this generalized theory also only yields weak solutions in a first step, the increased freedom, we have in choosing the right underlying space, namely all smooth functions, is used to even obtain strong(!) solutions.

The remaining part of this section is devoted to the (generalized) Theorems of Browder & Minty and Brezis, [**Růž04**, Thms. 1.5, 2.10]. We roughly follow the reasoning as laid out in [**Růž04**], with appropriate modifications to obtain the generalizations, and also with minor modifications and corrections.

In the following, we consider operators A of the form $A\colon E \to E'$, where E is an l.c.s. or a Banach space.

Operator A is called *strongly (sequentially) continuous* iff for every weakly-convergent series $x_n \rightharpoonup x$ we have $A(x_n) \to A(x)$. It is called *(sequentially) demicontinuous* iff $x_n \to x$ implies $A(x_n) \rightharpoonup A(x)$. More general notions are obtained by replacing in the definitions sequences converging to x with filters converging to x. Operator A is *hemicontinuous* iff for all $x, y, z \in E$, the map $t \mapsto \langle A(x + t\cdot y), z\rangle$ is continuous in the interval $[0, 1]$.

By the above definitions, operator A being strongly (sequentially) continuous implies A being (sequentially) continuous, and A being (sequentially) continuous implies A being (sequentially) demicontinuous, and A being (sequentially) demicontinuous implies A being hemicontinuous. One step towards the Theorem of Browder and Minty is a to reverse the latter implication: if f is hemicontinuous and monotonic, then f is demicontinuous. We need a couple of propositions to prepare for the proof of this statement.

Operator $A\colon E \to E'$ is *bounded*, if $A\colon E \to E'_\beta$ is bounded, i.e., when considering the strong dual. It is *sequentially bounded*, if it maps bounded sequences to bounded sequences. Operator A is *locally (sequentially) bounded*, if it maps convergent sequences to bounded sequences, i.e., if for every convergent sequence $x_n \to x$ in E $(n \to \infty)$, sequence $(Ax_n)_n$ is bounded in the strong dual E'_β.

An operator is bounded iff it is sequentially bounded.[1] If an operator is sequentially bounded, then it is locally (sequentially) bounded.

Operator $A\colon E \to E'$ is *monotonic* iff $\langle A(x) - A(y), x - y\rangle \geq 0$ for all $x, y \in E$. It is *strictly monotonic* iff $\langle A(x) - A(y), x - y\rangle > 0$ for all $x, y \in E$, $x \neq y$. Operator A is called *maximally monotonic* iff for all $x \in E$ and $b \in E'$, $\langle b - A(y), x - y\rangle \geq 0$ for all $y \in E$ implies $A(x) = b$. Operator $A\colon E \to E'$ is *pseudomonotonic*, if from $x_n \rightharpoonup x$ in E $(n \to \infty)$ and $\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0$, it follows that for all $y \in E$ we have $\langle A(x), x - y\rangle \leq \liminf_{n\to\infty}\langle A(x_n), x_n - y\rangle$. Operator $A\colon E \to E'$ has *property (M)*, if from $x_n \rightharpoonup x$ in E $(n \to \infty)$, $A(x_n) \rightharpoonup b$ in E' $(n \to \infty)$, and $\limsup_{n\to\infty}\langle A(x_n), x_n\rangle \leq \langle b, x\rangle$, it follows that $A(x) = b$. In case E is a Banach space, we call f *strongly monotonic* iff there exists a constant $c > 0$ such that $\langle A(x) - A(y), x - y\rangle \geq c \cdot \|x - y\|_E^2$ for all $x, y \in E$.

By the above definitions, operator A being strongly monotonic implies A being strictly monotonic, and A being strictly monotonic implies A being monotonic.

LEMMA 40 (Convergence Principles). *Let E be a real, (T_0), and complete l.c.s.. Then it holds.*

(i) *In addition, let E be reflexive. Then every weakly-convergent sequence is bounded.*

(ii) *In addition, let E be reflexive. From $x_n \rightharpoonup x$ in E $(n \to \infty)$ and $f_n \to f$ in E'_β $(n \to \infty)$ it follows that $\langle f_n, x_n\rangle \to \langle f, x\rangle$ $(n \to \infty)$.*

(iii) *In addition, let E be reflexive. From $x_n \to x$ in E $(n \to \infty)$ and $f_n \overset{*}{\rightharpoonup} f$ in E'_σ $(n \to \infty)$ it follows that $\langle f_n, x_n\rangle \to \langle f, x\rangle$ $(n \to \infty)$.*

(iv) *In addition, let E be a weak W space. Let $(x_n)_n$ be a bounded sequence. If all weakly-convergent subsequences of $(x_n)_n$ weakly converge to the same limit x, then the sequence itself weakly converges to x, $x_n \rightharpoonup x$ in E $(n \to \infty)$.*

(v) *If every subsequene $(x_{n_k})_k$ of a sequence $(x_n)_n$ contains a subsequence $(x_{n_{k_l}})_l$, converging to the same limit x, then the whole sequence $(x_n)_n$ converges to x.*

(vi) *In addition, let E be a strong W space. Then weak convergence implies strong convergence.*

PROOF. Ad (i): Let $x_n \rightharpoonup x$ in E $(n \to \infty)$ and $f \in E'$ be arbitrary. Then $\langle f, x_n\rangle \to \langle f, x\rangle$ $(n \to \infty)$. Hence, $\langle f, x_n\rangle$ is bounded by a constant $c(f)$. This means that family $(j(x_n))_n$ is simply bounded in the strong bidual, where j is the topological isomorphism, existing due to reflexivity of E. As E is barreled due to reflexivity of E, family $(j(x_n))_n$ is equicontinuous and thus bounded in the topology of bounded convergence.[2] As j^{-1} is linear and continuous, it maps bounded sets to bounded sets. Hence, sequence $(x_n)_n = (j^{-1}(j(x_n)))_n$ is bounded.

Ad (ii): The set $B := \{x, x_n \mid n\}$ is bounded by (i), because the sequence $(x_n)_n$ is weakly convergent. We have $f_n - f \to 0$ uniformly on bounded sets. Hence, $|\langle f - f_n, x_n\rangle| \leq \sup_{\tilde{x}\in B}|\langle f - f_n, \tilde{x}\rangle| \to 0$ $(n \to \infty)$. Furthermore, $|\langle f, x_n - x\rangle| \to 0$ $(n \to \infty)$, because $f \in E'$ and $(x_n)_n$ is weakly convergent. Combined, we obtain

$$\begin{aligned}|\langle f_n, x_n\rangle - \langle f, x\rangle| &= |\langle f_n, x_n\rangle - \langle f, x_n\rangle + \langle f, x_n\rangle - \langle f, x\rangle| \\ &= |\langle f - f_n, \tilde{x}\rangle| + |\langle f, x_n - x\rangle| \to 0 \quad .\end{aligned}$$

[1] *Proof.* Let $A\colon E \to F$ be an operator. $\Rightarrow$ If sequence (x_n) is bounded, then set $B := \{x_n \mid n \in \mathbb{N}\}$ is bounded. By assumption, $A(B) = \{A(x_n) \mid n \in \mathbb{N}\}$. Hence, sequence $(A(x_n))$ is bounded. $\Leftarrow$ Assume for a contradiction that operator A is not bounded. Then there exists a bounded set B such that $A(B)$ is unbounded. Hence, there exist elements $y_n \in A(B)\backslash B(0, n)$ with $y = A(x_n)$, $x_n \in B$. Then bounded sequence (x_n) is mapped to the unbounded sequence (y_n), in contradiction to the assumption.

[2] [**SW99**, III.4.2, III.4.1 Cor.]

Ad (iii): The set $C := \{x, x_n \mid n\}$ is totally bounded, because the sequence $(x_n)_n$ is convergent. [3] As E is barreled, by the principle of uniform boundedness[4], $(f_n)_n$ does not only converge pointwise to f but also uniformly on precompact sets. Hence, $|\langle f - f_n, x_n\rangle| \leq \sup_{\tilde{x}\in C} |\langle f - f_n, \tilde{x}\rangle| \to 0$ $(n \to \infty)$. Furthermore, $|\langle f, x_n - x\rangle| \to 0$ $(n \to \infty)$, because f is continuous. Combined, we obtain

$$|\langle f_n, x_n\rangle - \langle f, x\rangle| = |\langle f - f_n, \tilde{x}\rangle| + |\langle f, x_n - x\rangle| \to 0 \quad .$$

Ad (iv): Assume for a contradiction that $(x_n)_n$ does not weakly converge to x. Then there exist $f \in E'$, $\epsilon > 0$, and a subsequence $(x_{n_k})_k$ such that for all k we have

$$|\langle f, x\rangle - \langle f, x_{n_k}\rangle| \geq \epsilon > 0 \quad .$$

Subsequence $(x_{n_k})_k$ is bounded, because $(x_n)_n$ is bounded by assumption. As E is a weak W space, $(x_{n_k})_k$ is relatively weakly sequentially compact. Hence, there exists a weakly-convergent subsequence $(x_{n_{k_l}})_l$, weakly converging to a limit, which must be x by assumption. But this is impossible by above inequality for f.

Ad (v): Assume the opposite for a contradiction. Then there exists a 0-neighborhood U, and a subsequence $(x_{n_k})_k$ such that $x_{n_k} \notin x + U$ for all k. But this subsequence has a subsequence $(x_{n_{k_l}})_l$, converging to x, a contradiction.

Ad (vi): Let $x_n \rightharpoonup x$ be an arbitrary, weakly-convergent sequence in E $(n \to \infty)$. As E is reflexive, by item (i), sequence $(x_n)_n$ is bounded. Let $(x_{n_k})_k$ be an arbitrary subsequence, also bounded. As every bounded set in E is sequentially compact, there exists a subsequence $(x_{n_{k_l}})_l$, converging to a limit $y \in E$. We must have $x = y$, because for arbitrary $f \in E'$, we have $\lim_{l\to\infty} f(x_{n_{k_l}}) = f(x)$ by assumption, and $\lim_{l\to\infty} f(x_{n_{k_l}}) = f(y)$, implying $f(y - x) = 0$ for all $f \in E'$. As every subsequence of $(x_n)_n$ has a subsequence, converging to the same limit x, the whole sequence converges to x, by item (v). $\square$

LEMMA 41 (Minty Trick). *Let E be a real, (T_0), complete, and reflexive l.c.s.. Let operator $A\colon E \to E'_\beta$ be monotonic and hemicontinuous. Then it holds:*

(i) *Operator A is maximally monotonic.*
(ii) *If $x_n \rightharpoonup x$ (in E), $A(x_n) \rightharpoonup b$ (in E'), and $\langle A(x_n), x_n\rangle \to \langle b, x\rangle$, then $A(x) = b$.*
(iii) *If either $x_n \rightharpoonup x$ (in E) and $A(x_n) \to b$ (in E'), or $x_n \to x$ (in E) and $A(x_n) \rightharpoonup b$ (in E'), then $A(x) = b$.*

PROOF. Ad (i): Let $x \in E$ and $b \in E'$ be given such that $\langle b - A(y), x - y\rangle \geq 0$ for all $y \in E$. Set $y := x - t \cdot z$, $t > 0$. Then $\langle b - A(x - t \cdot z), z\rangle \geq 0$ by assumption and linearity. As A is hemicontinuous, we can let $t \to 0$, obtaining $\langle b - A(x), z\rangle \geq 0$. Analogously, but replacing z with $-z$, we obtain $\langle b - A(x), z\rangle \leq 0$. Consequently, $\langle b - A(x), z\rangle = 0$ for all $z \in E$. As E is an l.c.s., we obtain $A(x) = b$.

Ad (ii): As A is monotonic, we have $0 \leq \langle A(x_n) - A(y), x_n - y\rangle$ for all $y \in E$. By linearity, we have

$$\langle A(x_n), x_n\rangle - \langle A(y), x_n\rangle - \langle A(x_n), y\rangle + \langle A(y), y\rangle \geq 0 \quad .$$

By assumption, $\langle A(x_n), x_n\rangle \to \langle b, x\rangle$, $\langle A(y), x_n\rangle \to \langle A(y), x\rangle$, and $\langle A(x_n), y\rangle \to \langle b, y\rangle$, respectively. Hence, for all $y \in E$ we have

$$\langle b - A(y), x - y\rangle = \langle b, x\rangle - \langle A(y), x\rangle - \langle b, y\rangle + \langle A(y), y\rangle \geq 0 \quad .$$

By (i), operator A is maximally monotonic. Hence, $A(x) = b$.

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[3] For every 0-neighborhood U, there exists n_0 such that $x_n \in x + U$ for all $n \geq n_0$. Set $B := \{x, x_1, \ldots, x_{n_0-1}\}$. Then $C \subseteq B + U$, showing that C is totally bounded.

[4] [**SW99**, III.4.6]

Ad (iii): We have $\langle A(x_n), x_n\rangle \to \langle b, x\rangle$, in the first case by Lemma 40, item (ii), in the second case by the same Proposition, item (iii). By (ii), we obtain $A(x) = b$. $\square$

For the following two properties it is not totally clear, how they can be lifted to general l.c.s.. Hence, we state them for the Banach space setting only. We conjecture that these statements hold in a much more general situation, e.g., barreled l.c.s..

LEMMA 42. *Let E be a Banach space, and let $A\colon E \to E'_\beta$ be an operator. Then it holds:*

(i) *If A is strongly (sequentially) continuous, then A is compact.*
(ii) *If A is monotonic, then A is locally (sequentially) bounded.*

PROOF. Ad (i): Let B be an arbitrary bounded set. We have to show that $A(B)$ is relatively compact. As compactness and sequential compactness coincide in Banach spaces (actually, in all Fréchet spaces), it suffices to show that every sequence $(A(x_n))_n$ in $A(B)$ contains a convergent subsequence, where $x_n \in B$. As B is bounded, there exists a weakly-convergent subsequence $x_{n_k} \rightharpoonup x$. As A is strongly (sequentially) continuous, then $A(x_{n_k}) \to A(x)$ in E' $(k \to \infty)$.

Ad (ii): Assume for a contradiction that A is not locally bounded. Then there exists a convergent sequence $x_n \to x$ with $\|A(x_n)\|_{E'} \to \infty$ for $n \to \infty$.

As A is monotonic, for all $y \in E$ we have

$$0 \le \langle A(x_n) - A(y), x_n - y\rangle = \langle A(x_n) - A(y), (x_n - x) + (x - y)\rangle \quad .$$

By linearity, we have

$$0 \le \langle A(x_n), x_n - x\rangle + \langle A(x_n), x - y\rangle + \langle -A(y), x_n - x\rangle + \langle -A(y), x_n - y\rangle \quad .$$

Rearranging yields

$$\langle A(x_n), y - x\rangle \le \langle A(x_n), x_n - x\rangle - \langle A(y), x_n - y\rangle \quad .$$

Set $c_n := (1 + \|A(x_n)\|_{E'} \cdot \|x_n - x\|_E)^{-1}$. Define

$$c(x, y) := \sup_n \frac{\|A(y)\|_{E'} \cdot (\|x_n\|_E + \|y\|_E)}{1 + \|A(x_n)\|_{E'} \cdot \|x_n - x\|_E} < \infty \quad .$$

Then

$$\begin{aligned} c_n \cdot \langle A(x_n), y - x\rangle &\le c_n \cdot (\|A(x_n)\|_{E'} \cdot \|x_n - x\|_E + \|A(y)\|_{E'} \cdot (\|x_n\|_E + \|y\|_E)) \\ &\le 1 + c(x, y) \quad . \end{aligned}$$

Again, as A is monotonic, for all $y \in E$ we have

$$0 \le \langle A(y) - A(x_n), y - x_n\rangle = \langle A(y) - A(x_n), (y - x) + (x - x_n)\rangle \quad .$$

By linearity, we have

$$0 \le \langle A(y), y - x\rangle + \langle -A(x_n), y - x\rangle + \langle A(y), x - x_n\rangle + \langle -A(x_n), x - x_n\rangle \quad .$$

Rearranging yields

$$-\langle A(x_n), y - x\rangle \le \langle -A(x_n), x - x_n\rangle + \langle A(y), y - x_n\rangle \quad .$$

Hence, we also have

$$\begin{aligned} -c_n \cdot \langle A(x_n), y - x\rangle &\le c_n \cdot (\|A(x_n)\|_{E'} \cdot \|x_n - x\|_E + \|A(y)\|_{E'} \cdot (\|x_n\|_E + \|y\|_E)) \\ &\le 1 + c(x, y) \quad . \end{aligned}$$

As $z = y - x$ is arbitrary, $\sup_n |\langle c_n \cdot A(x_n), z\rangle| \le \tilde{c}(x, z) < \infty$. We have shown that the family of linear forms $(c_n \cdot A(x_n))$ is pointwise bounded. By the principle of uniform boundedness, we obtain $\|c_n \cdot A(x_n)\|_{E'} \leq c(x) < \infty$. Set $d_n := \|f(x_n)\|_{E'}$. Then

$$d_n \leq \frac{c(x)}{c_n} = c(x) \cdot (1 + d_n \cdot \|x_n - x\|_E) \quad .$$

Rearranging yields

$$d_n \leq \frac{c(x)}{1 + c(x) \cdot \|x_n - x\|_E} \quad .$$

Let $\|x_n - x\|_E \to 0$. Then there exists n_0 such that for all $n \geq n_0$ we have $\|A(x_n)\|_{E'} = d_n \leq 2 \cdot c(x)$. But this is a contradiction to the assumption that $\|A(x_n)\|_{E'}$ is unbounded. □

We remark that the argument in the proof of [**Růž04**, p.62, Lemma 1.4] to repeat the reasoning, by replacing variable v with $2u - v$, seems to be flawed. Note that operator A is not linear. The above proof corrects this, making use of monotonicity again.

LEMMA 43. *Let E be a real, (T_0), complete, and reflexive l.c.s.. Let $A\colon E \to E'_\beta$ be an operator. Then it holds:*

(i) *If A is (sequentially) demicontinuous, then A is locally (sequentially) bounded.*
(ii) *In addition, let E'_β be a weak W space. If A is locally (sequentially) bounded, hemicontinuous, and monotonic, then A is (sequentially) demicontinuous.*

PROOF. Ad (i): Assume for a contradiction that A is not locally (sequentially) bounded. Then there exists a convergent sequence $x_n \to x$ $(n \to \infty)$ such that $(A(x_n))_n$ is unbounded. As A is demicontinuous, $A(x_n) \rightharpoonup A(x)$ in E'_β $(n \to \infty)$. But then by Lemma 40 (i), $(A(x_n))_n$ is bounded, a contradiction.

Ad (ii): Let $x_n \to x$ a convergent sequence in E. As A is locally (sequentially) bounded by assumption, $(A(x_n))_n$ is bounded in the strong dual E'_β. By assumption, E'_β is a weak W space. Hence, $(A(x_n))_n$ is relatively sequentially compact. Thus, there exists a convergent subsequence $A(x_{n_k}) \to b$ $(k \to \infty)$ for a $b \in E'$. As E is reflexive and A is monotonic and hemicontinuous, by Lemma 41 (iii), we have $A(x) = b$. In addition, all weakly-convergent subsequences of $(A(x_n))_n$ weakly converge to b. Otherwise, again by Lemma 41 (iii), we would have $A(x) = c$ for a $c \neq b$. Then, by Lemma 40 (iv), the whole sequence $(A(x_n))_n$ weakly converges to $b = A(x)$. This shows that A is (sequentially) demicontinuous. □

LEMMA 44. *Let E be a real, (T_0), and complete l.c.s., and let $A, g\colon E \to E'$ be operators. Then it holds:*

(i) *If A is monotonic and hemicontinuous, then A is pseudomonotonic.*
(ii) *In addition, let E be reflexive. If A is strongly (sequentially) continuous, then A is pseudomonotonic.*
(iii) *If A and B are pseudomonotonic, then $A + B$ is pseudomonotonic.*
(iv) *If A is pseudomonotonic, then A has property (M).*
(v) *In addition, let E be reflexive and E'_β be a weak W space. If A is pseudomonotonic and locally (sequentially) bounded, then A is demicontinuous.*

PROOF. Ad (i): Let $x_n \rightharpoonup x$ be an arbitrary sequence in E $(n \to \infty)$ with $\limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle \leq 0$. We have $\langle A(x_n) - A(x), x_n - x\rangle \geq 0$, because A is monotonic. Hence, $\liminf_{n\to\infty} \langle A(x_n), x_n - x\rangle \geq \langle A(x), x_n - x\rangle = 0$. Here, we used weak convergence of $x_n \rightharpoonup x$. Combined with the assumption, we obtain $\lim_{n\to\infty} \langle A(x_n), x_n - x\rangle = 0$.

Let $y \in E$ and $t > 0$ be arbitrary. Set $z := x + t \cdot (y - x)$. By monotonicity of A, $\langle A(x_n) - A(z), x_n - (x + t \cdot (y - x))\rangle \geq 0$, implying

$$t \cdot \langle A(x_n), x - y\rangle \geq -\langle A(x_n), x_n - x\rangle + \langle A(z), x_n - x\rangle + t \cdot \langle A(z), x - y\rangle \quad .$$

Hence, $\liminf_{n\to\infty}\langle A(x_n), x - y\rangle \geq \langle A(z), x - y\rangle$, where we have used $t > 0$, weak convergence of $x_n \rightharpoonup x$, and $\lim_{n\to\infty}\langle A(x_n), x_n - x\rangle = 0$. Using the latter limit again and the hemicontinuity of A for $t \to 0^+$, we obtain $\liminf_{n\to\infty}\langle A(x_n), x_n - y\rangle \geq \langle A(u), x - y\rangle$. This shows A to be pseudomonotonic.

Ad (ii): Let $x_n \rightharpoonup x$ be a weakly-convergent sequence in E ($n \to \infty$). By assumption, A is strongly (sequentially) continuous. Hence, $A(x_n) \to A(x)$ in E'_β ($n \to \infty$). As E is reflexive, by Lemma 40 (iii), for all $y \in E$ we obtain $\langle A(x), x - y\rangle = \lim_{n\to\infty}\langle A(x_n), x_n - y\rangle$. This shows A to be pseudomonotonic.

Ad (iii): Let $x_n \rightharpoonup x$ be a sequence in E ($n \to \infty$) with $\limsup_{n\to\infty}\langle A(x_n) + B(x_n), x_n - x\rangle \leq 0$. We claim that we have both $\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0$ and $\limsup_{n\to\infty}\langle B(x_n), x_n - x\rangle \leq 0$, respectively. Assume for a contradiction that $a := \limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle > 0$ and thus $\limsup_{n\to\infty}\langle B(x_n), x_n - x\rangle \leq -a$. As operator B is pseudomonotonic, we have $\langle B(x), x - y\rangle \leq \liminf_{n\to\infty}\langle B(x_n), x_n - y\rangle$ for all $y \in E$. For $y := x$ we obtain the contradiction $0 = \langle g(x), x - x\rangle \leq \liminf_{n\to\infty}\langle B(x_n), x_n - x\rangle \leq -a < 0$. Pseudomonotonicity of A and B now yields inequalities $\langle A(x), x - y\rangle \leq \liminf_{n\to\infty}\langle A(x_n), x_n - y\rangle$ and $\langle B(x), x - y\rangle \leq \liminf_{n\to\infty}\langle B(x_n), x_n - y\rangle$ for all $y \in E$, respectively. Combining these two inequalities shows that $A + B$ is pseudomonotonic.

Ad (iv): Let $x_n \rightharpoonup x$ be a sequence in E ($n \to \infty$) such that $A(x_n) \rightharpoonup b$ in E'_β ($n \to \infty$) and $\limsup_{n\to\infty}\langle A(x_n), x_n\rangle \leq \langle b, x\rangle$. As weak convergence implies $*$-weak convergence (via $j\colon E \to E''$, $\langle jx, A\rangle = \langle A, x\rangle$), we have $\lim_{n\to\infty}\langle A(x_n), x\rangle = \langle b, x\rangle$. Hence, $\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0$. As by assumption A is pseudomonotonic, for all $y \in E$, it holds

$$\langle A(x), x - y\rangle \leq \liminf_{n\to\infty}\langle A(x_n), x_n - y\rangle \leq \langle b, x\rangle - \langle b, y\rangle = \langle b, x - y\rangle \quad .$$

Replacing y by $2x - y$, for all $y \in E$, we have $\langle A(x), x - y\rangle = \langle b, x - y\rangle$. Hence, $A(x) = b$. This shows that A has property (M).

Ad (v): Let $x_n \to x$ be an arbitrary convergent sequence in E ($n \to \infty$). As A is locally (sequentially) bounded, $(A(x_n))_n$ is bounded. Let $(A(x_{n_k}))_k$ be an arbitrary (bounded) subsequence. As E'_β is a weak W space, there exists relatively weakly convergent subsequence $A(x_{n_{k_l}}) \rightharpoonup b$ in E' ($l \to \infty$), for a $b \in E'$. As E is reflexive, A is monotonic and hemicontinuous, $x_{n_{k_l}} \to x$, by Lemma 41 (iii), it follows that $b = A(x)$. As every subsequence of $(Ax_n)_n$ contains a sub-subsequence, weakly converging to the same limit $A(x)$, this holds for the whole sequence, i.e., $A(x_n) \rightharpoonup A(x)$. Hence, operator A is demicontinuous.

□

We now introduce a very general notion of coerciveness for continuous functions between t.v.s.. Let E and F be t.v.s., and let $f\colon E \to F$ be a continuous map. As E and F are fully regular, their Stone-Čech compactifications exist, and we get the extension $\beta f\colon \beta E \to \beta F$. We call f *coercive*, if βf maps $\beta E\backslash\beta(E)$ to $\beta F\backslash\beta(F)$.

Let $A\colon E \to E'$ be an operator from E in its dual E'. We call A *coercive*, if map

$$x \mapsto \langle A(x), x\rangle \quad : E \to \mathbb{R}$$

is coercive.[5] Note that this map must then be continuous.[6]

[5] In the submitted thesis, we defined coerciveness on a closed subspace D. This more fine-granular notion is not needed in the sequel.

[6] This remark has been added after thesis submission.

In case E is a Banach space, operator $A\colon E \to E'$ is traditionally called *coercive*, if for any $b \in E'$ we have

$$\lim_{\|x\|_E \to \infty} \frac{\langle A(x) - b, x\rangle}{\|x\|_E} = \infty \quad .$$

PROPOSITION 45. [7] *Let E be a reflexive Banach space. If an operator $A\colon E \to E'$ is (sequentially) demicontinuous and coercive in the Banach sense, then it is also coercive in the general sense.*

PROOF. It suffices to show that the map $x \mapsto g(x) := \langle A(x), x\rangle$ is (sequentially) continuous, if A is (sequentially) demicontinuous. Let $x_n \to x$ be an arbitrary, convergent sequence in E $(n \to \infty)$. We have to show that $g(x_n) \to g(x)$ in $\mathbb{R}$ $(n \to \infty)$. As A is (sequentially) demicontinuous, it is locally (sequentially) bounded, by Lemma 43 (i). We have

$$\begin{aligned} |g(x_n) - g(x)| &= |\langle A(x), x\rangle - \langle A(x_n), x_n - x\rangle - \langle A(x_n), x\rangle| \\ &\le |\langle A(x) - A(x_n), x\rangle| + |\langle A(x_n), x_n - x\rangle| \quad . \end{aligned}$$

Then $|\langle A(x) - A(x_n), x\rangle| \to 0$, because $A(x_n) \rightharpoonup A(x)$ implies $A(x_n) \overset{*}{\rightharpoonup} A(x)$ by reflexivity of E. Furthermore, $|\langle A(x_n), x_n - x\rangle| \le \|A(x_n)\| \cdot \|x_n - x\| \to 0$, because $\sup_n \|A(x_n)\|$ is bounded. □

PROPOSITION 46. *If E is an infinite-dimensional and separable t.v.s., then there exists a countable independent set $\{y_1, y_2, \ldots\}$ such that*

$$E = \overline{\bigcup_n E_n} \quad \text{, where } E_n := \operatorname{span}\{y_1, \ldots, y_n\} \quad .$$

PROOF. There exists a countable and dense set $X = \{x_1, x_2, \ldots\}$, $E = \overline{X}$, because E is separable by assumption. We define index k_n, element y_n, and set E_n inductively. For the induction base, set $k_1 := 2$, $y_1 := x_1$, and thus $E_1 := \operatorname{span}\{y_1\}$. For the induction step, k_n, y_n, and E_n are given. There is a smallest index $k > k_n$ such that $x_k \notin E_n$. This exists, because E is infinite-dimensional. Set $k_{n+1} := k+1$, $y_{n+1} := x_k$, and $E_{n+1} := \operatorname{span}\{y_1, \ldots, y_{n+1}\}$. We have $X \subseteq \bigcup_n E_n$, from which the statement follows. □

THEOREM 47 (Galerkin Method). [8] *Let E be a T_0, complete, separable, weak W space, let E'_β be a weak W space. Let $A\colon E \to E'$ be a bounded and demicontinuous operator, and let $b \in E'$. Furthermore, let map $x \mapsto A(x) - b$ be coercive. Then there exists $x \in E$ with $A(x) = b$.*

PROOF. We can write E as $E = \overline{\bigcup_n E_n}$, where $E_n := \operatorname{span}\{y_1, \ldots, y_n\}$, by Proposition 46. We search for approximative solutions $x_n \in E_n$ of the form $x_n = \sum_{k \in n} c_k^n \cdot y_k$, solving the *Galerkin system*

$$\langle A(x_n) - b, y_k\rangle = 0 \quad , k \in [n] \, . \tag{15}$$

Define a nonlinear system of equations, $g^n(c^n) = 0$, where $g^n := (g_1^n, \ldots, g_n^n)$, $g_k^n\colon \mathbb{R}^n \to \mathbb{R}$, and

$$c^n \mapsto g_k^n(c^n) := \langle A(x_n) - b, y_k\rangle \quad .$$

As weak and strong convergence coincide on finite-dimensional spaces E_n, and as A is demicontinuous, each g^n is continuous. By assumption, map $x \mapsto \langle A(x) - b, x\rangle$

[7] This proposition has been added after thesis submission for further clarification of the relationship between the notions of coerciveness.

[8] This Theorem and its proof has been corrected, compared to the submitted thesis. The original formulation required operator A to be coercive only on a subspace. Unfortunately, the argument fails in such a situation. Operator A has to be coercive on the whole space.

is coercive. Hence, there exists a nonempty, absolutely-convex, open, and bounded set $B \subseteq E$ such that $\langle A(x) - b, x\rangle > 0$ for all $x \in E\backslash B$. Define $B_n := B \cap E_n$. Then $B_n \subseteq \{x \in E_n \mid |x|_{E_n} < r_n\}$ for an $r_n > 0$. Here, $|\cdot|_{E_n}$ denotes the euclidean norm on E_n. Hence, for all $x \in E_n$ with $|x|_{E_n} \geq r_n$, we have $\langle A(x) - b, x\rangle \geq 0$, because $x \notin B_n$.[9] By an application of Brouwer's fixed-point theorem [**Růž04**, Lemma 2.26, p.17], there exists solution $x_n \in E_n$ with $|x_n|_{E_n} \leq r_n$ to system (15).

As E is a weak W space, bounded sequence $(x_n)_n$ contains a weakly-convergent subsequence, converging to a limit x. In the sequel, for notational simplicity, we denote this subsequence again with $(x_n)_n$.

For all $w \in \bigcup_n E_n$ there exists n_0 such that for all $n \geq n_0$ we have $\langle A(x_n), w\rangle = \langle b, w\rangle$. Hence,

$$\lim_{n\to\infty} \langle A(x_n), w\rangle = \langle b, w\rangle$$

for all $w \in \bigcup_n E_n$.

As A is bounded, sequence $(A(x_n))$ is bounded. Let $(A(x_{n_k}))_k$ be an arbitrary subsequence of $(A(x_n))_n$. As E'_β is a weak W space, there exists weakly-convergent sub-subsequence $(A(x_{n_{k_l}}))_l$, converging to a limit $c \in E'$. As weak convergence implies $*$-weak convergence (via $j\colon E \to E''$),

$$\langle A(x_{n_{k_l}}), w\rangle = \langle j(w), A(x_{n_{k_l}})\rangle \to \langle j(w), c\rangle = \langle c, w\rangle$$

for all $w \in \bigcup_n E_n$. Hence, $\langle b, w\rangle = \langle c, w\rangle$. As $\bigcup_n E_n$ is dense in E and b, c are continuous, we obtain $b = c$.

We showed that every subsequence of $(A(x_n))_n$ has a sub-subsequence, weakly converging to the same limit b. Thus, $A(x_n) \rightharpoonup b$ in E' $(n \to \infty)$ due to Lemma 40 (v).

We have $x_n \in E_n$. Hence, $\langle A(x_n), x_n\rangle = \langle b, x_n\rangle$. Then

$$\lim_{n\to\infty} \langle A(x_n), x_n\rangle = \lim_{n\to\infty} \langle b, x_n\rangle = \langle b, x\rangle \quad ,$$

because $x_n \rightharpoonup x$.

As E is reflexive, A is monotonic and hemicontinuous, $x_n \rightharpoonup x$, $A(x_n) \rightharpoonup b$, and $\lim_{n\to\infty}\langle A(x_n), x_n\rangle = \langle b, x\rangle$, we can apply Lemma 41 (iii) to obtain $A(x) = b$. □

We recover the original theorems, when we restrict the theory to Banach spaces.

THEOREM 48 (Browder & Minty, 1963). *Let E be a separable and reflexive Banach space. Let $A\colon E \to E'$ be a (sequentially) hemicontinuous, monotonic, and coercive operator. Then A is surjective. The solution set is convex, closed, and bounded. In case that A is strictly monotonic, then the solution is unique.*

PROOF. [10] As A is monotonic, it is locally (sequentially) bounded, by Lemma 42 (ii). As A is (sequentially) hemicontinuous, monotonic, and locally (sequentially) bounded, it is (sequentially) demicontinuous, by Lemma 43 (ii). As A is (sequentially) demicontinuous and coercive in the Banach sense, for every $b \in E'$, map $x \mapsto A(x) - b$ is coercive in the general sense. Hence, by Theorem 47, there exists a solution $x \in E$ with $A(x) = b$. As b was arbitrary, A is surjective.

Define the set of solutions $S := \{x \in E \mid A(x) = b\}$. We thus proved that S is nonempty.

Set S is closed: Let $x_n \to x$ be a convergent sequence with $x_n \in S$. We want to show that $x \in S$. For all $y \in E$, we have

$$\langle b - A(y), x - y\rangle = \lim_{n\to\infty} \langle A(x_n) - A(y), x_n - y\rangle \geq 0 \quad ,$$

[9] This would not hold, if the operator were coercive only on a subspace of E.

[10] The proof has been slightly changed, compared to the submitted thesis, to align with the correction of Theorem 47.

because A is monotonic. As A is monotonic and hemicontinuous, it is maximally monotonic by Lemma 41 (i). Hence, it follows that $A(x) = b$, i.e., $x \in S$.

Set S is bounded: Assume for a contradiction that S is unbounded. Then for every $R > 0$, there exists $x \in S$ with $\|x\|_E \geq R$. As A is coercive by assumption, there exists $R_0 > 0$ such that $\langle A(x), x\rangle \geq (1 + \|b\|_{E'}) \cdot \|x\|_E$ for all $\|x\|_E \geq R_0 > 0$. Hence, for $x \in S$ with $\|x\|_E \geq R_0$, we obtain the contradiction

$$0 = \langle A(x), x\rangle - \langle b, x\rangle \geq (1 + \|b\|_{E'}) \cdot \|x\|_E - \|b\|_{E'} \cdot \|x\|_E > 0 \quad .$$

Thus, S is bounded.

Set S is convex: Let $x_0, x_1 \in S$, and let $0 \leq \lambda \leq 1$. We want to show that the convex combination $z := \lambda \cdot x_0 + (1 - \lambda) \cdot x_1 \in S$. For all $y \in E$ we have

$$\begin{aligned}\langle b - A(y), z - y\rangle &= \langle b - A(y), \lambda \cdot (x_0 - y) + (1 - \lambda) \cdot (x_1 - y)\rangle \\ &= \lambda \cdot \langle A(x_0) - A(y), x_0 - y\rangle + (1 - \lambda) \cdot \langle A(x_1) - A(y), x_1 - y\rangle \\ &\geq 0 \quad ,\end{aligned}$$

because A is monotonic. As A is maximally monotonic, $A(z) = b$, i.e., $z \in S$.

If A is strictly monotonic, there is at most one solution: Assume for a contradiction that there exist two different solutions $x, y \in S$, $x \neq y$. Then by strict monotonicity,

$$0 < \langle A(x) - A(y), x - y\rangle = \langle b - b, x - y\rangle = 0 \quad ,$$

a contradiction. □

THEOREM 49 (Brezis, 1968). *Let E be a separable and reflexive Banach space. Let $A\colon E \to E'$ be a pseudomonotonic, locally (sequentially) bounded, and coercive operator. Then A is surjective.*

PROOF. [11] As A is pseudomonotonic and locally (sequentially) bounded, it is (sequentially) demicontinuous, by Lemma 44 (v). As A is (sequentially) demicontinuous and coercive in the Banach sense, for every $b \in E'$, map $x \mapsto A(x) - b$ is coercive in the general sense. Hence, by Theorem 47, there exists a solution $x \in D$ with $A(x) = b$. As b was arbitrary, A is surjective. □

3. Dugundji and Quasi-Extensions

Recall the Theorem of Tietze-Urysohn, which states that one can extend real-valued functions, defined on a closed subset of a normal space, to the whole space. Dugundji's Extension Theorem is a strict generalization of this, and is a fundamental tool in the theory of absolute neighborhood retracts (ANRs), see e.g., [**GD03**].

THEOREM 50 (Dugundji's Extension). *For every metrizable space E the following holds. For every l.c.s. F, every nonempty and closed subset $A \subseteq E$, and every continuous map $f\colon A \to F$, there exists a continuous extension $g\colon E \to C$ of f with $g(E) \subseteq \mathrm{co}(f(A))$.*

PROOF. Let d be a metric for E. Note that $d(x, A) > 0$ for every $x \in E\backslash A$, because A is closed. Then the family of balls $\{B(x, \frac{1}{2}d(x, A)) \mid x \in E\backslash A\}$ is an open covering of $E\backslash A$. By the Theorem of Stone, this covering has a neighborhood-finite open refinement $\{U_\iota \mid \iota \in I\}$ and a partition of unity $\{\chi_\iota \mid \iota \in I\}$, subordinate to this refinement. For each U_ι, there exists a point $x_\iota \in E\backslash A$ with $U_\iota \subseteq B(x_\iota, \frac{1}{2}d(x_\iota, A))$. For every x_ι, there exists a point $a_\iota \in A$ with $d(x_\iota, a_\iota) \leq 2d(x_\iota, A)$.

[11] The proof has been slightly changed, compared to the submitted thesis, to align with the correction of Theorem 47.

Define the extension g by

$$g(x) := \begin{cases} f(x) & , x \in A\,, \\ \sum_{\iota \in I} \chi_\iota(x) f(a_\iota) & , x \in E \backslash A\,. \end{cases}$$

We have $g(E) \subseteq \mathrm{co}(f(A))$, because the sum in the definition of g is always finite due to the neighborhood-finiteness of the refinement. Hence, $g(x)$ is always a convex combination of values of $f(A)$.

We need to show the continuity of g. The following calculations will be used.

First of all, for every $x \in U_\iota$, we have $d(x_\iota, A) \le 2d(x, A)$, because

$$d(x_\iota, A) \le d(x_\iota, x) + d(x, A) \le \frac{1}{2} d(x_\iota, A) + d(x, A) \le 2d(x, A) \quad .$$

Furthermore, for every $x \in U_\iota$ and $a \in A$, we have $d(a, a_\iota) \le 6d(a, x)$, because

$$\begin{aligned} d(a, a_\iota) &\le d(a, x) + d(x, x_\iota) + d(x_\iota, a_\iota) \le d(a, x) + \frac{1}{2} d(x_\iota, A) + 2d(x_\iota, A) \\ &\le d(a, x) + d(x, A) + 4d(x, A) \le 6d(a, x) \quad . \end{aligned}$$

For $x \in E \backslash A$, f is continuous as a finite sum of continuous functions χ_ι. For $x \in A$, we prove the continuity in the following steps. As F is an l.c.s., there exists a convex and open neighborhood C of $f(x)$. As f is continuous on A, there exists a $\delta > 0$ such that $f(B(x, \delta) \cap A) \subseteq C$. We prove that $g(B(x, \delta/6)) \subseteq C$, showing the continuity of g in x. Let z be any point in $B(x, \delta/6) \backslash A$. There are finitely many sets $\{U_\iota \mid \iota \in I_0\}$ containing z. Then $d(x, a_\iota) < \delta$ for all $\iota \in I_0$, because of $d(z, x) < \delta/6$ and the above calculation. Then all a_ι, $\iota \in I_0$, are contained in $B(a, \delta) \cap A$, implying $f(a_\iota) \in C$, $\iota \in I_0$. By definition of g, $g(x) = \sum_{\iota \in I_0} \chi_\iota(x) f(a_\iota)$ is a convex combination of these points and thus contained in C, proving the continuity of G in point $x \in A$. □

As the values of the extension are in the convex hull of the image values of the original function, an immediate consequence is the following theorem. A metrizable t.v.s. E with such a property is called an *absolute retract*.

THEOREM 51 (Retraction). *For every metrizable t.v.s. E the following holds. For every l.c.s. F, every nonempty and convex subset $C \subseteq F$, every nonempty and closed subset $A \subseteq E$, and every continuous map $f\colon A \to C$, there exists a continuous extension $g\colon E \to C$ of f.*

In particular, for every metrizable t.v.s. E, and every nonempty and convex subset $C \subseteq E$, there exists a retraction from E onto C.

While extension theorems are available in the setting of metrizable spaces, no such results of the type of Dugundji are known for general l.c.s.. However, one can prove such results for quasi-extensions, where the new map coincides with the original one only approximately.

Let E and F be l.c.s., let $C \subseteq E$ be a compact subset, let $f\colon C \to F$ be a continuous map, let p be a continuous seminorm on F, and let $\epsilon > 0$. A map $g\colon E \to F$ is a (p, ϵ) *quasi-extension of* f, if $p(f(x) - g(x)) < \epsilon$ for all $x \in C$, and if $g(E) \subseteq \overline{\mathrm{co}}(f(C))$. The following result is taken from [**AKP$^+$92**, 3.6.1], and slightly extended.

THEOREM 52. *Let E and F be l.c.s., let $C \subseteq E$ be a compact subset, let $f\colon C \to F$ be a continuous map, let p be a continuous seminorm on F, and let $\epsilon > 0$. Then there exists a (p, ϵ) quasi-extension of f.*

PROOF. It suffices to prove the statement for $\epsilon = 1$, because p/ϵ is a continuous seminorm for every $\epsilon > 0$. As f is uniformly-continuous on compact set C, we can find a continuous seminorm p_0 on E with $p(f(x) - f(y)) \le 1/2$ for every $x, y \in C$

with $p_0(x-y) \leq 1$. Let Q be a finite (1/2)-net for C with respect to seminorm p_0. Fix $\delta > 0$ to be defined later. For each $y \in Q$ define the continuous positive function $\mu_y \colon E \to \mathbb{R}$ by $\mu_y(x) := \delta + 1 - p_0(x-y)$, if $p_0(x-y) \leq 1$, and $\mu_y(x) := \delta$, if $p_0(x-y) \geq 1$. Define map $g \colon E \to F$ by

$$g(x) := \left(\sum_{y\in Q} \mu_y(x)\right)^{-1} \sum_{y \in Q} \mu_y(x) f(y) \quad .$$

By construction, g is continuous (as a finite sum of continuous functions), finite-dimensional (the vectors $\{f(y) \mid y \in Q\}$ span a finite-dimensional space), and it maps E into $\overline{\mathrm{co}}(f(Q)) \subseteq \overline{\mathrm{co}}(f(C))$.

Let $x \in C$ be arbitrary. We want to bound $p(f(x) - g(x))$. For this, define $Q_1 := \{y \in Q \mid p_0(x-y) \leq 1\}$ and $Q_2 := Q \backslash Q_1$, respectively.

For the sum with Q_1 we have

$$\begin{aligned}
&\left(\sum_{y\in Q} \mu_y(x)\right)^{-1} \sum_{y \in Q_1} \mu_y(x) p(f(x) - f(y)) \\
&\leq \left(\sum_{y\in Q} \mu_y(x)\right)^{-1} \sum_{y \in Q_1} \mu_y(x) \cdot \frac{1}{2} \leq \frac{1}{2} \cdot \left(\sum_{y\in Q} \mu_y(x)\right)^{-1} \sum_{y \in Q} \mu_y(x) = \frac{1}{2} \quad .
\end{aligned}$$

Choose $\delta := 1/(4dn)$, where n denotes the number of elements in Q, and where d denotes the diameter of $f(C)$ with respect to p. For x there is a $y \in Q$ with $p_0(x-y) \leq 1/2$, because Q is a (1/2)-net. Hence, $\mu_y(x) \geq 1/2 + \delta$ and $\left(\sum_{y\in Q} \mu_y(x)\right)^{-1} \leq 2$. In addition, $\mu_y(x) = \delta$ for all $y \in Q_2$. Then for the sum with Q_2 we have

$$\left(\sum_{y\in Q} \mu_y(x)\right)^{-1} \sum_{y \in Q_2} \mu_y(x) p(f(x) - f(y)) \leq 2dn\delta \leq \frac{1}{2} \quad .$$

Combining the above two sums, we obtain

$$\begin{aligned}
p(f(x) - g(x)) &\leq \left(\sum_{y\in Q} \mu_y(x)\right)^{-1} \sum_{y \in Q} \mu_y(x) p(f(x) - f(y)) \\
&= \left(\sum_{y\in Q} \mu_y(x)\right)^{-1} \left(\sum_{y \in Q_1} \mu_y(x) p(f(x) - f(y)) + \sum_{y \in Q_2} \mu_y(x) p(f(x) - f(y))\right) \\
&\leq \frac{1}{2} + \frac{1}{2} = 1 \quad .
\end{aligned}$$

□

4. Measures of Noncompactness

An important part of Functional Analysis is concerned with measures of noncompactness and condensing operators. See [**AKP+92**, 3.6.1] for a systematic exposition of this topic. A measure of noncompactness quantifies the deviation of a bounded subset of a space from being compact. Hence, this notion does not make sense in Montel spaces.

The most general definition is as follows: Let E be a l.c.s., and let $(Q, \leq)$ be a partially-ordered set. A map $\chi \colon 2^E \to Q$ is called a *measure of noncompactness (NMC)*, if for all subsets $A \subseteq E$ we have $\chi(A) = \chi(\overline{\mathrm{co}}(A))$. See also [**AKP+92**, 1.2.1]. We note that going beyond l.c.s. to general t.v.s. does not make sense, because only for l.c.s. it is ensured that the convex hull of a compact set stays compact.

We will introduce the Hausdorff NMC α as a typical example. Another example, not treated here, is the Kuratowksi NMC, which is actually equivalent to the Hausdorff NMC, see [**AKP$^+$92**, 1.1.1, 1.1.7]. A condensing operator is a map under which the image of any set is more compact than the set itself. Via α, we will define a characteristic $[f]_A$, quantifying this condensation property. Such operators will play a distinguished role in the design of the FMV and Feng spectra, defined in the next chapter.

Let E be a Fréchet space, and let $M \subseteq E$. The *(Hausdorff) measure of noncompactness of* M is defined by

$$\alpha(M) := \inf \{\epsilon > 0 \mid M \text{ has a finite } \epsilon\text{-net in } E\} \quad . \tag{16}$$

For some function spaces, explicit formulas are known to compute the Hausdorff NMC, see e.g., [**AKP$^+$92**, 1.1.9–1.1.13] or [**AV05**, 3.6–3.9].

As a short digression, we show how one could lift the definition of MNC to limit spaces: Let $E := \varinjlim_\iota E_\iota$ be the strict inductive limit of a directed family of Banach spaces E_ι, which are not Montel spaces. An example is the space of Radon measures. On each E_ι, the Hausdorff MNC α_ι is defined. We have the relationship $\alpha_\iota(M) = \alpha_\kappa(M \cap E_\iota)$ for all $E_\iota \subseteq E_\kappa$ and bounded subsets $M \subseteq E_\iota$. Hence, on E we can define a limit NMC $\alpha\colon E \to \mathbb{R}$ by $\alpha(M) := \sup_\iota \alpha_\iota(M)$ for all bounded subsets $M \subseteq E$.

The measure of noncompactness has the following properties.

Proposition 53. *For sets $M, N \subseteq E$, $z \in E$, and $\lambda \in \mathbb{K}$ we have*

(i) $\alpha(M) \leq \alpha(N)$ *for* $M \subseteq N$.
(ii) $\alpha(\overline{M}) = \alpha(M)$.
(iii) $\alpha(z + M) = \alpha(M)$, *i.e.,* α *is translation-invariant.*
(iv) $\alpha(\lambda \cdot M) = |\lambda| \cdot \alpha(M)$, *i.e.,* α *is homogeneous.*
(v) $\alpha(M) = 0$ *iff* M *is precompact.*
(vi) $|\alpha(M) - \alpha(N)| \leq \alpha(M + N) \leq \alpha(M) + \alpha(N)$. *The first inequality only holds in case both subsets are nonempty.*
(vii) $\alpha(M \cup N) = \max\{\alpha(M), \alpha(N)\}$.
(viii) $\alpha(\mathrm{co}(M)) = \alpha(M)$.
(ix) $\alpha(B(z, 1)) = 1$, *if* E *is infinite-dimensional, and zero otherwise.*
(x) *If* $M_1 \supseteq M_2 \supseteq \ldots$ *is a decreasing sequence of closed sets in* E *with* $\alpha(M_n) \to 0$ *for* $n \to \infty$, *then the intersection* $M_\infty := \bigcap_n M_n$ *is nonempty and compact.*

Proof. We give the straight-forward proof for the sake of completeness.

(i) Every finite ϵ-net for N is one for M.
(ii) Inclusion $\alpha(M) \leq \alpha(\overline{M})$ follows from (i). For the other direction, note that every finite $(\epsilon + \delta)$-net for M is a finite ϵ-net for $\overline{M}$ for all $\delta > 0$.
(iii) If $\{x_1, \ldots, x_s\}$ is a finite ϵ-net for M, then $\{z + x_1, \ldots, z + x_s\}$ is a finite ϵ-net for $z + M$.
(iv) If $\{x_1, \ldots, x_s\}$ is a finite ϵ-net for M, then $\{\lambda \cdot x_1, \ldots, \lambda \cdot x_s\}$ is a finite $(|\lambda|\epsilon)$-net for $\lambda \cdot M$.
(v) We have M precompact iff $\overline{M}$ compact iff $\alpha(\overline{M}) = 0$ iff $\alpha(M) = 0$. If $\alpha(\overline{M}) = 0$, then $\overline{M}$ is totally bounded and complete. Hence, by [**SV06**, 5.1.17], $\overline{M}$ is compact, and vice versa.
(vi) For the first inequality, $\alpha(M) \leq \alpha(M + N)$ by items (i) and (iii). Then $\alpha(M) \leq \alpha(M + N) + \alpha(N)$. For the second inequality, note that if

$\{x_1, \ldots, x_s\}$ is a finite ϵ-net for M and $\{y_1, \ldots, y_t\}$ is a finite δ-net for N, then $\{x_1 + y_1, \ldots, x_s + y_t\}$ is a finite $(\epsilon + \delta)$-net for $M + N$.

(vii) The inclusion $\max\{\alpha(M), \alpha(N)\} \leq \alpha(M \cup N)$ follows from item (i). For the other direction, note that the union of a finite ϵ-net for M with a finite δ-net for N gives a finite $\max(\epsilon, \delta)$-net for $M \cup N$.

(viii) As $M \subseteq \mathrm{co}(M)$, then $\alpha(M) \leq \alpha(\mathrm{co}(M))$ by item (i). For the other direction, let N be a finite η-net for M, $\eta > 0$. Define $C := \overline{\mathrm{co}}(N)$. We have $d(x, z) \leq \eta$ for all $x \in \mathrm{co}(M)$ and $z \in C$. This can be seen as follows. Point z is a convex combination $z = \sum_i \lambda_i \cdot z_i$ with $z_i \in N$, $\lambda_i \in [0, 1]$, and $\sum_i \lambda_i = 1$. Now, the subtle issue comes: Making use of Theorem 21 in the third inequality, we have

$$\begin{aligned} d(x,z) &= d\left((\sum_i \lambda_i) \cdot x, \sum_i \lambda_i \cdot z_i\right) \leq \sum_i d\left(\lambda_i \cdot x, \lambda_i \cdot z_i\right) \\ &\leq \sum_i \lambda_i \cdot d(x, z_i) \leq \sum_i \lambda_i \cdot \eta = (\sum_i \lambda_i) \cdot \eta = 1 \cdot \eta = \eta \quad . \end{aligned}$$

In addition, set C is compact, because it is a closed and bounded set in a finite-dimensional space $\mathrm{span}(N)$. As C is compact, for every $\epsilon > 0$, there exists a finite ϵ-net K for C. Then K is a finite $(\eta + \epsilon)$-net for $\mathrm{co}(M)$.

(ix) It suffices to prove the statement for the unit ball $B := B(0, 1)$. In case $\dim E < \infty$, ball B is compact. Hence, $\alpha(B) = 0$ by item (v). Let us assume $\dim E = \infty$. The trivial estimate is $\alpha(B) \leq 1$ by taking B itself as a covering. Assume for a contradiction that $\alpha(B) \leq \epsilon < 1$. Then there exists a finite ϵ-net of closed balls of radius ϵ. Each of these balls in turn can be covered by finitely-many balls of radius ϵ^2, which gives a finite ϵ^2-net for B. We can cover the balls of radius ϵ^2 by finitely-many balls of radius ϵ^3, and so on. Hence, for every $n \geq 1$, there is a finite ϵ^n-net for B, showing $\alpha(B) \leq \epsilon^n \to 0$ for $n \to \infty$. By item (v), B would be compact, a contradiction to the fact that the unit ball is not compact in infinite-dimensional spaces.

(x) As $M_\infty \subseteq M_n$, by item (i) we have $\alpha(M_\infty) \leq \alpha(M_n) \to 0$ for $n \to \infty$. Hence, $\alpha(M_\infty) = 0$, and M_∞ is precompact by item (v). It is closed as an arbitrary intersection of closed sets and thus compact. We need to show that M_∞ is nonempty. Choose an element x_n from each set M_n. Build sets $N_m := \{x_n \mid n \geq m\}$. Then $N_m \subseteq M_n$ and precompactness of each N_n follows from (i) and (v). Hence, there exists a converging subsequence with limit x_∞. This limit belongs to M_∞.

□

From item (v) it follows that the measure of noncompactness only makes sense in infinite-dimensional spaces. Otherwise, it is zero.

For metrizable t.v.s., we will define two characteristics $[\cdot]_a$ and $[\cdot]_A$, respectively, based on the measure of noncompactness. Let $f \colon E \to F$ be a bounded operator between metrizable t.v.s. E and F, respectively. We define the *lower and upper characteristics of noncompactness* by

$$[f]_a := \sup\{\gamma > 0 \mid \alpha(f(M)) \geq \gamma \cdot \alpha(M), M \text{ bounded}\} \quad , \tag{17}$$

$$[f]_A := \inf\{\gamma > 0 \mid \alpha(f(M)) \leq \gamma \cdot \alpha(M), M \text{ bounded}\} \quad . \tag{18}$$

For infinite-dimensional and metrizable t.v.s., there exist bounded sets M with positive measure of noncompactness. Hence, we can rewrite above equations to

$$[f]_a := \inf_{\alpha(M)>0} \frac{\alpha(f(M))}{\alpha(M)} \quad , \tag{19}$$

$$[f]_A := \sup_{\alpha(M)>0} \frac{\alpha(f(M))}{\alpha(M)} \quad . \tag{20}$$

By the properties of α, we obtain

PROPOSITION 54. *For all bounded operators $f, g\colon E \to F$ between infinite-dimensional, metrizable t.v.s. E and F, and all $\lambda \in \mathbb{K}$, the following properties hold.*

(i) $[\lambda \cdot f]_a = |\lambda| \cdot [f]_a$, *i.e.,* $[\cdot]_a$ *is homogeneous.*
(ii) $[f+g]_A \le [f]_A + [g]_A$ *and* $[\lambda \cdot f]_A = |\lambda| \cdot [f]_A$, *i.e.,* $[\cdot]_A$ *is a seminorm.*
(iii) $[f]_a - [g]_A \le [f+g]_a \le [f]_a + [g]_A$.
(iv) $|[f]_a - [g]_a| \le [f-g]_A$. *In particular,* $[f-g]_A = 0$ *implies* $[f]_a = [g]_a$.
(v) $[f^{-1}]_A = [f]_a^{-1}$ *for f a homeomorphism.*
(vi) $[f]_a \le [f]_A$.
(vii) $[f]_A \le \|f\|$ *in case E and F are Banach spaces and f is linear.*

PROOF. We give a proof for the sake of completeness.

(i) We have

$$[\lambda \cdot f]_a = \inf_{\alpha(M)>0} \frac{\alpha((\lambda \cdot f)(M))}{\alpha(M)} = \inf_{\alpha(M)>0} \frac{|\lambda| \cdot \alpha(f(M))}{\alpha(M)} = |\lambda| \cdot [f]_a \quad .$$

(ii) The proof of $[\lambda \cdot f]_A = |\lambda| \cdot [f]_A$ is analogous to the one for $[\cdot]_a$.

$$\begin{aligned}[f+g]_A &= \sup_{\alpha(M)>0} \frac{\alpha((f+g)(M))}{\alpha(M)} \le \sup_{\alpha(M)>0} \frac{\alpha(f(M)) + \alpha(g(M))}{\alpha(M)} \\ &\le \sup_{\alpha(M)>0} \frac{\alpha(f(M))}{\alpha(M)} + \sup_{\alpha(M)>0} \frac{\alpha(g(M))}{\alpha(M)} = [f]_A + [g]_A \quad .\end{aligned}$$

(iii) The second inequality is proved by

$$\begin{aligned}[f+g]_a &= \inf_{\alpha(M)>0} \frac{\alpha((f+g)(M))}{\alpha(M)} \le \inf_{\alpha(M)>0} \frac{\alpha(f(M)) + \alpha(g(M))}{\alpha(M)} \\ &\le \inf_{\alpha(M)>0} \frac{\alpha(f(M))}{\alpha(M)} + \sup_{\alpha(M)>0} \frac{\alpha(g(M))}{\alpha(M)} = [f]_a + [g]_A \quad .\end{aligned}$$

The first inequality is a consequence of the second with

$$[f]_a = [(f+g) - g]_a \le [f+g]_a + [-g]_A = [f+g]_a + [g]_A \quad .$$

(iv) We have $[f]_a - [g-f]_A \le [f + (g-f)]_a = [g]_a$. Hence, $[f]_a - [g]_a \le [g-f]_a = [f-g]_A$. We then also have $[g]_a - [f]_a \le [g-f]_a = [f-g]_A$.

(v) As f is a homeomorphism, we have $\alpha(M) > 0$ iff $\alpha(f(M)) > 0$. Hence, we can argue

$$\begin{aligned}[f^{-1}]_A &= \sup_{\alpha(N)>0} \frac{\alpha(f^{-1}(N))}{\alpha(N)} = \sup_{\alpha(f(M))>0} \frac{\alpha(f^{-1}(f(M)))}{\alpha(f(M))} \\ &= \sup_{\alpha(M)>0} \frac{\alpha(M)}{\alpha(f(M))} = \left(\inf_{\alpha(M)>0} \frac{\alpha(f(M))}{\alpha(M)} \right)^{-1} = [f]_a^{-1} \quad .\end{aligned}$$

(vi) By definition, we have

$$[f]_a = \inf_{\alpha(M)>0} \frac{\alpha(f(M))}{\alpha(M)} \le \sup_{\alpha(M)>0} \frac{\alpha(f(M))}{\alpha(M)} = [f]_A \quad .$$

(vii) If $\{z_1, \dots, z_n\}$ is a finite ϵ-net for M, then $\{f(z_1), \dots, f(z_n)\}$ is a finite $\|f\| \cdot \epsilon$-net for $f(M)$. Then $\alpha(f(M)) \leq \|f\| \cdot \alpha(M)$.

$\square$

An operator f is *condensing*, if $[f]_A \leq 1$. It is *α-contractive*, if $[f]_A < 1$.

THEOREM 55 (Darbo). *Let E be an infinite-dimensional Banach or Fréchet space, let $C \subseteq E$ be a nonempty, convex, closed, and bounded subset, and let $f\colon C \to C$ be α-contractive. Then f has a fixed point.*

PROOF. We inductively define a sequence of sets by $C_0 := C$, and $C_{m+1} := \overline{\text{co}}(f(C_m))$. By construction, each set C_m is nonempty, convex, and closed. Define set $C_\infty := \bigcap_m C_m$. It is convex, closed, and bounded as the intersection of such sets. In addition, it is f-invariant, $f(C_\infty) \subseteq C_\infty$. We need to show that C_∞ is nonempty. For this, fix $\gamma \in ([f]_A, 1)$. The sequence $(C_m)_m$ is monotonically decreasing with respect to inclusion. We have $\alpha(C_m) \leq \gamma^m \alpha(C) < \infty$. Hence, $\alpha(C_m) \to 0$ for $m \to \infty$. This implies that C_∞ is nonempty and compact. Now, by Theorem 36 (Tychonoff) applied to $f\colon C_\infty \to C_\infty$, f has a fixed point. $\square$

We close this section with the following insight. The characteristics of noncompactness help to find an invariant compact set, given an invariant bounded set. This is exploited in the construction of the FMV and Feng spectra.

LEMMA 56. *Let E and F be infinite-dimensional Banach (or Fréchet) spaces, let $\Omega \subseteq E$ be a nonempty subset, and let $f, g\colon \Omega \to F$ be continuous operators with $[g]_A < [f]_a$. Then for every nonempty, convex, closed, and bounded subset $B \subseteq \Omega$ with*

$$f^{-1}(\overline{\text{co}}(g(B) \cup \{0\})) \subseteq B \quad , \tag{21}$$

there exists a nonempty, convex, (closed,) and compact subset $C \subseteq B$, also fulfilling above relation (21).

PROOF. By Lemma 57, there exists a set C such that $f^{-1}(\overline{\text{co}}(g(C) \cup \{0\})) = C$. We have $\alpha(C) < \infty$, because $C \subseteq B$ and B is bounded. Furthermore, we have

$$\begin{aligned}[f]_a \cdot \alpha(C) &\leq \alpha(f(C)) \leq \alpha(\overline{\text{co}}(g(C) \cup \{0\})) = \alpha(\text{co}(g(C) \cup \{0\})) \\ &= \alpha(g(C) \cup \{0\}) = \alpha(g(C)) \leq [g]_A \cdot \alpha(C) \quad .\end{aligned}$$

By assumption $[g]_A < [f]_a$, this can only happen with $\alpha(C) = 0$. Thus, C is compact. $\square$

LEMMA 57. *Let E and F be t.v.s., let $\Omega \subseteq E$ be a nonempty, convex, and closed subset with $0 \in \Omega$. Let $f, g\colon \Omega \to F$ be two operators, with f continuous. Then there exists a smallest (in the order of inclusion), nonempty, convex, and closed set $U \subseteq \Omega$ with*

$$f^{-1}(\overline{\text{co}}(g(U) \cup \{0\})) \subseteq U \quad . \tag{22}$$

For this smallest U_0, above relation (22) holds with equality.

PROOF. Define set $\mathcal{U}$ as the set of all nonempty, convex, and closed sets $U \subseteq \Omega$ with $0 \in U$, fulfilling the relation in (22). The set $\mathcal{U}$ is not empty, because $\Omega \in \mathcal{U}$. Define $U_0 := \bigcap \mathcal{U}$. By construction, U_0 is nonempty ($0 \in U_0$). It is convex as the intersection of convex sets.

Let $U \in \mathcal{U}$ be arbitrary. Define $U_1 := f^{-1}(\overline{\text{co}}(g(U_0) \cup \{0\}))$. Then $U_1 \subseteq f^{-1}(\overline{\text{co}}(g(U) \cup \{0\})) \subseteq U$. As U was arbitrary, $U_1 \subseteq U_0$. Hence, $f^{-1}(\overline{\text{co}}(g(U_1) \cup \{0\})) \subseteq f^{-1}(\overline{\text{co}}(g(U_0) \cup \{0\})) = U_1$. Thus, $U_1 \in \mathcal{U}$, implying $U_0 = U_1$. Consequently, the relation in (22) holds for U_0 with equality.

Finally, as $V_0 := \overline{\text{co}}(g(U_0) \cup \{0\})$ is closed and f is continuous, $U_0 = f^{-1}(V_0)$ is closed, too. $\square$

Interestingly, in the above lemma, no assumption is made on the continuity of the map g.

5. Michael Selection

Given a set-valued map $t\colon X \to 2^Y$, a *selection* is defined as a map $s\colon X \to Y$ such that $s(x) \in t(x)$ for all $x \in X$. In the setting of Banach spaces, the existence of selections was proved first by Michael [**Mic56**], starting a new branch in topology.

Theorem 58 (Michael Selection). *Let X be a paracompact space, and let E be a Banach space. Let $t\colon X \to 2^E$ be a lower-semicontinuous, set-valued map with nonempty, convex, and closed values. Then there exists a continuous selection $s\colon X \to E$ for t.*

The same phenomenon as for Dugundji's extension theorem occurs when trying to lift the selection theorem to general l.c.s.. One then has to accept approximate instead of exact solutions. Such an approximate selection theorem was established by Xu [**Xu01**].

Let X be a subset of a T_2 topological space E, let Y be an l.c.s. with origin 0. Denote with $\mathcal{O}_Y(y)$ the collection of all neighborhoods of y in space Y. Let $t\colon X \to 2^E$ be a set-valued map with nonempty values. Map t is *almost lower semicontinuous (a.l.s.c.)*, at point $x \in X$, if for each $V \in \mathcal{O}_E(0)$ there exists $U \in \mathcal{O}_X(x)$ such that $\bigcap\{t(x) + V \mid x \in U\} \neq \emptyset$. We say that t has *continuous, approximate selections*, if for each $V \in \mathcal{O}_E(0)$, there exists a continuous map $s\colon X \to E$ such that $s(x) \in t(x) + V$ for all $x \in X$.

Theorem 59 (Xu). *Let X be paracompact, E an l.c.s., and let $s\colon X \to 2^E$ be a set-valued map with nonempty and convex values. Then s is a.l.s.c. iff s has continuous, approximate selections.*

CHAPTER 3

Existing Spectra

This chapter contains the main theme of this work. It introduces several important existing nonlinear spectra and analyzes their properties. The selection of properties is based on the known properties of the linear spectrum, which we recap in the first section. The nonlinear spectra under consideration are the Rhodius, Dörfner, Neuberger, Kačurovskiĭ, Furi-Martelli-Vignoli (FMV), and Feng spectra, respectively. The analyzed properties are nonemptiness, closedness, boundednes, and semicontinuity. The material has been mainly taken from [**ADPV04**]. Hence, nothing presented here is new. Nevertheless, the presentation differs in two aspects. First of all, the material is structured according to the properties, not the spectra. Secondly, we claim that the main results, namely the closed- and boundedness of FMV and Feng spectra, are presented in a more structured, simple, and elegant way.

1. Linear Spectrum and Properties

1.1. Definitions. Given a t.v.s. E and a linear operator $u \in \mathcal{L}(E)$, its *linear resolvent set* is defined as

$$\rho(u) = \{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - u) \text{ is bijective}\} \quad . \tag{23}$$

For $\lambda \in \rho(u)$ the *linear resolvent operator* of u at λ is denoted by $r(u, \lambda) := (\lambda \cdot \mathrm{id}_E - u)^{-1}$. It is a linear operator. In case of E being a barreled space and u being linear and continuous, then $r(u, \lambda)$ is also continuous for all $\lambda \in \rho(u)$.

The *linear spectrum* of u is defined as the complement of the linear resolvent set, i.e.,

$$\sigma(u) = \{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - u) \text{ is not bijective}\} \quad . \tag{24}$$

Furthermore, the *linear spectral radius* of u is the number

$$r(u) := \sup\{|\lambda| \mid \lambda \in \sigma(u)\} \quad . \tag{25}$$

We list several important subspectra of the linear spectrum. Value λ belongs to the *(linear) point spectrum*, $\sigma_p(u)$, if $\ker(\lambda \cdot \mathrm{id}_E - u)$ is nontrivial. Value λ belongs to the *(linear) continuous spectrum*, $\sigma_c(u)$, if resolvent operator $r(u, \lambda)$ is defined on a dense subspace of E and is not continuous. Value λ belongs to the *(linear) residual spectrum*, $\sigma_r(u)$, if $r(u, \lambda)$ exists on a domain of definition, which is not dense in E.

In addition, the *(linear) defect spectrum*, $\sigma_\delta(u)$, is defined as the set of λ such that operator $(\lambda \cdot \mathrm{id}_E - u)$ is not surjective. The *(linear) compression spectrum*, $\sigma_{co}(u)$, is defined as the set of λ such that $\overline{(\lambda \cdot \mathrm{id}_E - u)(E)} \neq E$.

1.2. Properties in t.v.s. Let E be a t.v.s.. By definition, we have the inclusion

$$\sigma_p(u) \uplus \sigma_c(u) \uplus_r (u) \subseteq \sigma(u) \quad . \tag{26}$$

Proposition 60. *Let $u, v \colon E \to E$ be linear and continuous operators, and let $\lambda, \mu \in \mathbb{K}$. Then we have*

(i) *The linear resolvent is continuous in the first parameter and satisfies*

$$r(u,\lambda)-r(v,\lambda)=r(u,\lambda)\circ(u-v)\circ r(v,\lambda) \quad . \tag{27}$$

(ii) *The linear resolvent is also continuous in the second parameter and satisfies*

$$r(u,\lambda)-r(u,\mu)=-(\lambda-\mu)\cdot r(u,\lambda)\circ r(u,\mu) \quad . \tag{28}$$

Proof. The proofs are by simple expansion of identity and distributivity of composition of operators.

(i) We have

$$\begin{aligned}
& r(u,\lambda)-r(v,\lambda) \\
= \;& r(u,\lambda)\circ \mathrm{id}_E - \mathrm{id}_E \circ r(v,\lambda) \\
= \;& r(u,\lambda)\circ(\lambda\cdot\mathrm{id}_E - v)\circ r(v,\lambda) - r(u,\lambda)\circ(\lambda\cdot\mathrm{id}_E - u)\circ r(v,\lambda) \\
= \;& r(u,\lambda)\circ[(\lambda\cdot\mathrm{id}_E - v)-(\lambda\cdot\mathrm{id}_E - u)]\circ r(v,\lambda) \\
= \;& r(u,\lambda)\circ(u-v)\circ r(v,\lambda) \quad .
\end{aligned}$$

(ii) We have

$$\begin{aligned}
& r(u,\lambda)-r(u,\mu) \\
= \;& r(u,\lambda)\circ \mathrm{id}_E - \mathrm{id}_E \circ r(u,\mu) \\
= \;& r(u,\lambda)\circ(\mu\cdot\mathrm{id}_E - u)\circ r(u,\mu) - r(u,\lambda)\circ(\lambda\cdot\mathrm{id}_E - u)\circ r(u,\mu) \\
= \;& r(u,\lambda)\circ[(\mu\cdot\mathrm{id}_E - u)-(\lambda\cdot\mathrm{id}_E - u)]\circ r(u,\mu) \\
= \;& -(\lambda-\mu)\cdot r(u,\lambda)\circ r(u,\mu) \quad .
\end{aligned}$$

□

Let $p\colon \mathbb{K}\to\mathbb{K}$ be a polynomial

$$p(z) := \sum_{m=0}^{n} a_m\cdot z^m = c\cdot\prod_{m=0}^{n}(\lambda_m - z) \quad .$$

For $u\in\mathcal{L}(E)$, we define $p(u) := \sum_{m=0}^{n} a_m\cdot u^m$, where u^m denotes m-fold composition of u with itself. As each a_m is the m-th symmetric polynomial s_m in the roots λ_i, i.e., $a_m = s_m(\lambda_1,\ldots,\lambda_n)$, the factorization of $p(z)$ carries over to $p(u) = c\cdot\prod_{m=0}^{n}(\lambda_m\cdot\mathrm{id}_E - u)$.

Above argument yields that linear operators $(\lambda\cdot\mathrm{id}_E - u)$ and $(\mu\cdot\mathrm{id}_E - u)$ commute. Of course, this can also be seen by noting that the coefficients in the computed expression of their product are symmetric in λ and μ.

$$(\lambda\cdot\mathrm{id}_E - u)\circ(\mu\cdot\mathrm{id}_E - u) = (\lambda\cdot\mu)\cdot\mathrm{id}_E - (\lambda+\mu)\cdot u + u\circ u \quad .$$

Consequently, $p(u)$ is invertible iff each factor $(\lambda_m\cdot\mathrm{id}_E - u)$ is invertible. This can be seen as follows. Clearly, $p(u)$ is invertible, if it is a composition of invertible factors. For the other direction, assume there is a factor $(\lambda_k\cdot\mathrm{id}_E - u)$, which is not invertible. If this factor is not injective, then $p(u) = c\cdot\prod_{m=0,\neq k}^{n}(\lambda_m\cdot\mathrm{id}_E - u)\circ(\lambda_k\cdot\mathrm{id}_E - u)$ is not injective. If this factor is not surjective, then $p(u) = c\cdot(\lambda_k\cdot\mathrm{id}_E - u)\circ\prod_{m=0,\neq k}^{n}(\lambda_m\cdot\mathrm{id}_E - u)$ is not surjective.

Theorem 61 (Spectral Mapping). *Let $u\in\mathcal{L}(E)$. Then for every polynomial $p\colon\mathbb{K}\to\mathbb{K}$ we have*

$$\sigma(p(u)) = p(\sigma(u)) \quad . \tag{29}$$

Proof. First of all, for all polynomials p, we have the following equivalences: $0 \in \rho(p(u))$. $\iff$ Operator $p(u)$ is invertible. $\iff$ Operator $(\lambda_m \cdot \mathrm{id}_E - u)$ is invertible for all m. $\iff$ $\lambda_m \in \rho(u)$ for all m. $\iff$ $0 \in \lambda_m - \rho(u)$ for all m. Hence, $0 \in \sigma(p(u))$. $\iff$ There exists m with $0 \notin \lambda_m - \rho(u)$. $\iff$ There exists m with $0 \in \lambda_m - \sigma(u)$. $\iff$ $0 \in p(\sigma(u))$.

Secondly, we have the following equivalences: $\lambda \in \sigma(p(u))$. $\iff$ Operator $(\lambda \cdot \mathrm{id}_E - p(u))$ is not invertible. $\iff$ Operator $\tilde{p}(u)$ is not invertible, where $\tilde{p}(z) := \lambda - p(z)$. $\iff$ $0 \in \sigma(\tilde{p}(u))$. $\iff$ $0 \in \tilde{p}(\sigma(u))$. $\iff$ $-0 \in -\lambda + p(\sigma(u))$. $\iff$ $\lambda \in \lambda - \lambda + p(\sigma(u))$. $\iff$ $\lambda \in p(\sigma(u))$. □

By above theorem, in this very general setting, we obtain $r(u^k) = (r(u))^k$, because of $\sup\left\{|\lambda| \mid \lambda \in \sigma(u^k)\right\} = \sup\left\{|\lambda| \mid \lambda \in (\sigma(u))^k\right\}$. Analogously, one can prove $r(\alpha \cdot u) = |\alpha| \cdot r(u)$.

1.3. Properties in l.c.s. For barreled l.c.s., we obtain the following partition of the linear spectrum:

$$\sigma(u) = \sigma_p(u) \uplus \sigma_c(u) \uplus_r (u) \quad . \tag{30}$$

The case that $r(u, \lambda)$ exists and is unbounded cannot occur due the the closed-graph theorem.

Let E be an l.c.s., and let G be a nonempty open subset of $\mathbb{C}$. An E-valued map $f\colon G \to E$ is called *holomorphic* at point $\zeta_0 \in G$, if there exists an open neighborhood Z of ζ_0 such that for all $x' \in E'$, function $\zeta \mapsto x'(f(\zeta))$ is holomorphic in Z, and if for each $\zeta \in Z$, the linear form $x' \mapsto \partial_\zeta x'(f(\zeta))$ is $*$-weak continuous.

One can show that an E-valued map f, holomorphic and uniformly bounded on the entire complex plane, is constant by the Theorem of Liouville.[1] As map $\lambda \mapsto r(u, \lambda)$ is holomorphic on each point, where it is defined, the linear spectrum is nonempty.

1.4. Properties in Banach Spaces. Let $(E, \|\cdot\|_E)$ be a Banach space, and let $u\colon E \to E$ be a linear and continuous operator.

We call a sequence $(x_n)_n$ in E a *Weyl sequence* for u, if $\|x_n\|_E = 1$ and $\|u(x_n)\|_E \to 0$ for $n \to \infty$. The *(linear) approximate point spectrum*, $\sigma_q(u)$, is defined as the set of all λ such that there exists a Weyl sequence for operator $(\lambda \cdot \mathrm{id}_E - u)$.

We obtain the following (not necessarily disjoint) subdivisions:

$$\sigma(u) = \sigma_q(u) \cup \sigma_\delta(u) = \sigma_q(u) \cup \sigma_{co}(u) \quad . \tag{31}$$

Proposition 62. *For every compact set $\Sigma \subseteq \mathbb{K}$, there exists a linear and continuous operator $u = u(\Sigma)$ such that $\sigma(u) = \Sigma$.*

Proof. There exists a countable and dense subset $\{s_m \mid m \geq 1\}$ of compact set Σ. Let $E := \ell^2$, and define linear operator $u\colon E \to E$ by

$$u(x_1, x_2, x_3, \ldots) := (s_1 \cdot x_1, s_2 \cdot x_2, s_3 \cdot x_3, \ldots) \quad .$$

Operator u is continuous, because it is bounded.

$$\begin{aligned}\|u(x)\|_2 &= \sqrt{\sum_{m\geq 1} |s_m|^2 \cdot |x_m|^2} \leq \sqrt{\sup\{|s_m|^2 \mid m \geq 1\}} \cdot \sqrt{\sum_{m \geq 1} |x_m|^2} \\ &\leq (\sup|\Sigma|)^2 \cdot \|x\|_2 \quad .\end{aligned}$$

On the one hand, by definition, operator $(s_m \cdot \mathrm{id}_E - u)$ is not invertible. Hence, we have $\{s_m \mid m \geq 1\} \subseteq \sigma(u)$. Then $\Sigma = \overline{\{s_m \mid m \geq 1\}} \subseteq \sigma(u)$ by closedness of $\sigma(u)$.

[1] [**SW99**, IV.Ex.39]

On the other hand, every $\lambda \notin \Sigma$ has positive distance to closed set Σ. Consequently, there exists a $\delta > 0$ with $|\lambda - s_m| \geq \delta > 0$ for all m. Hence, operator $(\lambda \cdot \mathrm{id}_E - u)$ is invertible with inverse

$$r(u,\lambda)(x_1,x_2,x_3,\ldots) = \big((\lambda - s_1)^{-1} \cdot x_1, (\lambda - s_2)^{-1} \cdot x_2, (\lambda - s_3)^{-1} \cdot x_3, \ldots\big) \quad .$$

□

PROPOSITION 63 (Resolvent Bound). *The linear resolvent operator $r(u,\lambda)$ is continuous and hence bounded. For $|\lambda| > \rho(u)$, it is bounded explicitly by*

$$\|r(u,\lambda)\|_{E\to E} \leq \frac{1}{|\lambda| - \|u\|_{E\to E}} \quad . \tag{32}$$

PROOF. For $|\lambda| > r(u)$, operator $r(u,\lambda)$ can be expanded in a convergent von-Neumann series.

$$r(u,\lambda) = \frac{1}{\lambda} \cdot \left(\mathrm{id}_E - \frac{1}{\lambda} \cdot u\right)^{-1} = \frac{1}{\lambda} \cdot \sum_{k\geq 0} \frac{1}{\lambda^k} \cdot u^k \quad .$$

Hence, it follows that

$$\|r(u,\lambda)\|_{E\to E} \leq \frac{1}{|\lambda|} \cdot \sum_{k\geq 0} \frac{1}{|\lambda|^k} \cdot \|u\|^k_{E\to E} \leq \frac{1}{|\lambda|} \cdot \frac{1}{1 - \left(\frac{\|u\|_{E\to E}}{|\lambda|}\right)} \quad .$$

□

THEOREM 64 (Gelfand Formula). [2] *The linear spectral radius is bounded by the classical* Gel'fand formula

$$r(u) = \lim_{m\to\infty} \sqrt[m]{\|u^m\|_{E\to E}} = \inf_m \sqrt[m]{\|u^m\|_{E\to E}} \quad . \tag{33}$$

In particular, we have $r(u) \leq \|u\|_{E\to E}$.

THEOREM 65. *The spectrum $\sigma(u)$ is closed and bounded, and thus compact.*[3]

Let $(\mathcal{M}, p)$ be a vector space with a seminorm. We call a set-valued map $\sigma\colon \mathcal{M} \to 2^{\mathbb{K}}$ *upper semicontinuous*, if for all $f \in \mathcal{M}$ and all open $V \subseteq \mathbb{K}$, there exists $\delta > 0$ such that for all $g \in \mathcal{M}$ with $p(y - x) < \delta$, we have $\sigma(g) \subseteq V$.

The term lower semicontinuous is defined analogously.

THEOREM 66 (Semicontinuity). *The set-valued map $u \mapsto \sigma(u)$ is upper semicontinuous. In general, it is not lower semicontinuous.*[4]

THEOREM 67 (Spectrum of Compact Linear Operator). *Let $u\colon E \to E$ be a linear, continuous, and compact operator. Then we have*[5]

(i) *Set $\sigma(u)\backslash\{0\}$ is discrete and bounded.*
(ii) $\sigma_p(u) \subseteq \sigma(u) \subseteq \sigma_p(u) \cup \{0\}$.
(iii) *If E is infinite-dimensional, then $0 \in \sigma(u)$. Hence, $\sigma(u) = \sigma_p(u) \cup \{0\}$.*

Linear Spectral Theory is mostly developed in the setting of Hilbert and Banach spaces. For example, the question of how to extend the Gelfand formula (Theorem 64) beyond Banach spaces, was a research topic in the 2000's, see e.g., [**BM98, Tro01**].

[2] [**Wer00**, VI.1.6]
[3] [**ADPV04**, Chapter 1, Theorem 1.1 (f)]
[4] [**ADPV04**, Chapter 1, Theorem 1.1 (i), Example 1.1]
[5] [**ADPV04**, Chapter 1, Theorem 1.3]

2. Spectra Under Consideration

2.1. Classical, Rhodius, and Linear Spectrum. Let us bring to mind parts of classical physics, for example, Newton's Mechanics. Such classical physical theories provide deterministic, reversible, and continuous models of the world, see e.g., [**SH13**, p.2–3]. Consequently, their equations of motion (dynamics) need to be uniquely solvable to fulfill reversibility, and continuity demands that the better one can approximatively measure the initial conditions of such a modelled system, the better one can predict its future. These *classical* requirements are reflected in the definition of the spectrum, defined below.

Let $f_\cdot = \{f_\lambda \colon E \to F\}_{\lambda\in\Lambda}$ be a family of operators (operator pencil) between topological spaces X and Y over a parameter space Λ, which is an open subset $\Lambda \subseteq \mathbb{K}$. We call family $f_\cdot$ *continuously resolvable* for parameter λ, if the following two conditions hold:

- **Existence and uniqueness of solutions:** For every $y \in Y$, there exists exactly one $x \in X$ with $f_\lambda(x) = y$.
- **Continuity of solutions:** For every $x \in X$ and $y \in Y$ with $f_\lambda(x) = y$, there exist open neighborhoods U of x and V of y, and a continuous operator $r_\lambda \colon V \to U$ such that $f_\lambda \circ r_\lambda$ equals id_Y on V.

Operator r_λ is called *(local) resolvent operator for parameter* λ. The *resolvent set* is defined as the set of parameters

$$\rho\,(f_\cdot) := \{\lambda \in \mathbb{K} \mid f_\lambda \text{ is continuously resolvable}\} \quad . \tag{34}$$

Its complement

$$\sigma\,(f_\cdot) := \mathbb{K} \setminus \rho\,(f_\cdot) \tag{35}$$

is called the *(classical) spectrum* of $f_\cdot$. Furthermore, the *(classical) spectral radius* of $f_\cdot$ is the number

$$r\,(f_\cdot) := \sup\{|\lambda| \mid \lambda \in \sigma\,(f_\cdot)\} \quad . \tag{36}$$

Given a single operator $f \colon E \to E$ on a t.v.s. E, we always consider the associated family $f_\cdot := \{\lambda \cdot \mathrm{id}_E - f\}$ over $\Lambda = \mathbb{K}$. In this case, the resolvent set, $\rho(f) := \rho(f_\cdot)$, equals

$$\rho(f) = \{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - f) \text{ is bijective and } (\lambda \cdot \mathrm{id}_E - f)^{-1} \text{ is continuous}\} \quad .$$

Hence, for the *(classical) spectrum*, $\sigma(u) := \sigma(u_\cdot)$, we have $\sigma(f) = \{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - f)$ is not bijective, or it is bijective but operator $(\lambda \cdot \mathrm{id}_E - f)^{-1}$ is not continuous $\}$.

For $\lambda \in \rho(f)$ the *(global) resolvent operator* of f at λ is denoted by $r(f, \lambda) := (\lambda \cdot \mathrm{id}_E - f)^{-1}$.

For continuous operators $u \in \mathcal{C}(E)$ over a Banach space E, such a classical spectrum was defined by Rhodius [**Rho84**] in 1984. Clearly, this is a straightforward generalization of the classical linear spectrum.

In case of E being a barreled space and u being linear and continuous, also $r(u, \lambda)$ is linear and continuous for all $\lambda \in \rho(u)$. Then the resolvent set and spectrum even further simplify to the linear resolvent set and linear spectrum, respectively. The above argument sheds light on the reason, why the linear spectrum is often defined as the simplification stated above.

2.2. Mapping Spectrum. Conversely to the preceding line of thought, one can generalize the linear spectrum with focus on the mapping properties of the operator. Given an operator $f \colon E \to E$ defined on a t.v.s. E, the *mapping spectrum* is defined as

$$\Sigma(f) := \{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - f) \text{ is not bijective}\} \quad . \tag{37}$$

Analogously, the *injectivity spectrum* $\Sigma_i(f)$ and the *surjectivity spectrum* $\Sigma_s(f)$ are defined as

$$\Sigma_i(f) := \{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - f) \text{ is not injective}\} \quad , \tag{38}$$

$$\Sigma_s(f) := \{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - f) \text{ is not surjective}\} \quad , \tag{39}$$

respectively. Clearly, $\Sigma(f) = \Sigma_i(f) \cup \Sigma_s(f)$.

The *mapping spectral radius* is the number

$$r_\Sigma(f) := \sup\{|\lambda| \mid \lambda \in \Sigma(f)\} \quad . \tag{40}$$

2.3. Point Spectrum. Given an operator $f\colon E \to E$ defined on a t.v.s. E, the *point spectrum* is defined as

$$\sigma_p(f) := \{\lambda \in \mathbb{K} \mid \exists x \neq 0\colon f(x) = \lambda \cdot x\} \quad . \tag{41}$$

Such a point $x \neq 0$ is called an *eigenvector* of f for *eigenvalue* λ.

The *point spectral radius* is the number

$$r_p(f) := \sup\{|\lambda| \mid \lambda \in \sigma_p(f)\} \quad . \tag{42}$$

It goes without saying that the point spectrum is one of the most important spectra concerning applications of linear spectral theory.

We give a generalization, which is more tailored to the nonlinear case: Given two operators $j, f\colon E \to F$, we call number λ a (generalized) *eigenvalue* of (j, f), if equation $f(x) = \lambda \cdot j(x)$ has a nontrivial solution. We define

$$\sigma_p(j, f) := \{\lambda \in \mathbb{K} \mid \lambda \text{ is an eigenvalue for } (j, f)\} \quad . \tag{43}$$

By definition, $\sigma_p(\mathrm{id}_E, f) = \sigma_p(f)$ for all $f\colon E \to E$.

2.4. Spectra Defined Via Seminorms.

2.4.1. *A General Method.* Besides mapping properties or existence of nontrivial solutions, seminorms also give rise to spectra. In the sequel, let E and F be t.v.s., and let p and q be fixed seminorms on E and F, respectively. Every such pair (p, q) of seminorms gives rise to ten *characteristics* of an operator $f\colon E \to F$. These are categorized in five lower and five upper ones, denoted by lower- and uppercase letters, respectively.

$$[f]_s := \inf\{q(f(x)) \mid x \in E, p(x) = 1\} \quad , \tag{44}$$

$$[f]_S := \sup\{q(f(x)) \mid x \in E, p(x) = 1\} \quad , \tag{45}$$

$$[f]_d := \inf\{q(f(x)) \mid x \in E, 0 < p(x) \leq 1\} \quad , \tag{46}$$

$$[f]_D := \sup\{q(f(x)) \mid x \in E, 0 < p(x) \leq 1\} \quad , \tag{47}$$

$$[f]_{db} := \inf\{q(f(x))/p(x) \mid x \in E, 0 < p(x) \leq 1\} \quad , \tag{48}$$

$$[f]_{DB} := \sup\{q(f(x))/p(x) \mid x \in E, 0 < p(x) \leq 1\} \quad , \tag{49}$$

$$[f]_b := \inf\{q(f(x))/p(x) \mid x \in E, 0 < p(x) < \infty\} \quad , \tag{50}$$

$$[f]_B := \sup\{q(f(x))/p(x) \mid x \in E, 0 < p(x) < \infty\} \quad , \tag{51}$$

$$[f]_{lip} := \inf\{q(f(x) - f(y))/p(x - y) \mid x, y \in E, 0 < p(x - y) < \infty\} \quad , \tag{52}$$

$$[f]_{Lip} := \sup\{q(f(x) - f(y))/p(x - y) \mid x, y \in E, 0 < p(x - y) < \infty\} \quad . \tag{53}$$

We also give a generalization, which is more tailored to homogeneous operators. For this, recall that an operator $f\colon E \to F$ is called *τ-homogeneous* ($\tau > 0$), if $f(t \cdot x) = t \cdot f(x)$ for all $x \in E$ and $t > 0$.

$$[f]_s^\tau := \inf \{q(f(x)) \mid x \in E, p(x)^\tau = 1\} \quad , \tag{54}$$

$$[f]_S^\tau := \sup \{q(f(x)) \mid x \in E, p(x)^\tau = 1\} \quad , \tag{55}$$

$$[f]_d^\tau := \inf \{q(f(x)) \mid x \in E, 0 < p(x)^\tau \le 1\} \quad , \tag{56}$$

$$[f]_D^\tau := \sup \{q(f(x)) \mid x \in E, 0 < p(x)^\tau \le 1\} \quad , \tag{57}$$

$$[f]_{db}^\tau := \inf \{q(f(x))/p(x)^\tau \mid x \in E, 0 < p(x)^\tau \le 1\} \quad , \tag{58}$$

$$[f]_{DB}^\tau := \sup \{q(f(x))/p(x)^\tau \mid x \in E, 0 < p(x)^\tau \le 1\} \quad , \tag{59}$$

$$[f]_b^\tau := \inf \{q(f(x))/p(x)^\tau \mid x \in E, 0 < p(x)^\tau < \infty\} \quad , \tag{60}$$

$$[f]_B^\tau := \sup \{q(f(x))/p(x)^\tau \mid x \in E, 0 < p(x)^\tau < \infty\} \quad , \tag{61}$$

$$[f]_{lip}^\tau := \inf \{q(f(x) - f(y))/p(x-y)^\tau \mid x, y \in E, 0 < p(x-y)^\tau < \infty\} \quad , \tag{62}$$

$$[f]_{Lip}^\tau := \sup \{q(f(x) - f(y))/p(x-y)^\tau \mid x, y \in E, 0 < p(x-y)^\tau < \infty\} \quad . \tag{63}$$

By definition, for $\tau = 1$, these notions coincide with the usual ones.

The defined characteristics form a hierarchy.

Proposition 68. *For every operator $f\colon E \to F$ we have*

$$[f]_b \le [f]_{db} \le [f]_d \le [f]_s \le [f]_S \le [f]_D \le [f]_{DB} \le [f]_B \quad . \tag{64}$$

Trivially, $[f]_{lip} \le [f]_{Lip}$. In addition, in case $f(0) = 0$ we have

$$[f]_{lip} \le [f]_b \le \dots \le [f]_B \le [f]_{Lip} \quad . \tag{65}$$

Proof. We prove the inequalities for the ones defined by suprema. The reasoning is analogous for the ones defined by infima. Inequalities $[f]_S \le [f]_D$ and $[f]_{DB} \le [f]_B$ follow from the simple observation that the supremum is taken over increasing sets, respectively. Inequality $[f]_D \le [f]_{DB}$ is a consequence of $q(f(x)) \le q(f(x))/p(x)$ for every x with $0 < p(x) \le 1$. Inequality $[f]_B \le [f]_{Lip}$ follows from $q(f(x))/p(x) = q(f(x) - f(0))/p(x - 0) \le \sup_{x \ne y} \{q(f(x) - f(y))/p(x-y)\}$. $\square$

An analogous statement holds for homogeneous operators.

Proposition 69. *For every τ-homogeneous operator $f\colon E \to F$ we have*

$$[f]_b^\tau \le [f]_{db}^\tau \le [f]_d^\tau \le [f]_s^\tau \le [f]_S \le [f]_D^\tau \le [f]_{DB}^\tau \le [f]_B^\tau \quad . \tag{66}$$

Trivially, $[f]_{lip}^\tau \le [f]_{Lip}^\tau$. In addition, in case $f(0) = 0$ we have

$$[f]_{lip}^\tau \le [f]_b^\tau \le \dots \le [f]_B^\tau \le [f]_{Lip}^\tau \quad . \tag{67}$$

Proposition 70. *The characteristics coincide on additive and 1-homogeneous operators $u\colon E \to F$.*

$$[u]_{lip} = [u]_b = [u]_{db} = [u]_d = [u]_s \le [u]_S = [u]_D = [u]_{DB} = [u]_B = [u]_{Lip} \quad . \tag{68}$$

In particular, this holds for linear operators.

PROOF. We prove the inequalities for the ones defined by suprema. The reasoning is analogous for the ones defined by infima. We have

$$\begin{aligned} [u]_{Lip} &= \sup\{q(u(x)-u(y))/p(x-y) \mid x,y \in E, 0 < p(x-y) < \infty\} \\ &= \sup\{q(u((x-y)/p(x-y))) \mid x,y \in E, 0 < p(x-y) < \infty\} \\ &= \sup\{q(u(z/p(z))) \mid z \in E, 0 < p(z) < \infty\} \\ &= \sup\{q(u(x)) \mid x \in E, p(x) = 1\} = [u]_S \quad . \end{aligned}$$

□

In particular, we have $[\mathrm{id}_E]_Z = [\mathrm{id}_E]_z = 1$ for all lower and upper characteristics.

PROPOSITION 71. *Let $[\cdot]_z$ be a lower and $[\cdot]_Z$ be the corresponding upper characteristic. Then we have the following properties for all operators $f, g\colon E \to F$ and $\lambda \in \mathbb{K}$.*

(i) $[\lambda \cdot f]_z = |\lambda| \cdot [f]_z$*, i.e., $[\cdot]_z$ is homogeneous.*
(ii) $[f+g]_Z \leq [f]_Z + [g]_Z$ *and* $[\lambda \cdot f]_Z = |\lambda| \cdot [f]_Z$*, i.e., $[\cdot]_Z$ is a seminorm.*
(iii) $[f]_z - [g]_Z \leq [f+g]_z \leq [f]_z + [g]_Z$.
(iv) $|[f]_z - [g]_z| \leq [f-g]_Z$*. In particular,* $[f-g]_Z = 0$ *implies* $[f]_z = [g]_z$.
(v) $[f]_{Lip} = [f^{-1}]_{lip}^{-1}$ *for f a bijection between Banach spaces E and F, respectively, with $f(0) = 0$.*

PROOF. Item (i) follows from $q(\lambda \cdot f(x)) = |\lambda| \cdot q(f(x)$ and $\inf(|\lambda| \cdot a) = |\lambda| \cdot (\inf a)$. Item (ii) follows from $q(f(x)+g(x)) \leq q(f(x)) + q(g(x))$, and $\sup(a+b) \leq (\sup a) + (\sup b)$ and $\sup(|\lambda| \cdot a) = |\lambda| \cdot (\sup a)$. Item (iii) follows from $(\inf a) - (\sup b) \leq (\inf a) + (\inf b) \leq \inf(a+b) \leq (\inf a) + (\sup b)$. Item (iv) is a consequence of (iii) due to $[f]_z - [f-g]_Z \leq [f-(f-g)]_z = [g]_z$ and $[g]_z - [f-g]_Z = [g]_z - |-1| \cdot [g-f]_Z \leq [g-(g-f)]_z = [f]_z$, respectively. Item (v) is proved by noting that in Banach spaces $q(f(x)-f(y)) > 0$ iff $f(x) \neq f(y)$ iff $x \neq y$ iff $p(x-y) > 0$. Furthermore, $\sup a = (\inf a^{-1})^{-1}$. □

An analogous statement holds for homogeneous operators.

PROPOSITION 72. *Let $[\cdot]_z^\tau$ be a lower and $[\cdot]_Z^\tau$ be the corresponding upper characteristic. Then we have the following properties for all τ-homogeneous operators $f, g\colon E \to F$ and $\lambda \in \mathbb{K}$.*

(i) $[\lambda \cdot f]_z^\tau = |\lambda| \cdot [f]_z^\tau$*, i.e., $[\cdot]_z^\tau$ is homogeneous.*
(ii) $[f+g]_Z^\tau \leq [f]_Z^\tau + [g]_Z^\tau$ *and* $[\lambda \cdot f]_Z^\tau = |\lambda| \cdot [f]_Z^\tau$*, i.e., $[\cdot]_Z^\tau$ is a seminorm.*
(iii) $[f]_z^\tau - [g]_Z^\tau \leq [f+g]_z^\tau \leq [f]_z^\tau + [g]_Z^\tau$.
(iv) $|[f]_z^\tau - [g]_z^\tau| \leq [f-g]_Z^\tau$*. In particular,* $[f-g]_Z^\tau = 0$ *implies* $[f]_z^\tau = [g]_z^\tau$.
(v) $[f]_{Lip}^\tau = \left(\frac{1}{[f^{-1}]_{lip}^{1/\tau}}\right)^\tau$ *for f a bijection between Banach spaces E and F, respectively, with $f(0) = 0$.*

Given a lower characteristic $[\cdot]_z$, based on a fixed pair of seminorms over a t.v.s., one can define a corresponding spectrum $\sigma_z(f) := \{\lambda \in \mathbb{K} \mid [\lambda \cdot \mathrm{id}_E - f]_z = 0\}$. This gives five spectra for each of the lower characteristics.

$$\begin{aligned} \sigma_s(f) &:= \{\lambda \in \mathbb{K} \mid [\lambda \cdot \mathrm{id}_E - f]_s = 0\} \quad , \\ \sigma_d(f) &:= \{\lambda \in \mathbb{K} \mid [\lambda \cdot \mathrm{id}_E - f]_d = 0\} \quad , \\ \sigma_{db}(f) &:= \{\lambda \in \mathbb{K} \mid [\lambda \cdot \mathrm{id}_E - f]_{db} = 0\} \quad , \\ \sigma_b(f) &:= \{\lambda \in \mathbb{K} \mid [\lambda \cdot \mathrm{id}_E - f]_b = 0\} \quad , \\ \sigma_{lip}(f) &:= \{\lambda \in \mathbb{K} \mid [\lambda \cdot \mathrm{id}_E - f]_{lip} = 0\} \quad . \end{aligned}$$

Analogous spectra $\sigma_z^\tau(j,f) := \{\lambda \in \mathbb{K} \mid [\lambda \cdot j - f]_z^\tau = 0\}$ can be defined for τ-homogeneous operators $j, f\colon E \to F$.

$$\sigma_s^\tau(j,f) := \{\lambda \in \mathbb{K} \mid [\lambda \cdot j - f]_s^\tau = 0\} \quad ,$$
$$\sigma_d^\tau(j,f) := \{\lambda \in \mathbb{K} \mid [\lambda \cdot j - f]_d^\tau = 0\} \quad ,$$
$$\sigma_{db}^\tau(j,f) := \{\lambda \in \mathbb{K} \mid [\lambda \cdot j - f]_{db}^\tau = 0\} \quad ,$$
$$\sigma_b^\tau(j,f) := \{\lambda \in \mathbb{K} \mid [\lambda \cdot j - f]_b^\tau = 0\} \quad ,$$
$$\sigma_{lip}^\tau(j,f) := \left\{\lambda \in \mathbb{K} \mid [\lambda \cdot j - f]_{lip}^\tau = 0\right\} \quad .$$

By definition, $\sigma_z(f) = \sigma_z^1(\mathrm{id}_E, f)$.

2.4.2. *Noncompactness.* We define the *spectrum of noncompactness* by

$$\sigma_a(f) := \{\lambda \in \mathbb{K} \mid [\lambda \cdot \mathrm{id}_E - f]_a = 0\} \quad . \tag{69}$$

Completely similar to the other characteristics, we have

$$\sigma_a(f) \subseteq \{\lambda \in \mathbb{K} \mid [f]_a \leq |\lambda| \leq [f]_A\} \quad . \tag{70}$$

The *noncompactness spectral radius* is the number

$$r_a(f) := \sup\{|\lambda| \mid \lambda \in \sigma_a(f)\} \quad . \tag{71}$$

We obtain an analogous spectrum for τ-homogeneous operators $j, f\colon E \to F$.

$$\sigma_a^\tau(j,f) := \{\lambda \in \mathbb{K} \mid [\lambda \cdot j - f]_a^\tau = 0\} \quad . \tag{72}$$

By definition, $\sigma_a(f) = \sigma_a^1(\mathrm{id}_E, f)$.

2.4.3. *Quasi-Boundedness.* Let E and F be Banach spaces, and let $f \in \mathcal{C}(E,F)$ be a continuous operator. We consider the following characteristics

$$[f]_q := \liminf_{\|x\|_E \to \infty} \frac{\|f(x)\|_F}{\|x\|_E} \quad , \tag{73}$$

$$[f]_Q := \limsup_{\|x\|_E \to \infty} \frac{\|f(x)\|_F}{\|x\|_E} \quad . \tag{74}$$

This means that for every sequence $(x_n)_n$ in E with $\|x_n\|_E \to \infty$ $(n \to \infty)$, the corresponding sequence $y_n := \|f(x_n)\|_F / \|x_n\|_E$ has existing limes inferior or limes superior respectively, all equal to $[f]_q = \liminf_{n\to\infty} y_n$ or $[f]_Q = \limsup_{n\to\infty} y_n$ respectively.

We call f *quasi-bounded*, if $[f]_Q < \infty$.

From $[f]_q > 0$ it follows that there exists $\gamma > 0$ such that $\|f(x)\| \geq \gamma \cdot \|x\|$ for $\|x\|$ sufficiently large. Hence $[f]_q > 0$ implies f being coercive.

Analogous characteristics can be defined for τ-homogeneous operators $f\colon E \to F$.

$$[f]_q^\tau := \liminf_{\|x\|_E \to \infty} \frac{\|f(x)\|_F}{\|x\|_E^\tau} \quad , \tag{75}$$

$$[f]_Q^\tau := \limsup_{\|x\|_E \to \infty} \frac{\|f(x)\|_F}{\|x\|_E^\tau} \quad . \tag{76}$$

By definition, for $\tau = 1$, these notions coincide with the usual ones.

PROPOSITION 73. *For all continuous operators $f, g\colon E \to F$ between Banach spaces E and F, respectively, and all $\lambda \in \mathbb{K}$ the following properties hold.*

(i) $[\lambda \cdot f]_q = |\lambda| \cdot [f]_q$, *i.e.*, $[\cdot]_q$ *is homogeneous.*
(ii) $[f+g]_Q \leq [f]_Q + [g]_Q$ *and* $[\lambda \cdot f]_Q = |\lambda| \cdot [f]_Q$, *i.e.*, $[\cdot]_Q$ *is a seminorm.*
(iii) $[f]_q - [g]_Q \leq [f+g]_q \leq [f]_q + [g]_Q$.
(iv) $|[f]_q - [g]_q| \leq [f-g]_Q$. *In particular,* $[f-g]_Q = 0$ *implies* $[f]_q = [g]_q$.
(v) $[f]_Q = [f^{-1}]_q^{-1}$ *for* f *a homeomorphism.*
(vi) $[f]_b \leq [f]_q \leq [f]_Q \leq [f]_B$.

PROOF. We give a proof for the sake of completeness.

(i) We have

$$[\lambda\cdot f]_q = \liminf_{\|x\|_E\to\infty}\frac{\|\lambda\cdot f(x)\|_F}{\|x\|_E} = \liminf_{\|x\|_E\to\infty}|\lambda|\cdot\frac{\|f(x)\|_F}{\|x\|_E}$$
$$= |\lambda|\cdot\liminf_{\|x\|_E\to\infty}\frac{\|f(x)\|_F}{\|x\|_E} = |\lambda|\cdot[f]_q \quad .$$

(ii) The proof of $[\lambda\cdot f]_Q = |\lambda|\cdot[f]_Q$ is analogous to the one for $[\cdot]_q$.

$$[f+g]_Q = \limsup_{\|x\|_E\to\infty}\frac{\|(f+g)(x)\|_F}{\|x\|_E}$$
$$\leq \limsup_{\|x\|_E\to\infty}\frac{\|f(x)\|_F+\|g(x)\|_F}{\|x\|_E} \leq \limsup_{\|x\|_E\to\infty}\left(\frac{\|f(x)\|_F}{\|x\|_E}+\frac{\|g(x)\|_F}{\|x\|_E}\right)$$
$$\leq \limsup_{\|x\|_E\to\infty}\frac{\|f(x)\|_F}{\|x\|_E}+\limsup_{\|x\|_E\to\infty}\frac{\|g(x)\|_F}{\|x\|_E} = [f]_Q+[g]_Q \quad .$$

(iii) The second inequality is proved by

$$[f+g]_q = \liminf_{\|x\|_E\to\infty}\frac{\|(f+g)(x)\|_F}{\|x\|_E} \leq \liminf_{\|x\|_E\to\infty}\left(\frac{\|f(x)\|_F}{\|x\|_E}+\frac{\|g(x)\|_F}{\|x\|_E}\right)$$
$$\leq \liminf_{\|x\|_E\to\infty}\frac{\|f(x)\|_F}{\|x\|_E}+\limsup_{\|x\|_E\to\infty}\frac{\|g(x)\|_F}{\|x\|_E} = [f]_q+[g]_Q \quad .$$

The first inequality is a consequence of the second with

$$[f]_q = [(f+g)-g]_q \leq [f+g]_q+[-g]_Q = [f+g]_q+[g]_Q \quad .$$

(iv) We have $[f]_q-[g-f]_Q \leq [f+(g-f)]_q = [g]_q$. Hence, $[f]_q-[g]_q \leq [g-f]_q = [f-g]_Q$. We then also have $[g]_q-[f]_q \leq [g-f]_q = [f-g]_Q$.

(v) As f is a homeomorphism, we have $\|x\|_E\to\infty$ iff $\|f(x)\|_F\to\infty$. Otherwise, there would exist a bounded sequence such that f^{-1} is unbounded on this sequence. But by assumption, f^{-1} is continuous and hence bounded. Thus, we can argue

$$[f^{-1}]_Q = \limsup_{\|y\|_F\to\infty}\frac{\|f^{-1}(y)\|_E}{\|y\|_F} = \limsup_{\|f(x)\|_F\to\infty}\frac{\|f^{-1}(f(x))\|_E}{\|f(x)\|_F}$$
$$= \limsup_{\|x\|_E\to\infty}\frac{\|x\|_E}{\|f(x)\|_F} = \left(\liminf_{\|x\|_E\to\infty}\frac{\|f(x)\|_F}{\|x\|_E}\right)^{-1} = [f]_q^{-1} \quad .$$

(vi) This just follows from $\inf \leq \liminf \leq \limsup \leq \sup$.

$\square$

An analogous statement holds for τ-homogeneous operators.

PROPOSITION 74. *For all continuous operators $f,g\colon E\to F$ between Banach spaces E and F, respectively, and all $\lambda\in\mathbb{K}$, the following properties hold.*

(i) $[\lambda\cdot f]_q^\tau = |\lambda|\cdot[f]_q^\tau$, *i.e., $[\cdot]_q^\tau$ is homogeneous.*
(ii) $[f+g]_Q^\tau \leq [f]_Q^\tau+[g]_Q^\tau$ *and* $[\lambda\cdot f]_Q^\tau = |\lambda|\cdot[f]_Q^\tau$, *i.e., $[\cdot]_Q^\tau$ is a seminorm.*
(iii) $[f]_q^\tau-[g]_Q^\tau \leq [f+g]_q^\tau \leq [f]_q^\tau+[g]_Q^\tau$.
(iv) $\left|[f]_q^\tau-[g]_q^\tau\right| \leq [f-g]_Q^\tau$. *In particular, $[f-g]_Q^\tau = 0$ implies $[f]_q^\tau = [g]_q$.*
(v) $[f]_Q^\tau = \left(\frac{1}{[f^{-1}]_q^{1/\tau}}\right)^\tau$ *for f a homeomorphism.*
(vi) $[f]_b^\tau \leq [f]_q^\tau \leq [f]_Q^\tau \leq [f]_B^\tau$.

We define the *spectrum of quasi-boundedness* by

$$\sigma_q(f) := \{\lambda \in \mathbb{K} \mid [\lambda \cdot \mathrm{id}_E - f]_q = 0\} \quad . \tag{77}$$

Completely similar to the other characteristics, we have

$$\sigma_q(f) \subseteq \{\lambda \in \mathbb{K} \mid [f]_q \le |\lambda| \le [f]_Q\} \quad . \tag{78}$$

The *quasi-boundedness spectral radius* is the number

$$r_q(f) := \sup\{|\lambda| \mid \lambda \in \sigma_q(f)\} \quad . \tag{79}$$

An analogous spectrum can be defined for τ-homogeneous operators $j, f\colon E \to F$ by

$$\sigma_q^\tau(j, f) := \{\lambda \in \mathbb{K} \mid [\lambda \cdot j - f]_q = 0\} \quad . \tag{80}$$

By definition, $\sigma_q(f) = \sigma_q^1(\mathrm{id}_E, f)$.

The Q-, B-, and Lip-characteristics can be used to implement a very simple idea. Combining operators with different growth-rates yields the existence of an invariant and bounded set. This idea is formalized in the following lemma.

LEMMA 75. *Let E and F be Banach (or Fréchet) spaces, and let $f, g\colon E \to F$ be continuous operators. If $[f]_q > [g]_Q$, then there exists a nonempty, convex, closed, and bounded set $B \subseteq E$ such that*

$$f^{-1}(\overline{\mathrm{co}}(g(B) \cup \{0\})) \subseteq B \quad . \tag{81}$$

PROOF. Let b and c be real numbers such that $[g]_Q < b < c < [f]_q$. Hence, $\|g(x)\|_F \le b \cdot \|x\|_E$ and $\|f(x)\|_F \ge c \cdot \|x\|_E$ for all $x \in E$ with $\|x\|_E \ge r$ for a suitable $r > 0$. Set $g(B_r(E))$ is bounded, because $B_r(E)$ is bounded and g is continuous. Thus, there exists $R > 0$ such that $g(B_r(E)) \subseteq B_R(F)$. Combined, we obtain $\|g(x)\|_F \le R + b \cdot \|x\|_E$ for all $x \in E$. Set $\rho := R/(c - b)$. Then $g(B_\rho(E)) \subseteq B_{R+b\cdot\rho}(F)$. Let $x \in E$ with $f(x) \in B_{R+b\cdot\rho}(F)$. If $\|x\|_E > \rho$, then $R + b \cdot \rho \ge \|f(x)\|_F \ge c \cdot \|x\|_E > c \cdot \rho$, in contradiction to our choice of ρ. Hence, for $B := B_\rho(E)$ we have

$$f^{-1}(\overline{\mathrm{co}}(g(B) \cup \{0\})) \subseteq f^{-1}(\overline{\mathrm{co}}(B_{R+b\cdot\rho}(F))) = f^{-1}(B_{R+b\cdot\rho}(F)) \subseteq B_\rho(E) = B \quad .$$

Clearly, ball B is nonempty, convex, closed, and bounded. □

We only proved the lemma for the Q-characteristic. The proof is completely analogous for the B- and Lip-characteristics.

2.5. Dörfner Spectrum. In his Ph.D. thesis [**D9̈7**], Dörfner introduced and studied a spectrum for linearly-bounded operators.

Given two Banach spaces E and F, an operator $f\colon E \to F$ is *linearly bounded*, if $[f]_B < \infty$. Denote with $\mathcal{B}_{lin}(E, F)$ the class of linearly-bounded operators between E and F, and define $\mathcal{B}_{lin}(E) := \mathcal{B}_{lin}(E, E)$.

Given $f \in \mathcal{B}_{lin}(E)$, the *Dörfner resolvent set* is defined as

$$\rho_D(f) := \{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - f) \text{ is invertible and } r(f, \lambda) \in \mathcal{B}(E)\} \quad . \tag{82}$$

The *Dörfner spectrum* is defined as

$$\sigma_D(f) := \mathbb{K} \backslash \rho_D(f) \quad . \tag{83}$$

We note that $\sigma_D(f) = \Sigma(f) \cup \sigma_B(f)$ for $f \in \mathcal{B}_{lin}(E)$, because $[r(f, \lambda)]_B < \infty$ iff $[\lambda \cdot \mathrm{id}_E - f]_b = 1/[r(f, \lambda)]_B > 0$.

The *Dörfner spectral radius* is the number

$$r_D(f) := \sup\{|\lambda| \mid \lambda \in \sigma_D(f)\} \quad . \tag{84}$$

The original Dörfner spectrum is only defined for Banach spaces. We give one possible extension to arbitrary t.v.s. E. Define an operator $f\colon E \to E$ to be *linearly bounded*, if for all zero neighborhoods U there exists a $t > 0$ such that for all bounded sets B and $r > 0$ with $B \subseteq t \cdot U$, we have $f(B) \subseteq t \cdot r \cdot B$. Clearly, such f is bounded, because an arbitrary neighborhood U absorbs $f(B)$ for every given bounded set B. For E a Banach space, the two notions of linear boundedness coincide: f is linearly bounded in this sense iff $[f]_B < \infty$. We define $\mathcal{B}_{lin}(E)$ as the class of linearly-bounded operators, defined on an arbitrary t.v.s. E. Now, the notions of Dörfner resolvent set, spectrum, and spectral radius extend by using our newly-defined class $\mathcal{B}_{lin}(E)$.

2.6. Kačurovskiĭ Spectrum. Let E and F be Banach spaces, and let $f\colon E \to F$ be a continuous operator, i.e., $f \in \mathcal{C}(E,F)$. We call f *Lipschitz-continuous*, if $[f]_{Lip} < \infty$. We denote the class of Lipschitz-continuous operators by $\mathcal{C}_{Lip}(E,F)$, and define $\mathcal{C}_{Lip}(E) := \mathcal{C}_{Lip}(E,E)$ for short.

Kačurovskiĭ [**Kač69**] introduced his spectrum for Lipschitz-continuous operators $f \in \mathcal{C}_{Lip}(E)$ over Banach spaces E in 1969. The *Kačurovskiĭ resolvent set* is defined by

$$\rho_K(f) := \{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - f) \text{ is bijective and } r(f,\lambda) \in \mathcal{C}_{Lip}(E)\} \quad . \tag{85}$$

Its complement

$$\sigma_K(f) := \mathbb{K} \backslash \rho_K(f) \tag{86}$$

is called *Kačurovskiĭ spectrum*.

The *Kačurovskiĭ spectral radius* is the number

$$r_K(f) := \sup\{|\lambda| \mid \lambda \in \sigma_K(f)\} \quad . \tag{87}$$

2.7. Neuberger Spectrum. Let E and F be Banach spaces, and let $f\colon E \to F$ be an operator. Recall that f is *differentiable* at $x_0 \in E$, if there exists a linear and continuous operator $g\colon E \to F$ such that

$$\lim_{\|\Delta\|_E \to 0} \frac{\|f(x_0+\Delta) - f(x_0) - g(\Delta)\|_F}{\|\Delta\|_E} = 0 \quad . \tag{88}$$

As g is uniquely defined by above equation, we denote it with $f'|_{x_0}$. If f is differentiable at every point $x \in E$, and if the map $x \mapsto f'|_x\colon E \to \mathcal{C}(E,F)$ is continuous, then we say that f is *continuously differentiable*. We denote the class of continuously-differentiable operators by $\mathcal{C}^1(E,F)$, and $\mathcal{C}^1(E) := \mathcal{C}^1(E,E)$ for short.

For continuously-differentiable operators $f \in \mathcal{C}^1(E)$ over Banach spaces E, Neuberger [**Neu69**] introduced his spectrum in 1969. The *Neuberger resolvent set* is defined by

$$\rho_N(f) := \left\{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - f) \text{ is bijective and } r(f,\lambda) \in \mathcal{C}^1(E)\right\} \quad . \tag{89}$$

Its complement

$$\sigma_N(f) := \mathbb{K} \backslash \rho_N(f) \tag{90}$$

is called *Neuberger spectrum*.

The *Neuberger spectral radius* is the number

$$r_N(f) := \sup\{|\lambda| \mid \lambda \in \sigma_N(f)\} \quad . \tag{91}$$

2.8. FMV Spectrum. A well-known spectrum in Nonlinear Spectral Theory is the one introduced by Furi, Martelli, and Vignoli (FMV for short) in 1978 [**FMV78**]. Before we give its definition and prove basic properties, we consider proper and stably-solvable operators.

2.8.1. *Proper Operators.* Recall that $f\colon E \to F$ is *proper*, if the pre-image $f^{-1}(M)$ of each compact set M in F is compact in E. Operator F is called *proper on bounded and closed sets*, if the pre-image $f^{-1}(M)$ of each closed and bounded set M in F is compact in E. Furthermore, f is called *ray-proper*, if the pre-image $f^{-1}([0,y])$ of each segment $[0,y] := \{ty \mid t \in [0,1]\} \subseteq F$ is compact in E. Operator f is called *ray-invertible*, if for each $y \in F$ there exists a continuous path $\gamma\colon [0,1] \to E$ such that $f(\gamma(t)) = ty$ for $t \in [0,1]$.

The proofs of the following statements can be found in [**ADPV04**, Chapter 3, Thms. 3.1, 3.2 and Prop. 3.1].

PROPOSITION 76. *An operator is proper iff it is closed and ray-proper. In particular, every proper operator is ray-proper.*

PROPOSITION 77. *Let $f\colon E \to F$ be an operator with $f(0) = 0$. Then the following statements are equivalent.*

(i) *f is a global homeomorphism.*
(ii) *f is a local homeomorphism and proper.*
(iii) *f is a local homeomorphism and ray-proper.*
(iv) *f is a local homeomorphism and closed.*
(v) *f is a local homeomorphism and ray-invertible.*

PROPOSITION 78. *For a continuous operator $f\colon E \to F$ between Banach spaces E and F, respectively, we have*

(i) *If $[f]_a > 0$, then f is proper on closed and bounded sets.*
(ii) *If $[f]_a > 0$ and $[f]_q > 0$, then f is proper.*

2.8.2. *AQ-Stably-Solvable Operators.* The characteristics $[\cdot]_a$ and $[\cdot]_A$ have already been defined in Chapter 2, Eq. (19) and (20), respectively. Analogous notions can be defined for τ-homogeneous operators $f\colon E \to F$.

$$[f]_a^\tau := \sup\{\gamma > 0 \mid \alpha(f(M)) \geq \gamma \cdot \alpha(M)^\tau, M \text{ bounded}\} \quad , \tag{92}$$

$$[f]_A^\tau := \inf\{\gamma > 0 \mid \alpha(f(M)) \leq \gamma \cdot \alpha(M)^\tau, M \text{ bounded}\} \quad . \tag{93}$$

By definition, $[f]_a^1 = [f]_a$ and $[f]_A^1 = [f]_A$, respectively.

Their properties are similar to the usual ones.

PROPOSITION 79. *For all bounded, τ-homogeneous operators $f, g\colon E \to F$ between infinite-dimensional, metrizable t.v.s. E and F, and all $\lambda \in \mathbb{K}$, the following properties hold.*

(i) *$[\lambda \cdot f]_a^\tau = |\lambda| \cdot [f]_a^\tau$, i.e., $[\cdot]_a^\tau$ is homogeneous.*
(ii) *$[f+g]_A^\tau \leq [f]_A^\tau + [g]_A^\tau$ and $[\lambda \cdot f]_A^\tau = |\lambda| \cdot [f]_A^\tau$, i.e., $[\cdot]_A^\tau$ is a seminorm.*
(iii) *$[f]_a^\tau - [g]_A^\tau \leq [f+g]_a^\tau \leq [f]_a^\tau + [g]_A^\tau$.*
(iv) *$|[f]_a^\tau - [g]_a^\tau| \leq [f-g]_A^\tau$. In particular, $[f-g]_A^\tau = 0$ implies $[f]_a^\tau = [g]_a^\tau$.*
(v) *$[f]_A^\tau = \left([f^{-1}]_a^{1/\tau}\right)^\tau$ for f a homeomorphism.*
(vi) *$[f]_a^\tau \leq [f]_A^\tau$.*

Let us combine the A- and Q-characteristics. Define $[f]_{AQ} := \max\{[f]_A, [f]_Q\}$, and $[f]_{aq} := \max\{[f]_a, [f]_q\}$, respectively. These will be nice shorthands in the following definitions.

The notion of stably-solvable operator was introduced by FMV [**FMV76**] in 1976. Let E and F be two Banach spaces. We call an operator $f\colon E \to F$ *k-AQ stably solvable*, if f is continuous, and if for every operator $g\colon E \to F$ with $[g]_{AQ} \leq k$, the equation $f(x) = g(x)$ has a solution $x \in E$. In particular, for $k = 0$, g is compact, and we say that f is *AQ stably solvable*.

Define a *measure of solvability* of f by

$$\mu(f) := \inf\{k \geq 0 \mid f \text{ is not } k\text{-AQ stably solvable }\} \quad . \tag{94}$$

The corresponding *spectrum of AQ stable solvability* is defined as

(95)
$$\begin{aligned}\sigma_\delta(f) &:= \{\lambda \in \mathbb{K} \mid \mu(\lambda \cdot \mathrm{id}_E - f) = 0\} \\ &= \{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - f) \text{ is not AQ stably solvable } \} \quad .\end{aligned}$$

Analogous notions can be defined for τ-homogeneous operators $f \colon E \to F$. First of all, define $[f]^\tau_{AQ} := \max\{[f]^\tau_A, [f]^\tau_Q\}$, and $[f]^\tau_{aq} := \max\{[f]^\tau_a, [f]^\tau_q\}$, respectively. Secondly, f is called (k,τ)-*AQ stably solvable*, if f is continuous, and if for every operator $g \colon E \to F$ with $[g]^\tau_{AQ} \le k$, the equation $f(x) = g(x)$ has a solution $x \in E$. Again, for $k = 0$, g is compact, and we say that f is (τ)-*AQ stably solvable*.

The corresponding (τ)-*measure of solvability* of f is defined by

(96)
$$\mu^\tau(f) := \inf\{k \ge 0 \mid f \text{ is not } (k,\tau)\text{-AQ stably solvable } \} \quad .$$

By definition, $\mu(f) = \mu^1(f)$.

For τ-homogeneous operators $j, f \colon E \to F$, the corresponding *spectrum of* (τ)-*AQ-stable solvability* is defined as

(97)
$$\begin{aligned}\sigma^\tau_\delta(j,f) &:= \{\lambda \in \mathbb{K} \mid \mu^\tau(\lambda \cdot j - f) = 0\} \\ &= \{\lambda \in \mathbb{K} \mid (\lambda \cdot j - f) \text{ is not } (\tau)\text{-AQ stably solvable } \} \quad .\end{aligned}$$

By definition, $\sigma_\delta(f) = \sigma^1_\delta(\mathrm{id}_E, f)$.

2.8.3. *FMV-Regular Operators.* Based on their prior work on stably-solvable operators, FMV introduced FMV-regular operators and a corresponding spectrum in [**FMV78**]. An operator $f \colon E \to F$ between Banach spaces E and F, respectively, is called *FMV-regular*, if f is AQ-stably solvable and if $[f]_{aq} > 0$.

The *FMV resolvent set* is defined by

(98)
$$\rho_{FMV}(f) := \{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - f) \text{ is FMV regular } \} \quad .$$

Its complement

(99)
$$\sigma_{FMV}(f) := \mathbb{K} \backslash \rho_{FMV}(f)$$

is called *FMV spectrum*.

The *FMV spectral radius* is the number

(100)
$$r_{FMV}(f) := \sup\{|\lambda| \mid \lambda \in \sigma_{FMV}(f)\} \quad .$$

By definition, we have the following subdivision of the FMV spectrum.

(101)
$$\sigma_{FMV}(f) = \sigma_a(f) \cup \sigma_q(f) \cup \sigma_\delta(f)$$

An analogous spectrum can be defined for τ-homogeneous operators $j, f \colon E \to F$. Operator f is called (τ)-*FMV-regular*, if f is (τ)-AQ-stably solvable and if $[f]^\tau_{aq} > 0$. Then define the (τ)-*FMV spectrum*

(102)
$$\sigma^\tau_{FMV}(j,f) := \{\lambda \in \mathbb{K} \mid (\lambda \cdot j - f) \text{ is not } (\tau)\text{-FMV regular } \} \quad .$$

By definition, $\sigma_{FMV}(f) = \sigma^1_{FMV}(\mathrm{id}_E, f)$.

As above, we have the following subdivision of the (τ)-FMV spectrum:

$$\sigma^\tau_{FMV}(j,f) = \sigma^\tau_a(j,f) \cup \sigma^\tau_q(j,f) \cup \sigma^\tau_\delta(j,f) \quad .$$

2.9. Feng Spectrum. Instead of taking the Q-characteristic as in the FMV spectrum, one can also use the B-one to control growth. Both characteristics are used in order to obtain invariant, bounded sets. The construction with the B-characteristic gives another spectrum, similarly-defined as the FMV spectrum, but containing the eigenvalues of the operator. There is only one complication regarding the B-characteristic one has to cope with, when defining stably solvability, namely the case that $[g]_B = 0$ means $g \equiv 0$, which is not very useful. The definition due to Feng overcomes this hurdle by localization. Unfortunately, this introduces a complication via a boundary-value condition.

2.9.1. *Epi Operators.* Let us combine the A- and B-characteristics this time. Define $[f]_{AB} := \max\{[f]_A, [f]_B\}$, and $[f]_{ab} := \max\{[f]_a, [f]_b\}$, respectively. Again, these will be nice shorthands in the following definitions.

Analogously to AQ stably solvability, one could define: Let E and F be two Banach spaces. We call an operator $f\colon E \to F$ *k-AB stably solvable*, if f is continuous, and if for every operator $g\colon E \to F$ with $[g]_{AB} \leq k$, the equation $f(x) = g(x)$ has a solution $x \in E$. In particular, for $k = 0$, g is compact.

Unfortunately, this definition has a problem: $g \equiv 0$ in case $[g]_{AB} = 0$. Hence, a different – local – approach is needed, see below.

For a Banach space E denote with $\mathcal{O}(E)$ the family of all nonempty, open, bounded, and connected subsets of E. Given $U \in \mathcal{O}(E)$, we call an operator $f\colon \overline{U} \to F$ *k-epi* on $\overline{U}$, if f is continuous, $f(x) \neq 0$ on ∂U, and if for every operator $g\colon \overline{U} \to F$ with $g(x) = 0$ on ∂U and $[g]_A \leq k$, the equation $f(x) = g(x)$ has a solution $x \in U$. In particular, for $k = 0$, g is compact, and we say that f is *epi* on $\overline{U}$.

Define a *measure of solvability* of f by

$$\nu(f) := \inf_{U \in \mathcal{O}(E)} \nu_U(f) \text{ , where} \tag{103}$$

$$\nu_U(f) := \inf\{k \geq 0 \mid f \text{ is not } k\text{-epi on } \overline{U}\} \quad . \tag{104}$$

Proposition 80. *For all continuous operators $f\colon E \to F$, we have*

$$\mu(f) \leq \nu(f) \quad . \tag{105}$$

Proof. In case $\nu(f) = \infty$, there is nothing to prove. Let k be arbitrary with $\nu(f) < k$. Let $U \in \mathcal{O}(E)$ and operator $g\colon \overline{U} \to F$ be arbitrary such that $[g]_A \leq k$, $g(x) = 0$ on ∂U, and $f(x) \neq g(x)$ for all $x \in \overline{U}$. Define the extension $\tilde{g}\colon E \to F$ by $\tilde{g}(x) := g(x)$ on $\overline{U}$ and $\tilde{g}(x) := 0$ on $\overline{E \backslash U}$. It satisfies $[\tilde{g}]_A \leq k$, $[\tilde{g}]_Q = 0$, and $f(x) \neq \tilde{g}(x)$ for all $x \in E$, because $f(x) \neq 0$ for $x \neq 0$. This gives $\mu(f) \leq k$. □

This defines a spectrum, given by

$$\sigma_\nu(f) := \{\lambda \in \mathbb{K} \mid \nu(\lambda \cdot \mathrm{id}_E - f) = 0\} \quad . \tag{106}$$

Analogous notions can be defined for τ-homogeneous operators $j, f\colon E \to F$. Given $U \in \mathcal{O}(E)$, we call an operator $f\colon \overline{U} \to F$ *(k,τ)-epi* on $\overline{U}$, if f is continuous, $f(x) \neq 0$ on ∂U, and if for every operator $g\colon \overline{U} \to F$ with $g(x) = 0$ on ∂U and $[g]_A^\tau \leq k$, the equation $f(x) = g(x)$ has a solution $x \in U$. In particular, for $k = 0$, g is compact, and we say that f is *(τ)-epi* on $\overline{U}$.

The corresponding *(τ)-measure of solvability* of f is defined by

$$\nu^\tau(f) := \inf_{U \in \mathcal{O}(E)} \nu_U^\tau(f) \text{ , where} \tag{107}$$

$$\nu_U^\tau(f) := \inf\{k \geq 0 \mid f \text{ is not } (k,\tau)\text{-epi on } \overline{U}\} \quad . \tag{108}$$

By definition, $\nu(f) = \nu^1(f)$.

Proposition 81. *For all continuous and τ-homogeneous operators $f\colon E \to F$, we have*

$$\mu^\tau(f) \leq \nu^\tau(f) \quad . \tag{109}$$

This also defines a spectrum, given by

$$\sigma_\nu^\tau(j, f) := \{\lambda \in \mathbb{K} \mid \nu^\tau(\lambda \cdot j - f) = 0\} \quad . \tag{110}$$

By definition, $\sigma_\nu(f) = \sigma_\nu^1(\mathrm{id}_E, f)$.

2.9.2. *Feng-Regular Operators.* An operator $f\colon E \to F$ between Banach spaces E and F, respectively, is called *Feng-regular*, if $[f]_{ab} > 0$ and if $f\colon \overline{U} \to F$ is epi on all $\overline{U}$ for all $U \in \mathcal{O}(E)$.

The *Feng resolvent set* is defined by

$$\rho_F(f) := \{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - f) \text{ is Feng-regular }\} \quad . \tag{111}$$

Its complement

$$\sigma_F(f) := \mathbb{K} \backslash \rho_F(f) \tag{112}$$

is called *Feng spectrum*.

The *Feng spectral radius* is the number

$$r_F(f) := \sup\{|\lambda| \mid \lambda \in \sigma_F(f)\} \quad . \tag{113}$$

By definition, we have the following subdivision of the Feng spectrum.

$$\sigma_F(f) = \sigma_a(f) \cup \sigma_b(f) \cup \sigma_\nu(f) \quad . \tag{114}$$

As $\sigma_q(f) \subseteq \sigma_b(f)$ and $\sigma_\delta(f) \subseteq \sigma_\nu(f)$, we have

$$\sigma_{FMV}(f) \subseteq \sigma_F(f) \quad . \tag{115}$$

Analogous notions can be defined for τ-homogeneous operators $j, f\colon E \to F$. Operator f is called (τ)-*Feng-regular*, if $[f]^\tau_{ab} > 0$ and if $f\colon \overline{U} \to F$ is (τ)-epi on all $\overline{U}$ for all $U \in \mathcal{O}(E)$.

The (τ)-*Feng spectrum* is defined as

$$\sigma^\tau(j, f) := \{\lambda \in \mathbb{K} \mid (\lambda \cdot j - f) \text{ is not } (\tau)\text{-Feng-regular }\} \quad . \tag{116}$$

By definition, $\sigma_F(f) = \sigma^1_F(\mathrm{id}_E, f)$.

As above, we have the following subdivision of the (τ)-Feng spectrum.

$$\sigma^\tau_F(j, f) = \sigma^\tau_a(j, f) \cup \sigma^\tau_b(j, f) \cup \sigma^\tau_\nu(j, f) \quad . \tag{117}$$

As $\sigma^\tau_q(j, f) \subseteq \sigma^\tau_b(j, f)$ and $\sigma^\tau_\delta(j, f) \subseteq \sigma^\tau_\nu(j, f)$, we have

$$\sigma^\tau_{FMV}(j, f) \subseteq \sigma^\tau_F(j, f) \quad . \tag{118}$$

The following proposition eases a proof that an operator is epi.

Proposition 82. *For a τ-homogeneous operator $f\colon E \to F$, we have that f is epi on every $U \in \mathcal{O}(E)$ iff f is epi on some $U \in \mathcal{O}(E)$.*

Proof. Assume that f is epi on some $\overline{V}$, $V \in \mathcal{O}(E)$. By definition, $f(x) \neq 0$ on ∂V, and for all compact operators $g\colon \overline{V} \to F$ with $g(x) = 0$ on ∂V, the equation $f(x) = g(x)$ has a solution in V. Choose $r > 0$ with $B(0, r) \subseteq V$. This is possible, because V is open. Then f is also epi on $\overline{B(0, r)}$ (one just uses trivial extensions in the argument). Let $U \in \mathcal{O}(E)$ be arbitrary. Now choose $R > 0$ such that $U \subseteq \underline{B(0, R)}$. This is possible, because every U is bounded. We show that f is epi on $\overline{B(0, R)}$. Then it is also epi on subset U. For this, let $h\colon \overline{B(0, R)} \to F$ be an arbitrary compact operator with $h(x) = 0$ on the sphere $\partial B(0, R) = S(0, R)$. Define operator $g\colon \overline{B(0, r)} \to F$ by

$$g(x) := \left(\frac{r}{R}\right)^\tau \cdot h\left(\frac{R}{r} \cdot x\right) \quad .$$

Then g is compact as h is compact, and $g(x) = 0$ on $S(0, r)$. Hence, equation $f(x) = g(x)$ has a solution $\tilde{x}$ in $B(0, r)$. Then $\frac{r}{R} \cdot \tilde{x}$ is a solution to $f(x) = h(x)$ in $B(0, R)$, because f is τ-homogeneous. □

For τ-homogeneous operators, we can say much more on the relationship of the different spectra.

THEOREM 83. *Given τ-homogeneous operators $j, f\colon E \to F$, we have*

$$\sigma_p(j,f) \subseteq \sigma^\tau_{FMV}(j,f) = \sigma^\tau_F(j,f) \quad . \tag{119}$$

PROOF. We already know that $\sigma^\tau_{FMV}(j,f) \subseteq \sigma^\tau_F(j,f)$. As operators j and f are τ-homogeneous operators, we also obtain the inclusion $\sigma_p(j,f) \subseteq \sigma^\tau_q(j,f)$: given nontrivial solution x_0 for $\lambda \cdot j(x) - f(x) = 0$, consider the sequence $(n \cdot x_0)$. This shows that $[\lambda \cdot j - f]_q = 0$. The inclusion implies $\sigma_p(j,f) \subseteq \sigma^\tau_{FMV}(j,f)$.

The proof of $\sigma^\tau_F(j,f) \subseteq \sigma^\tau_{FMV}(j,f)$ is done in two steps. We use the help of the *large τ phantom* $\Theta^\tau(j,f)$. We do not deep-dive into Väth's phantom theory. For more information, see e.g., [**ADPV04**, Ch. 8]. We only prove that $\sigma^\tau_F(j,f) \subseteq \Theta^\tau(j,f)$ and $\Theta^\tau(j,f) \subseteq \sigma^\tau_{FMV}(j,f)$

It holds $\sigma^\tau_F(j,f) \subseteq \Theta^\tau(j,f)$: Suppose that $\lambda \notin \Theta^\tau(j,f)$. Then there exists $U \in \mathcal{O}(E)$ such that operator $(\lambda \cdot j - f)$ is (τ)-epi and thus epi on $\overline{U}$ and $[(\lambda \cdot j - f)|_{\overline{U}}]^\tau_a > 0$. By Proposition 82, we conclude that $(\lambda \cdot j - f)$ is also epi on the open unit ball $B(0,1)$ and $[(\lambda \cdot j - f)|_{\overline{B(0,1)}}]^\tau_a > 0$. The statement is proved if we show that $\lambda \notin \sigma^\tau_b(j,f)$.

Assume for a contradiction that $\lambda \in \sigma^\tau_b(j,f)$. Then there exists a sequence $(x_n)_n$ in $E\backslash\{0\}$ such that $\|(\lambda \cdot j - f)(x_n)\|_F \le \|x_n\|^\tau_E/n$. Normalizing this sequence, i.e., defining $e_n := x_n/\|x_n\|_E$, we obtain

$$\|(\lambda \cdot j - f)(e_n)\|_F = \frac{\|(\lambda \cdot j - f)(x_n)\|_F}{\|x_n\|^\tau_E} \le \frac{1}{n} \to 0 \quad ,$$

because operator $(\lambda \cdot j - f)$ is τ-homogeneous. Define set $M := \{e_1, e_2, \ldots\}$. Then $[(\lambda \cdot j - f)|_{\overline{B(0,1)}}]^\tau_a \cdot \alpha(M)^\tau \le \alpha((\lambda \cdot j - f)(M)) = 0$, also implying $\alpha(M) = 0$. As M is precompact and E is a Banach space, M is sequentially compact. Hence, there exists a strongly-convergent subsequence $e_{n_k} \to e$ in the sphere $S(0,1)$ $(k \to \infty)$. But then $(\lambda \cdot j - f)(e) = 0$ by continuity, a contradiction that $(\lambda \cdot j - f)$ is epi on $B(0,1)$.

It holds $\Theta^\tau(j,f) \subseteq \sigma^\tau_{FMV}(j,f)$: If $(\lambda \cdot j - f)$ is (τ)-FMV regular, then $[\lambda \cdot j - f]_a > 0$ and $[\lambda \cdot j - f]_q > 0$, and $(\lambda \cdot j - f)$ is (τ)-AQ stably solvable. We need to show that it is (τ)-epi on $\overline{U}$ for all $U \in \mathcal{O}(E)$. By Proposition 82, it suffices to show this for an open ball $U = B(0,r)$ for some $r > 0$. As $[\lambda \cdot j - f]_q > 0$, there exists an $r > 0$ with $(\lambda \cdot j - f)(x) \neq 0$ for all $\|x\|^\tau \ge r$. Let $g\colon \overline{U} \to F$ be compact with $g(x) = 0$ on $S(0,r) = \partial U$. Let $\tilde{g}$ be the trivial extension of g to E. As $(\lambda \cdot j - f)$ is (τ)-AQ stably solvable, there exists a solution x to equation $(\lambda \cdot j - f)(x) = \tilde{g}(x)$. We must have $\|x\|^\tau < r$ because of $(\lambda \cdot j - f)(x) \neq 0$ for $\|x\|^\tau \ge r$. Hence, $x \in U$ and x is even a solution of $(\lambda \cdot j - f)(x) = g(x)$. As g was arbitrary, operator $(\lambda \cdot j - f)$ is (τ)-epi on U. ☐

Recall that an *odd* operator f is one which satisfies $f(-x) = -f(x)$ for all x.

THEOREM 84. *Given odd and τ-homogeneous operators $j, f\colon E \to F$ such that j is a homeomorphism with $[j]_a > 0$ and f is compact, then we even have*

$$\sigma_p(j,f)\backslash\{0\} = \sigma^\tau_{FMV}(j,f)\backslash\{0\} = \sigma^\tau_F(j,f)\backslash\{0\} \quad . \tag{120}$$

PROOF. By the preceding Theorem 83, if suffices to prove that $\sigma^\tau_F(j,f)\backslash\{0\} \subseteq \sigma_p(j,f)\backslash\{0\}$.

Let $\lambda \neq 0$ be in the complement of $\sigma_p(j,f)$. As f is compact and $[j]^\tau_a > 0$, we have $[\lambda \cdot j - f]^\tau_a = |\lambda| \cdot [j]^\tau_a > 0$. The same argument as in the preceding theorem shows that $[\lambda \cdot j - f]^\tau_b > 0$. We need to show that that operator $(\lambda \cdot j - f)$ is epi on $\overline{U}$ for every $U \in \mathcal{O}(E)$. By Proposition 82, it suffices to show this for a specific U, we choose open unit ball $U := B(0,1)$ in E. Let $h\colon \overline{U} \to Y$ be an arbitrary compact operator with $h(x) = 0$ on the unit sphere $\partial U = S(0,1)$. Define operators $g_0, g_1\colon \overline{U} \to Y$ by $g_0 := j^{-1}((1/\lambda) \cdot f)$ and $g_1 := j^{-1}((1/\lambda) \cdot (f + h))$, respectively.

Both operators are compact, because f and $f+h$ are compact. Operator g_0 is odd as a composition of odd operators. In addition, $g_0(x) \neq x$ and $g_0(x) = g_1(x)$ for all $x \in \partial U = S(0,1)$. By the Antipodal Theorem of Borsuk and the homotopy property of the Leray-Schauder degree[6], we have

$$\deg(\mathrm{id}_E - g_1, U, 0) = \deg(\mathrm{id}_E - g_0, U, 0) \equiv 1 \pmod 2 \quad . \tag{121}$$

Hence, there exists $x \in U$ with $(\mathrm{id}_E - g_1)(x) = 0$, i.e., $\lambda \cdot j(x) = f(x) + h(x)$. But this means that equation $(\lambda \cdot j - f)(x) = h(x)$ has a solution for arbitrary compact h. Hence, operator $(\lambda \cdot j - f)$ is epi on $\overline{U}$. □

Reformulating above theorem, we obtain a nonlinear Fredholm alternative.

Theorem 85 (Nonlinear Fredholm Alternative). *Let E and F be infinite-dimensional Banach spaces, let $j, f\colon E \to F$ be odd, τ-homogeneous operators ($\tau > 0$) such that j is a homeomorphism and f is compact. Then the following statements are equivalent.*

(i) *The eigenvalue problem $f(x) = \lambda \cdot j(x)$ has only the trivial solution.*
(ii) *Operator $(\lambda \cdot j - f)$ is (τ)-FMV-regular, i.e., it is (τ)-AQ stably solvable, and $[\lambda \cdot j - f]^{\tau}_{aq} > 0$.*
(iii) *Operator $(\lambda \cdot j - f)$ is (τ)-Feng-regular, i.e., it is epi on $\overline{U}$ for all $U \in \mathcal{O}(E)$, and $[\lambda \cdot j - f]^{\tau}_{aq} > 0$.*

3. Restriction to Linear Operators

3.1. Rhodius Spectrum. The Rhodius spectrum equals the classical spectrum for linear operators: Let $u\colon E \to E$ be linear and continuous. On the one hand, if $\lambda \in \rho_R(u)$, then $r(u,\lambda)$ exists. Hence, $\lambda \in \rho(u)$. On the other hand, if $\lambda \in \rho(u)$, then $r(u,\lambda)$ exists. As $(\lambda \cdot \mathrm{id}_E - u)$ is bijective and continuous, $r(u,\lambda)$ is continuous by the Continuous-Inverse property of Banach / barreled spaces (Def. 8.5). Hence, $\lambda \in \rho_R(u)$.

3.2. Dörfner Spectrum. The Dörfner spectrum equals the classical spectrum for linear operators: Let $u\colon E \to E$ be linear and linearly bounded. On the one hand, if $\lambda \in \rho_D(u)$, then $r(u,\lambda)$ exists. Hence, $\lambda \in \rho(u)$. On the other hand, if $\lambda \in \rho(u)$, then $r(u,\lambda)$ exists. Linear operator $(\lambda \cdot \mathrm{id}_E - u)$ is bijective and continuous, because E is bornological as a Banach space, and u is continuous as a linearly-bounded operator in a bornological space E. Then $r(u,\lambda)$ is continuous by the Continuous-Inverse property of Banach / barreled spaces (Def. 8.5). Thus, $r(u,\lambda)$ is linearly-bounded, showing $\lambda \in \rho_R(u)$.

3.3. Kačurovskiĭ Spectrum. The Kačurovskiĭ spectrum equals the classical spectrum for linear operators: Let $u\colon E \to E$ be linear and Lipschitz continuous. On the one hand, if $\lambda \in \rho_K(u)$, then $r(u,\lambda)$ exists. Hence, $\lambda \in \rho(u)$. On the other hand, if $\lambda \in \rho(u)$, then $r(u,\lambda)$ exists. Linear operator $(\lambda \cdot \mathrm{id}_E - u)$ is bijective and continuous, because u is continuous as a Lipschitz-continuous linear operator. Then $r(u,\lambda)$ is continuous by the Continuous-Inverse property of Banach / barreled spaces (Def. 8.5). Operator $r(u,\lambda)$ is linearly-bounded and thus Lipschitz continuous, showing $\lambda \in \rho_K(u)$.

[6]We did not present the Leray-Schauder degree deg in this thesis, because this would have led us too far astray. The tedious and lengthy construction of these degrees, up to l.c.s. and even abstract neighborhood retracts, can be found in [**GD03**, §8–§17]. A quick overview is presented in [**ADPV04**, Sec. 3.5]. These degrees generalize the Schauder-Tychonoff fixed-point theorem in the sense of quantifying the number of solutions.

3.4. Neuberger Spectrum. The Neuberger spectrum equals the classical spectrum for linear operators: Let $u\colon E \to E$ be linear and continuously differentiable. On the one hand, if $\lambda \in \rho_N(u)$, then $r(u,\lambda)$ exists. Hence, $\lambda \in \rho(u)$. On the other hand, if $\lambda \in \rho(u)$, then $r(u,\lambda)$ exists. As $(\lambda \cdot \mathrm{id}_E - u)$ is bijective and continuously differentiable, $r(u,\lambda)$ is continuously differentiable by the Inverse Function Theorem in Banach spaces.[7] Hence, $\lambda \in \rho_N(u)$.

3.5. FMV Spectrum. The FMV spectrum equals the classical spectrum for linear operators: If u is linear and AQ stably solvable, then u is surjective. This follows easily from the fact that all constant maps $g\colon x \mapsto y$ obey $[g]_{AQ} = 0$, and thus $u = g$ has a solution.

Let $u\colon E \to F$ be linear and surjective. By Theorem 58 (Michael Selection), there exists a continuous selection $s\colon F \to E$ such that $u \circ s = \mathrm{id}_F$. Let $g\colon E \to F$ be compact. Then $g \circ s$ is compact, too. By Theorem 35 (Schauder-Tychonoff), map $g \circ s\colon F \to F$ has a fixed point $y \in F$. Define $x := s(y)$. Then $u(x) = u(s(y)) = y = (g \circ s)(y) = g(x)$. Hence, u is AQ stably solvable.

3.6. Feng Spectrum. The Feng spectrum equals the classical spectrum for linear operators. If u is linear, then it is 1-homogeneous. Hence, by Theorem 83, $\sigma_F(u) = \sigma_F^1(\mathrm{id}_E, u) = \sigma_{FMV}^1(\mathrm{id}_E, u) = \sigma_{FMV}(u) = \sigma(u)$.

4. Nonemptyness

4.1. Mapping Spectrum. The mapping spectrum may be empty. To see this, let $E := \mathbb{C}^2$, and consider the following (nonlinear but additive) operator $f\colon E \to E$, $f(z,w) := (\overline{w}, \mathrm{i}\overline{z})$. For every $\lambda \in \mathbb{C}$, map $(\lambda \cdot \mathrm{id}_E - f)(z,w) = (\lambda z - \overline{w}, \lambda w - \overline{z})$ is a bijection on E with inverse

$$(\lambda \cdot \mathrm{id}_E - f)^{-1}(\zeta,\omega) := \left(\frac{\overline{\lambda}\zeta + \overline{\omega}}{\mathrm{i} + |\lambda|^2}, \frac{\overline{\lambda}\omega + \mathrm{i}\overline{\zeta}}{\mathrm{i} - |\lambda|^2} \right) \quad . \tag{122}$$

This follows from the simple calculation

$$\begin{aligned}
(\lambda \cdot \mathrm{id}_E - f)^{-1}\left((\lambda \cdot \mathrm{id}_E - f)(z,w)\right) &= (\lambda \cdot \mathrm{id}_E - f)^{-1}\left(\lambda z - \overline{w}, \lambda w - \overline{z}\right) \\
&= \left(\frac{\overline{\lambda}(\lambda z - \overline{w}) + \overline{(\lambda w - \overline{z})}}{\mathrm{i} + |\lambda|^2}, \frac{\overline{\lambda}(\lambda w - \overline{z}) + \mathrm{i}\overline{(\lambda z - \overline{w})}}{\mathrm{i} - |\lambda|^2} \right) \\
&= \left(\frac{|\lambda|^2 z - \overline{\lambda}\overline{w} + \overline{\lambda}\overline{w} + \mathrm{i}z}{\mathrm{i} + |\lambda|^2}, \frac{|\lambda|^2 w - \overline{\lambda}\mathrm{i}\overline{z} + \overline{\lambda}\mathrm{i}\overline{z} - \mathrm{i}w}{\mathrm{i} - |\lambda|^2} \right) = \mathrm{id}_E(z,w)
\end{aligned}$$

The other identity $(\lambda \cdot \mathrm{id}_E - f)\left((\lambda \cdot \mathrm{id}_E - f)^{-1}(\zeta,\omega)\right) = \mathrm{id}_E(\zeta,\omega)$ is computed analogously. Hence, $\rho_\Sigma(f) = \mathbb{C}$, and finally $\Sigma(f) = \emptyset$.

4.2. Point Spectrum. The point spectrum may be empty. For operator f, defined as above, with $f(0) = 0$, we have $\sigma_p(f) \subseteq \Sigma_i(f) \subseteq \Sigma(f) = \emptyset$.

4.3. Spectra Defined Via Seminorms. Operator f, defined above, is additive and 1-homogeneous. Hence, all the spectra coincide by Proposition 70. As $[f]_b > 0$, we obtain $\sigma_z(f) = \emptyset$ for $z \in \{s, d, db, b, lip\}$.

4.4. Rhodius Spectrum. The Rhodius spectrum may be empty. To see this, again consider operator $f\colon E \to E$, $f(z,w) := (\overline{w}, \mathrm{i}\overline{z})$. Then $(\lambda \cdot \mathrm{id}_E - f)$ is invertible, and $r(f,\lambda)$ is continuous, because it equals (122). Hence $\rho_R(f) = \mathbb{C}$, and finally $\sigma_R(f) = \emptyset$.

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[7] [**Růž04**, Chapter 2, Theorems 2.17, 2.22]

4.5. Dörfner Spectrum. The Dörfner spectrum may be empty. To see this, again consider operator $f\colon E \to E$, $f(z,w) := (\overline{w}, \mathrm{i}\overline{z})$. We have $\sigma_D(f) = \Sigma(f) \cup \sigma_b(f) = \emptyset \cup \emptyset = \emptyset$ by the results above.

4.6. Kačurovskiĭ Spectrum. The Kačurovskiĭ spectrum may be empty. To see this, again (and again) consider operator $f\colon E \to E$, $f(z,w) := (\overline{w}, \mathrm{i}\overline{z})$. We have $\sigma_K(f) = \Sigma(f) \cup \sigma_{lip}(f) = \emptyset \cup \emptyset = \emptyset$ by the results above.

4.7. Neuberger Spectrum. In contrast to many of the other spectra, the Neuberger spectrum is always nonempty in case $\mathbb{K} = \mathbb{C}$. Let $\pi(f) := \{\lambda \in \mathbb{K} \mid (\lambda \cdot \mathrm{id}_E - f) \text{ is not proper }\}$. Then for $f \in \mathcal{C}^1(E)$, we have

$$\sigma_N(f) = \pi(f) \cup \bigcup_{x \in E} \sigma\left(f'|_x\right) \quad . \tag{123}$$

For the proof, consider an arbitrary λ. For the one direction, let λ be in the complement of the set of the right side. Then f is proper. Consequently, operator $(\lambda \cdot \mathrm{id}_E - f)$ is proper. As each $(\lambda \cdot \mathrm{id}_E - f)'|_x$, $x \in E$, is continuously invertible, $(\lambda\cdot\mathrm{id}_E - f)$ is a homeomorphism. Combined, $(\lambda\cdot\mathrm{id}_E - f)$ is a global homeomorphism. By assumption $(\lambda \cdot \mathrm{id}_E - f)$ is differentiable. In addition, $r(f,\lambda) = (\lambda \cdot \mathrm{id}_E - f)^{-1}$ is differentiable. Hence, λ is not in $\sigma_N(f)$. For the other direction, let λ be in the complement of the left side. Then $(\lambda \cdot \mathrm{id}_E - f)$ is a diffeomorphism. Consequently, it is proper (its inverse map is continuous, and thus maps compact sets on compact sets), and all its derivatives are invertible.

From the above, we deduce that the Neuberger spectrum is not empty, because the usual spectrum for linear operators is not empty.

4.8. FMV Spectrum. The FMV spectrum may be empty. As operator $f(z,w) := (\overline{w}, \mathrm{i}\overline{z})$, is defined over a finite-dimensional space, it cannot be used directly to show that the FMV spectrum is empty. But just use a countably-finite number of copies, i.e.,

$$f_\infty\left((z_1,w_1),(z_2,w_2),\ldots\right) := \left((\overline{w_1}, \mathrm{i}\overline{z_1}), (\overline{w_2}, \mathrm{i}\overline{z_2}), \ldots\right) \tag{124}$$

As $[f_\infty]_b > 0$, we have $[f_\infty]_a \geq [f_\infty]_{lip} \geq [f_\infty]_b > 0$ and $[f_\infty]_q \geq [f_\infty]_b > 0$. As f is surjective, it is stably solvable. Thus, $\sigma_{FMV}(f_\infty) = \sigma_a(f_\infty) \cup \sigma_q(f_\infty) \cup \sigma_\mu(f_\infty) = \emptyset$.

4.9. Feng Spectrum. The Feng spectrum may be empty. Operator f_∞, defined above, is also additive and 1-homogeneous. Hence, by Theorem 83, $\sigma_F(f_\infty) = \sigma_F^1(\mathrm{id}_E, f_\infty) = \sigma_{FMV}^1(\mathrm{id}_E, f_\infty) = \sigma_{FMV}(f_\infty) = \emptyset$.

5. Closedness

5.1. Mapping Spectrum. The mapping spectrum may not be closed. This can be seen by the following example. Let $E := \mathbb{R}$, and consider operator $f(x) := x^3$. Clearly, f is a bijection. Hence, $0 \notin \Sigma(f)$. For $\lambda > 0$, operator $(\lambda \cdot \mathrm{id}_E - f)$ is not injective, take e.g., $x_1 = 0, x_2 = \sqrt{\lambda}$. Thus, $\Sigma(f) \supseteq (0, \infty)$.

5.2. Rhodius Spectrum. The Rhodius spectrum may not be closed. Let $E := \mathbb{R}$, and again consider operator $f(x) := x^3$. Clearly, f is even a homeomorphism. Hence, $0 \notin \sigma_R(f)$. Again, for $\lambda > 0$, $(\lambda \cdot \mathrm{id}_E - f)$ is not injective. Thus, $\sigma_R(f) \supseteq (0, \infty)$.

5.3. Dörfner Spectrum. The Dörfner spectrum may not be closed. This can be seen by the following example. Let $E := \mathbb{R}$, and let $(\alpha_n)_n$ be a strictly-increasing sequence in $[0,1]$ with $\alpha_0 = 0$ and $\lim_{n\to\infty} \alpha_n = 1$. Define $c_n := \sum_{k=1}^{n} \frac{\alpha_k - \alpha_{k-1}}{k}$. Then $(c_n)_n$ is a Cauchy sequence, because $(\alpha_n)_n$ is one, as can be seen by

$$c_{n+p} - c_n = \sum_{k=n+1}^{n+p} \frac{\alpha_k - \alpha_{k-1}}{k} \leq \frac{1}{n+1} \sum_{k=n+1}^{n+p} \alpha_k - \alpha_{k-1} = \frac{1}{n+1}(\alpha_{n+p} - \alpha_n) \quad .$$

Hence, $c := \sum_{k=1}^{\infty} \frac{\alpha_k - \alpha_{k-1}}{k} = \lim_{n\to\infty} c_n < \infty$.

We build a piecewise-linear operator $f\colon E \to E$ by

$$f(x) := \begin{cases} c_n + \frac{x-\alpha_n}{n+1} & , \quad x \leq 1, \quad x \in [\alpha_n, \alpha_{n+1}] \\ c \cdot x & , \quad x \geq 1 \quad . \end{cases} ,$$

As f is strictly increasing, it is bijective. Its inverse is

$$f^{-1}(y) = \begin{cases} \alpha_k + (k+1)(y - c_k) & , \quad y \in [c_k, c_k + \frac{\alpha_{k+1}-\alpha_k}{k+1}] \\ \frac{y}{c} & , \quad y \in [c, \infty) \quad . \end{cases} ,$$

We have $[f^{-1}]_B \leq \sup\{\frac{1}{c}, \frac{\alpha_k + (k+1)(y-c_k)}{y} \mid y \in [c_k, c_k + \frac{\alpha_{k+1}-\alpha_k}{k+1}]\} < \infty$ as

$$\frac{\alpha_k + (k+1)(y - c_k)}{y} \leq \frac{\alpha_k + (k+1)(\frac{\alpha_{k+1}-\alpha_k}{k+1})}{c_n} \leq \frac{\alpha_k + 1}{c_n} \leq \frac{2}{\alpha_1} \quad .$$

Hence, on the one hand, $0 \notin \sigma_D(f)$. On the other hand, the sequence $(\frac{1}{k})_k$ is in $\sigma_D(f)$, because $(\frac{1}{k} \cdot \mathrm{id}_E - f)$ is constant on interval $[\alpha_k, \alpha_{k+1}]$.

5.4. Kačurovskiĭ Spectrum. The closedness of the Kačurovskiĭ spectrum is a consequence of the Banach fixed-point theorem.

PROPOSITION 86. *Let E be a Banach (or Fréchet) space, and let $f\colon E \to E$ be Lipschitz-continuous. If $[f]_{Lip} < 1$, then $\mathrm{id}_E - f$ is a lipeomorphism with*

$$\left[(\mathrm{id}_E - f)^{-1}\right]_{Lip} \leq \frac{1}{1 - [f]_{Lip}} \quad . \tag{125}$$

PROOF. First of all, $(\mathcal{C}_{Lip}(E), [\cdot]_{Lip})$ is a complete metric space. Secondly, for $f_z(x) := f(x) + z$, we have $[f_z]_{Lip} = [f]_{Lip} < 1$, because the Lip-characteristic is translation-invariant. Hence, by Theorem 28 (Banach Fixed-Point), f_z has a fixed point. Thus, the equation $(\mathrm{id}_E - f_z)(x) = 0$ has a unique solution and $(\mathrm{id}_E - f_z)^{-1}$ exists. We have

$$(\mathrm{id}_E \circ (\mathrm{id}_E - f)^{-1})(x) - (f \circ (\mathrm{id}_E - f)^{-1})(x) = \mathrm{id}_E(x) = x \quad .$$

Consequently, for all $y, z \in E$ we obtain

$$\begin{aligned} &\|(\mathrm{id}_E - f)^{-1}(z) - (\mathrm{id}_E - f)^{-1}(y)\|_E \\ &\leq \|f((\mathrm{id}_E - f)^{-1}(z)) - f((\mathrm{id}_E - f)^{-1}(y))\|_E + \|z - y\|_E \\ &\leq [f]_{Lip} \cdot \|(\mathrm{id}_E - f)^{-1}(z) - (\mathrm{id}_E - f)^{-1}(y)\|_E + \|z - y\|_E \quad , \end{aligned}$$

implying

$$\frac{\|(\mathrm{id}_E - f)^{-1}(z) - (\mathrm{id}_E - f)^{-1}(y)\|_E}{\|z - y\|_E} \leq \frac{1}{1 - [f]_{Lip}} \quad .$$

□

THEOREM 87. *Let E be a Banach (or Fréchet) space, and let $f, g\colon E \to E$ be Lipschitz-continuous. If f is a lipeomorphism and $[g]_{Lip} < [f]_{lip}$, then $f + g$ is a lipeomorphism with*

$$[(f + g)^{-1}]_{Lip} \leq \frac{1}{[f]_{lip} - [g]_{Lip}} \quad .$$

PROOF. First of all, $f + g = (\mathrm{id}_E - (-g \circ f^{-1})) \circ f$. Secondly, we have

$$[-g \circ f^{-1}]_{Lip} \leq [g]_{lip} \cdot [f^{-1}]_{Lip} \leq [g]_{Lip} \cdot [f^{-1}]_{Lip} < 1 \quad .$$

Hence, by Proposition 86, $(\mathrm{id}_E + (g \circ f^{-1}))^{-1}$ exists, and we have

$$[(\mathrm{id}_E + (g \circ f^{-1}))^{-1}]_{Lip} \leq \frac{1}{1 - [g]_{Lip} \cdot [f^{-1}]_{Lip}} \quad .$$

Finally, $(f + g)^{-1} = f^{-1} \circ (\mathrm{id}_E + (g \circ f^{-1}))^{-1}$, and

$$\begin{aligned}
[(f+g)^{-1}]_{Lip} &\leq [f^{-1}]_{lip} \cdot [(\mathrm{id}_E + (g \circ f^{-1}))^{-1}]_{Lip} \\
&\leq [f^{-1}]_{Lip} \cdot \frac{1}{1 - [g]_{Lip} \cdot [f^{-1}]_{Lip}} \\
&= \frac{1}{[f^{-1}]_{Lip}^{-1} - [g]_{Lip}} = \frac{1}{[f]_{lip} - [g]_{Lip}} \quad .
\end{aligned}$$

□

COROLLARY 88. *Let E be a Banach (or Fréchet) space, and let $f \in \mathcal{C}_{Lip}(E)$ be Lipschitz-continuous. Then the Kačurovskiĭ spectrum $\sigma_K(f)$ is closed.*

PROOF. Let $\lambda \in \rho_K(f)$ be arbitrary. Then $(\lambda \cdot \mathrm{id}_E - f)$ is a lipeomorphism with $[\lambda \cdot \mathrm{id}_E - f]_{lip} = [(\lambda \cdot \mathrm{id}_E - f)^{-1}]_{Lip}^{-1} = [r(f,\lambda)]_{Lip}^{-1} > 0$. Let $\mu \in \mathbb{K}$ be arbitrary with $|\mu - \lambda| < [r(f,\lambda)]_{Lip}^{-1}$. Then $(\mu - \lambda) \cdot \mathrm{id}_E$ is Lipschitz-continuous with $[(\mu - \lambda) \cdot \mathrm{id}_E]_{Lip} < [\lambda \cdot \mathrm{id}_E - f]_{lip}$. By Theorem 87 applied to $(\lambda \cdot \mathrm{id}_E - f)$ and $(\mu - \lambda) \cdot \mathrm{id}_E$, operator $r(f,\mu) = ((\lambda \cdot \mathrm{id}_E - f) + (\mu - \lambda) \cdot \mathrm{id}_E)^{-1}$ exists and is Lipschitz-continuous. Hence, $\mu \in \rho_K(f)$, proving that $\rho_K(f)$ is open. □

5.5. Neuberger Spectrum. The Neuberger spectrum may not be closed. This can be seen by the following example. Let $E = \mathbb{R}$. Define operator $f\colon E \to E$ as follows, distinguishing between even and odd intervals ($k \in \mathbb{N}$).

$$f(x) := \begin{cases} x & , \quad x \in [0,1] \quad , \\ 4k^2 - 1 + \frac{1}{2k} \cdot x & , \quad x \in [2k, 2k+1] \quad , \\ f_{2k+1}(x) & , \quad x \in [2k+1, 2(k+1)] \quad , \\ -f(-x) & , \quad x \leq 0 \quad . \end{cases}$$

Here, f_{2k+1} is a $\mathcal{C}^1$ operator, strictly increasing, with strictly positive derivative, extending f smoothly to odd intervals. As f is strictly increasing, f is a bijection. It is $\mathcal{C}^1$ by construction. Its inverse is $\mathcal{C}^1$, too, because f has strictly positive derivative. Hence, on the one hand, $0 \notin \sigma_N(f)$. On the other hand, the zero sequence $(\frac{1}{2k})_k$ is in $\sigma_N(f)$, because $\frac{1}{2k} \cdot \mathrm{id}_E - f$ is constant on interval $[2k, 2k+1]$.

5.6. FMV Spectrum. Showing the closedness of the FMV spectrum is based on a perturbation result. We sketch the proof on a high level first: From different growth-rates (Q-characteristic) of operators f and g, we obtain an invariant and bounded set. This set contains an invariant and compact subset, which we find with the help of the noncompactness measure (A-characteristic). Stability of f is now obtained via the Theorem of Dugundji and the existence of retractions, enforcing the solution of $f = g$ to lie in this compact set.

THEOREM 89. *Let E, F be infinite-dimensional Banach (or Fréchet) spaces, $k \geq 0$, and let $f\colon E \to F$ be continuous. If f is 0-AQ stably solvable and $[f]_{aq} > 0$, then f is $([f]_{aq} - \epsilon)$-AQ stably solvable for every $\epsilon > 0$.*

Proof. Let $g\colon E \to F$ be continuous with $[g]_{AQ} \le [f]_{aq} - \epsilon < [f]_{aq}$. We have to show that $f = g$ has a solution. As $[f]_q > [g]_Q$, by Lemma 75 applied to f and g, there exists a nonempty, convex, closed, and bounded subset $B \subseteq E$ such that $f^{-1}(\overline{\mathrm{co}}(g(B) \cup \{0\})) \subseteq B$. As $[f]_a > [g]_A$, by Lemma 56, there exists a nonempty, convex, and compact subset $C \subseteq B$ such that $f^{-1}(\overline{\mathrm{co}}(g(C) \cup \{0\})) \subseteq C$.

Define $D := \overline{\mathrm{co}}(g(C) \cup \{0\})$. Then $D \subseteq F$ is nonempty, convex, and compact. By Theorem 51 (Dugundji's Extension Theorem), there exists a retraction $r\colon F \to D$ of F onto D. Define $g_0\colon E \to F$ by $g_0(x) := r(g(x))$. Then g_0 is compact by construction and bounded in growth, $[g_0]_{AQ} = 0$. Operators f and g_0 have a common solution $\hat{x}$ by 0-AQ stably solvability of f. As $f(\hat{x}) \in g_0(E) \subseteq D$, we have $\hat{x} \in f^{-1}(D) \subseteq C$. Then $g(\hat{x}) \in D$, implying $g(\hat{x}) = r(g(\hat{x})) = g_0(\hat{x}) = f(\hat{x})$. Hence, $\hat{x}$ is also a solution for $f = g$. This shows that f is $([f]_{aq} - \epsilon)$-AQ stably solvable. □

Corollary 90. *We have $\mu(f) \ge [f]_{ab}$.*

Proposition 91. *Let E, F be infinite-dimensional Banach (or Fréchet) spaces, and let $f, g\colon E \to F$ be continuous operators.*

(i) *If f is k-AQ stably solvable for $k \ge [g]_{AQ}$, then $f + g$ is k'-AQ stably solvable for $k' \le k - [g]_{AQ}$.*
(ii) *If f is FMV-regular with $[g]_{AQ} < [f]_{aq}$, then $f + g$ is FMV-regular.*

Proof. For item (i), let $h\colon E \to F$ be continuous with $[h]_{AQ} \le k'$. We want to show that equation $f + g = h$ has a solution. We have $[h - g]_{AQ} \le [h]_{AQ} + [g]_{AQ} \le (k - [g]_{AQ}) + [g]_{AQ} = k$. As f is k-AQ stably solvable, $f = g - h$ has a solution.

In case, f is FMV-regular, we have $[f + g]_{aq} \ge [f]_{aq} - [g]_{AQ} > 0$. Furthermore, as f is 0-AQ stably solvable, f is even $([f]_{aq} - \epsilon)$-AQ stably solvable for every $\epsilon > 0$. Choose ϵ such that $k' := [f]_{aq} - [g]_{AQ} - \epsilon > 0$. Then $f + g$ is k'-AQ stably solvable and thus FMV-regular, proving item (ii). □

Theorem 92. *Let E be an infinite-dimensional Banach (or Fréchet) space, and let $f\colon E \to E$ be continuous. Then the FMV spectrum $\sigma_{FMV}(f)$ is closed.*

Proof. We show that $\rho_{FMV}(f)$ is open. Let $\lambda \in \rho_{FMV}(f)$. Define $\delta := \frac{1}{2}[f]_{aq} > 0$. Let $\mu \in \mathbb{K}$ with $|\lambda - \mu| < \delta$. We show that $\mu \in \rho_{FMV}(f)$. By choice of λ, operator $(\lambda \cdot \mathrm{id}_E - f)$ is FMV-regular with $[\lambda \cdot \mathrm{id}_E - f]_{aq} > 0$. As $[\mu \cdot \mathrm{id}_E]_{AQ} = |\mu| = \delta < [f]_{aq}$, operator $(\mu \cdot \mathrm{id}_E - f) = (\mu - \lambda) \cdot \mathrm{id}_E + (\lambda \cdot \mathrm{id}_E - f)$ is FMV-regular. Hence, $\mu \in \rho_{FMV}(f)$. □

An alternative proof goes as follows.

Proof. We show that $\rho_{FMV}(f)$ is open. Let $\lambda \in \rho_{FMV}(f)$. Then $(\lambda \cdot \mathrm{id}_E - f)$ is FMV-regular. Hence, $[\lambda \cdot \mathrm{id}_E - f]_{aq} > 0$ and $\mu(\lambda \cdot \mathrm{id}_E - f) > 0$. Choose $\delta(\lambda) := \frac{1}{2} \cdot [\lambda \cdot \mathrm{id}_E - f]_{aq} > 0$. Then for all η with $|\eta - \lambda| < \delta(\lambda)$, we have

$$[\eta \cdot \mathrm{id}_E - f]_{aq} \ge [\lambda \cdot \mathrm{id}_E - f]_{aq} - |\eta - \lambda| \ge [\lambda \cdot \mathrm{id}_E - f]_{aq} - \delta(\lambda) > 0 \quad .$$

Furthermore, we have

$$\mu(\eta \cdot \mathrm{id}_E - f) \ge [\eta \cdot \mathrm{id}_E - f]_{aq} > 0 \quad .$$

Hence, operator $(\eta \cdot \mathrm{id}_E - f)$ is FMV-regular, i.e., $\mu \in \rho_{FMV}(f)$. □

5.7. Feng spectrum. We prove the boundedness of the Feng spectrum via the properties of μ and ν.

Theorem 93. *Let E be an infinite-dimensional Banach (or Fréchet) space, and let $f\colon E \to E$ be continuous. Then the Feng spectrum $\sigma_F(f)$ is closed.*

PROOF. We show that $\rho_F(f)$ is open. Let $\lambda \in \rho_F(f)$. Then $(\lambda \cdot \mathrm{id}_E - f)$ is Feng-regular. Hence, $[\lambda \cdot \mathrm{id}_E - f]_{ab} > 0$ and $\nu(\lambda \cdot \mathrm{id}_E - f) > 0$. Choose $\delta(\lambda) := \frac{1}{2} \cdot [\lambda \cdot \mathrm{id}_E - f]_{ab} > 0$. Then for all η with $|\eta - \lambda| < \delta(\lambda)$, we have

$$[\eta \cdot \mathrm{id}_E - f]_{ab} \geq [\lambda \cdot \mathrm{id}_E - f]_{ab} - |\eta - \lambda| \geq [\lambda \cdot \mathrm{id}_E - f]_{ab} - \delta(\lambda) > 0 \quad .$$

Furthermore, we have

$$\nu(\eta \cdot \mathrm{id}_E - f) \geq \mu(\eta \cdot \mathrm{id}_E - f) \geq [\eta \cdot \mathrm{id}_E - f]_{aq} \geq [\eta \cdot \mathrm{id}_E - f]_{ab} > 0 \quad .$$

Hence, operator $(\eta \cdot \mathrm{id}_E - f)$ is Feng-regular, i.e., $\mu \in \rho_F(f)$. □

6. Boundedness

6.1. Mapping Spectrum. The mapping spectrum may be unbounded. This can be seen by the following example. Let $E = \mathbb{R}$, and define $f(x) := x^2$. Then $(\lambda \cdot \mathrm{id}_E - f)$ is not injective for all λ. For $\lambda = 0$, take $x_1 = 1, x_2 = -1$. For $\lambda \neq 0$, take $x_1 = 0, x_2 = \lambda$. Hence, $\mathbb{R} = \Sigma_i(f) = \Sigma(f)$.

6.2. Point Spectrum. The point spectrum may be unbounded. Take the same example as above for the mapping spectrum.

6.3. Spectra Defined Via Seminorms. Based on the spectra defined for the lower characteristics, one can also define corresponding *spectral radii for the lower characteristics*:

$$\begin{aligned} r_s(f) &:= \sup\{|\lambda| \mid \lambda \in \sigma_s(f)\} \quad , \\ r_d(f) &:= \sup\{|\lambda| \mid \lambda \in \sigma_d(f)\} \quad , \\ r_{db}(f) &:= \sup\{|\lambda| \mid \lambda \in \sigma_{db}(f)\} \quad , \\ r_b(f) &:= \sup\{|\lambda| \mid \lambda \in \sigma_b(f)\} \quad , \\ r_{lip}(f) &:= \sup\{|\lambda| \mid \lambda \in \sigma_{lip}(f)\} \quad . \end{aligned}$$

We have the following inclusions in case of identical seminorms $p = q$, because $[\lambda \cdot \mathrm{id}_E - f]_z = 0$ implies both $[f]_z - [\lambda \cdot \mathrm{id}_E]_Z \leq 0$ and $[\lambda \cdot \mathrm{id}_E]_z - [f]_Z \leq 0$. Consequently, $[f]_z \leq [\lambda \cdot \mathrm{id}_E]_z = |\lambda| = [\lambda \cdot \mathrm{id}_E]_Z \leq [f]_Z$. This gives

$$\sigma_z(f) \subseteq \{\lambda \in \mathbb{K} \mid [f]_z \leq |\lambda| \leq [f]_Z\} \quad . \tag{126}$$

Hence, for $f \in \mathcal{C}(E)$ with $[f]_Z < \infty$, the spectra for the lower characteristics $[\cdot]_z$ are bounded, and for each z/Z we have

$$r_z(f) \leq [f]_Z \quad . \tag{127}$$

6.4. Rhodius Spectrum. The Rhodius spectrum may be unbounded. Again, let $E = \mathbb{R}$, and consider $f(x) := x^2$. We have $f \in \mathcal{C}(E)$. Then $\mathbb{R} = \Sigma(f) \subseteq \sigma_R(f)$.

6.5. Dörfner Spectrum. The Dörfner spectrum may be unbounded. This can be seen by the following example. Consider operator $f\colon E \to E$ defined on $E := \mathbb{R}$ by $f(x) := 0$, if $x \leq 1$, and $f(x) := \sqrt{x-1}$, if $x \geq 1$.

First of all, we show $f \in \mathcal{B}_{lin}(E)$. We have

$$[f]_B = \sup\{|f(x)|/|x| \mid 0 < x < \infty\} = \sup\left\{\sqrt{x}/(x+1) \mid 0 < x < \infty\right\} \quad .$$

Its extremum is at $x = 1$, computed by

$$0 = \left(\frac{\sqrt{x}}{x+1}\right)' = \frac{(\sqrt{x})'(x+1) - (\sqrt{x})(x+1)'}{(x+1)^2} \Leftrightarrow \frac{1}{2\sqrt{x}}(x+1) = \sqrt{x} \quad .$$

Thus, $[f]_B = \frac{1}{2} < \infty$.

Secondly, we prove that $\sigma_D(f)$ is unbounded. Consider $f_\lambda = \lambda \cdot \mathrm{id}_E - f$. We distinguish three cases on λ. If $\lambda = 0$, then $0 \in \sigma_D(f)$, because $f_0 \leq 0$ is not surjective. For $\lambda < 0$, operator f_λ is strictly monotonically decreasing with

$f_\lambda(x) \to \pm\infty$ for $x \to \mp\infty$. Hence, f_λ is bijective in this case. Furthermore, $[f_\lambda^{-1}]_B = [f_\lambda]_b \leq \lambda \cdot 1 + [f]_B = \lambda + 1/2 < \infty$. Thus, $(-\infty, 0) \subseteq \rho_B(f)$. For $\lambda > 0$, operator f_λ has a local minimum at $x_0 = 1 + 1/4\lambda^2$, computed by $0 = f_\lambda'(x) = \lambda - \frac{1}{2\sqrt{x-1}}$. We have $f_\lambda(1) > f_\lambda(x_0)$ and $f_\lambda(x_0) < f_\lambda(1 + 1/2\lambda^2)$. Hence, f_λ is strictly monotonically decreasing in $[1, x_0)$ and strictly monotonically increasing in (x_0, ∞). Finally, combining all three cases, we obtain $\sigma_D(f) = [0, \infty)$.

6.6. Kačurovskiĭ Spectrum.

Proposition 94. *Let E be a Banach (or Fréchet) space, and let $f\colon E \to E$ be Lipschitz-continuous. Then the Kačurovskiĭ spectrum $\sigma_K(f)$ is bounded.*

Proof. Let $\lambda \in \rho_K(f)$ be arbitrary with $|\lambda| > [f]_{Lip}$. Operator $(\lambda \cdot \mathrm{id}_E)$ is a lipeomorphism. As $[-f]_{Lip} = [f]_{Lip} < |\lambda| = [\lambda \cdot \mathrm{id}_E]_{lip}$, by Theorem 87, operator $(\lambda \cdot \mathrm{id}_E - f)$ is a lipeomorphism. Hence, $\lambda \notin \sigma_K(f)$. □

As a consequence, we obtain that the Kačurovskiĭ spectral radius is bounded by

$$r_K(f) \leq [f]_{Lip} \quad . \tag{128}$$

6.7. Neuberger Spectrum. The Neuberger spectrum may be unbounded. Again, let $E = \mathbb{R}$, and consider $f(x) := x^2$. We have $f \in \mathcal{C}^1(E)$. Then $\mathbb{R} = \Sigma(f) \subseteq \sigma_N(f)$.

6.8. FMV Spectrum. The boundedness of the FMV spectrum follows from the following perturbation result, which itself relies on the Darbo fixed-point theorem.

Proposition 95. *Let E be an infinite-dimensional Banach (or Fréchet) space, and let $h\colon E \to E$ be continuous with $[h]_{AQ} < 1$. Then operator $(\mathrm{id}_E - h)$ is surjective. In particular, h has a fixed point.*

Proof. Let $y \in E$ be arbitrary. Define the translate $h_y(x) := h(x) + y$. Then $[h_y]_{AQ} = [h]_{AQ} < 1$, because both A- and Q-characteristic are translation-invariant. Fix $q \in ([h]_Q, 1)$. Then for a suitable $b > 0$, $\|h_y(x)\|_E \leq q \cdot \|x\|_E + b$ for all $x \in E$. For $R \geq 1/(1-q)$, h_y maps ball $B(0, R)$ into itself. By Theorem 55 (Darbo Fixed-Point), h_y has a fixed point. □

Theorem 96. *Let E be an infinite-dimensional Banach (or Fréchet) space, and let $f\colon E \to E$ be continuous with $[f]_{AQ} < \infty$. Then the FMV spectrum $\sigma_{FMV}(f)$ is bounded.*

Proof. For $\lambda \in \mathbb{K}$ with $|\lambda| > [f]_{AQ}$ we have $[\lambda \cdot \mathrm{id}_E - f]_{aq} \geq |\lambda| - [f]_{AQ} > 0$. We show that $(\lambda \cdot \mathrm{id}_E - f)$ is 0-AQ stably solvable. Let $g\colon E \to E$ be continuous with $[g]_{AQ} = 0$. Then $h := (f + g)/\lambda$ satisfies $[h]_{AQ} < 1$. Hence, by Proposition 95, there exists $z \in E$ with $(\mathrm{id}_E - h)(z) = 0$. Thus, $(\lambda \cdot \mathrm{id}_E - f)(z) = g(z)$. □

An alternative proof goes as follows.

Proof. For $\lambda \in \mathbb{K}$ with $|\lambda| > [f]_{AQ}$, we have $[\lambda \cdot \mathrm{id}_E - f]_{aq} \geq |\lambda| - [f]_{AQ} > 0$. We have

$$\mu(\lambda \cdot \mathrm{id}_E - f) \geq [\lambda \cdot \mathrm{id}_E - f]_{aq} > 0 \quad .$$

Hence, operator $(\lambda \cdot \mathrm{id}_E - f)$ is FMV-regular. □

As a consequence, we obtain that the FMV spectral radius is bounded by

$$r_{FMV}(f) \leq [f]_{AQ} \quad . \tag{129}$$

6.9. Feng Spectrum. We prove the boundedness of the Feng spectrum via the properties of μ and ν.

THEOREM 97. *Let E be an infinite-dimensional Banach (or Fréchet) space, and let $f\colon E \to E$ be continuous with $[f]_{AB} < \infty$. Then the Feng spectrum $\sigma_F(f)$ is bounded.*

PROOF. For $\lambda \in \mathbb{K}$ with $|\lambda| > [f]_{AB}$, we have $[\lambda \cdot \mathrm{id}_E - f]_{ab} \geq |\lambda| - [f]_{AB} > 0$. We have

$$\nu(\lambda \cdot \mathrm{id}_E - f) \geq \mu(\lambda \cdot \mathrm{id}_E - f) \geq [\lambda \cdot \mathrm{id}_E - f]_{aq} \geq [\lambda \cdot \mathrm{id}_E - f]_{ab} > 0 \quad .$$

Hence, operator $(\lambda \cdot \mathrm{id}_E - f)$ is Feng-regular. $\square$

As a consequence, we obtain that the Feng spectral radius is bounded by

$$r_F(f) \leq [f]_{AB} \quad . \tag{130}$$

7. Upper Semicontinuity

Upper-semicontinuity seems to be a delicate issue. We will show that it holds for the Kačurovskiĭ, FMV, and Feng spectra under suitable conditions. For the Rhodius and Neuberger spectra, it seems to be an open question, [**ADPV04**, Table 7.1, p.178].

In addition, it is claimed in [**ADPV04**, Chapter 5, Example 5.9] that the Dörfner spectrum is not upper semicontinuous. It is shown that σ_D is not graph-closed. Probably, the argument then implicitly uses the fact that semicontinuity implies a closed graph. But such an argument only holds in case that the set-valued map has closed values. This is not the case here, $\sigma_D(f)$ may be open, see Subsection 5.3.

THEOREM 98. *Let $(\mathcal{M}, p)$ be a seminormed space, and let $\sigma\colon \mathcal{M} \to 2^{\mathbb{K}}$ be a set-valued map, which has closed graph and bounded values such that*

$$\sup_{\lambda \in \sigma(f)} |\lambda| \leq p(f) \quad f \in \mathcal{M} \quad . \tag{131}$$

Then σ is upper semicontinuous.

PROOF. Let $f \in \mathcal{M}$, and let $V \subseteq \mathbb{K}$ with $\sigma(f) \subseteq V$. Define $\epsilon > 0$ such that $\sigma(B(f,\epsilon)) \backslash V \neq \emptyset$. Here, $B(f,\epsilon) := \{g \in \mathcal{M} \mid (f-g) < \epsilon\}$. Hence, for $g \in B(f,\epsilon)$, we have $\sup_{\lambda\in\sigma(g)} |\lambda| \leq p(g) \leq p(f) + \epsilon$. Define set $C := \overline{\sigma(B(f,\epsilon))} \backslash V$. As $\sigma(B(f,\epsilon))$ is bounded, C is compact.

For every $\lambda \in C$, there exists an open set $V_\lambda \subseteq \mathbb{K}$ with $\lambda \in V_\lambda$ and $\delta(\lambda) > 0$ such that $\sigma(B(f,\delta(\lambda))) \cap V_\lambda = \emptyset$, because map σ is graph-closed. Family $\{V_\lambda \mid \lambda \in C\}$ is an open covering of C. Hence, there exist finitely-many $\lambda_1, \ldots, \lambda_m \in C$ such that $C \subseteq V_{\lambda_1} \cup \cdots \cup V_{\lambda_m}$. Define $\delta := \min\{\epsilon, \delta(\lambda_1), \ldots, \delta(\lambda_m)\}$. Then $\sigma(B(f,\delta)) \subseteq V$, showing σ to be upper semicontinuous. $\square$

7.1. Kačurovskiĭ Spectrum. The Kačurovskiĭ spectrum $\sigma_K\colon \mathcal{C}_{Lip}(E) \to 2^{\mathbb{K}}$, $f \mapsto \sigma_K(f)$, is upper semicontinuous. This follows from Theorem 98 and Proposition 99.

PROPOSITION 99. *The Kačurovskiĭ spectrum is graph-closed and has bounded values.*

PROOF. As $r_K(f) \leq [f]_{Lip}$, σ_K has bounded values. We show that σ_K is graph-closed. Let $(f_n)_n$ and $(\lambda_n)_n$ be sequences with $\lambda_n \in \sigma_K(f_n)$ such that $[f_n - f]_{Lip} \to 0$ and $\lambda_n \to \lambda$ for $n \to \infty$. Then we have

$$[(\lambda_n \cdot \mathrm{id}_E - f_n) - (\lambda \cdot \mathrm{id}_E - f)]_{Lip} \leq |\lambda_n - \lambda| + [f_n - f]_{Lip} \to 0 \quad (n \to \infty)$$

We have to show that $\lambda \in \sigma_K(f)$. Assume the opposite for a contradiction, i.e., $\lambda \in \rho_K(f)$. Then $[\lambda \cdot \mathrm{id}_E - f]_{lip} > 0$. Due to convergence, one can choose n_0 such that for all $n \geq n_0$ we have

$$\max\{[f_n - f]_{Lip}, |\lambda_n - \lambda|\} < \frac{1}{2} \cdot [\lambda \cdot \mathrm{id}_E - f]_{lip} \quad .$$

Hence, for operator $g_n := (\lambda_n - \lambda) \cdot \mathrm{id}_E + f - f_n$, we get

$$[g_n]_{Lip} \leq [f_n - f]_{Lip} + |\lambda_n - \lambda| < [\lambda \cdot \mathrm{id}_E - f]_{lip} \quad .$$

Then $(\lambda_n \cdot \mathrm{id}_E - f_n) = (\lambda \cdot \mathrm{id}_E - f) + g_n$ is also a lipeomorphism, contradicting $\lambda_n \in \sigma_K(f_n)$. □

We note that $[\cdot]_{Lip}$ and $[\cdot]_{lip}$ have been mixed up in the proof in [**ADPV04**, Theorem 5.3].

7.2. FMV Spectrum. Denote with $\mathcal{A}(E)$ and $\mathcal{Q}(E)$ the spaces of operators $f\colon E \to E$ such that $[f]_A < \infty$ and $[f]_Q < \infty$, respectively.

We consider space $\mathcal{A}(E) \cap \mathcal{Q}(E)$ with seminorm $p_{AQ}(f) := [f]_{AQ}$. The topology, induced by p_{AQ}, is often called *FMV topology*.

The FMV spectrum $\sigma_{FMV}\colon \mathcal{A}(E) \cap \mathcal{Q}(E) \to 2^{\mathbb{K}}$, $f \mapsto \sigma_{FMV}(f)$ is upper semicontinuous. This follows from Theorem 98 and Proposition 100.

Proposition 100. *The FMV spectrum is graph-closed and has bounded values.*

Proof. As $r_{FMV}(f) \leq [f]_{AQ}$, σ_F has bounded values. We show that σ_{FMV} is graph-closed by looking at its complement.

Fix $\lambda \in \rho_{FMV}(f)$. Then $[\lambda \cdot \mathrm{id}_E - f]_{aq} > 0$ and $\mu(\lambda \cdot \mathrm{id}_E - f) > 0$. As $\rho_{FMV}(f)$ is open by Theorem 92, we have $\delta(\lambda) := \frac{1}{4} \cdot [\lambda \cdot \mathrm{id}_E - f]_{aq} > 0$. Choose $g \in \mathcal{A}(E) \cap \mathcal{Q}(E)$ with $[g - f]_{AQ} < \delta(\lambda)$. Then for all η with $|\eta - \lambda| < \delta(\lambda)$, we have

$$\begin{aligned} [\eta \cdot \mathrm{id}_E - g]_{aq} &\geq [\eta \cdot \mathrm{id}_E - f]_{aq} - [f - g]_{AQ} \\ &\geq [\lambda \cdot \mathrm{id}_E - f]_{aq} - |\eta - \lambda| - [f - g]_{AQ} \\ &\geq [\lambda \cdot \mathrm{id}_E - f]_{aq} - 2 \cdot \delta(\lambda) > 0 \quad . \end{aligned}$$

Furthermore, we obtain

$$\mu(\eta \cdot \mathrm{id}_E - g) \geq [\eta \cdot \mathrm{id}_E - g]_{aq} > 0 \quad .$$

Thus, the complement of the graph of $\sigma_{FMV}(f)$ is open. □

7.3. Feng Spectrum. We consider the space $\mathcal{A}(E) \cap \mathcal{B}(E)$ of operators, together with seminorm $p_{AB}(f) := [f]_{AB}$. The topology, induced by p_{AB}, is sometimes called *Feng topology*.

The Feng spectrum $\sigma_F\colon \mathcal{A}(E) \cap \mathcal{B}(E) \to 2^{\mathbb{K}}$, $f \mapsto \sigma_F(f)$, is upper semicontinuous. This follows from Theorem 98 and Proposition 101.

Proposition 101. *The Feng spectrum is graph-closed and has bounded values.*

Proof. As $r_F(f) \leq [f]_{AB}$, σ_F has bounded values. We show that σ_F is graph-closed by looking at its complement.

Fix $\lambda \in \rho_F(f)$. Then $[\lambda \cdot \mathrm{id}_E - f]_{ab} > 0$ and $\nu(\lambda \cdot \mathrm{id}_E - f) > 0$. As $\rho_F(f)$ is open by Theorem 93, we have $\delta(\lambda) := \frac{1}{4} \cdot [\lambda \cdot \mathrm{id}_E - f]_{ab} > 0$. Choose $g \in \mathcal{A}(E) \cap \mathcal{B}(E)$ with $[g - f]_{AB} < \delta(\lambda)$. Then for all η with $|\eta - \lambda| < \delta(\lambda)$, we have

$$\begin{aligned} [\eta \cdot \mathrm{id}_E - g]_{ab} &\geq [\eta \cdot \mathrm{id}_E - f]_{ab} - [f - g]_{AB} \\ &\geq [\lambda \cdot \mathrm{id}_E - f]_{ab} - |\eta - \lambda| - [f - g]_{AB} \\ &\geq [\lambda \cdot \mathrm{id}_E - f]_{ab} - 2 \cdot \delta(\lambda) > 0 \quad . \end{aligned}$$

Furthermore, we obtain

$$\nu(\eta \cdot \mathrm{id}_E - g) \geq \mu(\eta \cdot \mathrm{id}_E - g) \geq [\eta \cdot \mathrm{id}_E - g]_{aq} \geq [\eta \cdot \mathrm{id}_E - g]_{ab} > 0 \quad .$$

Thus, the complement of the graph of $\sigma_F(f)$ is open. $\square$

CHAPTER 4

Applications

1. Nemyckii Operator

This section is just a preparation for the analysis of the p-Laplace operator, introduced in the next section. We show properties of so-called *Nemyckii* operators F, defined as follows.

$$(Fu)(x) := f(x, u(x)) \quad ,$$

where $u := (u^1, \ldots, u^{d'}) \colon \Omega \to \mathbb{R}^{d'}$, $\Omega \subseteq \mathbb{R}^d$ is a domain, and function $f \colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following conditions:

(i) *Carathéodory Condition*: Function f is measurable in the first and continuous in the second argument, i.e., for each $y \in \mathbb{R}^{d'}$, $x \mapsto f(x, y)$ is measurable, and for each $x \in \Omega$, $y \mapsto f(x, y)$ is continuous.

(ii) *Growth Condition*: There exist constants $b > 0$, $1 \leq q, p_i < \infty$, $i \in [d']$, and $a \in \mathcal{L}^q(\Omega)$ such that

$$|f(x, y)| \leq |a(x)| + b \cdot \sum_{i \in [d']} |y^i|^{p_i/q} \quad .$$

Theorem 102. *Under the above assumptions, the (nonlinear) Nemyckii operator is defined between $F \colon \prod_{i \in [d']} \mathcal{L}^{p_i}(\Omega) \to \mathcal{L}^q(\Omega)$, is bounded, and continuous. For all u in the domain of F it holds*

$$\|Fu\|^q_{\mathcal{L}^q(\Omega)} \leq (d' + 1)^{q-1} \cdot \left(\|a\|^q_{\mathcal{L}^q(\Omega)} + b^q \cdot \sum_{i \in [d']} \|u^i\|^{p_i}_{\mathcal{L}^{p_i}(\Omega)} \right) \quad .[1]$$

The proof of above theorem, given in [**Růž04**, Section 3.1.2, Lemma 1.19], is problematic in several aspects. First of all, it suggests that the above statement also holds in case $q = 1$ or one of the $p_i = 1$, which is actually true. But $\mathcal{L}^1(\Omega)$ is not reflexive in case of a bounded domain Ω. Hence, the arguments, using [**Růž04**, Chapter 3, Lemma (0.3) (iv)], in the proof fail in such a non-reflexive case. The cited lemma demands reflexivity. Secondly, mentioned Lemma (0.3) (iv) only gives a weakly-convergent sequence. In contrast, here we need a strongly-convergent sequence.

Proof. *Each Fu is measurable:* As $u \in \mathcal{L}^p(\Omega)$, it is Lebesgue-measurable, implying that there exists a sequence of step functions $(u_n)_n$, $u_n = \sum_{i \in [m_n]} c_{n,i} \cdot \chi_{G_{n,i}}$, converging to u pointwise almost everywhere, i.e., $u_n(x) \to u(x)$ almost everywhere ($n \to \infty$). As f is continuous in the second argument, we obtain $Fu(x) = f(x, u(x)) = \lim_{n\to\infty} f(x, u_n(x))$ almost everywhere. Each

$$f(x, u_n(x)) = \sum_{i \in [m_n]} f(x, c_{n,i}) \cdot \chi_{G_{n,i}}(x)$$

[1]The exponents q have been corrected, compared to the submitted version of this thesis.

is measurable as a sum-product of measurable functions. Note that each $f(x, c_{n,i})$ is measurable by the Carathéodory condition. Finally, Fu is measurable as the limit of measurable functions.

Operator F is bounded: First note that $(a_1+\cdots+a_{d'})^r \leq (d')^{r-1}\cdot(a_1^r+\cdots+a_{d'}^r)$ for $r > 1$.[2] We have

$$\begin{aligned}
\|Fu\|^q_{\mathcal{L}^q(\Omega)} &= \int |f(x,u(x))|^q \,\mathrm{d}\lambda \leq \int \left(|a(x)| + b\cdot \sum_{i\in[d']} |u^i|^{p_i/q}\right)^q \mathrm{d}\lambda \\
&\leq (d'+1)^{q-1}\left(\int |a(x)|^q \,\mathrm{d}\lambda + b^q \cdot \sum_{i\in[d']}\int |u^i|^{p_i}\,\mathrm{d}\lambda\right) \\
&\leq (d'+1)^{q-1}\cdot\left(\|a\|^q_{\mathcal{L}^q(\Omega)} + b^q\cdot\sum_{i\in[d']}\|u^i\|^{p_i}_{\mathcal{L}^{p_i}(\Omega)}\right) \quad .
\end{aligned}$$

Operator F is continuous: For notational simplicity, we prove the statement for $d' = 1$, $p = p_1$. Let $u_n \to u$ in $\mathcal{L}^p$ be an arbitrary convergent sequence. We consider an arbitrary subsequence $(Fu_{n_k})_k$. For convergent sequence $(u_{n_k})_k$, there exists a subsequence $(u_{n_{k_l}})_l$, converging pointwise almost everywhere to x, by Theorem 24. Hence, subsequence $(Fu_{n_{k_l}})_l$ also converges pointwise almost everywhere to Fu, because f is continuous in the second argument by the Carathéodory condition. We then have $(Fu_{n_{k_l}} - Fu)(x) \to 0$ pointwise almost everywhere.

Define measurable functions

$$\begin{aligned}
h_l(x) &:= C\cdot\left(|a(x)|^q + b^q\cdot|u_{n_{k_l}}(x)|^p + |f(x,u(x))|^q\right) \quad , \\
h(x) &:= C\cdot\left(|a(x)|^q + b^q\cdot|u(x)|^p + |f(x,u(x))|^q\right) \quad .
\end{aligned}$$

Then $h_l(x) \to h(x)$ pointwise almost everyhere $(l\to\infty)$. In addition, as $u_{n_{k_l}} \to u$ in $\mathcal{L}^p(\Omega)$, we have

$$\int h_l \,\mathrm{d}\lambda \to \int h\,\mathrm{d}\lambda \quad (l\to\infty) \quad .$$

These functions serve as majorants, because $\left|Fu_{n_{k_l}} - Fu\right| \leq h_l$ by construction. Hence, we have everything to apply Theorem 23 (generalized Majorized Convergence), obtaining

$$\|Fu_{n_k} - Fu\|^q_{\mathcal{L}^q(\Omega)} \to 0 \quad (k\to\infty) \quad .$$

Hence, every subsequence of $(Fu_n)_n$ contains a subsequence, converging (in norm) to the same limit Fu. By Lemma 40 (v), then the whole sequence $(Fu_n)_n$ converges (in norm) to Fu. This shows that F is sequentially continuous. As sequential continuity is equivalent to continuity in Banach spaces, F is continuous. □

2. p-Laplace Operator

The literature on the p-Laplace operator is vast. We refer the reader to the interesting notes of Lindqvist [**Lin17**] and also to [**ADPV04**, Sections 12.5 and 12.6] and the references mentioned therein. As the p-Laplacian is a nonlinear generalization of the linear, ordinary Laplacian, it is a beautiful object of study to develop and sharpen the tools in (Nonlinear) Functional Analysis. Arguably even more than the original Laplacian ($p = 2$), it brings together different fields of mathematics like Calculus of Variations, Partial Differential Equations, Potential Theory, Function Theory, Mathematical Physics, and even Statistics and Game

[2]Use convexity $(r > 1)$ of $x \mapsto x^r$. Then e.g., the point $(\frac{a+b}{2})^r$ on the curve is below the point $\frac{a^r+b^r}{2}$ on the secant between a^r and b^r. The general case is an instance of Jensen's inequality.

Theory. The *p-Laplace operator* is formally defined as $\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for functions $u = u(x_1, \ldots, x_d)$. For fixed real s, $2 \leq p < \infty$, and a bounded domain $\Omega \subseteq \mathbb{R}^d$ with Lipschitz-continuous boundary $\partial\Omega$, we consider equations of the form

$$\Delta_p u = s \cdot |u|^{p-2}u \quad , \text{ in } \Omega \quad , \tag{132}$$

$$u = 0 \quad , \text{ in } \partial\Omega \quad . \tag{133}$$

As usual, divergence and gradient are defined by $\operatorname{div}(v_1, \ldots, v_d) := \partial_1 v_1 + \cdots + \partial_d v_d$ and $\nabla u := (\partial_1 u, \ldots, \partial_d u)$, respectively.

According to above equation, a strong solution u would need to be in class $\mathcal{C}^1$, compactly supported in Ω, and with partial derivatives $\partial_i u$ bounded in $\mathcal{L}^p(\Omega)$. Hence, for a weak solution u, it suffices to take $E := \mathcal{W}_0^{1,p}(\Omega)$ as the underlying space. Note that E and E' are separable and reflexive Banach spaces. Due to the Lipschitz conditions on the boundary of Ω, the norms $\|u\|_E$ and $\|\nabla u\|_{\mathcal{L}^p(\Omega)}$ are equivalent by Poincare's inequality.

The weak formulation of (132) reads as

$$\int_\Omega -\operatorname{div}(|\nabla u|^{p-2}\nabla u)\phi \,\mathrm{d}\lambda = s \cdot \int_\Omega |u|^{p-2}u\phi \,\mathrm{d}\lambda$$

for all $\phi \in E$. By Green's first identity, the first integral changes to

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla\phi\rangle_{\mathbb{R}^d} \,\mathrm{d}\lambda = s \cdot \int_\Omega |u|^{p-2}u\phi \,\mathrm{d}\lambda \quad .$$

For $u, \phi \in E$, define operators $Ju, Bu \in E'$ by

$$\langle Ju, \phi\rangle := (Ju)(\phi) := \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla\phi\rangle_{\mathbb{R}^d} \,\mathrm{d}\lambda \quad , \tag{134}$$

$$\langle Bu, \phi\rangle := (Bu)(\phi) := \int_\Omega |u|^{p-2}u\phi \,\mathrm{d}\lambda \quad . \tag{135}$$

Before we proceed to prove certain properties of these operators using the Theory of Monotonic Operators, we need a small preparation via the following inequality.

Proposition 103. *There is a constant $c > 0$ such that for all $s, t \in \mathbb{R}$ and $p \geq 2$ we have*

$$(|s|^{p-2}s - |t|^{p-2}t)(s - t) \geq c|s - t|^p \quad . \tag{136}$$

Proof. The cases $p = 2$, $s = 0$, $t = 0$, or $s = t$ are clear. In case one of the arguments is negative, e.g., s, consider $\tilde{s} = -s$. If $s < t$, switch roles. Hence, we can assume that $s > t > 0$ and $p > 2$. Define $m := \frac{s+t}{2}$. Then $s > m > t > 0$. We have

$$\begin{aligned}
&|s|^{p-2} \cdot \left(\frac{s}{s-t}\right) + |t|^{p-2} \cdot \left(\frac{-t}{s-t}\right) \\
&= |s|^{p-2} \cdot \left(\frac{s-m}{s-t}\right) + |s|^{p-2} \cdot m - |t|^{p-2} \cdot m + |t|^{p-2} \cdot \left(\frac{m-t}{s-t}\right) \\
&\geq |s|^{p-2} \cdot \left(\frac{s-m}{s-t}\right) + |t|^{p-2} \cdot \left(\frac{m-t}{s-t}\right) \\
&\geq \left| s \cdot \left(\frac{s-m}{s-t}\right) + t \cdot \left(\frac{m-t}{s-t}\right) \right|^{p-2} \\
&= \frac{1}{2} \cdot |s-t|^{p-2} \quad .
\end{aligned}$$

Here, we used the convexity of $x \mapsto |x|^{p-2}$. $\square$

Lemma 104. *Operator $J\colon E \to E'$ is odd, $(p-1)$-homogeneous, bounded, continuous, and coercive. If $p \geq 2$, J is strictly monotonic and a topological isomorphism.*

The proof of strict monotonicity of operator J, given in [**Růž04**, Chapter 3, Lemma 1.28], is wrong. It suggests that this property even holds for the range $]1,2[$. But the restriction to $p \geq 2$ seems to be necessary, see e.g., [**ADPV04**, Section 12.5, p.368]. The calculation presented is wrong, e.g., the differentiation should yield $|\zeta|^{p-3}$ and not $|\zeta|^{p-4}$ as stated. Hence, one does not get the lower bound $\min(1, p-1) \cdot |\zeta|^{p-2} \cdot |\eta|^2$. Rather one only gets

$$\sum_{i,j=1}^{d} \frac{\partial g^i}{\partial \zeta_j}(\zeta)\eta^i\eta^j = |\zeta|^2 \left(|\eta|^2 + (p-2) \cdot \frac{(\zeta \cdot \eta)^2}{|\zeta|} \right) \quad ,$$

which can become negative for $p < 2$. For example, set $\zeta = \eta$ with $|\zeta| < 2 - p$. We correct these flaws in the proof below.

Proof. Operator J is odd and $(p-1)$-homogeneous by definition. For every $u, \phi \in E$, we have

$$\begin{aligned} |\langle Ju, \phi\rangle| &\leq \int_\Omega |\nabla u|^{p-1} \cdot |\nabla \phi| \,\mathrm{d}\lambda \\ &\leq \left(\int_\Omega |\nabla u|^{(p-1)p'} \,\mathrm{d}\lambda \right)^{\frac{1}{p'}} \cdot \left(\int_\Omega |\nabla \phi|^p \,\mathrm{d}\lambda \right)^{\frac{1}{p}} \\ &= \|\nabla u\|_{\mathcal{L}^p(\Omega)}^{p-1} \cdot \|\nabla \phi\|_{\mathcal{L}^p(\Omega)} \quad , \end{aligned}$$

using Hölder's inequality and the dual parameter $p' := p/(p-1)$. Then for every $u \in E$, $\|Ju\|_{E'} \leq c \cdot \|u\|_E$. Hence, $Ju \in E'$ and J is bounded.

Operator J is continuous: Let $u_n \to u$ $(n \to \infty)$ be arbitrary. Then $\nabla u_n \to \nabla u$ in $\mathcal{L}^p(\Omega)$. Set F as $\zeta \mapsto |\zeta|^{p-1}\zeta$. Then F is a vector-valued Nemyckii operator. By Theorem 102, operator $F\colon (\mathcal{L}^p(\Omega))^d \to (\mathcal{L}^{p'}(\Omega))^d$ is continuous. Hence, $F(\nabla u_n) \to F(\nabla u)$ in $(\mathcal{L}^{p'}(\Omega))^d$. We obtain

$$\begin{aligned} \langle Ju_n - Ju, \phi\rangle &= \int_\Omega \langle F(\nabla u_n) - F(\nabla u), \nabla\phi\rangle_{\mathbb{R}^d} \,\mathrm{d}\lambda \\ &\leq \|F(\nabla u_n) - F(\nabla u)\|_{\mathcal{L}^{p'}(\Omega)} \cdot \|\nabla \phi\|_{\mathcal{L}^p(\Omega)} \\ &\leq c \cdot \|F(\nabla u_n) - F(\nabla u)\|_{\mathcal{L}^{p'}(\Omega)} \cdot \|\phi\|_E \quad . \end{aligned}$$

This shows that $Ju_n \to Ju$ in E' $(n \to \infty)$. Hence, J is sequentially continuous. As sequential and general continuity are equivalent in Banach spaces, J is also continuous.

Operator J is coercive: For all $u \in E$, we have

$$\langle Ju, u\rangle = \int_\Omega |\nabla u|^p \,\mathrm{d}\lambda = \|\nabla u\|_{\mathcal{L}^p(\Omega)}^p \geq c \cdot \|u\|_E^p \quad .$$

For all $p > 1$, it follows

$$\frac{\langle Ju, u\rangle}{\|u\|_E} \geq \|u\|_E^{p-1} \to \infty \quad (|u|_E \to \infty) \quad .$$

Operator J is a topological isomorphism: As J is (sequentially) demicontinous and coercive, it is surjective by Theorem 48 (Browder-Minty). It is injective and thus bijective by strict monotonicity. It is continuous, and its inverse is continuous. □

Lemma 105. *Operator $B\colon E \to E'$ is odd, $(p-1)$-homogeneous, and bounded. For $p < d$, it is strongly (sequentially) continuous, and thus compact.*

The proof in [**Růž04**, Section 3.2.2, Lemma 2.17] of the strong (sequential) continuity is not correct, because it applies [**Růž04**, Chapter 3, Lemma (0.3)], written is item (iii) but meant is item (iv), which only yields a weakly- and not a strongly-convergent sequence. In addition, the condition $p < d$, needed for the used embedding $E \to \mathcal{L}^p(\Omega)$ to be compact, was overlooked in [**ADPV04**, Section 12.5, Lemma 12.3]. It is unclear, if this condition is really necessary in the statement, or if it is only a limitation of the proof method. In this context, we note that the Sobolev embedding theorems are formulated in a way such that they are provably best-possible.

Proof. Operator B is odd and $(p-1)$-homogeneous by definition. For every $u, \phi \in E$ we have

$$|\langle Bu, \phi\rangle| \leq \int_\Omega |u|^{p-1} \cdot |\phi| \,\mathrm{d}\lambda \leq \left(\int_\Omega |u|^{(p-1)p'} \,\mathrm{d}\lambda\right)^{\frac{1}{p'}} \cdot \left(\int_\Omega |\phi|^p \,\mathrm{d}\lambda\right)^{\frac{1}{p}}$$
$$= \|u\|^{p-1}_{\mathcal{L}^p(\Omega)} \cdot \|\phi\|_{\mathcal{L}^p(\Omega)} \quad ,$$

using Hölder's inequality and the dual parameter $p' := p/(p-1)$. Then for every $u \in E$, $\|Bu\|_{E'} \leq c \cdot \|u\|_E$. Hence, $Bu \in E'$ and B is bounded.

Operator B is strongly (sequentially) continuous: Let $u_n \rightharpoonup u$ be a weakly-convergent sequence in E $(n \to \infty)$. We have to show that $Bu_n \to Bu$ in E' $(n \to \infty)$. Let $(Bu_{n_k})_k$ be an arbitrary subsequence.

As $u_{n_k} \rightharpoonup u$ in E $(k \to \infty)$, $(u_{n_k})_k$ is bounded by Lemma 40 (i). As the embedding $E \to \mathcal{L}^r(\Omega)$ is compact for all $r < \frac{dp}{d-p}$, in particular for $r = p-1$, sequence $(u_{n_k})_k$ is relatively compact. As compactness and sequential compactness coincide in Banach spaces, sequence $(u_{n_k})_k$ is relatively sequentially compact. Hence, there exists a subsequence $(u_{n_{k_l}})_l$, strongly converging to a limit, which must be u. Now, Nemyckij operator $Fu := |u|^{p-2}u$ is sequentially continuous. Thus, $Fu_{n_{k_l}} \to Fu$ in E' $(l \to \infty)$. From

$$\begin{aligned}
&\sup_{\phi \in E, \|\phi\|_E \leq 1} |\langle Bu_{n_{k_l}} - Bu, \phi\rangle| \\
&\leq \sup_{\phi \in E, \|\phi\|_E \leq 1} \int_\Omega |Fu_{n_{k_l}} - Fu| \cdot |\phi| \,\mathrm{d}\lambda \\
&\leq \sup_{\phi \in E, \|\phi\|_E \leq 1} \int_\Omega \|Fu_{n_{k_l}} - Fu\|_{\mathcal{L}^{p'}(\Omega)} \cdot \|\phi\|_{\mathcal{L}^p(\Omega)} \,\mathrm{d}\lambda \\
&\leq c \cdot \|Fu_{n_{k_l}} - Fu\|_{\mathcal{L}^{p'}(\Omega)} \to 0 \quad (l \to \infty) \quad ,
\end{aligned}$$

we obtain $Bu_{n_{k_l}} \to Bu$ in E' $(l \to \infty)$. Hence, every subsequence of $(Bu_n)_n$ contains a strongly-convergent subsequence. By Lemma 40 (v), we have $Bu_n \to Bu$ in E' $(n \to \infty)$, showing that B is strongly (sequentially) continuous.

As sequential compactness implies compactness for Banach space E' by Lemma 42 (i), we are done. □

Combining Theorem 85 with Lemmas 104 and 105, we obtain the following results.

Theorem 106 (Discreteness for Δ_p). *Let $p < d$, $s \neq 0$, and $\lambda := 1/s$. The $(p-1)$-FMV and $(p-1)$-Feng spectrum of the associated operator $(B - \lambda \cdot J)$ coincide with the classical point spectrum of problem (132).*

Theorem 107 (Fredholm Alternative for Δ_p). *Let $p < d$, $s \neq 0$, and $\lambda := 1/s$. If s is not a classical eigenvalue of problem (132), then there exists $k > 0$ such that the associated operator $(B - \lambda \cdot J)$ is both $(k, p-1)$-AQ stably solvable and $(k, p-1)$-epi on $\overline{U}$ for all $U \in \mathcal{O}(\mathcal{W}^{1,p}_0(\Omega))$.*

3. Navier-Stokes Equations

[3] The equations, named after Claude Louis Marie Henri Navier and George Gabriel Stokes, are fundamental equations of Hydrodynamics. They represent a mathematical model for the velocity of a viscous fluid (liquid, gas). We refer the reader to the comprehensive book of Lemarié-Rieusset [**Lem16**] and the references therein. We first consider the stationary version, defined over a smooth and bounded domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$.

$$-\Delta u + [\nabla u].u + \nabla p = f \quad , \quad \text{in } \Omega \quad , \tag{137}$$

$$\operatorname{div} u = 0 \quad , \quad \text{in } \Omega \quad . \tag{138}$$

Here, for $d \geq 1$, $u := (u^1, \dots, u^d)\colon \Omega \to \mathbb{R}^d$ denotes the *velocity* of a viscous fluid, $p\colon \Omega \to \mathbb{R}$ denotes the *pressure*, and $f\colon \Omega \to \mathbb{R}^d$ is an *external force*. In addition, we have the (vector-valued) *Laplace operator*, $\Delta u := (\sum_{j\in[d]} \partial_j^2 u^i)_{i\in[d]}$, the nonlinear (vector-valued) *turbulence term*, $[\nabla u].u := (\sum_{j\in[d]} u^j \partial_j u^i)_{i\in[d]}$, the (vector-valued) *divergence*, $\operatorname{div} u := (\sum_{j\in[d]} \partial_j u^i)_{i\in[d]}$, and the *gradient* $\nabla p := (\partial_1 p, \dots, \partial_d p)$. For (vector-valued) operators u, we have their *gradient* defined as $\nabla u := (\partial_i u^j)_{i,j\in[d]} = (\nabla u^j)_{j\in[d]}$.

As $\operatorname{div} u = 0$ implies $\operatorname{div}(-\Delta u + [\nabla u].u) = 0$, a necessary condition for a solution to exist is that $\operatorname{div}(f - \nabla p) = 0$.

We seek strong solutions of above equation, which are smooth and compactly supported in Ω. Hence, we define

$$E := \ker \operatorname{div} := \{u \in (\mathcal{D}(\Omega))^d \mid \operatorname{div} u = 0\} \tag{139}$$

as underlying space. We have a look at the weak formulation of (137). For $u, \phi \in E$, we define operators

$$\begin{aligned} \langle A_1 u, \phi\rangle := (A_1 u)(\phi) &:= \int_{\mathbb{R}^d} \langle -\Delta u, \phi\rangle_{\mathbb{R}^d} \,\mathrm{d}\lambda \\ &= \int_{\mathbb{R}^d} \langle \nabla u, \nabla\phi\rangle_{\mathbb{R}^{d\times d}} \,\mathrm{d}\lambda \quad , \end{aligned} \tag{140}$$

$$\langle A_2 u, \phi\rangle := (A_2 u)(\phi) := \int_{\mathbb{R}^d} \langle [\nabla u].u, \phi\rangle_{\mathbb{R}^d} \,\mathrm{d}\lambda \quad , \tag{141}$$

$$A := A_1 + A_2 \quad , \tag{142}$$

$$\langle b, \phi\rangle := b(\phi) := \int_{\mathbb{R}^d} \langle f - \nabla p, \phi\rangle_{\mathbb{R}^d} \,\mathrm{d}\lambda \quad . \tag{143}$$

LEMMA 108. *We have $A_1, A_2, A\colon E \to E'$ and $b \in E'$.*

(i) *Operator A_1 is linear, and (sequentially) continuous, hemicontinuous, (sequentially) demicontinuous, bounded, strictly monotonic, and pseudomonotonic.*
(ii) *Operator A_2 is strongly (sequentially) continuous, hemicontinuous, (sequentially) demicontinuous, bounded, and pseudomonotonic.*
(iii) *Operator A is (sequentially) continuous, hemicontinuous, (sequentially) demicontinuous, bounded, and pseudomonotonic.*

PROOF. For fixed $u \in E$, all operators $A_1 u$, $A_2 u$, Au, and b are in E^*, because $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ is linear in the second argument. To show continuity, it suffices to

[3] This section has been changed, compared to the submitted thesis. Reason is the correction of Theorem 47. In the original version, the stationary Navier-Stokes equations were solved for smooth, fast-decaying right-hand sides. The correction of the error made it necessary to restrict the class of functions further to bounded domains and to use an inequality from Brezis-Marcus.

show boundedness. We demonstrate this only for $A_1 u$, because the other cases are completely analogous.

$$\begin{aligned}|\langle A_1 u, \phi\rangle| &\leq \int_\Omega |\langle \nabla u, \nabla\phi\rangle_{\mathbb{R}^{d\times d}}| \,\mathrm{d}\lambda \leq \sqrt{\int_\Omega |\nabla u|^2 \,\mathrm{d}\lambda} \cdot \sqrt{\int_\Omega |\nabla\phi|^2 \,\mathrm{d}\lambda} \\ &\leq c_u \cdot \| |\nabla\phi| \|_{\mathcal{L}^2(\Omega)} \quad .\end{aligned}$$

Note that $|\nabla u|^2 = \langle \nabla u, \nabla u\rangle_{\mathbb{R}^{d\times d}} = \sum_{i,j\in[d]} (\partial_i u^j)^2 \in E$, same for $|\nabla\phi|^2$. Hence, both integrals are bounded.

Ad (i): By definition, A_1 is linear, because $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ is linear in the first argument, and the Laplace operator Δ is linear. It suffices to show sequential continuity. Let $u_n \to u$ be a convergent sequence in E $(n \to \infty)$. We need to show that $A_1 u_n \to A_1 u$ in E'_β $(n \to \infty)$. It suffices to show this convergence pointwise in distribution space E'. For arbitrary $\phi \in E$, we have

$$\begin{aligned}&|\langle A_1 u_n, \phi\rangle - \langle A_1 u, \phi\rangle| \\ &= |\langle A_1(u_n - u), \phi\rangle| \leq \int_\Omega |\langle \nabla(u_n - u), \nabla\phi\rangle_{\mathbb{R}^{d\times d}}| \,\mathrm{d}\lambda \\ &\leq \sqrt{\int_\Omega |\nabla(u_n - u)|^2 \,\mathrm{d}\lambda} \cdot \sqrt{\int_\Omega |\nabla\phi|^2 \,\mathrm{d}\lambda} \\ &\leq c_\phi \cdot \| |\nabla(u_n - u)| \|_{\mathcal{L}^2(\Omega)} \quad \longrightarrow \quad 0 \quad (n \to \infty) \quad .\end{aligned}$$

Hence, A_1 is (sequentially) continuous, and thus hemicontinuous and (sequentially) demicontinuous. It is bounded as a linear and continuous operator.

For arbitrary $u \in E$, we have

$$\langle A_1 u, u\rangle = \int_\Omega \langle \nabla u, \nabla u\rangle_{\mathbb{R}^{d\times d}} \,\mathrm{d}\lambda = \int_\Omega |\nabla u|^2 \,\mathrm{d}\lambda \geq 0 \quad ,$$

with $|\langle A_1 u, u\rangle| = 0$ iff $u = 0$. Hence, A_1 is strictly monotonic. As operator A_1 is monotonic and hemicontinuous, it is pseudomonotonic by Lemma 44 (i).

Ad (ii): Let $u_n \rightharpoonup u$ be a weakly-convergent sequence in E. As E is a strong W space, we even have $u_n \to u$, by Lemma 40 (vi). It suffices to show the pointwise convergence of $A_2 u_n \to A_2 u$ in E'_β $(n \to \infty)$. For arbitrary $\phi \in E$, we have

$$\begin{aligned}&|\langle A_2 u_n, \phi\rangle - \langle A_2 u, \phi\rangle| \\ &= \left| \int_\Omega \langle [\nabla u_n].u_n, \phi\rangle_{\mathbb{R}^d} - \langle [\nabla u].u, \phi\rangle_{\mathbb{R}^d} \,\mathrm{d}\lambda \right| \\ &= \left| \int_\Omega \langle [\nabla u_n].(u_n - u), \phi\rangle_{\mathbb{R}^d} + \langle [\nabla(u_n - u)].u, \phi\rangle_{\mathbb{R}^d} \,\mathrm{d}\lambda \right| \\ &\leq \|u_n - u\|_{\mathcal{L}^4(\Omega)} \cdot \| |\nabla u_n| \|_{\mathcal{L}^2(\Omega)} \cdot \|\phi\|_{\mathcal{L}^4(\Omega)} \\ &\quad + \left| \int_\Omega \langle [\nabla(u_n - u)].u, \phi\rangle_{\mathbb{R}^d} \,\mathrm{d}\lambda \right| \quad \longrightarrow \quad \infty \quad (n \to \infty) \quad .\end{aligned}$$

Note that sequence $(\nabla u_n)_n$ is bounded. This shows A_2 to be strongly (sequentially) continuous. Thus, A_2 is also hemicontinuous and (sequentially) demicontinuous. As every bounded subset B of E is compact, $A_2(C)$ is also compact by continuity. Hence, A_2 is bounded.

As operator A_2 is strongly (sequentially) continuous, it is pseudomonotonic by Lemma 44 (ii).

Ad (iii): Operator $A = A_1 + A_2$ is (sequentially) continuous, hemicontinuous, (sequentially) demicontinuous, and bounded, because A_1 and A_2 are (sequentially) continuous, hemicontinuous, (sequentially) demicontinuous, and bounded.

In addition, operator A is pseudomonotonic by Lemma 44 (iii), because A_1 and A_2 are pseudomonotonic. From this, we again obtain that A is (sequentially) demicontinuous, by Lemma 44 (v). □

We will use the famous inequality of Brezis and Marcus, see [**BM97**, Eq. (0.10)].

THEOREM 109 (Brezis & Marcus, 1997). *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$, be a bounded domain of class $\mathcal{C}^2$. For every $\lambda \leq \lambda^*(\Omega)$ and every $u \in \mathcal{C}_0^\infty$, we have*

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}\lambda - (1/4) \cdot \int_\Omega |u/\delta|^2 \,\mathrm{d}\lambda \geq \lambda \cdot \int_\Omega |u|^2 \,\mathrm{d}\lambda \quad . \tag{144}$$

The best-possible constant $\lambda^*(\Omega)$ can be lower-bounded by

$$\lambda^*(\Omega) \geq 1/(4 \operatorname{diam}^2(\Omega)) > 0 \quad . \tag{145}$$

Answering a question of Brezis and Marcus in the affirmative, the following bound has been achieved for best-possible $\lambda^*(\Omega)$ in [**HOHOL02**, Theorem 3.2].

THEOREM 110 (M. and Th. Hoffmann-Ostenhof & Laptev). *Denote with $\mathbb{S}^{d-1}$ the $(d-1)$-dimensional unit sphere. For a convex and bounded domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$, in class $\mathcal{C}^2$, in (144), the best-possible constant $\lambda^*(\Omega)$ can be lower-bounded by*

$$\lambda^*(\Omega) \geq \frac{d^{(d-2)/d} \cdot (\operatorname{vol} \mathbb{S}^{d-1})^{2/d}}{4 \cdot (\operatorname{vol} \Omega)^{2/d}} \quad . \tag{146}$$

The reader may be irritated by the formulation of the theorem below. In the past, one first obtained a weak solution u for the velocity, and then the pressure p was obtained by results, based on a theorem of De Rham, see e.g., [**Růž04**, Theorem 2.35, p.85]. But this is a very fancy way of showing that $p = 0$ does the job. For incompressible flows, there is no a priori relation between the velocity u and the pressure p. These are independent variables of the motion. This is pointed out several times in the book [**Ari89**, p.129]. Hence, the pressure p is just an input parameter like the external force, and the velocity u is computed according to this exterted sum force.

THEOREM 111. *For every external force $f \in (\mathcal{D}(\Omega))^d$ and pressure $p \in \mathcal{D}(\Omega)$ such that $\operatorname{div}(f - \nabla p) = 0$, there exists a globally-defined, strong, smooth, and compactly-supported solution $u \in E$ for the stationary Navier-Stokes equations (137) in the smooth and bounded domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$.*

PROOF. Space E is a strong W space, space E'_β is also a strong W space, by Theorem 26. Thus, they are also weak W spaces. In addition, E is separable.

By Lemma 108 (iii), operator A is bounded and (sequentially) demicontinuous. We have to show that map $u \mapsto \langle Au - b, u\rangle$ is coercive. Let $\mathcal{P}$ be a family of seminorms, generating the topology of l.c.s. E. A sequence u_n converges to ∞ in the compactification βE iff for all seminorms $p \in \mathcal{P}$ of l.c.s. E, for all $r > 0$, there exists $n_0 = n_0(r)$ such that $p(u_n) > r$ for all $n \geq n_0$. We choose the kinetic energy as norm $K(u) := \int_\Omega |u|^2 \,\mathrm{d}\lambda = \int_\Omega \sum_{i \in [d]} (u^i)^2 \,\mathrm{d}\lambda$. Then $K(u_n) \to \infty$ for this specific sequence.

For map $u \mapsto \langle A_1 u, u\rangle = (A_1 u)(u)$ and arbitrary $u \in E$, we have

$$\langle A_1 u, u\rangle = \int_\Omega |\nabla u|^2 \,\mathrm{d}\lambda \geq \lambda \cdot \int_\Omega |u|^2 \,\mathrm{d}\lambda = \lambda \cdot K(u) \quad ,$$

by the Brezis-Marcus inequality 109.

For map $u \mapsto \langle A_2 u, u\rangle = (A_2 u)(u)$ and arbitrary $u \in E$, we have

$$\begin{aligned}\langle A_2 u, u\rangle &= \int_\Omega \langle [\nabla u].u, u\rangle_{\mathbb{R}^d} \,\mathrm{d}\lambda = \int_\Omega \sum_{i,j=1}^d u^j \partial_{x_j} u^i \cdot u^i \,\mathrm{d}\lambda \\ &= \int_\Omega \sum_{j=1}^d u^j \partial_{x_j} \left(\frac{|u|^2}{2}\right) \mathrm{d}\lambda = -\int_\Omega (\operatorname{div} u) \frac{|u|^2}{2} \,\mathrm{d}\lambda = 0 \quad .\end{aligned}$$

For map $u \mapsto \langle b, u\rangle = b(u)$, by Cauchy-Schwartz, we have

$$\langle b, u\rangle \leq \sqrt{\int_\Omega |f - \nabla p|^2 \,\mathrm{d}\lambda} \cdot \sqrt{\int_\Omega |u|^2 \,\mathrm{d}\lambda} = c_b \cdot K(u) \quad .$$

Combined, we obtain that map $u \mapsto \langle Au - b, u\rangle$ is coercive.

$$\begin{aligned}\langle Au_n - b, u_n\rangle &= \langle A_1 u_n, u_n\rangle + \langle A_2 u_n, u_n\rangle - \langle b, u_n\rangle \\ &\geq \lambda \cdot K(u_n) + 0 - c_b \cdot \sqrt{K(u_n)} \quad \longrightarrow \infty \quad (n \to \infty) \quad .\end{aligned}$$

Hence, by Theorem 47, applied to A and b, there exists an $u \in E$ such that $Au = b$ in E'. For all $\phi \in E$, this means

$$\int_{\mathbb{R}^d} \langle -\Delta u + [\nabla u].u + \nabla p - f, \phi\rangle_{\mathbb{R}^d} \,\mathrm{d}\lambda = 0 \tag{147}$$

Successively, set $\phi = (\psi_1, 0, \dots, 0), (0, \psi_2, 0, \dots, 0), \dots (0, \dots, 0, \psi_d)$ for arbitrary functions $\psi_i \in \mathcal{D}(\Omega)$, $\operatorname{div} \psi_i = 0$. In particular, as for $Tu := (-\Delta u + [\nabla u].u + \nabla p - f)$ we have $\operatorname{div} Tu = 0$, we can take the i-th component $\psi_i := (Tu)_i$. Then one obtains

$$-\Delta u + [\nabla u].u + \nabla p - f = 0 \quad , \tag{148}$$

i.e., u is even a strong solution for the original equation (148), not only for the averaged one (147)! □

Imagine, we would have chosen a Sobolev space for E. Where does the argument break down in this case? First of all, the methods of the Theory of Monotonic Operators would yield an $u \in E = \mathcal{W}^{1,2}$ such that (147) holds in a weak sense. The notation hides that we do not have $\partial^\alpha u$ as ordinary derivatives of u. Hence, even plugging in bump functions ϕ, we can never come to (148), because the derivatives of u are just not defined. We only get a relation between the weak derivatives of u.

Finally, we mention the *Clay Millenium Prize Problem* of the Navier-Stokes equations. We refer the reader to [**Lem16**, Chapter 1] for a detailed description of the problem. The non-stationary version of the equations is given by

$$\partial_t u - \nu \cdot \Delta u + [\nabla u].u + \nabla p = f \quad , \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d \quad , \tag{149}$$

$$\operatorname{div} u = 0 \quad , \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d \quad . \tag{150}$$

Here, this time all time-dependent, $u(x,t) := (u^1(x,t), \dots, u_d(x,t)) \colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ denotes the velocity of a viscous fluid, $p \colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ the pressure, and $f \colon \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}^d$ is an external force, $d \geq 1$. In addition, constant $\nu > 0$ denotes the viscosity of the fluid.

Basically, we conjecture that the Theory of Maximally Monotonic Operators, mainly developed in the setting of Banach spaces, may help in the solution of this problem, when lifted to more general l.c.s., e.g., weak W spaces. Next steps could be: Generalize the Theorem of Browder to weak W spaces, see [**Růž04**, Chapter 3, Theorem 3.43]. Use the result of Rockafellar [**Roc66**] on the maximal monotonicity of the subdifferential, already partially formulated for general t.v.s.. Look specifically at the subdifferential of a generalization of the duality map. Define a *generalized time derivative* $\frac{\mathrm{d}u}{\mathrm{d}t}$ in space $E' = \mathcal{L}^1\left(\mathbb{R}^d_{\geq 0}, (\mathcal{S}^d)'\right)$, and prove that there exists $h\colon E \to \mathbb{R}_{\geq 0}$ such that

$$\int_s^t \left\langle \frac{\mathrm{d}u(\tau)}{\mathrm{d}t}, u(\tau) \right\rangle \mathrm{d}\lambda = h(u(t)) - h(u(s)) \quad .$$

Bibliography

[Ada70] Norbert Adasch, *Tonnelierte Räume und zwei Sätze von Banach*, Mathematische Annalen **186** (1970), 209–214 (ger).

[Ada83] ———, *Ein optimaler Satz über offene Abbildungen*, Mathematische Annalen **265** (1983), 113–114.

[Ada86] ———, *Der Satz über offene Abbildungen in topologischen Vektorräumen*, Mathematische Zeitschrift **191** (1986), 645–648.

[Ada03] Fournier Adams, *Sobolev Spaces*, 2nd ed., Pure and Applied Mathematics, no. 140, Elsevier Ltd., 2003.

[ADPV04] Jürgen Appell, Espedito De Pascale, and Alfonso Vignoli, *Nonlinear Spectral Theory*, De Gruyter Series in Nonlinear Analysis and Applications, vol. 10, Walter de Gruyter & Co., Berlin, 2004.

[AKP$^+$92] R. R. Akhmerov, Mikhail I. Kamenskiĭ, Alexandre S. Potapov, Alexandra Rodkina, and Boris N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel Boston Berlin, 1992.

[Alt06] Hans Wilhelm Alt, *Lineare Funktionalanalysis*, 5th ed., Springer, Berlin, Heidelberg, New York, 2006.

[Ari89] Rutherford Aris, *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*, Dover Publications Inc., 1989.

[AV05] Jürgen Appell and Martin Väth, *Elemente der Funktionalanalysis*, Vieweg, Wiesbaden, 2005.

[BM97] Haim Brezis and Moshe Marcus, *Hardy's inequalities revisited*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **Ser. 4, 25** (1997), no. 1–2, 217–237.

[BM98] Fernando Bonales and Rigoberto Mendoza, *Extending the formula to calculate the spectral radius of an operator*, Proceedings of the American Mathematical Society **126** (1998), no. 1, 97–103.

[Bou98a] Nicolas Bourbaki, *General topology. Chapters 1–4*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation.

[Bou98b] ———, *General topology. Chapters 5–10*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation.

[D9̈7] Michael Dörfner, *Spektraltheorie für nichtlineare Operatoren*, Ph.D. thesis, Universität Würzburg, 1997.

[Esk] Grigoriĭ Eskin, *Lectures on Linear Partial Differential Equations.*

[FMV76] Massimo Furi, Mario Martelli, and Alfonso Vignoli, *Stably-solvable operators in Banach spaces*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **60** (1976), no. 1, 21–26.

[FMV78] M. Furi, M. Martelli, and A. Vignoli, *Contributions to the spectral theory for nonlinear operators in Banach spaces*, Ann. Mat. Pura Appl. (4) **118** (1978), 229–294.

[FNSS73] Svatopluk Fučík, Jindřich Nečas, Jiří Souček, and Vladimír Souček, *Spectral analysis of nonlinear operators*, Lecture Notes in Mathematics, Vol. 346, Springer-Verlag, Berlin-New York, 1973.

[Fri63] Avner Friedman, *Generalized functions and partial differential equations*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963.

[Fur01] T. Furuta, *Invitation to Linear Operators: From Matrices to Bounded Linear Operators on a Hilbert Space*, Series in mathematical analysis and applications, Taylor & Francis, 2001.

[GD03] Andrzej Granas and James Dugundji, *Fixed point theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[HM62] Taqdir Husain and Mark Mahowald, *Barrelled spaces and the open mapping theorem*, Proc. Amer. Math. Soc. **13** (1962), 423–424.

[HOHOL02] M. Hoffmann-Ostenhof, Th. Hoffmann-Ostenhof, and Ari Laptev, *A geometrical version of Hardy's inequality*, Journal of Functional Analysis **189** (2002), no. 2, 539–548.

[Hol77] Ralf Hollstein, *Über die Tonneliertheit von lokalkonvexen Tensorprodukten*, Manuscripta mathematica **22** (1977), 7–12.

[Hus62] Taqdir Husain, *S-spaces and the open mapping theorem*, Pacific Journal of Mathematics **12** (1962), no. 1, 41–74.

[Hus64a] ———, *B(J)-Spaces and the Closed Graph Theorem*, Mathematische Annalen **153** (1964), 293–298.

[Hus64b] ———, *B(T)-Spaces and the Closed Graph Theorem II*, Mathematische Annalen **156** (1964), 334–336.

[Kač69] R. I. Kačurovskiĭ, *Regular points, spectrum and eigenfunctions of nonlinear operators*, Dokl. Akad. Nauk SSSR **188** (1969), 274–277.

[Kat04] Yitzhak Katznelson, *An Introduction to Harmonic Analysis*, 3rd ed., Cambridge Mathematical Library, Cambridge University Press, 2004.

[Kha82] S.M. Khaleelulla, *Counterexamples in Topological Vector Spaces*, Lecture Notes in Mathematics, no. 936, Springer-Verlag, 1982.

[Kri71] Vasagamoorthi Krishnasamy, *On the Closed Graph and Open Mapping Theorems*, Ph.D. thesis, McMaster University, November 1971.

[Kun93] Michael Kunzinger, *Barrelledness, Baire-like- and (LF)-spaces*, Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow, copublished in the United States with John Wiley & Sons, Inc., New York, 1993.

[Lem16] Lemarié-Rieusset, Pierre Gilles, *The Navier-Stokes Problem in the 21st Century*, CRC Press, Taylor & Francis Group, 2016.

[Lin17] Peter Lindqvist, *Notes on the p-Laplace Equation*, Tech. Report 161, University of Jyväskylä, 2017.

[Mat98] Martin Mathieu, *Funktionalanalysis*, Spektrum Akademischer Verlag, Berlin, Heidelberg, 1998.

[Mic56] Ernest Michael, *Continuous selections. I*, Ann. of Math. (2) **63** (1956), 361–382.

[Neu69] J. W. Neuberger, *Existence of a spectrum for nonlinear transformations*, Pacific J. Math. **31** (1969), 157–159.

[PC87] Pedro Pérez Carreras, *Some aspects of the theory of barreled spaces*, Časopis pro pěstování matematiky **112** (1987), no. 2, 123–161.

[Ptá58] Vlastimil Pták, *Completeness and the open mapping theorem*, Bulletin de la Société Mathématique de France **86** (1958), 41–74.

[Ptá59] ———, *On the closed graph theorem*, Czechoslovak Mathematical Journal **09** (1959), no. 4, 523–527.

[Ptá60] ———, *Über einen Typ von Sätzen über abgeschlossene Abbildungen*, Commentationes Mathematicae Universitatis Carolinae **001** (1960), no. 4, 3–7.

[Ptá62] ———, *The principle of uniform boundedness and the closed graph theorem*, Czechoslovak Mathematical Journal **12** (1962), no. 4, 523–528.

[Ptá65] ———, *Some Open Mapping Theorems in LF-Spaces and their Application to Existence Theorems for Convolution Equations*, Mathematica Scandinavica **16** (1965), 75–93.

[Ptá66] ———, *Some Metric Aspects of the Open Mapping and Closed Graph Theorems*, Mathematische Annalen **163** (1966), 95–104.

[Ptá69] ———, *Openness of linear mappings in LF-spaces*, Czechoslovak Mathematical Journal **19** (1969), no. 3, 547–552.

[Ptá74] ———, *A Theorem of the Closed Graph Type*, Manuscripta mathematica **13** (1974), 109–130.

[Rho84] Adolf Rhodius, *Über numerische Wertebereiche und Spektralwertabschätzungen*, Acta. Sci. Math. **47** (1984), 465–470.

[Roc66] R. T. Rockafellar, *Characterization of the subdifferentials of convex functions.*, Pacific J. Math. **17** (1966), no. 3, 497–510.

[Rod91] Brian Rodrigues, *Some new classes of topological vector spaces with closed graph theorems*, Commentationes Mathematicae Universitatis Carolinae **32** (1991), no. 2, 287–296.

[Rou05] George G. Roussas, *An Introduction to Measure-theoretic Probability*, Elsevier Ltd., 2005.

[Růž04] Michael Růžička, *Nichtlineare Funktionalanalysis*, Masterclass, Springer-Verlag, Berlin, Heidelberg, 2004.

[Sax74] Stephen A. Saxon, *Some Normed Barrelled Spaces which are not Baire*, Mathematische Annalen **209** (1974), 153–160.

[Sch57] Laurent Schwartz, *Théorie des distributions á valeurs vectorielles. I*, Annales de l'Institut Fourier **7** (1957), 1–141.

[Sch58] ———, *Théorie des distributions á valeurs vectorielles. II*, Annales de l'Institut Fourier **8** (1958), 1–209.

[SH13] Leonard Susskind and George Hrabovsky, *Classical mechanics*, Basic Books, New York, 2013, The theoretical minimum.

[SR89] S. Simons and Brian Rodrigues, *The Open Mapping and Closed Range Theorems*, Mathematische Annalen **283** (1989), no. 1, 87–96.

[SV06] Satish Shirali and Harkrishan L. Vasudeva, *Metric spaces*, Springer-Verlag London, Ltd., London, 2006.

[SW99] H. H. Schaefer and M. P. Wolff, *Topological Vector Spaces*, 2nd ed., Graduate Texts in Mathematics, vol. 3, Springer-Verlag, New York, 1999.

[Trè06] François Trèves, *Topological Vector Spaces, Distributions and Kernels*, Dover Publications, Inc., Mineola, NY, 2006, Unabridged republication of the 1967 original.

[Tro01] Vladimir G. Troitsky, *Spectral radii of bounded operators on topological vector spaces*, 2001.

[Val78] Manuel Valdivia, *On the Closed Graph Theorem in Topological Spaces*, Manuscripta mathematica **23** (1977–78), 173–184.

[Val71a] ———, *Absolutely convex sets in barrelled spaces*, Annales de l'Institut Fourier **21** (1971), no. 2, 3–13.

[Val71b] ———, *A hereditary property in locally convex spaces*, Annales de l'Institut Fourier **21** (1971), no. 2, 1–2.

[Val72a] ———, *On nonbornological barrelled spaces*, Annales de l'Institut Fourier **22** (1972), no. 2, 27–30.

[Val72b] ———, *Some examples on quasi-barrelled spaces*, Annales de l'Institut Fourier **22** (1972), no. 2, 21–26.

[Val73] ———, *On weak compactness*, Studia Mathematica **49** (1973), no. 1, 35–40.

[Val75] ———, *On B_r-completeness*, Annales de l'Institut Fourier **25** (1975), no. 2, 235–248.

[Val77] ———, *The space $D(U)$ is not B_r-complete*, Annales de l'Institut Fourier **27** (1977), no. 4, 29–43.

[Val79] ———, *On certain barrelled normed spaces*, Annales de l'Institut Fourier **29** (1979), no. 3, 39–56.

[Val84] ———, *B_r-Complete Spaces which are not B-Complete*, Mathematische Zeitschrift **185** (1984), 253–260.

[VC81] Manuel Valdivia and Pedro Pérez Carreras, *On Totally Barreled Spaces*, Mathematische Zeitschrift **178** (1981), 263–270.

[VD72] D. Van Dulst, *Barreledness of subspaces of countable codimension and the closed graph theorem*, Compositio Mathematica **24** (1972), no. 2, 227–234.

[vQ01] Boto von Querenburg, *Mengentheoretische Topologie*, 3rd ed., Springer Verlag, Berlin, Heidelberg, 2001.

[Wer00] Dirk Werner, *Funktionalanalysis*, extended ed., Springer-Verlag, Berlin, 2000.

[Xu01] Yuguang Xu, *A note on a continuous approximate selection theorem. Comment on: "Continuous selections and approximate selection for set-valued mappings and applications to metric projections" [SIAM J. Math. Anal.* **14** *(1983), no. 1, 185–194] by F. Deutsch and P. Kenderov*, J. Approx. Theory **113** (2001), no. 2, 324–325.

Index

Erklärung

Ich erkläre, dass ich die vorliegende Diplomarbeit mit dem Thema

Nichtlineare Spektraltheorie

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