Bachelorarbeit
Fakultät für Mathematik und Informatik
FernUniversität in Hagen
Lehrgebiet Analysis
Wintersemester 2021

The Connes-Thom Isomorphism

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Matrikelnummer: 8800600
Vorgelegt am 07.02.2022

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1 Introduction

A C^*-algebra A may be thought of as a complete subspace of the bounded linear operators $\mathcal{B}(H)$ on a Hilbert space H that is closed under taking adjoints and composition of operators. This viewpoint is made precise by the Gelfand-Naimark-Segal Theorem. We interpret the composition as a multiplication which is not commutative in general. In some cases however, it is.
The Gelfand-Naimark Theorem states that the multiplication is commutative if and only if there is a locally compact Hausdorff space X such that $A \cong C_0(X)$. The space $C_0(X)$ refers to the set of continuous functions which vanish at infinity. The isomorphism is a map that respects all operations from A and $C_0(X)$. For example, multiplication on $C_0(X)$ is given by point-wise multiplication. Thus, C^*-algebras formally describe functions on a "noncommutative" space that vanish at infinity.
One can then try to generalize results about commutative spaces to noncommutative ones. Concerning this viewpoint, we will mostly be concerned with K-theory. In K-theory, one assigns two groups K_0 and K_1 to a C^*-algebra A (a locally compact Hausdorff space in the commutative case). The groups K_0 and K_1 are constructed so that certain equivalences are respected. For example $K_j(A) \cong K_j(B)$ if A can be continuously deformed into B.
The theory of C^*-algebras largely depends on analyzing examples. Crossed product are a very powerful way to construct new C^*-algebras from old ones. If G is a group and A a C^*-algebra, the crossed product $A \rtimes G$ may informally be thought of as a C^*-algebra that contains both A and G where G is implemented through unitary operators.
As many interesting C^*-algebras arise from crossed products, it is desirable to have tools that allow the computation of K-theory for these algebras. The basic techniques to achieve that are Connes' Thom isomorphism and the Pimsner-Voiculescu sequence. The Connes-Thom isomorphism states that

$$K_0(A \rtimes \mathbb{R}) \cong K_1(A)$$

and

$$K_1(A \rtimes \mathbb{R}) \cong K_0(A).$$

The Pimsner-Voiculescu sequence allows to compute K-theory for many crossed products with $\mathbb{Z}$.
We will conduct the necessary results from the theory of C^*-algebras, K-theory and crossed products. Afterwards, we proceed with a very detailed description of the proof for Connes' Thom isomorphism which first appeared in [1] and is based on both, Connes' original proof from [2] and the proof of the Pimsner-Voiculescu sequence in [3].
Furthermore, we sketch how the Pimsner-Voiculescu sequence can be derived from Connes' Thom isomorphism. We end with the calculation of K-theory for some C^*-algebras using the Pimsner-Voiculescu sequence.

2 C^*-Algebras

2.1 Basic Theory

Definition 2.1. *Let A be a complex algebra. An involution on A is an antilinear map $^* : A \longrightarrow A$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for $a, b \in A$. If A admits an involution, we say that A is a *-algebra. Whenever A is a *-algebra, A is called a Banach-*-algebra if it is equipped with a norm $||\cdot||$ such that $||ab|| \leq ||a||\ ||b||$ for $a, b \in A$ holds, making A complete. A C^*-algebra A is a Banach-*-algebra, for which all $a \in A$ satisfy the C^*-identity $||a^*a|| = ||a||^2$.*

Whenever A and B are *-algebras, we call a linear map $\varphi : A \longrightarrow B$ a *-homomorphism if for $a, b \in A$ $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$ hold. A *-isomorphism is a bijective *-homomorphism. In that case, it's inverse φ^{-1} is a *-isomorphism too. If $\varphi : A \longrightarrow A$ is a *-isomorphism, then φ is called a *-automorphism. We will frequently write $A \cong B$ if there is a *-isomorphism between A and B. The algebra A is called unital if $(A, \cdot)$ admits a (multiplicative) unit and commutative if $ab = ba$ for all $a, b \in A$. We say that $a \in A$ is self-adjoint if $a^* = a$. If A, B are unital *-algebras and $\varphi : A \longrightarrow B$ is a *-homomorphism, then φ is called unital if $\varphi(1_A) = 1_B$.
Whenever X is a locally compact Hausdorff space and Y a topological space, the set of continuous functions $f : X \longrightarrow Y$ will be denoted by $C(X, Y)$.

Example 2.2.

(1) Let H be a Hilbert space. The Banach space $\mathcal{B}(H)$ is a C^*-algebra with composition as multiplication and adjoining of operators being it's involution.

(2) Let A be a C^*-algebra and X a locally compact Hausdorff space. For any $f \in C(X, A)$, we set $K_{f,\varepsilon} = \{x \in X \mid ||f(x)|| \geq \varepsilon\}$. Let $C_0(X, A) = \{f \in C(X, A) \mid K_{f,\varepsilon}$ compact for all *$\varepsilon > 0\}$. $C_0(X, A)$ is a C^*-algebra equipped with the pointwise operations from A and $||f|| = \sup_{x \in X} ||f(x)||_A$ for $f \in C_0(X, A)$.*

In this section, A will from now on denote a C^*-algebra. The starting point of the theory of C^*-algebras is the Gelfand-Naimark theorem. By $C_0(X)$, we will denote the C^*-algebra $C_0(X, \mathbb{C})$ for a given locally compact Hausdorff space.

Theorem 2.3 (Gelfand-Naimark)**.** *If A is a commutative C^*-algebra, then there is a locally compact Hausdorff space X with*

$$A \cong C_0(X).$$

Proof. See [4, Theorems 1.3.6 and 2.1.10]. □

This theorem gives us an intuition on how to think about noncommutative C^*-algebras. It is the space of continuous functions on a "noncommutative" topological space and we can try to generalize results from topology to C^*-algebras.
The following result about *-homomorphisms is frequently useful.

Theorem 2.4. *Suppose that $\varphi : A \longrightarrow B$ is a *-homomorphism of C^*-algebras.*

(1) The map φ is norm-decreasing, i.e. $||\varphi(a)||_B \leq ||a||_A$ for $a \in A$.

(2) If φ is injective, it is also isometric.

Proof. See [5, Theorem 1.5.7]. □

Theorem 2.4 implies that there is at most one norm turning a *-algebra into a C^*-algebra. Furthermore, this theorem replaces the fundamental theorems for operators on Banach spaces for *-homomorphisms. A sub-C^*-algebra of A is a subset $B \subseteq A$ that is a C^*-algebra with the operations inherited from A.

Definition 2.5. *Suppose that A is a C^*-algebra. An ideal of A is a closed linear subspace I such that for all $a \in A$ $aI \subseteq I$ and $Ia \subseteq I$.*

If I is an ideal of A, it is also a sub-C^*-algebra of A and the quotient A/I is a C^*-algebra [4, Theorems 3.1.3 and 3.1.4]. The multiplication is given by $(a+I)(b+I) = ab+I$ and the involution by $(a+I)^* = a^*+I$.

Proposition 2.6. *If $\varphi : A \longrightarrow B$ is a *-homomorphism of C^*-algebras, then $\varphi(A) \subseteq B$ is a C^*-algebra.*

Proof. The proof is taken from [4, Theorem 3.1.7].
Set $I = \ker(\varphi)$, so that I is an ideal of A. As A/I is a C^*-algebra, the map $a+I \mapsto \varphi(a)$ is an injective *-homomorphism of C^*-algebras and thus isometric by Theorem 2.4. But the image of isometric operators between Banach spaces is complete. □

We say that A is simple if it's only ideals are 0 and A itself. The next example from [4, Example 3.2.2] is a C^*-algebraic version of the finite range approximation theorem.

Example 2.7. *Given a Hilbert space H, the sub-C^*-algebra $\mathcal{K}(H)$ of $\mathcal{B}(H)$ is simple.*

Proof. Given an ideal $0 \neq I \subseteq \mathcal{K}(H)$, there is $K \in I$ with $K \neq 0$. We can multiply K with operators $T, S \in \mathcal{K}(H)$ to obtain any projection $P = TKS$ onto a 1-dimensional subspace of H. But I is assumed to be closed, so the spectral Theorem for compact operators yields $I = \mathcal{K}(H)$. That is, because we can always wright $K = \frac{K+K^*}{2} + i\frac{K-K^*}{2i}$ and both summands are up to a factor i self-adjoint. □

Definition 2.8. *Let B be a *-algebra. A *-representation of B is a pair (π, H), where H is a Hilbert space and $\pi : B \longrightarrow \mathcal{B}(H)$ is a *-homomorphism. If π is injective, (π, H) is said to be faithful.*

Theorem 2.9 (Gelfand-Naimark-Segal)**.** *Suppose A is a C^*-algebra. There is a faithful *-representation (π, H) of A.*

Proof. See [4, Theorem 3.4.1]. □

We will now introduce techniques that are mandatory for K-theory, following [6, Chapter 1].
In K-theory, the matrix-algebras of a C^*-algebra are important. Given $n \in \mathbb{N}$, the set $M_n(A)$ of $n \times n$-matrices with entries in A equipped with entry-wise addition and scalar-multiplication is a vector space. For $a, b \in M_n(A)$, we define $ab = (\sum\limits_{j=1}^{n} a_{ij}b_{jk})$ and $a^* = (a^*_{ji})_{ij}$. With these operations, $M_n(A)$ is a *-algebra.
There is a unique norm on $M_n(A)$ that turns $M_n(A)$ into a C^*-algebra [6, section 1.3]. Oftentimes, one needs to adjoin a unit to A using the following Proposition. We define $\tilde{A} = \{(a, z) \mid a \in A, z \in \mathbb{C}\}$. With the multiplication $(a, \alpha)(b, \beta) = (ab + a\beta + \alpha b, \alpha\beta)$ and the involution $(a, \alpha)^* = (a^*, \overline{\alpha})$, $\tilde{A}$ is a *-algebra with unit $(0, 1)$. We identify $\{(a, 0) \mid a \in A\}$ with A, turning A to a subset of $\tilde{A}$. Instead of $(0, 1)$, we will from now on write $1_{\tilde{A}}$.

Proposition 2.10. *Let A be a C^*-algebra.*

(1) The *-algebra $\tilde{A}$ carries a C^*-norm.

(2) It contains A as an ideal and $\tilde{A}/A \cong \mathbb{C}$.

Proof. See [5, Proposition 1.1.3]. □

Note that if A is already unital, the unit of $\tilde{A}$ does not agree with the original unit from A. If $\varphi : A \longrightarrow B$ is a *-homomorphism of C^*-algebras, the map $\tilde{\varphi} : \tilde{A} \longrightarrow \tilde{B}$, $a + \alpha 1_{\tilde{A}} \mapsto \varphi(a) + \alpha 1_{\tilde{B}}$ is a unital *-homomorphism.

Definition 2.11. *A sequence of C^*-algebras and *-homomorphisms*

$$... \longrightarrow A_n \xrightarrow{\varphi_n} A_{n+1} \xrightarrow{\varphi_{n+1}} A_{n+2} \longrightarrow ...$$

is said to be exact at A_{n+1} if $\varphi_n(A_n) = \ker(\varphi_{n+1})$, *The sequence is called exact if it is exact at all the A_n. If B, I are C^*-algebras and $\varphi : I \longrightarrow A$, $\psi : A \longrightarrow B$ are *-homomorphisms, then the sequence*

$$0 \xrightarrow{0_{0,I}} I \xrightarrow{\varphi} A \xrightarrow{\psi} B \xrightarrow{0_{B,0}} 0 \tag{1}$$

is called a short exact sequence if it as an exact sequence. In the above sequence, 0 denotes the C^*-algebra $\{0\}$. The maps $0_{0,I}$ and $0_{B,0}$ denote the zero-maps from 0 to I and from B to 0, respectively.

Thus, φ is always injective and ψ surjective in a short exact sequence. Since $0_{0,I}$ and $0_{B,0}$ are the only maps from 0 to I and from B to 0, respectively, we will often omit these terms above the corresponding arrows and write

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

instead. If there is a *-homomorphism $\lambda : B \longrightarrow A$ such that $\psi \circ \lambda = \mathrm{Id}_B$, the sequence is called split exact. That is, B is not only a quotient of A, but also a sub-C^*-algebra and the map ψ restricts to the identity on B.

Example 2.12. *Let A be a C^*-algebra.*

(1) If I is an ideal of A, then

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0$$

is a short exact sequence. The map π is given by the canonical projection, i.e. $\pi(a) = [a] = a + I$. The symbol ι denotes the inclusion map of $A \subset \tilde{A}$, i.e. $\iota(a) = a$ for $a \in A$.

(2) The special case of adjoining a unit gives rise to a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] & \tilde{A} \arrow[r, "\pi"] & \mathbb{C} \arrow[r] \arrow[l, bend left, "\lambda"] & 0. \end{tikzcd}$$

with $\lambda(z) = 1_{\tilde{A}} z$.

Proof. Let I be an ideal of A. If $x \in I$, then $\pi(x) = x + I = I$, showing $\iota(A) \subseteq \ker(\pi)$. Conversely, if $x \in \ker(\pi)$, then $I = \pi(x) = x + I$, thus $x \in I$ which shows $\ker(\pi) \subseteq \iota(A)$. Since ι is injective and π is surjective, the above sequence is exact.
Turning to the case of adjoining a unit, the above sequence is exact by Proposition 2.10. λ is a *-homomorphism identifying $\mathbb{C}$ with $\mathbb{C}1_{\tilde{A}}$. We have $\pi(\lambda(\alpha)) = \pi(\alpha 1_{\tilde{A}}) = \alpha 1_{\tilde{A}} + I = \alpha(1_{\tilde{A}} + I) = \alpha$. □

2.2 Tensor Products

Tensor products play a big role in both, K-theory and the study of crossed products.

Definition 2.13. *Let V and W be complex vector spaces. An algebraic tensor product of V and W is any complex linear space T such that there exists a bilinear $\tau : V \times W \longrightarrow T$ map satisfying the following universal property:*
Given a bilinear map $B : V \times W \longrightarrow S$ into another complex vector space S, there is a linear map $L : T \longrightarrow S$ with $B = L \circ \tau$.

Proposition 2.14. *Suppose V and W are $\mathbb{C}$-vector spaces. The following assertions hold:*

(1) There exists a tensor product of V and W.

(2) Any two tensor products of V and W are isomorphic.

Proof. See [7, Page 299, Satz 2]. □

By the second assertion, we may take any tensor product T of V and W and write $V \otimes W$ instead of T. The elements $\tau(v, w)$ are denoted by $v \otimes w$ for $v \in V$ and $w \in W$. These vectors span $V \otimes W$ linearly, see [7, Page 300].

Now, suppose that B and C are *-algebras. Then the algebraic tensor product $B \otimes C$ carries a *-algebra structures too, see [4, Pages 188 and 189]. For $b, b' \in C$ and $c, c' \in C$, the multiplication is given by $(b \otimes c)(b' \otimes c') = bb' \otimes cc'$ and the involution by $(b \otimes c)^* = b^* \otimes c^*$. Both operations are then extended linearly.

Definition 2.15. *If C is a *-algebra, then any norm γ on C that satisfies $\gamma(cc') \leq \gamma(c)\gamma(c')$ and $\gamma(c^*c) = \gamma(c)^2$ is called a C^*-norm on C.*

In particular, we do not insist C^*-norms to be complete. In fact, whenever a C^*-norm on C is complete, it must be the unique norm turning C into a C^*-algebra. The next proposition guaranties the existence of C^*-norms on algebraic tensor products of C^*-algebras.

Proposition 2.16. *Let A, B be C^*-algebras. There exists a C^*-norm $||\cdot||_{\max}$ on $A \otimes B$ such that given any C^*-norm γ on $A \otimes B$ the inequality*

$$\gamma(c) \leq ||c||_{\max} \tag{2}$$

holds for any $c \in A \otimes B$. Furthermore, $||a \otimes b||_{\max} \leq ||a|| \ ||b||$ for $a \in A$ and $b \in B$.

Proof. See [4, Theorem 6.3.5, Corollary 6.3.6 and Page 193]. □

The norm $||\cdot||_{\max}$ is called the maximal norm on $A \otimes B$. It's completion is called the maximal tensor product of A and B. It is denoted by $A \otimes_{\max} B$.

Definition 2.17. *We say that a C^*-algebra A is nuclear if for any C^*-algebra B, there is a unique C^*-norm on $A \otimes B$.*

We will from now on only be concerned with taking tensor products of C^*-algebras where one of the factors is nuclear.

Example 2.18. *Let H be a Hilbert space. Then $\mathcal{K}(H)$ is nuclear.*

Proof. See [4, Example 6.3.2]. □

Proposition 2.19. *If A is a commutative C^*-algebra, then it is nuclear too.*

Proof. See [4, Theorem 6.4.15]. □

Proposition 2.20. *Let X denote a locally compact Hausdorff space and A a C^*-algebra. The bilinear map $B : C_0(X) \times A \longrightarrow C_0(X, A)$, $B(f, a)(x) = f(x)a$ induces a *- homomorphism*

$$\varphi : C_0(X) \otimes A \longrightarrow C_0(X, A)$$

satisfying $\varphi(f \otimes a) = B(f, a)$. The map φ extends to a *-isomorphism $\varphi : C_0(X) \otimes_{\max} A \longrightarrow C_0(X, A)$.

Proof. See [4, Theorem 6.4.17] □

The following Proposition states that taking maximal tensor products is associative.

Proposition 2.21. *Let A, B, C be C^*-algebras. Then there is a *-isomorphism*

$$(A \otimes_{\max} B) \otimes_{\max} \cong A \otimes_{\max} (B \otimes_{\max} C)$$

mapping $(a \otimes b) \otimes c$ to $a \otimes (b \otimes c)$.

Proof. See [4, Page 215, Excercise 9]. □

Tensor products are also commutative.

Proposition 2.22. *Let A and B denote C^*-algebras. There is a *-isomorphism $A \otimes_{\max} B \cong B \otimes_{\max} A$ mapping $a \otimes b$ to $b \otimes a$.*

Proof. The map $A \times B \longrightarrow B \otimes A$, $(a, b) \mapsto b \otimes a$ is bilinear. Thus, it extends to a linear map $A \otimes B \longrightarrow B \otimes A$, $a \otimes b \mapsto b \otimes a$ by the definition of a tensor product. Repeating this with A and B interchanged, we observe that $B \otimes A \longrightarrow A \otimes B$, $b \otimes a \mapsto a \otimes b$ is it's inverse. These two maps are *-homorphisms. Now, any C^*-norm on $A \otimes B$ induces a C^*-norm on $B \otimes A$ and vice versa. Thus, $a \otimes b \mapsto b \otimes a$ extends to an isomorphism $A \otimes_{\max} B \cong B \otimes_{\max} A$. $\square$

3 K-theory

In K-theory, one assigns two groups, $K_0(A)$ and $K_1(A)$ to A. These groups reflect the properties of A in many ways. For example, a *-isomorphism of C^*-algebras A and B induces isomorphisms $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$.

3.1 The Semigroup $V(A)$

Definition 3.1. *Suppose that B is a *-algebra and that $p, u, v \in B$.*

(1) We call p a projection if $p = p^* = p^2$.

(2) We say that u is a unitary if $u^*u = uu^* = 1$.

(3) Whenever A is unital, v is said to be a partial isometry if $vv^*v = v$.

Topological properties of the set of projections in a given *-algebra play an important role. The topology will come from a C^*-norm.

Definition 3.2. *Given a *-algebra B that carries a C^*-norm, let $p, u, v \in B$. We write*

(1) $p \sim_h q$ if there is a continuous map $[0,1] \ni t \mapsto p_t \in B$ of projections such that $p_0 = p$ and $p_1 = q$,

(2) $p \sim_u q$ if there is a unitary $u \in \tilde{B}$ with $upu^* = q$,

(3) $p \sim q$ if $v^*v = p$ and $vv^* = q$ for a partial isometry $v \in B$.

The above relations $\sim_h, \sim_u, \sim$ are called homotopic equivalence, unitary equivalence and Murray-von Neumann equivalence, respectively. Each of the three relations is an equivalence relation [6, Page 21]. For any $p, q \in A$, we will write $\operatorname{diag}(p, q)$ instead of $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in M_2(A)$.

Proposition 3.3. *Let $p, q \in A$ be projections. Then the following implications hold:*

(1) If $p \sim_h q$ then $p \sim_u q$.

(2) If $p \sim_u q$ then $p \sim q$.

(3) If $p \sim q$ then $\operatorname{diag}(p, 0) \sim_u \operatorname{diag}(q, 0)$ in $M_2(A)$.

(4) If $p \sim_u q$ then $\operatorname{diag}(p, 0) \sim_h \operatorname{diag}(q, 0)$ in $M_2(A)$.

Proof. See [6, Propositions 2.2.7 and 2.2.8] $\square$

We set $M_\infty(A) = \{a \ |a \text{ is an } \mathbb{N} \times \mathbb{N} - \text{matrix with entries in } A \text{ and finitely many non-zero entries}\}$ and identify $M_n(A) \cong \{\operatorname{diag}(a, 0) \in M_{n+1} \mid a \in M_n(A)\} \subseteq M_{n+1}(A)$. With this notation, we have $M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A)$. $M_\infty(A)$ is a *-algebra inheriting the operations from the *-algebras $M_n(A)$. If $a \in M_n(A)$, then $||a||_{M_\infty(A)} = ||a||_{M_n(A)}$ is a norm on $M_\infty(A)$ as the right side is independent of n. Let $\mathcal{P}_\infty(A)$ denote it's projections, i.e. matrices $a \in M_n(A)$ for some $n \in \mathbb{N}$ with $a^* = a^2 = a$.

Definition 3.4. *Suppose that A is a C^*-algebra. We set $V(A) = \mathcal{P}_\infty(A)/\sim_h$.*

The set $(V(A), +)$ is a semigroup equipped with the addition $[p]_{\sim_h} + [q]_{\sim_h} = [\operatorname{diag}(p, q)]_{\sim_h}$. Furthermore, Proposition 3.3 implies that either of the relations $\sim_h, \sim_u, \sim$ agree with each other on $M_\infty(A)$. Thus, $V(A) = \mathcal{P}_\infty / \sim_u = \mathcal{P}_\infty / \sim$.
The following Proposition is based on [6, Proposition 2.3.2].

Proposition 3.5. *If A is a C^*-algebra, then the following assertions are true.*

(1) The semigroup $V(A)$ is Abelian.

(2) If $p, q \in M_\infty(A)$ are projections with $pq = qp = 0$, then $[p]_{\sim_h} + [q]_{\sim_h} = [p + q]_{\sim_h}$.

Proof. Fix projections $p \in M_n(A)$, $q \in M_m(A)$. We may suppose $n > m$, such that $q \in M_n(A)$. The matrix $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_n(\tilde{A})$ is a unitary. Thus, we have $\operatorname{diag}(p, q) \sim_u u^* \operatorname{diag}(p, q)u = \operatorname{diag}(q, p)$, which implies $[p]_{\sim_u} + [q]_{\sim_u} = [q]_{\sim_u} + [p]_{\sim_u}$.
Now assume $pq = qp = 0$ and let $v = \begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix} \in M_2(A)$. The equality $vv^* = \begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix} \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ implies $vv^*v = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix} = v$, so v is a partial isometry. But $p + q = v^*v \sim vv^* = \operatorname{diag}(p, q)$. □

The next example is a special case of [6, Example 3.3.3].

Example 3.6. *Let $H = \ell^2(\mathbb{N})$ and $\delta_{11} \in \mathcal{B}(H)$ be the operator defined by $\delta_{11}(x_i)_{i\in\mathbb{N}} = (\delta_{1i}x_i)_{i\in\mathbb{N}}$. Then $1_{\mathcal{B}(H)} - \delta_{11}$ is a projection and $1_{\mathcal{B}(H)} - \delta_{11} \sim 1_{\mathcal{B}(H)}$. In particular, we have $[1_{\mathcal{B}(H)}]_\sim + [\delta_{11}]_\sim = [1_{\mathcal{B}(H)}]_\sim$.*

Proof. $1_{\mathcal{B}(H)} - \delta_{11}$ is a projection with range $\{x \in H \mid x_1 = 0\}$. Let $v \in \mathcal{B}(H)$ be the unilateral right shift. Then $v^*v = 1_{\mathcal{B}(H)}$ and $vv^* = 1_\mathcal{B}(H) - \delta_{11}$, and we conclude $1_{\mathcal{B}(H)} - \delta_{11} \sim 1_{\mathcal{B}(H)}$. The equality $[1_{\mathcal{B}(H)}]_\sim + [\delta_{11}]_\sim = [1_{\mathcal{B}(H)}]_\sim$ now follows from Proposition 3.5 by noting that $(1_{\mathcal{B}(H)} - \delta_{11})\delta_{11} = 0$. □

Example 3.6 shows that V(A) does not have cancellation in general.

3.2 The K-groups

To define $K_0(A)$, one needs the so-called Grothendieck-construction, see [6, Paragraph 3.1.1]. For an Abelian semigroup S, we now define a relation on $S \times S$. Given $(r, s), (r', s') \in S \times S$, write $(r, s) \sim (r', s')$ if there is $t \in S$ such that $r + s' + t = r' + s + t$.

Lemma 3.7. *The relation $\sim$ is an equivalence relation and $G(S) = S \times S/\sim$ is an Abelian group.*

Proof. See [6, Paragraph 3.1.2]. □

The addition on $G(S)$ is given by $[(s, t)] + [(s', t')] = [(s + s', t + t')]$. We shall write $s - t$ instead of $[(s, t)]$ if $s, t \in S$. The map $\gamma_S : S \longrightarrow G(S)$, $s \mapsto s - 0$ is a semigroup homomorphism.

Lemma 3.8. *If $\varphi : S \longrightarrow T$ is a semigroup homomorphism, there exists a unique group homomorphism $G(\varphi) : G(S) \longrightarrow G(T)$ such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ \downarrow{\scriptstyle \gamma_S} & & \downarrow{\scriptstyle \gamma_T} \\ G(S) & \xrightarrow{G(\varphi)} & G(T) \end{array}$$

commutes, i.e. $G(\varphi)(s_1 - s_2) = \varphi(s_1) - \varphi(s_2)$.

Proof. See [6, Paragraph 3.1.2]. □

Definition 3.9. *Suppose A is a C^*-algebra. We set $K_{00}(A) = G(V(A))$ and write $[p]_{00}$ instead of $[p]_{\sim_h} - [0]_{\sim_h}$ for a projection $p \in M_\infty(A)$.*

The following Proposition shows that K_{00} respects *-homomorphisms, see [8, Sections 5.2, 5.3] and [6, Section 3.2.2, Proposition 3.2.4].

Proposition 3.10. *For a C^*-algebra B and a *-homomorphism $\varphi : A \longrightarrow B$, there is a group homomorphism $K_{00}(\varphi) : K_{00}(A) \longrightarrow K_{00}(B)$ such that the following properties hold:*

(1) We have $K_{00}(\mathrm{Id}_A) = \mathrm{Id}_{K_{00}(A)}$.

(2) If $\psi : B \longrightarrow C$ is another *-homomorphism into a C^*-algebra C, then $K_{00}(\psi\circ\varphi) = K_{00}(\psi)\circ K_{00}(\varphi)$.

It is given by

$$K_{00}(\varphi)([p]_{00} - [q]_{00}) = [\varphi(p)]_{00} - [\varphi(q)]_{00}. \tag{3}$$

Proof. If $p \sim_h q$ in $M_\infty(A)$, there is a path of projections p_t from p to q. Since the mapping $t \mapsto p_t$ is continuous, so is $t \mapsto \varphi(p_t)$, also $\varphi(p_0) = p$ and $\varphi(p_1) = q$. Thus, $\varphi(p) \sim_h \varphi(q)$ and the map $\alpha : V(A) \longrightarrow V(B)$, $\alpha[p]_{\sim_h} = [\varphi(p)]_{\sim_h}$ is well-defined. For projections $p, q \in A$, we have

$$\begin{aligned}\alpha([p]_{\sim_h} + [q]_{\sim_h}) &= \alpha[\mathrm{diag}(p,q)]_{\sim_h} \\ &= [\varphi(\mathrm{diag}(p,q))]_{\sim_h} \\ &= [\mathrm{diag}(\varphi(p),\varphi(q))]_{\sim_h} \\ &= [\varphi(p)]_{\sim_h} + [\varphi(q)]_{\sim_h} \\ &= \alpha[p]_{\sim_h} + \alpha[q]_{\sim_h}\end{aligned} \tag{4}$$

Thus, α is a semigroup homomorphism. Set $K_{00}(\varphi) = G(\alpha)$. Formula (3) holds by definition. We have

$$K_{00}(\mathrm{Id}_A)([p]_{00} - [q]_{00}) = [\mathrm{Id}_A(p)]_{00} - [\mathrm{Id}_A(q)]_{00} = [p]_{00} - [q]_{00} = \mathrm{Id}_{K_{00}(A)}([p]_{00} - [q]_{00})$$

and

$$K_{00}(\psi \circ \varphi)([p]_{00} - [q]_{00}) = [\psi(\varphi(p))]_{00} - [\psi(\varphi(q))]_{00} = K_{00}(\psi)(K_{00}(\varphi)([p]_{00} - [q]_{00})).$$

□

Lemma 3.11. *If A is a unital C^*-algebra, the split exact sequence of Example 2.12 (2) induces a split exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & K_{00}(A) \arrow[r, "K_{00}(\iota)"] & K_{00}(\tilde{A}) \arrow[r, "K_{00}(\pi)"] & K_{00}(\mathbb{C}) \arrow[r] \arrow[l, bend left, "K_{00}(\lambda)"] & 0. \end{tikzcd} \tag{5}$$

Proof. See [6, Proposition 3.2.8]. □

Now, we are able to define $K_0(A)$.

Definition 3.12. *Let $\pi : \tilde{A} \longrightarrow \mathbb{C}$ denote the canonical projection. Set $K_0(A) = \ker(K_{00}(\pi)) \subseteq K_{00}(\tilde{A})$.*

The following Proposition is based on the [6, Discussion below Definition 4.1.1].

Proposition 3.13. *If A is a unital C^*-algebra, then $K_{00}(A) \cong K_0(A)$.*

Proof. Let $\iota : A \longrightarrow \tilde{A}$ denote the inclusion $A \subseteq \tilde{A}$ and $\pi : \tilde{A} \longrightarrow \mathbb{C}$ the canonical projection. By definition, the sequence

$$0 \longrightarrow K_0(A) \xrightarrow{\ i\ } K_{00}(\tilde{A}) \xrightarrow{K_{00}(\pi)} K_{00}(\mathbb{C}) \longrightarrow 0.$$

is exact if $i : K_0(A) \longrightarrow K_{00}(\tilde{A})$ denotes the inclusion $K_0(A) \subseteq K_{00}(\tilde{A})$. If $p \sim_h q$ in A, then $p \sim_h q$ in $\tilde{A}$. Thus, the map $\alpha : K_{00}(A) \longrightarrow K_0(A)$, $K_{00}(A) \ni [p]_{00} \mapsto [p]_{00} \in K_{00}(\tilde{A})$ is well-defined. Indeed, $K_{00}(\pi)(\alpha([p]_{00} - [q]_{00})) = K_{00}(\pi)([\iota(p)]_{00} - [\iota(q)]_{00}) = K_{00}(\pi) \circ K_{00}(\iota)([p]_{00} - [q]_{00}) = 0$,

because the sequence (5) is exact at $K_{00}(\tilde{A})$. Since

$$K_{00}(\iota)([p]_{00} - [q]_{00}) = \alpha([p]_{00} - [q]_{00}),$$

the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & K_{00}(A) \arrow[r, "K_{00}(\iota)"] \arrow[d, "\alpha"] & K_{00}(\tilde{A}) \arrow[r, "K_{00}(\pi)"] \arrow[d, equal] & K_{00}(\mathbb{C}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_0(A) \arrow[r, "i"] & K_{00}(\tilde{A}) \arrow[r, "K_{00}(\pi)"] & K_{00}(\mathbb{C}) \arrow[r] & 0 \end{tikzcd} \tag{6}$$

Now, $K_{00}(\iota)$ is injective, because the upper sequence is exact at $K_{00}(A)$. Thus, $i \circ \alpha = K_{00}(\iota)$ implies that α is injective. α is surjective too, because i is injective and $K_{00}(\iota)(K_{00}(A)) = \ker(\pi) = i(K_0(A))$. □

If A is unital, we may identify $K_0(A)$ with $K_{00}(A)$ and write $[p]_0$ instead of $[p]_{00}$. Under this identification, the following Theorem holds.

Theorem 3.14 (Standard picture of $K_0(A)$)**.** *Let A be a C^*-algebra.*

$$K_0(A) = \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_\infty(\tilde{A}),\ \ p - q \in M_\infty(A)\} \tag{7}$$

Additionally, we have

(1) $K_0(A) = \{[p]_0 - [s(p)]_0 \mid p \in \mathcal{P}_\infty(\tilde{A})\}$, where $s = \lambda \circ \pi$, i.e. $s(a + 1_{\tilde{A}}\alpha) = 1_{\tilde{A}}\alpha$.

(2) Let $p, q \in M_\infty(\tilde{A})$. Then $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$ if and only if there are $r_1, r_2 \in \mathcal{P}_\infty(\tilde{A})$ with $s(r_1) = r_1$ and $s(r_2) = r_2$ with $\operatorname{diag}(p, r_1) \sim_h \operatorname{diag}(q, r_2)$. In this case, one may choose $r_1 = 1_k$, $r_2 = 1_l$ for some $k, l \in \mathbb{N}$.

(3) For $p, q \in M_\infty(\tilde{A})$ satisfying $p - q \in M_\infty(A)$, we have $[p]_0 - [q]_0 = 0$ if and only if there is $m \in \mathbb{N}$ such that $\operatorname{diag}(p, 1_m) \sim_h \operatorname{diag}(q, 1_m)$.

Proof. See [6, Proposition 4.2.2]. □

Let $\varphi : A \longrightarrow B$ be a *-homomorphism of C^*-algebras. If $[p]_0 - [q]_0 \in K_0(A)$, we calculate that $\tilde{\varphi}(p) - \tilde{\varphi}(q) = \tilde{\varphi}(p - q) = \varphi(p - q) \in M_\infty(B)$. So φ induces a homorphism $K_0(\varphi) : K_0(A) \longrightarrow K_0(B)$ with

$$K_0(\varphi)([p]_0 - [q]_0) = [\tilde{\varphi}(p)]_0 - [\tilde{\varphi}(q)]_0. \tag{8}$$

If A is unital, $K_0(\varphi)$ is given by $K_{00}(\varphi)$ under the identification of Proposition 3.13.
Suppose that H is a seperable Hilbert space and $E \in \mathcal{K}(H)$ is a projection of rank one, i.e. a projection such that there is $\xi \in H$ with $E\xi = \xi$ and $Eh = 0$ if $\xi \perp h$. Now, the map $\varphi : A \longrightarrow A \otimes \mathcal{K}(H)$, $\varphi(a) = \varphi(a) \otimes E$ is a *-homomorphism, because E is a projection.

Proposition 3.15. *The map $K_0(\varphi)$ is an isomorphism.*

Proof. See [9, Corollary 6.2.11]. □

We now follow [8, Section 8.1] to introduce $K_1(A)$.

Definition 3.16. *Suppose that A is a unital C^*-algebra and let $u, v \in A$ be unitaries.*

(1) We write $u \sim_h v$ if there is a continuous map $[0, 1] \ni t \mapsto u_t \in A$ of unitaries such that $u_0 = u$ and $u_1 = v$.

(2) We set $\mathcal{U}(A) = \{u \in A \mid u$ unitary$\}$ and $\mathcal{U}(A)_0 = \{u \in \mathcal{U}(A) \mid u \sim_h 1\}$.

(3) Given $n \in \mathbb{N}$, we let $\mathcal{U}_n(A)$ and $\mathcal{U}_n(A)_0$ denote the sets $\mathcal{U}_n(M_n(A))$ and $\mathcal{U}_n(M_n(A))_0$, respectively.

We will identify $u \in \mathcal{U}_n(\tilde{A})$ with $\operatorname{diag}(u, 1) \in \mathcal{U}_{n+1}(\tilde{A})$. This turns $\mathcal{U}_n(\tilde{A})$ into a subgroup of $\mathcal{U}_{n+1}(\tilde{A})$. Let $\mathcal{U}_\infty(\tilde{A}) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(\tilde{A})$. By $\mathcal{U}_\infty(\tilde{A})_0$ we denote the set $\bigcup_{n=1}^{\infty} \mathcal{U}_n(\tilde{A})_0 \subseteq \mathcal{U}_\infty(\tilde{A})$. Both, $\mathcal{U}_\infty(\tilde{A})$ and $\mathcal{U}_\infty(\tilde{A})_0$ are groups inheriting the group operations from the $\mathcal{U}_n(\tilde{A})$. If $u \in \mathcal{U}_n(\tilde{A})_0$ and $v \in \mathcal{U}_n(\tilde{A})$, there is a continuous map $t \mapsto u_t$ such that $u_0 = u$ and $u_1 = u$. But then $t \mapsto v^* u_t v$ is also continuous, which implies $v^* u v \sim_h v^* 1 v = 1$. Thus, $\mathcal{U}_\infty(\tilde{A})_0$ is a normal subgroup of $\mathcal{U}_\infty(\tilde{A})$.

Definition 3.17. *Given a C^*-algebra A, we set $K_1(A) = \mathcal{U}_\infty(\tilde{A}) / \mathcal{U}_\infty(\tilde{A})_0$.*

For $u \in \mathcal{U}_\infty(\tilde{A})$, let $[u]_1$ denote the equivalence class of u. So we get a map $[\ ]_1 : \mathcal{U}_\infty(\tilde{A}) \longrightarrow K_1(A)$, $u \mapsto [u]_1$. As for K_0, there is a standard picture of K_1.

Theorem 3.18 (Standard picture of $K_1(A)$)**.** *If A is a C^*-algebra, we have*

$$K_1(A) = \{[u]_1 \mid u \in \mathcal{U}_\infty(\tilde{A})\} \tag{9}$$

The group $K_1(A)$ satisfies the following properties:

(1) The map $[\ ]_1$ is a group homomorphism.

(2) Addition is given by$[\mathrm{diag}(u,v)] = [u]_1 + [v]_1$.

(3) The group $K_1(A)$ *is Abelian.*

(4) For $u, v \in \mathcal{U}_\infty(\tilde{A})$, *we have* $[u]_1 = [v]_1$ *if and only if* $u \sim_h v$.

Proof. See [6, Proposition 8.1.4]. □

Now, let $\varphi : A \longrightarrow B$ be a *-homomorphism of C^*-algebras. If $u, v \in M_n(\tilde{A})$ with $u \sim_h v$, we have $\tilde{\varphi}(u) \sim_h \tilde{\varphi}(v)$, so the map $K_1(\varphi) : K_1(A) \longrightarrow K_1(B)$, $[u]_1 \mapsto [\tilde{\varphi}(u)]_1$ is well-defined.

Proposition 3.19. *Let* $\varphi : A \longrightarrow B$ *and* $\psi : B \longrightarrow C$ *be* **-homomorphisms of* C^**-algebras. Then the following properties hold for* $j \in \{0, 1\}$*:*

(1) $K_j(\mathrm{Id}_A) = \mathrm{Id}_{K_j(A)}$

(2) $K_j(\psi \circ \varphi) = K_j(\psi) \circ K_j(\varphi)$

Proof. See [6, Propositions 4.1.3 and 8.2.2]. □

To state another important property of K-theory, we need the notion of a homotopy.

Definition 3.20. *Suppose that* $\varphi, \varphi' : A \longrightarrow B$ *and* $\psi : B \longrightarrow A$ *are* **-homomorphisms of* C^**-algebras.*

(1) φ *and* φ' *are called homotopic if there is a map* $[0,1] \ni t \mapsto \varphi_t$ *of* **-homomorphisms* $\varphi_t : A \longrightarrow B$ *such that* $\varphi_0 = \varphi$, $\varphi_1 = \varphi'$ *and* $t \mapsto \varphi_t(a)$ *continuous for all* $a \in A$ *in the norm.*

(2) The sequence (not necessarily exact) $A \xrightarrow{\varphi} B \xrightarrow{\psi} A$ *is called a homotopy if* $\varphi \circ \psi$ *is homotopic to* Id_B *and* $\psi \circ \varphi$ *to* Id_A.

Proposition 3.21. *Suppose that* $A \xrightarrow{\varphi} B \xrightarrow{\psi} A$ *is a homotopy of* C^**-algebras. Then for* $j = 0, 1$ *the map* $K_j(\varphi) : K_j(A) \longrightarrow K_j(B)$ *is an isomorphism with inverse* $K_j(\psi)$.

Proof. See [6, Proposition 4.1.4] for the proof if $j = 0$. The statement for $j = 1$ can be derived from that by using Proposition 3.28. □

Proposition 3.22. *Suppose that* H *is a seperable (not necessarily finite dimensional) Hilbert space. Then* $K_0(\mathcal{K}(H)) = \mathbb{Z}$ *and* $K_1(\mathcal{K}(H)) = 0$.

Proof. See [6, Corollary 6.4.2 and Example 8.1.8]. □

The isomorphism may described as follows. A projection in $\mathcal{K}(H)$ is a projection onto a finite dimensional subspace. One may identify $M_n(\mathcal{K}(H)) \cong \mathcal{K}(H^n)$. Now a projection $p \in \mathcal{K}(H^n)$ is mapped to the dimension of the subspace it projects onto. In particular, 1 is the image of the rank one projections.

3.3 The Cyclic Six-Term Exact Sequence

The following Proposition shows that K-theory respects exactness properties. The six-term exact sequence is then derived by defining so-called connecting maps δ_j between $K_j(B)$ and $K_{|j-1|}(I)$ for $j = 0, 1$.

Proposition 3.23. *Let*

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

be a short exact sequence of C^**-algebras. Then the induced sequence*

$$0 \longrightarrow K_j(I) \xrightarrow{K_j(\varphi)} K_j(A) \xrightarrow{K_j(\psi)} K_j(B) \longrightarrow 0$$

is exact at $K_j(A)$ *for* $j \in \{0, 1\}$.

Proof. See [6, Propositons 4.3.2 and 8.2.4] □

With the convention $arctan(\infty) = \pi/2$, the map $d(r,t) = |arctan(r) - arctan(t)|$ defines a metric on the set $\mathbb{R} \cup \{\infty\}$. This metric turns $\mathbb{R} \cup \{\infty\}$ into a locally compact Hausdorff space.

Definition 3.24. *We set*

$$SA = C_0(\mathbb{R}, A) \tag{10}$$

and

$$CA = \{f \in C_0(\mathbb{R} \cup \{\infty\}, A)\} \tag{11}$$

SA is called the suspension of A and CA the cone of A.

When talking about suspensions and cones of A, we will write f_t instead of $f(t)$ for $f \in CA$ or $f \in SA$. Also, $S\mathbb{C} = C_0(\mathbb{R})$ and $C\mathbb{C}$ will be denoted by S and C, respectively.
The maps $\iota : SA \longrightarrow CA$ given by continuous extension, i.e. by setting $\iota(f)_\infty = 0$ and $\pi : CA \longrightarrow A$, $\pi(f) = f(\infty)$, yield a short exact sequence

$$0 \longrightarrow SA \xrightarrow{\iota} CA \xrightarrow{\pi} \mathbb{C} \longrightarrow 0 \tag{12}$$

Furthermore, given a *-homomorphism $\varphi : A \longrightarrow B$ into a C^*-algebra B, then the map $S\varphi : SA \longrightarrow SB$ given by $S\varphi(f)_t = \varphi(f_t)$ is a *-homomorphism. Also, the map $C\varphi : CA \longrightarrow CB$, $C\varphi(f)_t = \varphi(f_t)$ is a *-homomorphism.
The following Lemma is a consequence of Proposition 2.20, see also [6, Lemma 10.1.1].

Lemma 3.25. *Let X be a locally compact Hausdorff space. Then elements $g \in C_0(X, A)$ of the form $g(x) = \sum_{i=1}^{n} f_i(x)a_i$ for $f_i \in C_0(X)$ and $a_i \in A$ are dense in $C_0(X, A)$.*

By choosing $X = \mathbb{R}$ and $X = \mathbb{R} \cup \{\infty\}$, we obtain the following Proposition which is taken from [6, Proposition 10.1.2].

Proposition 3.26. *If*

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

is a short exact sequence of C^*-algebras, then both sequences,

$$0 \longrightarrow SI \xrightarrow{S\varphi} SA \xrightarrow{S\psi} SB \longrightarrow 0$$

and

$$0 \longrightarrow CI \xrightarrow{C\varphi} CA \xrightarrow{C\psi} CB \longrightarrow 0$$

are short exact sequences too.

Proof. We will only proof the result for cones, the rest follows from replacing every C by an S.
$C\varphi$ is injective, because φ is and $C\varphi(CI) \subseteq \ker(C\psi)$, because $C\psi \circ C\varphi = C(\psi \circ \varphi) = C0 = 0$. If $f \in CA$ with $\psi(f) = 0$ is given, then $C\psi(f_t) = 0$ for all t so that $f_t = \varphi(g_t)$ for some $g_t \in I$. As $||\varphi(g_t)|| = f_t$, we infer $g \in CI$ and $C\varphi(g) = f$.
By Lemma 3.25, $\psi(A)$ contains a dense subspace of B and is thus surjective, because the image of a *-homomorphism is a C^*-algebra by Proposition 2.6. □

The most important tools for calculating the K-theory of a C^*-algebra are Bott periodicity and the cyclic six-term exact sequence. Let

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{13}$$

be a short exact sequence of C^*-algebras.

Proposition 3.27. *There is a group homomorphism $\delta_1 : K_1(B) \longrightarrow K_0(I)$, which makes the sequence*

$$K_1(I) \xrightarrow{K_1(\varphi)} K_1(A) \xrightarrow{K_1(\psi)} K_1(B) \xrightarrow{\delta_1} K_0(I) \xrightarrow{K_0(\varphi)} K_0(A) \xrightarrow{K_0(\psi)} K_0(B)$$

exact.

Proof. See [6, Lemma 9.1.1 and Proposition 9.3.3]. □

Note that by Propositon 3.23, the sequence above is already exact at $K_0(A)$ and $K_1(A)$. The map δ_1 is called the index map associated to the sequence (13).

Proposition 3.28. *There is an isomorphism $\theta_A : K_1(A) \longrightarrow K_0(SA)$. Furthermore if B is a C^*-algebra, the following diagram commutes for any *-homomorphism $\varphi : A \longrightarrow B$:*

$$\begin{tikzcd} K_1(A) \arrow[r, "\theta_A"] \arrow[d, "K_1(\varphi)"] & K_0(SA) \arrow[d, "K_0(S\varphi)"] \\ K_1(B) \arrow[r, "\theta_B"] & K_0(SB) \end{tikzcd} \tag{14}$$

Proof. See [6, Theorem 10.1.3]. □

Theorem 3.29. *[Bott periodicity] There is an isomorphism $\beta_A : K_0(A) \longrightarrow K_1(SA)$. Furthermore if B is a C^*-algebra, the following diagram commutes for any *-homomorphism $\varphi : A \longrightarrow B$:*

$$\begin{tikzcd} K_0(A) \arrow[r, "\beta_A"] \arrow[d, "K_0(\varphi)"] & K_1(SA) \arrow[d, "K_1(S\varphi)"] \\ K_0(B) \arrow[r, "\beta_B"] & K_1(SB) \end{tikzcd} \tag{15}$$

Proof. See [6, Chapter 11]. □

Remark 3.30. *Suppose that A is a C^*-algebra. Bott periodicity, Theorem 3.29 and Proposition 3.28 may be summarized as*

$$K_j(A) = K_{|j-1|}(SA)$$

for $j = 0, 1$.

Theorem 3.31 (six-term exact sequence)**.** *There is a group homomorphism $\delta_0 : K_0(B) \longrightarrow K_1(I)$ such that the six-term sequence*

$$\begin{tikzcd} K_0(I) \arrow[r, "K_0(\varphi)"] & K_0(A) \arrow[r, "K_0(\psi)"] & K_0(B) \arrow[d, "\delta_0"] \\ K_1(B) \arrow[u, "\delta_1"] & K_1(A) \arrow[l, "K_1(\psi)"] & K_1(I) \arrow[l, "K_1(\varphi)"] \end{tikzcd} \tag{16}$$

of Abelian groups is exact.

The map δ_0 is called the exponential map associated to the sequence (13). The six-term exact sequence is in fact a combination of Theorem 3.29 and Proposition 3.27. For that reason, the term Bott periodicity often refers to both the six-term sequence and Theorem 3.29 in the literature.

The most important aspect about the index and exponential map is that these make the six-term sequence exact. However, we will need two more facts about these maps as we proceed.

Proposition 3.32. *Suppose that*

$$0 \longrightarrow I' \xrightarrow{\varphi'} A' \xrightarrow{\psi'} B' \longrightarrow 0 \tag{17}$$

is a short exact sequence of C^*-algebras. Furthermore, suppose that we are given *-homomorphisms $\gamma : I \longrightarrow I'$, $\alpha : A \longrightarrow A'$ and $\beta : B \longrightarrow B'$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\varphi"] \arrow[d, "\gamma"] & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & I' \arrow[r, "\varphi'"] & A' \arrow[r, "\psi'"] & B' \arrow[r] & 0 \end{tikzcd} \tag{18}$$

We denote the index and exponential map of (6.8) by δ_0' and δ_1', respectively. In this situation both diagrams,

$$\begin{array}{ccc} K_1(B) & \xrightarrow{\delta_1} & K_0(I) \\ \Big\downarrow{\scriptstyle K_1(\beta)} & & \Big\downarrow{\scriptstyle K_0(\gamma)} \\ K_1(B') & \xrightarrow{\delta_1'} & K_0(I') \end{array} \tag{19}$$

and

$$\begin{array}{ccc} K_0(B) & \xrightarrow{\delta_0} & K_1(I) \\ \Big\downarrow{\scriptstyle K_0(\beta)} & & \Big\downarrow{\scriptstyle K_1(\gamma)} \\ K_0(B') & \xrightarrow{\delta_0'} & K_1(I') \end{array} \tag{20}$$

commute.

Proof. See [6, Proposition 9.1.5 and Proposition 12.2.1]. □

Proposition 3.33 (Second standard picture of the index map)**.** *Let A be a C^*-algebra. Suppose that we are given natural numbers $m \geq n$ and let $u \in \mathcal{U}_n(\tilde{B})$. Also assume that $v \in M_m(\tilde{A})$ is a partial isometry such that $\tilde{\psi}(v) = \operatorname{diag}(u, 0_{m-n})$. In that case, there are projections $p, q \in M_m(\tilde{I})$ satisfying $1_m - v^*v = \tilde{\varphi}(p)$ and $1_m - vv^* = \tilde{\varphi}(q)$ and*

$$\delta_1(u) = [p]_0 - [q]_0.$$

Proof. See [6, Proposition 9.2.2]. □

Note that since φ is injective, p and q are unique up to choice of v. Also, since δ_1 is well-defined, $[p]_0 - [q]_0$ does not depend on the choice of v.

4 C^*-Dynamics and Crossed Products

Crossed products are C^*-algebras that are made out of another C^*-algebra and a group.

Definition 4.1. *Let G be a group that is endowed with a topology $\mathcal{T}$.*

(1) The group G is called a topological group if both the maps $\cdot : G \times G \longrightarrow G$ and ${}^{-1} : G \longrightarrow G$ are continuous where $G \times G$ is equipped with the product topology.

(2) A locally compact group is a topological group that is locally compact and Hausdorff.

Throughout this paragraph, G will denote a locally compact group and A a C^*-algebra.

4.1 Haar Measure

To define crossed products of C^*-algebras, one needs the notion of Haar-measure.

Definition 4.2. *Let X be a locally compact Hausdorff space.*

(1) A Borel measure μ on X is a measure on X such that any Borel set of X is measurable. That is, any open set is measurable.

(2) Suppose μ is a Borel measure on X. If for any measurable set A of X and any open set $V \subseteq X$

$$\mu(V) = \sup\{\mu(C) \mid C \subseteq V \ , \ C \ \textit{compact}\} \tag{21}$$

and

$$\mu(A) = \inf\{\mu(U) \mid V \subseteq U, \ U \ \textit{open}\} \tag{22}$$

then μ is called a Radon measure on X.

(3) Suppose that G is a locally compact group. A (left) Haar measure on G is a Radon measure $\mu \neq 0$ such that

$$\mu(sA) = \mu(A) \tag{23}$$

for $A \subseteq G$ measurable and $s \in G$.

Theorem 4.3. *Suppose that G is a locally compact group.*

(1) There exists a Haar-measure μ on G.

(2) A Haar measure on G is unique up to multiplication by a positive real number.

Proof. See [10, Theorem 1.3.5]. □

We will write μ_G to denote any Haar-measure on G.
Suppose that V is a Banach space. For a function $f \in C(G, V)$, let $\text{supp}(f) = \overline{\{s \in G \mid f(s) \neq 0\}}$, set $C_c(G, V) = \{f \in C(G, V) \mid \text{supp}(f) \text{ compact}\}$ and $C_c(G) = C_c(G, \mathbb{C})$.

Proposition 4.4. *Let G be a locally compact group.*

1) For $1 \leq p < \infty$, we have $\overline{C_c(G)} = L^p(G, \mu_G)$ where $L^p(G, \mu_G)$ is the Banach space of p-integrable functions with respect to μ_G.

2) If $f \in C_c(G)$, then for every $\epsilon > 0$, there is a neighborhood V of $1_G \in G$ such that for all $s, t \in G$ with $s^{-1}t \in V$ or $t^{-1}s \in V$, we have $|f(s) - f(t)| < \epsilon$.

Proof. See [10, Proposition 1.3.3 and Lemma 1.3.7]. □

By the definition of Haar measure, $\int_G \chi_B(sr) d\mu_G(r) = \mu_G(s^{-1}B) = \mu_G(B) = \int_G \chi_B(r) d\mu_G(r)$ for fixed $s \in G$ and any μ_G-measurable set B. Passing to linear combinations of characteristic functions and taking a limit if necessary, yields

$$\int_G f(sr) d\mu_G(r) = \int_G f(r) d\mu_G(r) \tag{24}$$

for any $f \in L^p(G, \mu_G)$. This property is called left-invariance of μ_G.

Theorem 4.5. *There is a function $\Delta : G \longrightarrow \mathbb{R}_{>0}$ that satisfies the following properties:*

(1) The map Δ is a continuous group homomorphism into the group $(\mathbb{R}_{>0}, \cdot)$.

(2) For any $f \in L^1(G, \mu_G)$, we have $\int_G f(rs)d\mu_G(r) = \Delta(s) \int_G f(r)d\mu_G(r)$.

(3) If $f \in L^1(G, \mu_G)$, then $\int_G f(r)d\mu_G(r) = \int_G \Delta(r^{-1})f(r^{-1})d\mu_G(r)$.

Proof. See [10, Theorem 1.4.1] □

Property (2) of the preceeding Theorem determines Δ uniquely. Indeed, if Δ' also satisfies property (2), then $(\Delta(s) - \Delta'(s)) \int_G f(r)d\mu_G(r) = 0$ for any $f \in L^1(G, \mu_G)$. Thus, $\Delta(s) \neq \Delta'(s)$ for any $s \in G$ would yield a contradiction to $\mu_G \neq 0$. Since any other Haar measure of G is a constant multiple of μ_G, Δ does, again by property (2), not depend on the choice of μ_G. The homomorphism Δ is called the modular function of G.

4.2 Vector-valued Integration

We want to integrate functions that take values in an arbitrary C^*-algebra rather than $\mathbb{C}$ or $\mathbb{R}$.

Proposition 4.6. *Suppose that V is a complex Banach space. There is a linear map $C_c(G, V) \longrightarrow V$, $f \mapsto \int_G f(s)d\mu_G(s)$. Furthermore, this map has the following properties:*

(1) If $\phi \in C_c(G)$ and $x \in X$, then the integral of the function f given by $t \mapsto \phi(t)x$ may be evaluated by $\int_G f(s)d\mu_G(s) = x \int_G \phi(s)d\mu_G(s)$.

(2) Given a bounded linear Operator $T : X \longrightarrow Y$ into another Banach space Y, we have $\int_G Tf(s)d\mu_G(s) = T \int_G f(s)d\mu_G(s)$ for any $f \in C_c(G, V)$.

(3) $|| \int_G f(s)d\mu_G(s)|| \leq \int_G ||f(s)||d\mu_G(s)$ for $f \in C_c(G, V)$.

Proof. See [11, Lemma 1.91]. □

The above integral definition is sufficient in order to define crossed products and avoids the technical difficulties of defining $L^1(G, A)$, see [11, Appendix B].
However, we will need to take slightly more general integrals in the case $G = \mathbb{R}$. We want to integrate functions that are continuous on a compact subset of $\mathbb{R}$ (and do not vanish on the boundary). Integrals of this type will appear in the proof of Connes' cocycle Lemma.
A sufficient definition of such an integral is being defined in [12, Page 203]. Suppose V is a Banach space and $f : I \longrightarrow V$ a continuous function on a compact interval $I = [t, r]$. We can form the Riemann sum

$$I_n(f) = \sum_{j=0}^{n} \frac{1}{n} f\left(t + j \cdot \frac{r - t}{n}\right)$$

and take the limit

$$\int_t^r f(s)ds := \lim_{n\to\infty} I_n(f).$$

This can be seen by copying the proofs in the case $V = \mathbb{R}$. But as we assume that f is continuous, the theory becomes easier. We will now conduct the most important properties of this integral.

Proposition 4.7. *Let V be a Banach space. Also, suppose that $f : [t, r] \longrightarrow V$ is continuous.*

(1) The integral satisfies the triangle inequality $|| \int_t^r f(s)ds|| \leq \int_t^r ||f(s)||ds$.

(2) If $g : [t, r] \longrightarrow \mathbb{C}$ is continuous and $v \in V$, we have $\int_t^r f(s)ds = v \int_t^r g(s)ds$ for $f(s) = v \cdot g(s)$.

(3) If $T : V \longrightarrow W$ is a continuous operator between Banach spaces, then $\int\limits_t^r Tf(s)ds = T\int\limits_t^r f(s)ds$.

Sketch of the Proof. We need to apply the triangle inequality to the Riemann sums. □

4.3 The Universal Norm

Following [11, Chapter 2], we now introduce crossed products. These arise in the following way. We first give the algebra $C_c(G, A)$ a *-algebra structure (that depends on an action). The crossed product will then be the completion of $C_c(G, A)$ with respect to the universal norm.

Definition 4.8. *Suppose V is a complex Banach space. For $v \in V$, let $\rho_v : \mathcal{B}(V) \longrightarrow \mathbb{R}$ be defined by $\rho_v(T) = ||Tv||$. The strong topology is the coarsest topology on $\mathcal{B}(V)$ such that all of the maps ρ_v are continuous with respect to it.*

The set $\mathrm{Aut}(A)$ of *-automorphisms of A is a group equipped with composition being it's multiplication. We endow this group with the topology of strong convergence inherited from $\mathcal{B}(A)$.
For any Hilbert space H, $\mathcal{U}(H)$ will denote the group of unitary operators on H. We endow $\mathcal{U}(H)$ also with the topology of strong convergence inherited from $\mathcal{B}(H)$.

Definition 4.9. *A unitary representation (U, H) of G consists of a Hilbert space H and a group homomorphism $U : G \longrightarrow \mathcal{U}(H)$, $s \mapsto U_s$ that is continuous with respect to the strong topology.*

We now give an Example of a representation of G [11, Example 1.83].

Example 4.10. *Let G be a locally compact group and $H = L^2(G, \mu_G)$. The map $V : G \longrightarrow \mathcal{U}(H)$, $V_s f(r) = f(s^{-1}r)$ induces a unitary representation of G.*

Proof. We will first show that for fixed $s \in G$ the operator V_s is a unitary. Indeed, taking $f, g \in H$ we have

$$\langle V_s f, g\rangle = \int_G f(s^{-1}r)g(r)d\mu_G(r) = \int_G f(r)g(sr)d\mu_G(r) = \langle f, V_{s^{-1}}g\rangle \tag{25}$$

which shows that $V_{s^{-1}} = V_s^*$. We also have $V_{s^{-1}}V_s f = V_s V_{s^{-1}} f = f$ which shows that V_s is a unitary, because $V_s^* = V_{s^{-1}} = V_s^{-1}$. Furthermore, $V_{st}f(r) = f((st)^{-1}r) = f(t^{-1}s^{-1}r) = V_t f(s^{-1}r) = V_s V_t f(r)$, so V is a group homomorphism.
Now, let $\varepsilon > 0$, $f \in C_c(G)$ and set $C_f = 2 \cdot \mu_G(\mathrm{supp}(f)) < \infty$. Applying Proposition 4.4 (2) to the function f, there is a neighborhood $U \subseteq G$ of $1 \in G$ such that for all $t, r \in G$ with $s^{-1}rr^{-1}t = s^{-1}t \in U$, we have $|f(s^{-1}r) - f(t^{-1}r)| < \varepsilon$. Taking any t in the neighborhood sU of s, we calculate

$$\begin{aligned} ||V_s f - V_t f||^2 &= \int_G |f(s^{-1}r) - f(t^{-1}r)|^2 d\mu_G(r) \\ &= \int\limits_{(s\cdot \mathrm{supp}\, f)\cup(r\cdot \mathrm{supp}\, f)} |f(s^{-1}r) - f(t^{-1}r)|^2 d\mu_G(r) \\ &\leq \int\limits_{(s\cdot \mathrm{supp}\, f)\cup(r\cdot \mathrm{supp}\, f)} \varepsilon^2 \; d\mu_G(r) \\ &\leq \varepsilon^2 \cdot C_f. \end{aligned} \tag{26}$$

Proposition 4.4 (1) implies that V is continuous in the strong topology on $\mathcal{U}(H)$, because the family (V_s) is uniformly bounded as it consists of unitary operators. □

In the proof above, we used that $\mu_G(C)$ is finite for any compact subset $C \subseteq G$, see [10, Corollary 1.3.6].

Definition 4.11. *Suppose that A is a C^*-algebra and G a locally compact group.*

(1) A group-action of G on A is a group homomorphism $\alpha : G \longrightarrow \mathrm{Aut}(A)$ that is continuous in the strong topology.

(2) If $\alpha : G \longrightarrow \mathrm{Aut}(A)$ is an action, the tuple (A, G, α) is called a C^*-dynamical system.

We will form now on fix an action on $\alpha : G \longrightarrow \mathrm{Aut}(A)$ so that (A, G, α) is a C^*-dynamical system throughout the rest of this section.
The set $C_c(G, A, \alpha) = C_c(G, A)$ is a linear space equipped with the pointwise operations from A. Now, for $f, g \in C_c(G, A)$ let $(f * g)(s) = \int_G f(r)\alpha_s(g(s^{-1}r))d\mu_G(r)$. We call $f * g$ the convolution of f and g. Furthermore, for $f \in C_c(A, G)$, set $f^*(s) = \Delta(s^{-1})\alpha_s(f(s^{-1})^*)$. The function f^* is continuous, because each of the maps $\Delta : G \longrightarrow G$, ${}^{-1} : G \longrightarrow G$ and ${}^* : A \longrightarrow A$ are continuous. The set $\mathrm{supp}(f^*)$ is compact, because the inverse $X^{-1} = \{x^{-1} \mid x \in X\}$ of a compact set $X \subseteq G$ is compact too.
We call $C_c(G, A)$ the convolution algebra of the system (A, G, α).

Proposition 4.12. *Suppose that (A, G, α) is a C^*-dynamical system. The convolution algebra $C_c(G, A)$ is a *-algebra equipped with convolution as it's multiplication and * as it's involution.*

Proof. See [11, page 48] □

Definition 4.13. *Let H be a Hilbert space and (A, G, α) a C^*-dynamical system. If (π, H) is a *-representation of A and (U, H) a unitary representation of G, then the pair (π, U) is called a covariant representation of (A, G, α) if $\pi(\alpha_s(a)) = U_s\pi(a)U_s^*$ for all $a \in A$.*

The reason to consider covariant representations of the system (A, G, α) is that these induce *-representations of the convolution algebra $C_c(G, A)$.

Proposition 4.14. *Let (π, U) be a covariant representation of a C^*-dynamical system (A, G, α). Then the map $\pi \rtimes U : C_c(G, A) \longrightarrow \mathcal{B}(H)$, $\pi \rtimes U(f) = \int_G \pi(f(s))U_s d\mu_G(s)$ defines a *-representation of $C_c(G, A)$. Also, the following estimate holds:*

$$||\pi \rtimes U(f)|| \leq \int_G ||f(s)||d\mu_G(s) < \infty \tag{27}$$

Proof. See [11, Proposition 2.23]. □

The representation $\pi \rtimes U$ is called the integrated form of π and U.
The following Proposition will later be used to show that the norm of a crossed product is positively definite.

Lemma 4.15. *There exists a covariant representation (π, U) of (A, G, α) such that the integrated form $\pi \rtimes U$ is faithful.*

Proof. See [11, Example 2.14 and Lemma 2.26] □

For $f \in C_c(G, A)$, we set $||f||_{\mathrm{u}} = \sup_{(\pi,U)} ||\pi \rtimes U(f)||$. Estimate (27) in Proposition 4.14 implies $||f||_{\mathrm{u}} < \infty$. Also by Proposition 4.14, $|| \cdot ||_{\mathrm{u}}$ satisfies the C^*-identity. Lemma 4.15 yields that $|| \cdot ||_{\mathrm{u}}$ is positively definite. This norm is called the universal norm of the system (A, G, α).

Definition 4.16. *If (A, G, α) is a C^*-dynamical system, we set $A \rtimes_\alpha G = \overline{C_c(G, A)}$, where the closure is taken with respect to the universal norm. The C^*-algebra $A \rtimes_\alpha G$ is called the (universal) crossed product of (A, G, α).*

As crossed products are defined in terms of a dense subspace, it is often useful to know more dense subspaces. The following Lemma is an incarnation of the existence of partitions of unity for crossed products.

Lemma 4.17. *Suppose that (A, G, α) is a C^*-dynamical system. Functions $f \in C_c(G, A)$ of the form $t \mapsto f(t) = \sum_{i=1}^{k} a_i f_i(t)$ for $f_i \in C_c(G)$ are dense in the crossed product $A \rtimes_\alpha G$.*

Proof. See [11, Proof of Lemma 3.18]. □

Definition 4.18. *let (A, G, α) be a C^*-dynamical system suppose that $f \in C_c(G, A)$. We define $||f||_{L^1(G,A)} = ||f||_{L^1} = \int_G ||f(r)||dr$.*

The integral is well-defined, because the function $r \mapsto ||f(r)||$ is a continuous function with compact support and $||\cdot||_{L^1}$ defines a norm. The following Proposition is frequently useful to show boundedness of a homomorphism in the universal norm.

Proposition 4.19. *Suppose that (A, G, α) and (B, β, G) are C^*-dynamical systems. Furthermore, assume that $\varphi : C_c(G, A) \longrightarrow C_c(G, B)$ is a *-homomorphism that is bounded in the norm $||\cdot||_{L^1}$. Then φ extends to a *-homomorphism $\varphi : A \rtimes_\alpha G \longrightarrow B \rtimes_\beta G$.*

Proof. See [11, Corollary 2.46]. □

4.4 Equivariance

To apply the cyclic six-term exact sequence of K-theory to crossed products, we need a relation between crossed products and short exact sequences.

Definition 4.20. *Suppose that (A, G, α) and (B, G, β) are C^*-dynamical systems.*

(1) A *-homomorphism $\varphi : A \longrightarrow B$ is called G-equivariant or just equivariant if $\varphi(\alpha_s(a)) = \beta_s(\varphi(a))$ for any $a \in A$ and $s \in G$.

(2) An equivariant short exact sequence is a short exact sequence of equivariant *-homomorphisms.

Let (B, G, β) be another C^*-dynamical system. If $\varphi : A \longrightarrow B$ is equivariant, let $\hat{\varphi}(f)(s) = \varphi(f(s))$ for $f \in C_c(G, A)$. Equivariance yields that $\hat{\varphi} : C_c(G, A) \longrightarrow C_c(G, B)$ is a *-homomorphism.

Lemma 4.21. *Suppose (A, G, α) and (B, G, β) are C^*-dynamical systems and that $\varphi : A \longrightarrow B$ is an equivariant *-homomorphism. There is a unique extension $\hat{\varphi} : A \rtimes_\alpha G \longrightarrow B \rtimes_\beta G$ of $\hat{\varphi}$.*

Proof. See [11, Corollary 2.47]. □

The following Proposition shows that taking crossed products respects equivariant short exact sequences.

Proposition 4.22. *Let $(I, G, \gamma), (B, G, \beta)$ and (A, G, α) be C^*-dynamical systems and*

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

an equivariant short exact sequence. Then the sequence

$$0 \longrightarrow I \rtimes_\gamma G \xrightarrow{\hat{\varphi}} A \rtimes_\alpha G \xrightarrow{\hat{\psi}} B \rtimes_\beta G \longrightarrow 0$$

is exact too.

Proof. See [11, Proposition 3.19]. □

4.5 Crossed Products and Tensor Products

There is also a connection between crossed and tensor products, see [11, Remark 2.74].
Suppose that (B, G, β) is a C^*-dynamical system. For fixed $t \in G$, the map $(\alpha \otimes \beta)_t : A \otimes B \longrightarrow A \otimes_{\max} B$, $(\alpha \otimes \beta)_t(a \otimes b) = \alpha_t(a) \otimes \beta_t(b)$ is continuous with respect to the maximal norm on $A \otimes B$. Indeed, the map $\gamma_{\alpha_t} : A \otimes B \longrightarrow \mathbb{R}_+$, $\gamma_{\alpha_t}\left(\sum\limits_{i=1}^n a_i \otimes b_i\right) = \left\|\sum\limits_{i=1}^n \alpha_t(a_i) \otimes \beta_t(b_i)\right\|_{\max}$ defines a C^*-norm in $A \otimes B$. But then

$$\left\|\left(\alpha \otimes \beta)_t(\sum_{i=1}^n a_i \otimes b_i\right)\right\|_{\max} = \gamma_{\alpha_t}\left(\sum_{i=1}^n a_i \otimes b_i\right) \leq \left\|\sum_{i=1}^n a_i \otimes b_i\right\|_{\max} \tag{28}$$

by Propositon 2.16. This implies that $\alpha \otimes \beta$ extends to $A \otimes_{\max} B$.
Now, $(\alpha \otimes \beta) : G \longrightarrow \mathrm{Aut}(A \otimes_{\max} B)$, $t \mapsto (\alpha \otimes \beta)_t$ is a group homomorphism, because α is a group homomorphism. Also by Proposition 2.16,

$$||(\alpha \otimes \beta)_t(a \otimes b) - (\alpha \otimes \beta)_s(a \otimes b)||_{\max} \leq ||(\alpha_t(a) - \alpha_s(a))|| \ ||b|| + ||(\beta_t(b) - \beta_s(b))|| \ ||a|| \tag{29}$$

which shows that $\alpha \otimes \mathrm{Id}$ is continuous in the strong topology.
Now if B is a C^*-algebra, then $\mathrm{Id} : G \longrightarrow \mathrm{Aut}(B)$, $t \mapsto \mathrm{Id}_B$ is an action on B.

Proposition 4.23. *Suppose (A, G, α) is a C^*-dynamical system and that B is a C^*-algebra. There is an isomorphism*

$$(A \rtimes_\alpha G) \otimes_{\max} B \cong (A \otimes_{\max} B) \rtimes_{\alpha \otimes \mathrm{Id}} G.$$

Given $f \in C_c(G, A)$ and $b \in B$, this isomorphism maps $f \otimes b$ to the function $t \mapsto f(t) \otimes b$.

Proof. See [11, Lemma 2.75]. □

4.6 Abelian Groups and Takai Duality

We will now assume that G is Abelian. The circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is a group equipped with the multiplication from $\mathbb{C}$ ($z^{-1} = \overline{z}$).

Definition 4.24. *Let G be a locally compact Abelian group.*

(1) A character of G is a group homomorphism $\chi : G \longrightarrow \mathbb{T}$.

(2) The Pontryagin dual of G is the set of all characters of G. It is denoted by $\hat{G}$.

The Pontryagin dual is a group with multiplication $(\chi\chi')(t) = \chi(t)\chi'(t)$. We endow G with the weak-* topology coming from $L^1(G)$. This is the coarsest topology such that for any $f \in L^1(G)$, the maps $\chi \mapsto \int_G f(s)\chi(s)ds$ is continuous.

Proposition 4.25. *If G is a locally compact Abelian group, then $\hat{G}$ is a locally compact Abelian group.*

Proof. See [11, Corollary 1.79]. □

dualizing twice gives us the original group back.

Theorem 4.26 (Pontryagin duality)**.** *Suppose that G is a locally compact Abelian group. Then there is in isomorphism $G \cong \hat{\hat{G}}$ into the double dual group that is a homeomorphism.*

Proof. See [10, Theorem 3.5.5]. □

On the crossed product $A \rtimes_\alpha G$, there is an action $\hat{\alpha} : \hat{G} \longrightarrow \mathrm{Aut}(A \rtimes_\alpha G)$ given by $\hat{\alpha}_\chi(f)(r) = \overline{\chi(r)}f(r)$ for $r \in G$ and $f \in C_c(G, A)$, see [11, Section 7.1].

Theorem 4.27 (Takai duality)**.** *Suppose that (A, G, α) is a C^*-dynamical system with G Abelian. Then*

$$(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \cong A \otimes_{\max} \mathcal{K}(L^2(G)).$$

The Takai duality Theorem states that the operation of taking crossed products is it's own inverse up to tensoring with the compact operators.

Definition 4.28. *Let (A, G, α) be a C^*-dynamical system with G Abelian and I an ideal of A. If $\alpha_s(a) \in I$ for any $s \in G$ and $a \in I$, then I is called α-invariant.*

If I is an ideal of A and $\iota : I \longrightarrow A$ the inclusion map, we may identify $I \rtimes_\alpha G$ with it's image under $\hat{\iota}$ given by Proposition 4.22. So we have $\hat{\iota}(I \rtimes_\alpha G) = \overline{I \rtimes_\alpha G} \subseteq A \rtimes_\alpha G$. Then $I \rtimes_\alpha G$ is an ideal of $A \rtimes_\alpha G$. Now if $f \in C_c(G, I)$ and $\chi \in \hat{G}$, then $\hat{\alpha}_\chi(f)(s) = \overline{\chi(s)}f(s) \in I$ for any $s \in G$. We infer $\hat{\alpha}_\chi(f) \in I \rtimes_\alpha G$, so $I \rtimes_\alpha G$ is $\hat{\alpha}$-invariant. The next Proposition states that the correspondence of α-invariant ideals and $\hat{\alpha}$-invariant ideals in the crossed products is one to one.

Proposition 4.29. *Suppose that (A, G, α) is a C^*-dynamical system with G Abelian. If J is an $\hat{\alpha}$-invariant ideal of $A \rtimes_\alpha G$, then there is an α-invariant ideal I of A such that $J = I \rtimes_\alpha G$.*

Proof. See [13, Proposition 6.3.9]. □

Proposition 4.29 is a consequence of the Takai duality Theorem. One has to show that the invariant ideals of $A \otimes_{\max} \mathcal{K}(L^2(G))$ for the double dual action $\widehat{\hat{\alpha}}$ correspond to the α-invariant ideals of A.
The group $\mathbb{R}$ is it's own dual.

Example 4.30. *The map $\omega : \mathbb{R} \longrightarrow \hat{\mathbb{R}}$, $y \mapsto \omega_y$, where $\omega_y(x) = e^{-ixy}$ is both, a homeomorphism and an isomorphism.*

Proof. See [11, Example 1.80]. □

We will from now on identify $y \in \mathbb{R}$ with ω_y. Suppose now $A = \mathbb{C}$, then $\alpha = \mathrm{Id}$ is the only action on A.

Definition 4.31. *Let G be a locally compact group. The crossed product $\mathbb{C} \rtimes_{\mathrm{Id}} G$ is called the (universal) group C^*-algebra of G and is denoted by $C^*(G)$.*

For $f \in C_c(G, A) = C_c(G)$, we define the Fourier transform of f, $\hat{f} : \hat{G} \longrightarrow \mathbb{C}$ by $\hat{f}(\chi) = \int f(s)\chi(s)ds$. Now, $\hat{f}$ is an element of $C_0(\hat{G})$. Indeed f is continuous by definition of the topology on $\hat{G}$ and the set $\{\chi \in \hat{G} : \hat{f}(\chi) \geq \varepsilon\}$ is compact for any $\varepsilon > 0$ by the arguments in [4, Page 15].

Proposition 4.32. *Suppose that G is a locally compact Abelian group. The Fourier transformation extends to an isomorphism $\hat{}: C^*(G) \longrightarrow C_0(\hat{G})$.*

Proof. See [11, Proposition 3.1]. □

4.7 Smoothing

Suppose that $G = \mathbb{R}$. We want to use the smooth structure (i.e. being able to take derivatives) of $\mathbb{R}$. The following example motivates this procedure. The map $s \mapsto \tau_s$, $\tau_s(f)_t = f_{t-s}$ defines an action on $S = C_0(\mathbb{R})$. We will later see that τ is continuous in the strong topology.

Example 4.33. *Suppose $f \in C^1(\mathbb{R}) \cap C_0(\mathbb{R})$ and that*

$$(\tau_s(f) - f)/s$$

converges uniformly to a function $g \in C_0(\mathbb{R})$ as $s \to 0$, then $g = f'$.

Proof. We have $(\tau_s(f)_t - f_t)/s = (f(t-s) - f(t))/s$, so $(\tau_s(f) - f)/s$ converges to f' pointwise. □

The theory of smoothing is taken from [2, Appendix 3].
Our generalized notion of smoothness will depend on the action α.

Definition 4.34. *Suppose that V is a Banach space. A function $f : \mathbb{R} \longrightarrow V$ is called differentiable if the limit $f'(s_0) = \lim\limits_{s \to s_0} \frac{f(s) - f(s_0)}{s - s_0}$ exists with respect to the norm of V for any $s_0 \in \mathbb{R}$.*

We call a function $f : \mathbb{R} \longrightarrow V$ smooth if $f^{(n)} = (f^{(n-1)})'$ exists for any $n \in \mathbb{N}$ with $f^{(0)} = f$. As α is continuous in the strong topology, $s \mapsto \alpha_s(a)$ is a continuous map for any $a \in A$. The smooth elements of A are the ones for which this map is smooth.

Definition 4.35. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system.*

(1) Suppose that $a \in A$. We call a smooth if the map $s \mapsto \alpha_s(a)$ is smooth.

(2) We set $A^\infty = \{a \in A : a \text{ is smooth}\}$.

We can identify $M_n(A^\infty)$ with $M_n(A)^\infty$ by extending the action to every entry. Indeed, we have seen that $M_n(A)$ may be represented as a sub-C^*-algebra of $\mathcal{B}(H^n) \cong M_n(\mathcal{B}(H))$. But then being continuous in the strong topology just means being continuous in every entry with respect to the strong topology. Furthermore, we set $\delta(a) = \lim_{s\to 0} \frac{\alpha_s(a)-a}{s}$ for $a \in A^\infty$. In fact, as $\delta(\alpha_t(a)) = \alpha_t(a)$, a is differentiable if and only if $\delta(a)$ exists. We think of δ as a noncommutative analogue to the derivative.

Remark 4.36. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system . The set A^∞ is a *-algebra. Furthermore, if $a, b \in A^\infty$ and $z \in \mathbb{C}$, then*

(1) $\delta(a + b) = \delta(a) + \delta(b)$,

(2) $\delta(za) = z\delta(a)$,

(3) $\delta(a^*) = \delta(a)^*$ *and*

(4) $\delta(ab) = a\delta(b) + \delta(a)b$.

Proof. The hardest part is (4), we need to mimic the prove of the product rule in the commutative case. We have

$$\delta(ab) = \frac{\alpha_s(a)\alpha_s(b) - \alpha_s(a)b + \alpha_s(a)b - ab}{t}$$

and as $\alpha_s(a) \to a$ when $s \to 0$, we obtain the desired formula. □

The algebra A^∞ is not complete in general. For example if $A = C_0(\mathbb{R})$, we may approximate functions that are only continuous by smooth ones. We are mainly interested in A^∞, because we want to replace projections by smooth ones.

Lemma 4.37. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system. Any projection $p \in M_n(A)$ is homotopic to a projection $q \in M_n(A^\infty)$.*

We will now sketch the proof of this Lemma.

Lemma 4.38. *If $(A, \mathbb{R}, \alpha)$ is a C^*-dynamical system, then A^∞ is dense in A.*

Sketch of the Proof. Take a mollifier $\phi \in C^\infty(\mathbb{R})$. Given $a \in A$, set $b = \int_{\mathbb{R}} \alpha_s(a)\phi(s)ds$ which mimics the convolution formula in the commutative case. $\delta(b) = \int_{\mathbb{R}} \alpha_s(a)\phi'(s)ds$ and b is close to a. □

One needs the so-called holomorphic functional calculus, see [12, Section 3.3].

Definition 4.39. *Suppose that A is a C^*-algebra and $\mathcal{A}$ a dense *-subalgebra. $\mathcal{A}$ is called a local C^*-algebra if it is closed under holomorphic calculus.*

Lemma 4.40. *If $(A, \mathbb{R}, \alpha)$ is a C^*-dynamical system, then A^∞ is a local C^*-algebra.*

Sketch of the Proof. As α_t is an automorphism, $f(\alpha_t(a)) = \alpha_t(f(a))$ for functions holomorphic on a neighbourhood of $sp(a)$. That follows from holomorphic functional calculus being a contour integral. One then mimics a proof of the chain rule. □

Sketch of the Proof of Lemma 4.37. The proof is very similar to the one of [6, Lemma 6.3.1 (i)], so we describe the adjustments that have to be made. We cannot use continuous functional calculus as A^∞ is only closed under holomorphic functional calculus. One instead defines f to be 1 on the set $\{z \in \mathbb{C} : \mathrm{Re}(z) > 1 - 3\delta\}$ and 0 on $\{z \in \mathbb{C} : \mathrm{Re}(z) < 3\delta\}$. These domains do not intersect, so f is holomorphic. Now, approximate p by an element $a \in A^\infty$ and replace it by $\frac{a+a^*}{2}$ if it is not self-adjoint. Then $f(a)^2 = f(a)$ and $f(a)$ is close to both p and a by the last estimate in the proof of [12, Proposition 3.3.9]. But $f(a)$ is also close to a projection from A^∞, because it is close to a self-adjoint element, see [8, Proof of Proposition 4.6.2]. Projections that are close need to be homotopic, see [6, Proposition 2.2.4]. □

We will from now on only be concerned with differentiation of projections. Suppose that A is commutative and that $C_0(X) \cong A$ by Gelfand duality, Theorem 2.3. Take a projection $p \in C_0(X) \cong A$. It is a function satisfying $p = p^* = p^2$. Now, $p = p^2$ implies $p(x) \in \{0, 1\}$. On a connected component p has to be constant, because it is a continuous function.

Example 4.41. *Set* $A = M_2(\mathbb{C})$ *and let* $u_t = \frac{1}{\sqrt{t^2+1}} \cdot \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix}$. *Then* $\alpha_t(a) = u_t a u_t^*$ *defines an action on* A. *Furthermore,* $\delta(p)$ *exists for* $p = \operatorname{diag}(1, 0)$ *and is not* 0.

Proof. The map $t \mapsto u_t$ is continuous in t and consists of unitaries, so α is an action. We have

$$\alpha_t(p) = \begin{pmatrix} \frac{1}{t^2+1} & \frac{-t}{t^2+1} \\ \frac{-t}{t^2+1} & \frac{t^2}{t^2+1} \end{pmatrix}$$

But then $\delta(p) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. □

We will see in the proof of Connes' cocycle Lemma that $\delta(p) = 0$ if and only if $\alpha_s(p) = p$ for any $s \in \mathbb{R}$.

5 The Connes-Thom Isomorphism

5.1 Statement of the Theorem

Throughout this section, $(A, \mathbb{R}, \alpha)$ will denote a C^*-dynamical system. Haar measure on $(\mathbb{R}, +)$ is Lebesgue measure and the modular function is 1.
Our goal is to prove the following Theorem which first appeared in [2].

Theorem 5.1 (Connes-Thom isomorphism)**.** *Suppose A is a C^*-algebras and that $\alpha : \mathbb{R} \longrightarrow \mathrm{Aut}(A)$ is an action by the real line on A. There are isomorphisms*

$$\phi_A^1 : K_0(A \rtimes_\alpha \mathbb{R}) \longrightarrow K_1(A)$$

and

$$\phi_A^0 : K_1(A \rtimes_\alpha \mathbb{R}) \longrightarrow K_0(A)$$

such that given another C^*-dynamical system $(B, \beta, \mathbb{R})$ and an equivariant *-homomorphism $\varphi : A \longrightarrow B$, both diagrams,

$$\begin{array}{ccc} K_0(A \rtimes_\alpha \mathbb{R}) & \xrightarrow{\phi_A^1} & K_1(A) \\ \Big\downarrow{\scriptstyle K_0(\hat{\varphi})} & & \Big\downarrow{\scriptstyle K_1(\varphi)} \\ K_0(B \rtimes_\beta \mathbb{R}) & \xrightarrow{\phi_B^1} & K_1(B) \end{array} \tag{30}$$

and

$$\begin{array}{ccc} K_1(A \rtimes_\alpha \mathbb{R}) & \xrightarrow{\phi_A^0} & K_0(A) \\ \Big\downarrow{\scriptstyle K_1(\hat{\varphi})} & & \Big\downarrow{\scriptstyle K_0(\varphi)} \\ K_1(B \rtimes_\beta \mathbb{R}) & \xrightarrow{\phi_B^0} & K_0(B) \end{array} \tag{31}$$

commute.

The pair (ϕ_A^1, ϕ_A^0) is often just called the Thom isomorphism. Commutation of (30) and (31) is often referred to as naturality of the Thom-isomorphism.
Theorem 5.1 is a generalization of Bott periodicity, Theorem 3.29. If $\alpha = \mathrm{Id}$, then $A \rtimes_\alpha \mathbb{R} \cong SA$. That is, because

$$A \rtimes_\alpha \mathbb{R} \cong (A \otimes_{\max} \mathbb{C}) \rtimes_{\alpha \otimes \mathrm{Id}} \mathbb{R} \cong A \otimes_{\max} C^*(\mathbb{R}).$$

By the Fourier transformation, Proposition 4.32, $C^*(\mathbb{R}) \cong S$ and $A \otimes_{\max} S \cong SA$ by Proposition 2.20.

5.2 Outline of the Proof

The action α is homotopic to the trivial action Id on A. This follows from the associative structure of $\mathbb{R}$, set $\alpha_s^{(t)}(a) = \alpha_{t \cdot s}(a)$. We have $\alpha^{(0)} = \mathrm{Id}$ and $\alpha^{(1)} = \alpha$. However, $A \rtimes_\alpha \mathbb{R} \cong A \rtimes_{id} \mathbb{R}$ does not always hold, i.e. crossed products do not respect homotopies. However, as we have seen in Proposition 3.21, K-theory does and so one might suspect

$$K_j(A \rtimes_\alpha \mathbb{R}) \cong K_j(A \rtimes_{\mathrm{Id}} \mathbb{R}) \cong K_{|j-1|}(A).$$

The proof will however take a different route. We will in fact pass to a different C^*-dynamical system associated to $(A, \mathbb{R}, \alpha)$ in which we can remove the α-part from the action.

Example 5.2. *Let $A = S = C_0(\mathbb{R})$ and $\alpha : \mathbb{R} \longrightarrow \mathrm{Aut}(A)$ be given by right translation. Then $A \rtimes_\alpha \mathbb{R}$ is not isomorphic to $A \rtimes_{\mathrm{Id}} \mathbb{R}$.*

Proof. As A is commutative, convolution with respect to Id is commutative by the formula

$$(f * g)(r) = \int_{\mathbb{R}} f(s)g(r-s)ds = \int_{\mathbb{R}} f(r-s)g(s)ds = (g * f)(r).$$

Convolution with respect to α is not commutative as $\alpha \neq \mathrm{Id}$. So $A \rtimes_\alpha \mathbb{R}$ and $A \rtimes_{\mathrm{Id}} \mathbb{R}$ are not isomorphic. □

We will associate a short exact sequence, called the Wiener-Hopf extension to the system $(A, \mathbb{R}, \alpha)$. This sequence is of the form

$$0 \longrightarrow * \xrightarrow{*} * \xrightarrow{*} A \rtimes_\alpha \mathbb{R} \longrightarrow 0$$

The six-term sequence of K-theory, Theorem 3.31, then yields an exact sequence

$$\begin{tikzcd} K_0(B) \arrow[r, "*"] & * \arrow[r, "*"] & K_0(A \rtimes_\alpha \mathbb{R}) \arrow[d, "\delta_0"] \\ K_1(A \rtimes_\alpha \mathbb{R}) \arrow[u, "\delta_1"] & * \arrow[l, "*"] & K_1(B) \arrow[l, "*"] \end{tikzcd}$$

for some C^*-algebra B. We will show that the K-groups of B are $K_0(B) \cong K_1(A)$ and $K_1(B) \cong K_1(A)$. The next step is to show that δ_0 and δ_1 are isomorphisms. In order to achieve this, the problem will be reduced to showing that δ_1 as always surjective. We first examine the situation when A is unital and deduce the non-unital case from that. Naturality is achieved by investigating all the isomorphisms out of which $K_0(A \rtimes_\alpha \mathbb{R}) \cong K_1(A)$ and $K_1(A \rtimes_\alpha \mathbb{R}) \cong K_0(A)$ are composed.
This Prove first appeared in [1]. The Wiener-Hopf extension is an analogue of the Toeplitz extension from [3, Page 98]. In [8, Section 10.9] another variant of this proof due to unpublished work of M. Pimsner D. Voiculescu is given. The outline given above is the same, however the use of Connes' cocycle is replaced by various estimates. Much of our presentation supplements this proof with more details too.

5.3 The Wiener-Hopf Extension

Given $s \in \mathbb{R}$, the translation map $\tau_s(f)_t = f_{t-s}$ is an automorphism of S. Now, since $\sup_{t \in \mathbb{R}} |f_{t-s} - f_{t-r}|$ converges to 0 as r approaches s, τ is continuous in the strong topology, so τ defines an action on S. Thus, we obtain an action $\tau \otimes \alpha$ on the tensor product $S \otimes_{\max} A$. Under the isomorphism in Proposition 2.20, this action becomes $(\tau \otimes \alpha)_s(f)_t = \alpha_s(f_{t-s})$ for $f \in SA$. Repeating this argument with S replaced by C and setting $\infty - s = \infty$ for any $s \in \mathbb{R}$, we get an action $(\tau \otimes \alpha) : \mathbb{R} \longrightarrow CA$, $(\tau \otimes \alpha)_s(f)_t = \alpha_s(f_{t-s})$.

Lemma 5.3. *The short exact sequence*

$$0 \longrightarrow SA \xrightarrow{\iota} CA \xrightarrow{\pi} A \longrightarrow 0$$

is equivariant with respect to the actions described above.

Proof. For $f \in SA$, we have $(\tau \otimes \alpha)_s(\iota(f))_t = \alpha_s(\iota(f)_{t-s}) = \begin{cases} \alpha_s(0) & if \;\; t = \infty \\ \alpha_s(f_{t-s}) & otherwise \end{cases} = \iota((\tau \otimes \alpha)_s(f))_t$. Also, given $f \in CA$, we have

$$\pi((\tau \otimes \alpha)_s(f)) = (\tau \otimes \alpha)_s(f)_\infty = \alpha_s(f)_{\infty - s} = \alpha_s(f_\infty) = \alpha_s(\pi(f)) \tag{32}$$

$\square$

Now, Proposition 4.22 yields that the sequence

$$0 \longrightarrow SA \rtimes_{\tau \otimes \alpha} \mathbb{R} \xrightarrow{\hat{\iota}} CA \rtimes_{\tau \otimes \alpha} \mathbb{R} \xrightarrow{\hat{\pi}} A \rtimes_\alpha \mathbb{R} \longrightarrow 0$$

is exact. This sequence is called the Wiener-Hopf extension of $(A, \mathbb{R}, \alpha)$, see [1, Page 144]. Applying Theorem 3.31, we get a six-term exact sequence

$$\begin{tikzcd} K_0(SA \rtimes_{\tau \otimes \alpha} \mathbb{R}) \arrow[r, "K_0(\hat{\iota})"] & K_0(CA \rtimes_{\tau \otimes \alpha} \mathbb{R}) \arrow[r, "K_0(\hat{\pi})"] & K_0(A \rtimes_\alpha \mathbb{R}) \arrow[d, "\delta_0"] \\ K_1(A \rtimes_\alpha \mathbb{R}) \arrow[u, "\delta_1"] & K_1(CA \rtimes_{\tau \otimes \alpha} \mathbb{R}) \arrow[l, "K_1(\hat{\pi})"] & K_1(SA \rtimes_{\tau \otimes \alpha} \mathbb{R}) \arrow[l, "K_1(\hat{\iota})"] \end{tikzcd} \tag{33}$$

of K-groups.

Lemma 5.4. *If both, $K_0(CA \rtimes_{\tau\otimes\alpha} \mathbb{R})$ and $K_1(CA \rtimes_{\tau\otimes\alpha} \mathbb{R})$ are 0, then δ_0 and δ_1 are isomorphisms.*

Proof. We have

$$\ker(\delta_0) = K_0(\hat{\pi})(K_0(CA \rtimes_{\tau\otimes\alpha} \mathbb{R})) = K_0(\hat{\pi})(0) = 0 \tag{34}$$

and

$$\delta_0(K_0(A \rtimes_\alpha \mathbb{R})) = \ker(K_1(\hat{\iota})) = K_1(SA \rtimes_{\tau\otimes\alpha} \mathbb{R}) \tag{35}$$

which yields that δ_0 is an isomorphism. Similarly, δ_1 is an isomorphism. ☐

We will later show that the K-theory of $SA \rtimes_{\tau\otimes\alpha} \mathbb{R}$ is the same as of A. Because of this, it is natural to try to show that the assumptions of Lemma 5.4 are satisfied and then, to conclude that Theorem 5.1 follows from that. Indeed, this will be our argument. However, one needs to be cautious that naturality ((30) and (31)) are satisfied.
At this point, we should evaluate functions $f \in C_c(\mathbb{R}, SA)$ or $f \in C_c(\mathbb{R}, CA)$ like $f(r)(t) = f(r)_t$. But we will mostly write $f_t(r)$ instead of $f(r)_t$. Also, $f(r)$ will denote the function $t \mapsto f_t(r) = f(r)_t$.
The following Lemma is based on [1, Page 147] and [11, Lemma 7.4].

Lemma 5.5. *There is a *-isomorphism $\hat{\varphi} : SA \rtimes_{\tau\otimes\alpha} \mathbb{R} \longrightarrow SA \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R}$ mapping $f \in C_c(\mathbb{R}, SA)$ to*

$$\hat{\varphi}(f)_t(r) = \alpha_{-t}(f_t(r)).$$

Proof. Let $\varphi : SA \longrightarrow SA$ denote the map $\varphi(f)_t = \alpha_{-t}(f_t)$. We need to first verify that φ is well-defined. As $t \to \infty$ or $t \to -\infty$, we have $||\alpha_{-t}(f_t)|| = ||f_t|| \to 0$. The estimate

$$\begin{aligned} ||\alpha_{-t}(f_t) - \alpha_{-s}(f_s)|| &\le ||\alpha_{-t}(f_t) - \alpha_{-s}(f_t)|| + ||\alpha_{-s}(f_t) - \alpha_{-s}(f_s)|| \\ &= ||\alpha_{-t}(f_t) - \alpha_{-s}(f_t)|| + ||f_t - f_s|| \end{aligned} \tag{36}$$

yields continuity of $\varphi(f)$, because α is continuous in the strong topology.
We now show that φ is equivariant with respect to the actions $\tau \otimes \alpha$ and $\tau \otimes \mathrm{Id}$. Indeed,

$$\begin{aligned} (\tau \otimes \mathrm{Id})_s(\varphi(f))_t = \varphi(f)_{t-s} &= \alpha_{s-t}(f_{t-s}) \\ &= \alpha_{-t}(\alpha_s(f_{t-s})) \\ &= \alpha_{-t}((\tau \otimes \alpha)_s(f)_t) \\ &= \varphi((\tau \otimes \alpha)_s(f))_t \end{aligned} \tag{37}$$

which shows $(\tau \otimes \mathrm{Id})_s(\varphi(f)) = \varphi((\tau \otimes \alpha)_s(f))$. Applying Lemma 4.21, we obtain a *-homomorphism $\hat{\varphi} : SA \rtimes_{\tau\otimes\alpha} \mathbb{R} \longrightarrow SA \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R}$. Furthermore, φ is bijective, because it's inverse is given by

$$\varphi^{-1}(f)_t = \alpha_t(f_t)$$

Now, φ^{-1} is equivariant too and $\widehat{\varphi^{-1}}$ is the inverse of $\hat{\varphi}$. Given $f \in C_c(\mathbb{R}, SA)$, we calculate that $\hat{\varphi}(f)_t(r) = \varphi(f(r))_t = \alpha_{-t}(f_t(r))$. ☐

Lemma 5.5 gives an intuitive reason, why we are switching to the dynamical systems $(SA, \tau \otimes \alpha, \mathbb{R})$ and $(CA, \tau \otimes \alpha, \mathbb{R})$. The τ-part of the action carries all the information about the α-part.
For $f \in C_c(\mathbb{R}, S)$, we define $T_f \in \mathcal{B}(L^2(\mathbb{R}))$ by $T_f\xi(t) = \int_\mathbb{R} f_t(s)\xi(t-s)ds$ for $\xi \in L^2(\mathbb{R})$. If $h \in L^2(\mathbb{R})$, $||h||_{L^2(\mathbb{R})} = 1$, then the rank one projection onto h is given by $E_h\xi(t) = \int_\mathbb{R} h(t)h(t-s)\xi(t-s)ds$.

Lemma 5.6. *Suppose $h \in L^2(\mathbb{R})$, $||h||_{L^2(\mathbb{R})} = 1$. Then for any sequence $\phi^{(n)} \in S = C_0(\mathbb{R})$ with $||h - \phi^{(n)}||_{L^2(\mathbb{R})} \to 0$, the functions $f^{(n)} \in C_c(S)$ defined by $f_t^{(n)}(r) = \phi_t^{(n)}\phi_{t-r}^{(n)}$ satisfy*

$$T_{f^{(n)}} \to E_h$$

in $\mathcal{K}(L^2(\mathbb{R}))$.

Proof. Given $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ with $||\phi^{(n)} - h||_{L^2(\mathbb{R})} < \varepsilon$ for $n \geq n_0$. Given $\xi \in L^2(\mathbb{R})$, we infer

$$\begin{aligned}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\phi_t^{(n)}h(t-s)-h(t)h(t-s)|\ |\xi(t-s)|ds\right)^2 dt &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\phi_t^{(n)}-h(t)|\ |h(t-s)|\ |\xi(t-s)|ds\right)^2 dt\\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}}|\phi_t^{(n)}-h(t)|^2|h(t-s)|^2|dsdt\ \ ||\xi||^2_{L^2(\mathbb{R})}\\ &= ||\phi^{(n)}-h||^2_{L^2(\mathbb{R})}||h||^2_{L^2(\mathbb{R})}||\xi||^2_{L^2(\mathbb{R})}\\ &\leq \varepsilon^2||\xi||^2_{L^2(\mathbb{R})}\end{aligned} \tag{38}$$

and

$$\begin{aligned}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\phi_t^{(n)}h(t-s)-\phi_t^{(n)}\phi_{t-s}^{(n)}|\ |\xi(t-s)|ds\right)^2 dt &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\phi_t^{(n)}|\ \ |h(t-s)-\phi_{t-s}^{(n)}|\ \ |\xi(t-s)|ds\right)^2 dt\\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}}|\phi_t^{(n)}|^2\ \ |h(t-s)-\phi_{t-s}^{(n)}|^2dsdt\ \ ||\xi||^2_{L^2(\mathbb{R})}\\ &= ||\phi^{(n)}-h||^2_{L^2(\mathbb{R})}||\phi^{(n)}||^2_{L^2(\mathbb{R})}||\xi||^2_{L^2(\mathbb{R})}\\ &\leq \varepsilon^2||\phi^{(n)}||^2_{L^2(\mathbb{R})}||\xi||^2_{L^2(\mathbb{R})}\\ &\leq \epsilon^2(1+\varepsilon)^2||\xi||_{L^2(\mathbb{R})}\end{aligned} \tag{39}$$

which yields $T_{f^{(n)}} \longrightarrow E_h$. ☐

The following Example may be found in [11, Example 2.12 and the proof of Theorem 4.2.4].

Example 5.7. *Let $M : S \longrightarrow \mathcal{B}(L^2(\mathbb{R}))$ denote the multiplication operator, $M(f)h(t) = f_t h(t)$ for $f \in S$, $h \in L^2(\mathbb{R})$. Furthermore, let $V : \mathbb{R} \longrightarrow \mathcal{U}(L^2(\mathbb{R}))$ denote the (left-)regular representation of $\mathbb{R}$ from Example 4.10.*

(1) The the pair (M, V) is a covariant representation.

(2) The integrated form is given by $M \rtimes V(f)h(t) = \int_{\mathbb{R}} f_t(s)h(t-s)ds$.

(3) $\psi = M \rtimes V$ is bounded in the universal norm and it's image lies in $\mathcal{K}(L^2(\mathbb{R}))$.

Proof. Given $s, t \in \mathbb{R}$, we have

$$V_s M(f) V_s^* h(t) = M(f)V_s^* h(t-s) = f(t-s)V_s^* h(t-s) = f(t-s)h(t) = M(\tau_s(f))h(t). \tag{40}$$

Thus, (M, V) is a covariant representation. For $f \in C_c(\mathbb{R}, S)$, the integrated form satisfies

$$M \rtimes V(f)h(t) = \int_{\mathbb{R}} M(f(s))V_s ds h(t) = \int_{\mathbb{R}} M(f(s))V_s h(t)ds = \int_{\mathbb{R}} f_t(s)V_s h(t)ds = \int_{\mathbb{R}} f_t(s)h(t-s)ds$$

where we used Property (2) of Proposition 4.6 in the second equality.
As the integrated form of a covariant representation, $\psi = M \rtimes V$ is bounded by definition of the universal, the only assertion left to prove is (3).
We must show that for $f \in C_c(\mathbb{R}, S)$, $\psi(f)$ is compact. Using Lemma 4.17, we can approximate f by linear combinations of functions like $g_t(r) = \phi(r)\tilde{f}_t$ for $\phi \in C_c(\mathbb{R})$ and $f \in S$. As $|\tilde{f}_t| \to 0$ if $t \to -\infty, \infty$, we may take $\overline{f} \in C_c(\mathbb{R}) \subseteq S$ with $||\tilde{f} - \overline{f}||_\infty = \sup_{t\in\mathbb{R}} |\tilde{f}_t - \overline{f}_t| < \varepsilon$.
Setting $\overline{g}_t(r) = \phi(r)\overline{f}_t$, we have

$$||\overline{g} - g||_{L^1} = \int_{\mathbb{R}} ||\overline{f} - \tilde{f}||_\infty |g(s)|ds \leq \varepsilon \cdot C.$$

But the universal norm is dominated by $|| \cdot ||_{L^1}$, see Proposition 4.14, so we infer $||g - \overline{g}||_{\mathrm{u}} \leq \varepsilon \cdot C$. Furthermore, $T_{\overline{g}}$ is compact, because it's kernel lies in $L^2(\mathbb{R} \times \mathbb{R})$. As the compact operators are closed in $\mathcal{B}(L^2(\mathbb{R}))$, we conclude that $\psi(S \rtimes_\tau \mathbb{R}) \subseteq \mathcal{K}(L^2(\mathbb{R}))$. ☐

We call $\psi = M \rtimes V$ the natural representation of $S \rtimes_\tau \mathbb{R}$. Note that as ψ is bounded in the universal norm, we may extend ψ ot a *-homomorphism $\psi : S \rtimes_\tau \mathbb{R} \longrightarrow \mathcal{K}(L^2(\mathbb{R}))$.

Lemma 5.8. *The *-homomorphism ψ is surjective.*

Proof. Let $T \in \mathcal{K}(L^2(\mathbb{R}))$. Set $\mathrm{Re}(T) = \frac{T-T^*}{2}$ and $\mathrm{Im}(T) = \frac{T-T^*}{2i}$, so that $T = \mathrm{Re}(T) + i\mathrm{Im}(T)$. As $\mathrm{Re}(T)$ is a compact self-adjoint operator, we may apply the spectral Theorem. There is an orthonormal bases $(h_i)_{i\in\mathbb{N}}$ of $L^2(\mathbb{R})$ of eigenvectors of $\mathrm{Re}(T)$. The sum $\sum\limits_{i\in\mathbb{N}} \lambda_i P_i$ converges in the operator norm to $\mathrm{Re}(T)$, where λ_i is the eigenvalue to h_i and P_i is the projection onto h_i, i.e. $P_i h_j = \delta_{ij} h_i$. But as the image of a *-homomorphism is a C^*-algebra by Proposition 2.6, $P_i \in \psi(S \rtimes_\tau \mathbb{R})$ by Lemma 5.6 and thus $\mathrm{Re}(T) \in \psi(S \rtimes_\tau \mathbb{R})$. The same argument may be applied to $\mathrm{Im}(T)$, so that $T \in \psi(S \rtimes_\tau \mathbb{R})$. □

The following Lemma is known in much wider generality, see [11, Theorem 4.24]. In our situation however, it can be derived from the Takai duality Theorem, see [8, Lemma 10.9.1].

Lemma 5.9. *The *-homomorphism $\psi : S \rtimes_\tau \mathbb{R} \longrightarrow \mathcal{K}(L^2(\mathbb{R}))$ from Example 5.7 is injective.*

Proof. The Fourier transformation $\hat{}: C^*(\mathbb{R}) \longrightarrow S$, $\hat{f}(t) = \int_\mathbb{R} e^{-ist} f(s)ds$ for $f \in C_c(\mathbb{R})$ is an isomorphism by Proposition 4.32. We will now show that this isomorphism is equivariant with respect to the dual action $\hat{\mathrm{Id}}$ on $C^*(\mathbb{R}) = \mathbb{C} \rtimes_{\mathrm{Id}} \mathbb{R}$ and the action τ on S. Indeed, for $f \in C_c(\mathbb{R})$, we have

$$\widehat{\hat{\mathrm{Id}}_r(f)}(t) = \int_\mathbb{R} e^{-its} \hat{\mathrm{Id}}_r(f)(s)ds = \int_\mathbb{R} e^{-its} \overline{e^{-isr}}(f)(s)ds = \int_\mathbb{R} e^{-i(t-r)s}(f)(s) = \tau_r(\hat{f})(t) \tag{41}$$

which yields by denseness that $\hat{}$ is an equivariant isomorphism. But then $S \rtimes_\tau \mathbb{R} \cong (\mathbb{C} \rtimes_{\mathrm{Id}} \mathbb{R}) \rtimes_{\hat{\mathrm{Id}}} \mathbb{R}$. The second term is, by Takai duality, Theorem 4.27, isomorphic to

$$\mathbb{C} \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) \cong \mathcal{K}(L^2(\mathbb{R})).$$

Now, $S \rtimes_\tau \mathbb{R} \cong \mathcal{K}(L^2(\mathbb{R}))$ is simple by Example 2.7, but then the ideal $\ker(\psi)$ must be 0. □

By Propositions 4.23 and 2.20, we infer that

$$SA \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R} \cong (S \otimes_{\max} A) \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R} \cong A \otimes_{\max} (S \rtimes_\tau \mathbb{R}) \tag{42}$$

We will now calculate the isomorphism $SA \rtimes_{\tau\otimes\alpha} \mathbb{R} \cong A \otimes \mathcal{K}(L^2(\mathbb{R}))$, see [1, Page 145].

Lemma 5.10. *There is a *-isomorphism $\gamma : A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) \longrightarrow SA \rtimes_{\tau\otimes\alpha} \mathbb{R}$ such for $f \in C_c(\mathbb{R}, S)$ and $a \in A$, we have $\gamma(a \otimes T_f)_t(r) = \alpha_t(a) f_t(r)$.*

Proof. Let $\eta : A \otimes_{\max} (S \rtimes_\tau \mathbb{R}) \longrightarrow SA \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R}$ be the *-isomorphism from Proposition 4.23. It is given by $\eta(a \otimes f)_t(r) = af_t(r)$. Furthermore, fix the *-isomorphisms $\psi : S \rtimes_\tau \mathbb{R} \longrightarrow \mathcal{K}(L^2(\mathbb{R}))$ and $\hat{\varphi} : SA \rtimes_{\tau\otimes\alpha} \mathbb{R} \longrightarrow SA \rtimes_{\tau\otimes\alpha} \mathbb{R}$ from Example 5.7 and Lemma 5.5, respectively.
As $\psi^{-1} : \mathcal{K}(L^2(\mathbb{R})) \longrightarrow S \rtimes_\tau \mathbb{R}$ is a *-isomorphism, the map

$$\mathrm{Id} \otimes \psi^{-1} : A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) \longrightarrow A \otimes_{\max} (S \rtimes_\tau \mathbb{R})$$

given by $(\mathrm{Id} \otimes \psi^{-1})(a \otimes T_f)_t(r) = a \otimes \psi^{-1}(T_f) = a \otimes f$ is a *-isomorphism. Now, $\gamma = \hat{\varphi}^{-1} \circ \eta \circ (\mathrm{Id} \otimes \psi^{-1})$ is the desired isomorphism. Indeed,

$$\begin{aligned}
\gamma(a \otimes T_f)_t(r) &= \hat{\varphi}^{-1} \circ \eta \circ (\mathrm{Id} \otimes \psi^{-1})(a \otimes T_f)_t(r) \\
&= \alpha_t(\eta \circ (\mathrm{Id} \otimes \psi^{-1})(a \otimes T_f)_t(r)) \\
&= \alpha_t(\eta(a \otimes f)_t(r)) \\
&= \alpha_t(af_t(r)) \\
&= \alpha_t(a) f_t(r).
\end{aligned} \tag{43}$$

□

The next Lemma only serves the purpose to show naturality in Theorem 5.1.

Lemma 5.11. *Let $(B, \beta, \mathbb{R})$ be another C^*-dynamical system and suppose that $\varphi : A \longrightarrow B$ is an equivariant *-homomorphism.*

(1) The *-homomorphism $S\varphi$ is equivariant with respect to the actions $\tau \otimes \varphi$ and $\tau \otimes \beta$.

(2) If γ_A denotes the *-isomophism from Lemma 5.10 applied to $(A, \mathbb{R}, \alpha)$ and γ_B the one corresponding to the system $(B, \beta, \mathbb{R})$, then the following diagram commutes:

$$\begin{array}{ccc} A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) & \xrightarrow{\gamma_A} & SA \rtimes_{\tau\otimes\alpha} \mathbb{R} \\ \Big\downarrow{\scriptstyle \varphi\otimes\mathrm{Id}} & & \Big\downarrow{\scriptstyle \widehat{S\varphi}} \\ B \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) & \xrightarrow{\gamma_B} & SB \rtimes_{\tau\otimes\beta} \mathbb{R} \end{array} \tag{44}$$

Proof. We first verify the equivariance of $S\varphi$. For $f \in SA$, we have

$$\begin{aligned} S\varphi((\tau\otimes\alpha)_s(f))_t &= \varphi((\tau\otimes\alpha)_s(f)_t) \\ &= \varphi(\alpha_s(f_{t-s})) \\ &= \beta_s(\varphi(f_{t-s})) \\ &= \beta_s(S\varphi(f)_{t-s}) \\ &= (\tau\otimes\beta)_s(S\varphi(f))_t \end{aligned}$$

Suppose that $f \in C_c(\mathbb{R}, S)$ and $a \in A$. By equivariance if φ, we get:

$$\begin{aligned} \widehat{S\varphi}(\gamma_A(a\otimes T_f))_t(r) &= S\varphi(\gamma_A(a\otimes T_f)(r))_t \\ &= \varphi(\gamma_A(a\otimes T_f)_t(r)) \\ &= \varphi(\alpha_t(a)f_t(r)) \\ &= \beta_t(\varphi(a)f_t(r)) \\ &= \beta_t(\varphi(a))f_t(r) \end{aligned} \tag{45}$$

Also, $(\varphi \otimes \mathrm{Id})(a \otimes T_f) = \varphi(a) \otimes T_f$, so we get

$$\gamma_B((\varphi\otimes\mathrm{Id})(a\otimes T_f))_t(r) = \gamma_B(\varphi(a)\otimes f)_t(r) = \beta_t(\varphi(a))f_t(r) \tag{46}$$

which implies that the diagram commutes, because linear combinations of the form $a \otimes T_f$ are dense in $A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R}))$. ☐

For $h \in L^2(\mathbb{R})$, $||h||_{L^2(\mathbb{R})} = 1$, let $\varphi : A \longrightarrow A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R}))$, denote the *-homomorphism $\varphi(a) = a \otimes E_h$. Furthermore, let $\gamma : A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) \longrightarrow SA \rtimes_{\tau\otimes\alpha} \mathbb{R}$ be the *-isomorphism from Lemma 5.10. Set $\omega_E = \gamma \circ \varphi$. The next Lemma is based on [1, Page 146].

Lemma 5.12. *The map $K_0(\omega_E) : K_0(A) \longrightarrow K_0(SA \rtimes_{\tau\otimes\alpha} \mathbb{R})$ is an isomorphism.*

Proof. By Proposition 3.15, $K_0(\varphi) : K_0(A) \longrightarrow K_0(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})))$ is an isomorphism. The *-isomorphism γ induces an isomorphism $K_0(\gamma) : K_0(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R}))) \longrightarrow K_0(SA \rtimes_{\tau\otimes\alpha} \mathbb{R})$ by Proposition 3.19. By setting $\omega_E = \gamma \circ \varphi$, wee see that $K_0(\omega_E)$ is an isomorphism by Proposition 3.19. ☐

Lemma 5.13. *Let $(B, \beta, \mathbb{R})$ be another C^*-dynamical system. Suppose that ω_{EA} is the isomorphism, we get if we apply Lemma 5.12 to $(A, \mathbb{R}, \alpha)$ and ω_{EB} the one when applied to $(B, \mathbb{R}, \beta)$. The rank one projection E may be chosen arbitrarily. In that case, the following diagram commutes for any equivariant *-homomorphism $\varphi : A \longrightarrow B$:*

$$\begin{array}{ccc} A & \xrightarrow{\omega_{EA}} & SA \rtimes_{\tau\otimes\alpha} \mathbb{R} \\ \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \widehat{S\varphi}} \\ B & \xrightarrow{\omega_{EB}} & SB \rtimes_{\tau\otimes\beta} \mathbb{R} \end{array} \tag{47}$$

Proof. Let $\varphi_A : A \longrightarrow A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R}))$ and $\varphi_B : B \longrightarrow B \otimes_{\max} \mathcal{K}(L^2(\mathbb{R}))$ denote the *-homomorphisms $\varphi_A(a) = a \otimes E$ and $\varphi_B(b) = b \otimes E$. We obtain:

$$\begin{aligned}(\varphi \otimes \mathrm{Id})(\varphi_A(a)) &= (\varphi \otimes \mathrm{Id})(a \otimes E) \\ &= \varphi(a) \otimes E \\ &= \varphi_B(\varphi(a))\end{aligned} \tag{48}$$

Thus, the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi_A"] \arrow[d, "\varphi"] & A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) \arrow[d, "\varphi\otimes\mathrm{Id}"] \\ B \arrow[r, "\varphi_B"] & B \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) \end{tikzcd} \tag{49}$$

commutes. But then, the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi_A"] \arrow[d, "\varphi"] & A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) \arrow[r, "\psi_A"] \arrow[d, "\varphi\otimes\mathrm{Id}"] & SA \rtimes_{\tau\otimes\alpha} \mathbb{R} \arrow[d, "\widehat{S\varphi}"] \\ B \arrow[r, "\varphi_B"] & B \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) \arrow[r, "\psi_B"] & SB \rtimes_{\tau\otimes\beta} \mathbb{R} \end{tikzcd} \tag{50}$$

commutes by Lemma 5.11. Now, the top row is ω_{EA} and the bottom one is ω_{EB}. ☐

Lemma 5.14. *There is an isomorphism $\lambda_A : K_1(A) \longrightarrow K_1(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})))$ such that given another C^*-algebra B and a *-homomorphism $\varphi : A \longrightarrow B$, the diagram*

$$\begin{tikzcd} K_1(A) \arrow[r, "\lambda_A"] \arrow[d, "\varphi"] & K_1(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R}))) \arrow[d, "\varphi\otimes\mathrm{Id}"] \\ K_1(B) \arrow[r, "\lambda_B"] & K_1(B \otimes_{\max} \mathcal{K}(L^2(\mathbb{R}))) \end{tikzcd} \tag{51}$$

commutes.

Proof. We have $SA \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) \cong S(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})))$ given by $f \otimes T \mapsto g$ with $g_t = f_t \otimes T$ by Proposition 2.21. Furthermore, the diagram

$$\begin{tikzcd} SA \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) \arrow[r] \arrow[d, "S\varphi\otimes\mathrm{Id}"] & S(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R}))) \arrow[d, "S(\varphi\otimes\mathrm{Id})"] \\ SB \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) \arrow[r] & S(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R}))) \end{tikzcd}$$

commutes. We obtain isomorphisms

$$K_1(A) \cong K_0(SA) \cong K_0(SA \otimes_{\max} \mathcal{K}(L^2(\mathbb{R}))) \cong K_0(S(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})))) \cong K_1(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R}))).$$

The first and last one come from Proposition 3.28. The second one comes from Lemma 5.12 by choosing an arbitrary rank one projection E. Combining all the diagrams from Lemma 5.13 and Proposition 3.28, we obtain that diagram (51) commutes. ☐

Lemma 5.15. *There are isomorphisms*

$$\Theta_{A,0} : K_0(SA \rtimes_{\tau\otimes\alpha} \mathbb{R}) \longrightarrow K_0(A)$$

and

$$\Theta_{A,1} : K_1(SA \rtimes_{\tau\otimes\alpha} \mathbb{R}) \longrightarrow K_1(A)$$

such that given any C^*-dynamical system $(B,\beta,\mathbb{R})$ and an equivariant *-homomorphism $\varphi : A \longrightarrow B$, the both diagrams,

$$\begin{array}{ccc} K_0(SA \rtimes_{\tau\otimes\alpha}\mathbb{R}) & \xrightarrow{\Theta_{A,0}} & K_0(A) \\ \big\downarrow{\scriptstyle K_0(\widehat{S\varphi})} & & \big\downarrow{\scriptstyle K_0(\varphi)} \\ K_0(SB \rtimes_{\tau\otimes\beta}\mathbb{R}) & \xrightarrow{\Theta_{B,0}} & K_0(B) \end{array} \tag{52}$$

and

$$\begin{array}{ccc} K_1(SA \rtimes_{\tau\otimes\alpha}\mathbb{R}) & \xrightarrow{\Theta_{A,1}} & K_1(A) \\ \big\downarrow{\scriptstyle K_1(\widehat{S\varphi})} & & \big\downarrow{\scriptstyle K_1(\varphi)} \\ K_1(SB \rtimes_{\tau\otimes\beta}\mathbb{R}) & \xrightarrow{\Theta_{B,1}} & K_1(B) \end{array} \tag{53}$$

commute.

Proof. Set $\Theta_{A,0} = K_0(\omega_{EA})^{-1}$ for any rank one projection $E \in \mathcal{K}(L^2(\mathbb{R}))$, where ω_{EA} is given by Lemma 5.12. (52) commutes by Lemma 5.13.
Let $\lambda_A : K_1(A) \longrightarrow K_1(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})))$ be the the isomorphism from Lemma 5.14. Now, we set $\Theta_{A,1} = \lambda_A^{-1} \circ K_1(\gamma_A)^{-1}$, where $\gamma_A : A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R})) \longrightarrow SA \rtimes_{\tau\otimes\alpha}\mathbb{R}$ is the *-isomorphism from Lemma 5.10. Then $\Theta_{A,1}$ is an isomorphism and the corresponding diagram commutes like in Lemma 5.13. $\square$

We are now able to show that verifying the assumptions of Lemma 5.4 yields Theorem 5.1.
Suppose that $(B,\beta,\mathbb{R})$ is another C^*-dynamical system and that $\varphi : A \longrightarrow B$ is an equivariant *-homomorphism. The Wiener-Hopf extensions of $(A,\mathbb{R},\alpha)$ and $(B,\mathbb{R},\beta)$ connect in the following way:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SA \rtimes_{\tau\otimes\alpha}\mathbb{R} & \xrightarrow{\widehat{\iota_A}} & CA \rtimes_{\tau\otimes\alpha}\mathbb{R} & \xrightarrow{\widehat{\pi_A}} & A \rtimes_{\alpha}\mathbb{R} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \widehat{S\varphi}} & & \big\downarrow{\scriptstyle \widehat{C\varphi}} & & \big\downarrow{\scriptstyle \hat{\varphi}} & & \\ 0 & \longrightarrow & SB \rtimes_{\tau\otimes\beta}\mathbb{R} & \xrightarrow{\widehat{\iota_B}} & CB \rtimes_{\tau\otimes\beta}\mathbb{R} & \xrightarrow{\widehat{\pi_B}} & B \rtimes_{\beta}\mathbb{R} & \longrightarrow & 0 \end{array} \tag{54}$$

The upper row denotes the Wiener-Hopf extension of $(A,\mathbb{R},\alpha)$. The lower row is the Wiener-Hopf extension of $(B,\mathbb{R},\beta)$. Now, $\hat{\varphi}$ is well-defined by Proposition 4.21. $\widehat{S\varphi}$ is well-defined by the first calculation in Lemma 5.11. The same calculation however, shows that so is $\widehat{C\varphi}$. We want to apply Proposition 3.32, so we need to show that the diagram above commutes. Note that given $f \in SA$,

$$C\varphi(\iota_A(f))_t = \varphi(\iota_A(f)_t) = \begin{cases} \varphi(f_t) & \text{if } t \in \mathbb{R} \\ \varphi(0) & \text{if } t = \infty \end{cases} = \begin{cases} S\varphi(f)_t & \text{if } t \in \mathbb{R} \\ 0 & \text{if } t = \infty \end{cases} = \iota_B(S\varphi(f))_t.$$

But then

$$\widehat{C\varphi}(\widehat{\iota_A}(f))_t(r) = C\varphi(\widehat{\iota_A}(f)(r))_t = C\varphi(\iota_A(f(r)))_t = \iota_B(S\varphi(f(r)))_t = \iota_B(\widehat{S\varphi}(f)(r))_t = \widehat{\iota_B}(\widehat{S\varphi}(f))_t(r)$$

for any $f \in C_c(SA,\mathbb{R})$. Also,

$$\hat{\varphi}(\widehat{\pi_A}(f))(r) = \varphi(\pi_A(f(r)) = \varphi(f_\infty(r)) = C\varphi(f(r))_\infty = \widehat{C\varphi}(f)_\infty(r) = \pi_B(\widehat{C\varphi}(f)(r))_\infty = \widehat{\pi_B}(\widehat{C\varphi}(f))(r).$$

Since (54) commutes, we may apply Proposition 3.32. Let $\delta_{A,1}$ and $\delta_{A,0}$ denote the index and exponential map of the Wiener-Hopf extension for the system $(A,\mathbb{R},\alpha)$, respectively. Also, let $\delta_{B,1}$, $\delta_{B,0}$ denote the index and exponential map for $(B,\mathbb{R},\beta)$, respectively. Proposition 3.32 yields that both,

$$\begin{array}{ccc} K_1(A \rtimes_{\alpha}\mathbb{R}) & \xrightarrow{\delta_{A,1}} & K_0(SA \rtimes_{\tau\otimes\alpha}\mathbb{R}) \\ \big\downarrow{\scriptstyle K_1(\hat{\varphi})} & & \big\downarrow{\scriptstyle K_0(\widehat{S\varphi})} \\ K_1(B \rtimes_{\beta}\mathbb{R}) & \xrightarrow{\delta_{B,1}} & K_0(SB \rtimes_{\tau\otimes\beta}\mathbb{R}) \end{array} \tag{55}$$

and

$$\begin{array}{ccc} K_0(A \rtimes_\alpha \mathbb{R}) & \xrightarrow{\delta_{A,0}} & K_1(SA \rtimes_{\tau\otimes\alpha} \mathbb{R}) \\ \big\downarrow{\scriptstyle K_0(\hat{\varphi})} & & \big\downarrow{\scriptstyle K_1(\widehat{S\varphi})} \\ K_0(B \rtimes_\beta \mathbb{R}) & \xrightarrow{\delta_{B,0}} & K_1(SB \rtimes_{\tau\otimes\beta} \mathbb{R}) \end{array} \tag{56}$$

commute.
If we say that an assertion about the system $(A, \mathbb{R}, \alpha)$ "always holds", we mean that it holds whenever we replace $(A, \mathbb{R}, \alpha)$ by any other C^*-dynamical system $(B, \mathbb{R}, \beta)$.
Now if $K_0(CA \rtimes_{\tau\otimes\alpha} \mathbb{R}) = 0$ and $K_1(CA \rtimes_{\tau\otimes\alpha} \mathbb{R}) = 0$ always holds, we may apply Lemma 5.4 to infer that $\delta_{A,1}, \delta_{B,1}, \delta_{A,0}, \delta_{B,0}$ are isomorphisms. But then, applying Lemma 5.15, we see that $\delta_{A,1} \circ \Theta_{A,0}$, $\delta_{A,0} \circ \Theta_{A,1}$, $\delta_{B,1} \circ \Theta_{B,0}$ and $\delta_{B,0} \circ \Theta_{B,1}$ are isomorphisms.

Lemma 5.16. *If both, $K_0(CA \rtimes_{\tau\otimes\alpha} \mathbb{R}) = 0$ and $K_1(CA \rtimes_{\tau\otimes\alpha} \mathbb{R}) = 0$ always hold, then the maps*

$$\phi_A^0 = \Theta_{A,0} \circ \delta_{A,1} : K_1(A \rtimes_\alpha \mathbb{R}) \longrightarrow K_0(A)$$

and

$$\phi_A^1 = \Theta_{A,1} \circ \delta_{A,0} : K_0(A \rtimes_\alpha \mathbb{R}) \longrightarrow K_1(A)$$

satisfy the assertions of Theorem 5.1.

Proof. We have seen that both maps are isomorphisms. Thus, we are left proving naturality, see [1, Page 153]. Using Lemma 5.15, we obtain commutative diagrams

$$\begin{array}{ccccc} K_1(A \rtimes_\alpha \mathbb{R}) & \xrightarrow{\delta_{A,1}} & K_0(SA \rtimes_{\tau\otimes\alpha} \mathbb{R}) & \xrightarrow{K_0(\Theta_{A,0})} & K_0(A) \\ \big\downarrow{\scriptstyle K_1(\hat{\varphi})} & & \big\downarrow{\scriptstyle K_0(\widehat{S\varphi})} & & \big\downarrow{\scriptstyle K_0(\varphi)} \\ K_1(B \rtimes_\beta \mathbb{R}) & \xrightarrow{\delta_{B,1}} & K_0(SB \rtimes_{\tau\otimes\beta} \mathbb{R}) & \xrightarrow{K_0(\Theta_{B,0})} & K_0(B) \end{array} \tag{57}$$

and

$$\begin{array}{ccccc} K_0(A \rtimes_\alpha \mathbb{R}) & \xrightarrow{\delta_{A,0}} & K_1(SA \rtimes_{\tau\otimes\alpha} \mathbb{R}) & \xrightarrow{K_1(\Theta_{A,1})} & K_1(A) \\ \big\downarrow{\scriptstyle K_0(\hat{\varphi})} & & \big\downarrow{\scriptstyle K_1(\widehat{S\varphi})} & & \big\downarrow{\scriptstyle K_1(\varphi)} \\ K_0(B \rtimes_\beta \mathbb{R}) & \xrightarrow{\delta_{B,0}} & K_1(SB \rtimes_{\tau\otimes\beta} \mathbb{R}) & \xrightarrow{K_1(\Theta_{B,1})} & K_1(B) \end{array} \tag{58}$$

which implies naturality. □

So we need to show that $K_0(CA \rtimes_{\tau\otimes\alpha} \mathbb{R})$ and $K_1(CA \rtimes_{\tau\otimes\alpha} \mathbb{R})$ are 0.
The following Lemma is contained in [6, Example 4.1.5].

Lemma 5.17. *We have $K_0(CA) \cong 0$ and $K_1(CA) \cong 0$.*

Proof. We fix an arbitrary homeomorphism $h : (0, 1] \longrightarrow \mathbb{R} \cup \{\infty\}$. Define $h^T : C \longrightarrow C_0((0, 1])$, $C \ni f \mapsto f \circ h$. It's inverse is given by $f \mapsto f \circ h^{-1}$, so h^T is a *-isomorphism. Now, set $\gamma_s(f)_t = f_{ts}$. Then $C \xrightarrow{\alpha_0} 0 \longrightarrow 0$ is a homotopy. We infer $K_j(CA) = 0$ by Propositon 3.21. □

The following Lemma from [1, Lemma 1], reduces the problem even further.

Lemma 5.18. *If the index map δ_1 of the Wiener-Hopf extension is always surjective, then*

$$K_j(CA \rtimes_{\tau\otimes\alpha} \mathbb{R}) = 0$$

for $j = 0, 1$.

Proof. The proof consists of two steps:

1) If $K_1(A) = 0$, then $K_0(A \rtimes_\alpha \mathbb{R}) = 0$.
2) If $K_0(A) = 0$, then $K_1(A \rtimes_\alpha \mathbb{R}) = 0$.

We are insisting that $\delta_1 : K_1(A \rtimes_\alpha \mathbb{R}) \longrightarrow K_0(SA \rtimes_{\tau\otimes\alpha} \mathbb{R})$ is always surjective.

Step 1) If $K_1(A \rtimes_\alpha \mathbb{R}) \cong 0$, then $K_0(SA \rtimes_{\tau\otimes\alpha} \mathbb{R}) \cong 0$, but then $K_0(A) \cong 0$ by Lemma 5.15. So we have

$$K_1(A \rtimes_\alpha \mathbb{R}) \cong 0 \Rightarrow K_0(A) \cong 0 \tag{59}$$

Now if $K_1(A) \cong 0$, we have $K_1(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{R}))) \cong K_1(SA \rtimes_{\tau\otimes\alpha} \mathbb{R}) \cong 0$ by Lemma 5.9 and 5.15. By Takai duality, Theorem 4.27, we infer $K_1((A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}) \cong 0$. Formula (59) now implies $K_0(A \rtimes_\alpha \mathbb{R}) \cong 0$.
Thus, step 1)

$$K_1(A) = 0 \Rightarrow K_0(A \rtimes_\alpha \mathbb{R}) = 0$$

is completed.

Step 2) If $K_0(A) \cong 0$, we conclude by Bott periodicity, Theorem 3.29 that $K_1(SA) \cong 0$. So that $K_0(SA \rtimes_{\mathrm{Id}\otimes\alpha} \mathbb{R})$ by step 1). Now, $K_0(S(A \rtimes_\alpha \mathbb{R})) \cong 0$ by Proposition 4.23. But then

$$K_1(A \rtimes_\alpha \mathbb{R}) \cong 0$$

by Proposition 3.28.

Now, since $K_0(CA) \cong 0$ and $K_1(CA) \cong 0$ by Lemma 5.17, we conclude:
$K_0(CA \rtimes_{\tau\otimes\alpha} \mathbb{R}) \cong 0$ by step 1) and $K_1(CA \rtimes_{\tau\otimes\alpha} \mathbb{R}) \cong 0$ by step 2). □

In view of Lemma 5.16, we need to show that $\delta_1 : K_1(A \rtimes_\alpha \mathbb{R}) \longrightarrow K_0(SA \rtimes_{\tau\otimes\alpha} \mathbb{R})$ is surjective.
We will from now on suppose that A is unital and prove the general case later.
We set $B = M_m(A)$. By [4, Example 6.3.1], $M_m(\mathbb{C}) \otimes_{\max} A \cong B$. This isomorphism is given by sending $(\alpha_{ij})_{1\le i,j\le m} \otimes a$ to $(\alpha_{ij}a)_{1\le i,j\le m}$. Note that we may extend the action $\alpha : \mathbb{R} \longrightarrow A$ to $\alpha : \mathbb{R} \longrightarrow B$ by applying α to all the components.

5.4 Connes' Cocycle

We must show that given a projection $p \in B$, there is an element $h \in K_1(A \rtimes_\alpha \mathbb{R})$ with $\delta_1(h) = K_0(\omega_E)[p]_0$ for some rank one projection E. That is sufficient, because $K_0(\omega_E)$ is an isomorphism by Lemma 5.12. We can always replace p by a projection q with $p \sim_h q$.

Definition 5.19. *Suppose that A is a unital C^*-algebra and G a locally compact group. We say that two actions $\alpha, \beta : G \longrightarrow \mathrm{Aut}(A)$ are exterior equivalent if there is a map $s \mapsto u_s$ of unitaries satisfying the following conditions:*

(1) $u_{st} = u_s\alpha_s(u_t)$.

(2) $\beta_t(a) = u_t\alpha_t(a)u_t^*$.

(3) $t \mapsto u_t$ is continuous in the norm.

A function $u : G \longrightarrow \mathcal{U}(A)$ satisfying (1) is called a unitary (1-)cocycle and the equation (1) is called the (1-)cocycle identity. We will only be concerned with exterior equivalence in the situation $G = \mathbb{R}$.
The following Proposition is contained in [2, Proposition 4].

Proposition 5.20. *There is an action β on B that is exterior equivalent to α and a projection $q \in B^\infty$ with $p \sim_h q$ such that $\beta_t(q) = q$ for all $t \in \mathbb{R}$.*

We will now give a proof for Proposition 5.20. We must find a map of unitaries $t \mapsto u_t$ that is continuous in the norm on B and satisfies

$$u_{s+t} = u_s\alpha_s(u_t) \tag{60}$$

$$\alpha_t(p) = u_t^* p u_t. \tag{61}$$

We fix a projection $q \in M_n(B^\infty)$ that is homotopic to p by using Lemma 4.37. Now, set

$$\delta(q) = \lim_{t\to 0} \frac{\alpha_t(q) - q}{t}$$

and

$$P = i[\delta(q), q]$$

where $[a, a'] = aa' - a'a$. Furthermore, set

$$P_{n,t} = \int \cdots \int\limits_{0 \leq s_1 \leq \ldots \leq s_n \leq t} \alpha_{s_1}(P) \ldots \alpha_{s_n}(P) ds_1 \ldots ds_n$$

for $t \in \mathbb{R}_+$. This integral is well defined, because $P_{n,t} = \int\limits_0^t \int\limits_0^{s_n} \ldots \int\limits_0^{s_2} \alpha_{s_1}(P) ds_1 \alpha_{s_2}(P) ds_2 \ldots ds_{n-1} \alpha_{s_n}(P) ds_n$, so $P_{n,t}$ is an iterated Riemann integral with continuous integrand. In fact, we integrate over an n-dimensional triangle inscribed into the n-dimensional cube with side length t. With an $n - dimensional$ triangle, we mean a volume that looks like a triangle whenever we look at a 2-dimensional cross-section from all sides of the cube. Now, let

$$u_t = \sum_{n=0}^{\infty} P_{n,t}$$

for $t \geq 0$. We will first show that u_t has all the claimed properties for $t \geq 0$. Then the cocycle identity (60) tells us that for $t \leq 0$ we must have

$$u_0 = u_0 \alpha_0(u_0)$$

implying $u_0 = 1$, so we get

$$1 = u_{t-t} = u_t \alpha_t(u_{-t}).$$

The last equation may be solved for $u_{-t} = \alpha_{-t}(u_t^*)$ giving us the full cocycle.

Lemma 5.21. *The series* $u_t = \sum\limits_{n=0}^{\infty} P_{n,t}$ *converges absolutely.*

Proof. We have

$$||P|| = ||\delta(q)q - q\delta(q)|| \leq 2 \cdot C$$

with $C = \max(||\delta(q)||, ||q||)$. But then

$$\begin{aligned} ||u_t|| &\leq \sum_{n=0}^{\infty} ||P_{n,t}|| \\ &\leq \sum_{n=0}^{\infty} \int \cdots \int\limits_{0 \leq s_1 \leq \ldots \leq s_n \leq t} ||\alpha_{s_1}(P) \ldots \alpha_{s_n}(P)|| ds_1 \ldots ds_n \\ &\leq \sum_{n=0}^{\infty} (2C)^n \int \cdots \int\limits_{0 \leq s_1 \leq \ldots \leq s_n \leq t} 1 \ \ ds_1 \ldots ds_n \\ &= \sum_{n=0}^{\infty} \frac{(2Ct)^n}{n!} < \infty. \end{aligned} \tag{62}$$

Again, we are integrating over a triangle inscribed into the n-dimensional cube.
So we have $\int \cdots \int\limits_{0 \leq s_1 \leq \ldots \leq s_n \leq t} 1 \ \ ds_1 \ldots ds_n = \frac{t^n}{n!}$. This follows from the following induction step:

$$\begin{aligned} \int \cdots \int\limits_{0 \leq s_1 \leq \ldots \leq s_{n+1} \leq t} 1 \ \ ds_1 \ldots ds_{n+1} &= \int\limits_0^t P_{n,s_n} ds_n \\ &= \int\limits_0^t \frac{s_n^n}{n!} ds_n \\ &= \frac{t^{n+1}}{(n+1)!}. \end{aligned} \tag{63}$$

□

Next, we remark that $\delta(q)$ is self-adjoint. Indeed, we have

$$\delta(q)^* = \lim_{t\to 0}\left(\frac{\alpha_t(q)-q}{t}\right)^* = \frac{\alpha_t(q^*)-q^*}{t} = \frac{\alpha_t(q)-q}{t} = \delta(q).$$

But that implies $P^* = (i[\delta(q),q])^* = -i[q,\delta(q)] = i[\delta(q),q] = P$, so P is self-adjoint too. But then the adjoint of the integral $P_{n,t}$ is given by

$$P_{n,t}^* = \int\cdots\int_{0\le s_n\le\ldots\le s_1\le t} \alpha_{s_1}(P)\ldots\alpha_{s_n}(P)ds_1\ldots ds_n,$$

so we just need to reorder the integrand. To motivate the following procedures, we calculate a few terms of $u_t\alpha_t(u_s)$ which we want to be equal to u_{t+s}. The terms are $u_t = 1 + P_{1,t} + P_{2,t} + \ldots$ and $\alpha_t(u_s) = 1 + \alpha_t(P_{1,s}) + \alpha_t(P_{2,s}) + \ldots$, multiplying everything out, we get

$$u_t\alpha_t(u_s) = 1 + (P_{1,t} + \alpha_t(P_{1,s})) + (P_{2,t} + P_{1,t}\alpha_t(P_{1,s}) + \alpha_t(P_{2,s}))\ldots$$

We have

$$P_{1,t} + \alpha_t(P_{1,s}) = \int_0^t \alpha_{s_1}(P)ds_1 + \int_0^s \alpha_{s_1+t}(P)ds_1 = \int_0^{t+s}\alpha_{s_1}(P)ds_1$$

and

$$\begin{aligned} &P_{2,t} + P_{1,t}\alpha_t(P_{1,s}) + \alpha_t(P_{2,s}) \\ &= \iint_{0\le s_1\le s_2\le t} \alpha_{s_1}(P)\alpha_{s_2}(P)ds_1ds_2 + \int_{0\le s_1\le t}\alpha_{s_1}(P)ds_1 \int_{t\le s_2\le t+s}\alpha_{s_2}(P)ds_2 \\ &+ \iint_{t\le s_1\le s_2\le t+s}\alpha_{s_1}(P)\alpha_{s_2}(P)ds_1ds_2. \end{aligned} \tag{64}$$

To get $P_{2,t+s}$, we must fill the full triangle described by $0 \le s_1 \le s_2 \le t+s$. Looking at the 3 integrals above, we see that the first integral runs over the volume $0 \le s_1 \le s_2 \le t$, the second over $0 \le s_1 \le t \le s_2 \le t+s$ and the third one over $t \le s_1 \le s_2 \le t+s$. But these three conditions together form a partition of the condition $0 \le s_1 \le s_2 \le t+s$.

Lemma 5.22. *The u_t satisfies the cocycle identity $u_{t+s} = u_t\alpha_t(u_s)$ for all $t, s \ge 0$.*

Proof. Set $U_n = U_{n,t,s} = \sum_{k=0}^{n} P_{k,t}\alpha_t(P_{n-k,s})$, so that the series $\sum_{n=0}^{\infty} U_n$ converges to $u_t\alpha_t(u_s)$ in the norm. Looking at one of the terms U_n, we want to show that $U_n = P_{n,t+s}$.
The summand $P_{k,t}\alpha_t(P_{n-k,t})$ corresponds to the volume described by the two conditions

$$0 \le s_1 \le \ldots \le s_k \le t$$

and

$$t \le s_{k+1} \le \ldots \le s_n \le t+s.$$

But the combination of these two conditions for all k forms a partition of the condition

$$0 \le s_1 \le \ldots \le s_n \le t+s.$$

□

So we know that the cocycle identity is satisfied. The proof of Lemma 5.21 shows that $u_s \to 1 = u_0$ as $s \to \infty$. Applying the cocycle identity, we get $u_{t+s} = u_t\alpha_t(u_s)$. The right term converges to u_t as $s \to 0$. Thus, $t \mapsto u_t$ is continuous in the norm.

The only assertion left is to show that u_t is a unitary for all $t \geq 0$. We set $Q = [\delta(q), q] = P/i$. Our integral terms now become

$$P_{n,t} = i^n \int\limits_{0 \leq s_1 \leq ... \leq s_n \leq t} \cdots \int \alpha_{s_1}(Q)...\alpha_{s_n}(Q) ds_1...ds_n$$

and

$$P^*_{n,t} = (-i)^n \int\limits_{0 \leq s_n \leq ... \leq s_1 \leq t} \cdots \int \alpha_{s_1}(Q)...\alpha_{s_n}(Q) ds_1...ds_n.$$

If we expand the terms of $u_t u_t^* = \sum\limits_{n=0}^{\infty} \sum\limits_{k=0}^{n} P_{k,t} P^*_{n-k,t}$, we get 1 for the first,

$$-i \int\limits_{0 \leq s_1 \leq t} \alpha_{s_1}(Q) ds_1 + i \int\limits_{0 \leq s_1 \leq t} \alpha_{s_1}(Q) ds_1 = 0$$

for the second and

$$-\iint\limits_{0 \leq s_2 \leq s_1 \leq t} \alpha_{s_1}(Q)\alpha_{s_2}(Q) ds_1 ds_2 + \int\limits_{0 \leq s_1 \leq t} \alpha_{s_1}(Q) ds_1 \int\limits_{0 \leq s_2 \leq t} \alpha_{s_2}(Q) ds_2 - \iint\limits_{0 \leq s_1 \leq s_2 \leq t} \alpha_{s_1}(Q)\alpha_{s_2}(Q) ds_1 ds_2$$

for the third. If we compute the first difference in the third term, we see that the combination of the conditions

$$0 \leq s_1 \leq t$$

and

$$0 \leq s_2 \leq t$$

is weaker than the condition $0 \leq s_2 \leq s_1 \leq t$. Thus, all the terms from the first term cancel and we are left with the part from the second integral where $0 \leq s_2 \leq s_1 \leq t$ does not hold. But that just means $0 \leq s_1 \leq s_2 \leq t$, so the rest cancels with the third term.
If we say terms of $P_{k,t} P^*_{n-k,t}$, we mean the $\binom{n}{k}$ summands of

$$\begin{aligned}
P_{k,t} P^*_{n-k,t} &= i^k \int\limits_{0 \leq s_1 \leq ... \leq s_k \leq t} \cdots \int \alpha_{s_1}(Q)...\alpha_{s_k}(Q) ds_1...ds_k \ (-i)^{n-k} \int\limits_{0 \leq s_n \leq ... \leq s_{k+1} \leq t} \cdots \int \alpha_{s_{k+1}}(Q)...\alpha_{s_n}(Q) ds_1...ds_{n-k} \\
&= i^k (-i)^{n-k} \int\limits_{0 \leq s_n \leq ... \leq s_{k+1} \leq s_1 \leq ... \leq s_k \leq t} \cdots \int \alpha_{s_1}(Q)...\alpha_{s_n}(Q) ds_1...ds_n \\
&+ i^k (-i)^{n-k} \int\limits_{0 \leq s_n \leq ... \leq s_{k+2} \leq s_1 \leq s_{k+1} \leq ... \leq s_k \leq t} \cdots \int \alpha_{s_1}(Q)...\alpha_{s_n}(Q) ds_1...ds_n \\
&+ ... \\
&+ i^k (-i)^{n-k} \int\limits_{0 \leq s_n \leq ... \leq s_{k+2} \leq s_1 \leq ... \leq s_k \leq s_{k+1} \leq t} \cdots \int \alpha_{s_1}(Q)...\alpha_{s_n}(Q) ds_1...ds_n \\
&+ i^k (-i)^{n-k} \int\limits_{0 \leq s_n \leq ... \leq s_{k+3} \leq s_1 \leq s_{k+2} \leq ... \leq s_k \leq s_{k+1} \leq t} \cdots \int \alpha_{s_1}(Q)...\alpha_{s_n}(Q) ds_1...ds_n \\
&+ ... \\
&+ i^k (-i)^{n-k} \int\limits_{0 \leq s_1 \leq ... \leq s_k \leq s_n \leq ... \leq s_{k+1} \leq t} \cdots \int \alpha_{s_1}(Q)...\alpha_{s_n}(Q) ds_1...ds_n.
\end{aligned} \tag{65}$$

So we split the product into all the triangles that lie in the integration volume.

Lemma 5.23. *The element u_t is a unitary for any $t \geq 0$.*

Proof. We want to show that $\sum_{k=0}^{n} P_{k,t}P_{n-k,t}^* = 0$ for $n \geq 1$. As t is fixed, we may set $P_k = P_{k,t}$ Since the left factor has a factor i^k and the right one a factor $(-i)^{n-k}$, the sign alternates as we run over k. We will in fact show the following:
After we cancel the rest of $P_{k-1}P_{n-k+1}^*$ with the terms of $P_kP_{n-k}^*$, all the terms from $P_kP_{n-k}^*$ that are left are terms of $P_{k+1}P_{n-k-1}^*$. We prove this by induction.
Suppose that all the terms that were left in $P_{k-1}P_{n-k-1}^*$ were terms of $P_kP_{n-k}^*$. Thus, the terms of $P_kP_{n-k}^*$ that are left have to satisfy

$$0 \leq s_1 \leq ... \leq s_k \leq t,$$

$$0 \leq s_n \leq ... \leq s_{k+1} \leq t$$

and cannot satisfy both of the following conditions:

$$0 \leq s_1 \leq ... \leq s_{k-1} \leq t$$

$$0 \leq s_n \leq ... \leq s_k \leq t$$

We want these terms to satisfy:

$$0 \leq s_1 \leq ... \leq s_{k+1} \leq t$$

$$0 \leq s_n \leq ... \leq s_{k+2} \leq t$$

As the second condition from the first set of conditions implies the second from the last one, we must only show that $s_k \leq s_{k+1}$. Now, the first condition of the first set implies the first of the second. Combining the second condition with the first set, we are left with

$$0 \leq s_n \leq ... \leq s_{k+1} \leq t$$

and

$$s_{k+1} \leq s_k$$

That is, because $s_{k+1} \leq s_k$ is the only relation in which these differ. But then satisfying the first set and not the second means that $s_k \leq s_{k+1}$.
Now if we look at the last term $P_0P_n^* = P_n^*$, we see that this summand consists of only one term which has to cancel with a term of the previous summand, so the whole sum is 0. The identity $u_t^*u_t = 1$ may be shown by replacing $\leq$ with $\geq$ and vice versa. $\square$

The next Lemma states that a noncommutative function with derivative 0 is fixed by the action.

Lemma 5.24. *Suppose that $b \in M_n(B^\infty)$ with $\delta(b) = 0$. Then $\alpha_t(b) = b$ for all $t \in \mathbb{R}$.*

Proof. If $\delta(b) = 0$, the derivative of $s \mapsto \alpha_s(b)$ is 0 (for any s), because $\delta(\alpha_s(b)) = \alpha_s(\delta(b)) = 0$. But then

$$\lim_{t\to 0} \frac{\phi(\alpha_{t+s}(b)) - \phi(\alpha_s(b))}{t} = 0$$

for any continuous linear functional $\phi \in M_n(B)'$. So the map $s \mapsto \phi(\alpha_s(b))$ is constant. If $\alpha_s(b) \neq \alpha_r(b)$ for some $s, r \in \mathbb{R}$, we may apply the Hahn Banach Theorem to find some $\phi \in M_n(B)'$ with $||\phi(\alpha_s(b))|| = 1$ and $||\phi(\alpha_r(b))|| = 0$ which is a contradiction. $\square$

We have a map $t \mapsto u_t$ that satisfies all of the assertions claimed in Proposition 5.20 for $t, s \geq 0$. However, if it is possible to extend $t \mapsto u_t$ to a cocycle on $\mathbb{R}$, then we must have

$$1 = u_0 = u_{t-t} = u_t\alpha_t(u_{-t})$$

for $t \geq 0$. So set $v_t = u_t$ for $t \geq 0$ and $v_t = \alpha_{-t}(u_t^*)$ for $t \leq 0$.

Proof of Proposition 5.20. The element v_t is a unitary for all $t \in \mathbb{R}$ as u_t is one for $t \geq 0$. The map $t \mapsto v_t$ is also continuous in the norm as $t \mapsto u_t$ is. The cocycle identity follows from a case distinction. For example, if $t \geq 0$, $s \leq 0$ and $t \geq -s$, then

$$v_t\alpha_t(v_s) = u_t\alpha_{t-s}(u_s^*) = u_t(u_{t-s}\alpha_{t-s}(u_s))^*u_{t-s} = u_tu_t^*u_{t-s} = u_{t-s}.$$

We will now identify u_t and v_t by setting $u_t = v_t$ for $t \leq 0$. The only thing left is to show that $u_t\alpha_t(q)u_t^* = q$ for all $t \in \mathbb{R}$. Note that $\beta_t(a) = u_t\alpha_t(a)u_t^*$ is an action (exterior equivalent to α) on B. We calculate

$$\begin{aligned} u_t\alpha_t(q)u_t^* &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_{k,t}\alpha_t(q)P_{n-k,t}^* \\ &= \alpha_t(q) + (P_{1,t}\alpha_t(q) + \alpha_t(q)P_{1,t}^*) + \sum_{n=2}^{\infty}\sum_{k=0}^{n} P_{k,t}\alpha_t(q)P_{n-k,t}^* \end{aligned} \tag{66}$$

First, we provide an estimate for the last term. We have

$$\begin{aligned} \left\|\sum_{n=2}^{\infty}\sum_{k=0}^{n} P_{k,t}\alpha_t(q)P_{n-k,t}^*\right\| &\leq \sum_{n=2}^{\infty}\sum_{k=0}^{n} ||P_{k,t}||\ ||P_{n-k,t}|| \\ &\leq \sum_{n=2}^{\infty}\sum_{k=0}^{n} \frac{(2Ct)^n}{k!(n-k)!} \\ &\leq t^2\sum_{n=2}^{\infty}\sum_{k=0}^{n} \frac{(2C)^n t^{n-2}}{k!(n-k)!} \end{aligned} \tag{67}$$

where we used the estimates from Lemma 5.21. Now, we give an estimate for the second term, fix $\varepsilon > 0$ and $t > 0$ so small that $||\alpha_s(P) - P|| \leq \varepsilon$ for $0 \leq s \leq t$.

$$\begin{aligned} \left\|\frac{P_{1,t}\alpha_t(q)}{t} - iP\alpha_t(q)\right\| &= \frac{||i\int_0^t(\alpha_s(P) - P)ds\ \ \alpha_t(q)||}{t} \\ &\leq ||q|| \cdot \varepsilon \end{aligned} \tag{68}$$

As $\alpha_t(q) \to q$ whenever $t \to 0$, we obtain

$$\frac{P_{1,t}\alpha_t(q)}{t} \to iPq.$$

Now, by a similar estimate, we get

$$\frac{\alpha_t(q)P_{1,t}^*}{t} \to -iqP.$$

Combining all these estimates, we get

$$\frac{\beta_t(q)}{t} = \frac{u_t\alpha_t(q)u_t^*}{t} = \frac{u_t\alpha_t(q)u_t^* - \alpha_t(q) + \alpha_t(q)}{t} \to i[P,q] + \delta(q).$$

We will now calculate that $i[P,q] + \delta(q) = 0$. We have

$$i[P,q] = i(Pq - qP) = i(i((\delta(q)q - q\delta(q))q - q(\delta(q)q - q\delta(q))) = 2q\delta(q)q - q\delta(q) - \delta(q)q.$$

Applying the product rule, we get $\delta(q) = \delta(q^2) = q\delta(q) + \delta(q)q$. Thus,

$$i[P,q] + \delta(q) = 2q\delta(q)q = 2q(q\delta(q) + \delta(q)q)q = 4q\delta(q)q = 0.$$

And Lemma 5.24 applied to the action β yields the claim. ☐

It is possible to show that two exterior equivalent C^*-dynamical systems are *-isomorphic, see [11, Lemma 2.68]. However, we cannot simply switch from the action α to β in order to directly deduce that any projection p lies in the image of the index map δ_1. The reason for this is that the isomorphism $K_0(\omega_E) : K_0(A) \longrightarrow K_0(SA \rtimes_{\tau\otimes\alpha} \mathbb{R})$ from Lemma 5.12 depends on the action itself. Also, we would need to switch to another action for every $p \in \mathcal{P}_\infty(A)$. We must instead provide calculations with the cocycle to show $K_0(\omega_E)([p]_0) \in \delta_1(K_1(SA \rtimes_{\tau\otimes\alpha} \mathbb{R}))$ instead.

5.5 Surjectivity of the Index Map

Let $p \in B$ be a projection. We will from now on fix a cocycle u_t satisfying the assertions of Proposition 5.20. That is, $t \mapsto u_t$ is a map of unitaries that is continuous in the norm and satisfies

$$u_{s+t} = u_s \alpha_s(u_t) \tag{69}$$

$$\alpha_t(p) = u_t^* p u_t. \tag{70}$$

Note that we have switched to a projection $q \in B^\infty$ with $p \sim_h p$ and called it p. That is no problem, because K-theory respects homotopies by construction.
The following identifications are made in [2, Page 148]. There is an isomorphism $CB \cong M_m(CA)$ by sending a function $f : \mathbb{R} \longrightarrow CB$ to the matrix of functions $(f_{ij})_{1 \leq i,j \leq m}$. We may also extend $\tau \otimes \alpha$ to CB by $(\tau \otimes \alpha)_s(f)_t(r) = \alpha_s(f_{t-s}(r))$ where the right α is the extended version on B.
Furthermore, we have

$$CB \rtimes_{\tau \otimes \alpha} \mathbb{R} \cong M_m(CA \rtimes_{\tau \otimes \alpha} \mathbb{R})$$

by Proposition 4.23. Also, $M_m(SA) \cong SB$ and we may extend the action $\tau \otimes \alpha$ to SB. Applying Proposition 4.23 again, we obtain

$$M_m(SB \rtimes_{\tau \otimes \alpha} \mathbb{R}) \cong M_m(SA \rtimes_{\tau \otimes \alpha} \mathbb{R}).$$

We will also identify $f \in C_c(\mathbb{R}, SB)$ with $\hat{\imath}(f) \in C_c(\mathbb{R}, CB)$, where $\hat{\imath}(f)_t(r) = f_t(r)$ and extend this identification, so that $SB \rtimes_{\tau \otimes \alpha} \mathbb{R} \subseteq CB \rtimes_{\tau \otimes \alpha} \mathbb{R}$ and $S \rtimes_\tau \mathbb{R} \subseteq C \rtimes_\tau \mathbb{R}$
Suppose $E \in \mathcal{K}(L^2(\mathbb{R}))$ is a rank one projection. By Lemma 5.12, it is sufficient to show that $K_0(\omega_E)[p]_0$ lies in the image of δ_1 for any projection $p \in M_m(A)$. For this, we choose a specific rank one projection, see [1, Page 146].
Let E be the rank one projection onto $h \in L^2(\mathbb{R})$ given by

$$h(t) = e^{-t/2} \chi(t).$$

h is called the first Laguerre function. Here, $\chi = \chi_{[0,\infty]}$ is the characteristic function of the set $[0, \infty]$.
The following Lemma will later be useful to apply Proposition 3.33 to our situation. It is contained in [2, Lemma 2]. If $f, g \in C_c(\mathbb{R}, SA)$, then by Property (2) in Proposition 4.6, we have

$$(f * g)_t(r) = \left(\int_{\mathbb{R}} f(s)(\tau \otimes \alpha)_s(g(r-s))ds \right)_t = \int_{\mathbb{R}} f_t(s)(\tau \otimes \alpha)_s(g(r-s))_t ds = \int_{\mathbb{R}} f_t(s)\alpha_s(g_{t-s}(r-s))ds.$$

Lemma 5.25. *Given any $f \in C_c(\mathbb{R}, C)$, set $f_p \in C_c(\mathbb{R}, CB)$ to $f_{p,t}(r) = (f_p)_t(r) = pu_r f_t(r)$.*

(1) Then $f \mapsto f_p$ extends to a *-homomorphism $C \rtimes_\tau \mathbb{R} \longrightarrow CB \rtimes_{\tau \otimes \alpha} \mathbb{R}$.

(2) Furthermore, this *-homomorphism maps $S \rtimes_\tau \mathbb{R}$ into $SB \rtimes_{\tau \otimes \alpha} \mathbb{R}$.

Proof. Fix $f, g \in C_c(\mathbb{R}, C)$. We will first verify that $f \mapsto f_p$ is a homomorphism with respect to the convolution algebra operations. Indeed, the cocycle identity (69) implies that

$$u_s^* u_r = u_s^* u_{s+(r-s)} = u_s^* u_s \alpha_s(u_{r-s}) = \alpha_s(u_{r-s}).$$

Now, we calculate that

$$\begin{aligned}
(f_p * g_p)_t(r) &= \int_{\mathbb{R}} f_{p,t}(s)\alpha_s(g_{p,t-s}(r-s))ds \\
&= \int_{\mathbb{R}} pu_s f_t(s)\alpha_s(pu_{t-s}g_{r-s}(r-s))ds \\
&= \int_{\mathbb{R}} pu_s\alpha_s(p)\alpha_s(u_{r-t})f_{p,t}(s)g_{p,t-s}(r-s)ds \\
&= \int_{\mathbb{R}} pu_s\alpha_s(p)u_s^* u_r f_{p,t}(s)g_{p,t-s}(r-s)ds \\
&= \int_{\mathbb{R}} ppu_r f_{p,t}(s)g_{p,t-s}(r-s)ds \\
&= pu_r \int_{\mathbb{R}} f_{p,t}(s)g_{p,t-s}(r-s)ds \\
&= pu_r(f*g)_t(r) \\
&= (f*g)_{p,t}(r).
\end{aligned} \tag{71}$$

Furthermore,

$$u_r\alpha_r(u_{-r}) = u_{r-r} = u_0 = 1,$$

so that

$$\begin{aligned}
(f_p^*)_t(r) &= \alpha_r(f_{p,t-r}(-r)^*) \\
&= \alpha_r((pu_{-r}f_{t-r}(-r))^*) \\
&= (\alpha_r(pu_{-r}))^*\overline{f_{t-r}(-r)} \\
&= (\alpha_r(pu_{-r}))^* f_t^*(r) \\
&= (\alpha_r(p)\alpha_r(u_r)\alpha_r(u_{-r}))^* f_t^*(r) \\
&= (u_r^* pu_r\alpha_r(u_{-r}))^* f_t^*(r) \\
&= pu_r f_t^*(r).
\end{aligned} \tag{72}$$

We want to apply Proposition 4.19, so we must show that $f \mapsto f_p$ is bounded in $||\cdot||_{L^1}$. We calculate

$$\begin{aligned}
||f_p||_{L^1} &= \int_{\mathbb{R}} ||f_p(r)||_{CB}ds \\
&= \int_{\mathbb{R}} \sup_{t\in\mathbb{R}} ||f_{p,t}(r)||_B ds \\
&= \int_{\mathbb{R}} \sup_{t\in\mathbb{R}} ||pu_r f_t(r)||_B ds \\
&= \int_{\mathbb{R}} \sup_{t\in\mathbb{R}} ||pu_r||_B |f_p(r)| ds \\
&= \int_{\mathbb{R}} \sup_{t\in\mathbb{R}} ||pu_r u_r^* p^*||^{1/2} |f_p(r)| ds \\
&= \int_{\mathbb{R}} \sup_{t\in\mathbb{R}} ||p||_B^2 |f_p(r)| ds \\
&= \int_{\mathbb{R}} \sup_{t\in\mathbb{R}} |f_p(r)| ds \\
&= ||f||_{L^1}
\end{aligned} \tag{73}$$

where we used $||p||^2 = ||pp^*|| = ||p||$. The second assertion follows from $\lim_{t\to-\infty} f_{p,t}(r) = 0$ for all $f \in C_c(\mathbb{R}, S)$ and $r \in \mathbb{R}$. □

Let $e = \psi^{-1}(E)$, where ψ is the isomorphism from Lemmas 5.8 and 5.9. We infer that $e \in S \rtimes_\tau \mathbb{R}$ is a projection. The next Lemma will later be used to apply Proposition 3.33. It is contained in [1, Pages 149 and 150] and [14, Lemma 6].

Lemma 5.26. *There is an element $f \in C \rtimes_\tau \mathbb{R}$ satisfying*

$$f + f^* + f^* * f = 0$$

and

$$f + f^* + f * f^* = e.$$

We will now prepare the proof of Lemma 5.26 with the following calculations.
First, we define the following kernel operator. Let $\tilde{f}$ denote the function $\tilde{f}_t(r) = e^{-r/2}\chi(r)\chi(t-r)$. $F \in \mathcal{B}(L^2(\mathbb{R}))$ is now given by $Fh(t) = \int_{\mathbb{R}} \tilde{f}_t(s)h(t-s)ds$. We set $g(r) = e^{-r/2}\chi(r)$. The operator F is bounded, because

$$|Fh(t)| \leq \int \mathbb{R}\tilde{f}_t(s)|h(t-s)|ds = \int_{\mathbb{R}} g(s)\chi(t-s)|h(t-s)|ds = \int_{-\infty}^{t} g(s)|h(t-s)|ds$$

holds. Applying the Cauchy-Schwartz inequality, we obtain $||Fh||^2_{L^2(\mathbb{R})} \leq \int_{\mathbb{R}} \int_{-\infty}^{t} g(s)^2 dsdt\ \ ||h||^2_{L^2(\mathbb{R})}$. But $\int_{\mathbb{R}} \int_{-\infty}^{t} g(s)^2 dsdt \leq \int_{\mathbb{R}} e^{-t}dt = 2$. Now, let $\varepsilon > 0$ be given. Set $\chi_\varepsilon(t) = \begin{cases} 1 & \text{if } t \geq \varepsilon \\ t/\varepsilon & \text{if } 0 \leq t \leq \varepsilon \\ 0 & \text{if } t \leq 0 \end{cases}$, so that χ_ε is continuous and only differs from χ on the interval $[0, \varepsilon]$. Let F_ε denote the operator, where χ is replaced by χ_ϵ in the kernel function.
Furthermore, set $m_\varepsilon(t) = \sup_{0 \leq s \leq \varepsilon} |g(t-s)|$. We infer that

$$\int_{\mathbb{R}} m_\varepsilon(t)^2 dt = \int_{\mathbb{R}} \sup_{0 \leq s \leq \varepsilon} |g(t-s)|^2 ds = \int_{\mathbb{R}} \sup_{0 \leq s \leq \varepsilon} e^{s-t}\chi(t-s)ds \leq e^\varepsilon \int_{\mathbb{R}} e^{-t}dt = e^\varepsilon < \infty.$$

Also, $m_{\varepsilon'}(t) \leq m_\varepsilon(t)$ if $\varepsilon' \leq \varepsilon$ for all $t \in \mathbb{R}$. Now, $F_\varepsilon \to F$ in $\mathcal{B}(L^2(\mathbb{R}))$ as $\varepsilon \to 0$. Indeed, fixing $h \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
|(F_\varepsilon - F)h(t)| &\leq \int_{\mathbb{R}} g(s)|\chi(t-s) - \chi_\varepsilon(t-s)||h(t-s)|ds \\
&= \int_{\mathbb{R}} g(t-s)|\chi(s) - \chi_\varepsilon(s)||h(s)|ds \\
&= \int_0^\varepsilon g(t-s)|\chi(s) - \chi_\varepsilon(s)||h(s)|ds \\
&\leq \int_0^\varepsilon m_\varepsilon(t)|\chi(s) - \chi_\varepsilon(s)||h(s)|ds \\
&\leq \varepsilon \cdot m_\varepsilon(t)^2||h||_{L^2(\mathbb{R})}.
\end{aligned} \tag{74}$$

We can now infer that $||(F_\varepsilon - F)h||^2_{L^2(\mathbb{R})} \leq \varepsilon \int_{\mathbb{R}} m_\varepsilon(t)^2 dt||h||_{L^2(\mathbb{R})}$. But this yields $F_\varepsilon \to F$ in $\mathcal{B}(L^2(\mathbb{R}))$.
Now, let $g_\varepsilon = e^{-r/2}\chi_\varepsilon(r)\chi_\varepsilon(1/\varepsilon - r)$ and let F'_ε be the operator with kernel $g_\varepsilon(s)\chi_\varepsilon(t-s)$, i.e.

$$F'_\varepsilon h(t) = \int_{\mathbb{R}} g_\varepsilon(s)\chi_\varepsilon(t-s)h(t-s)ds.$$

We have

$$\langle (F'_\varepsilon - F_\varepsilon)h, \xi \rangle \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |g(s) - g_\varepsilon(s)|\ \ \chi(t-s)\ \ |h(t-s)|ds|\xi(t)|dt$$

But then by Fubini's Theorem

$$\begin{aligned}
\langle (F'_\varepsilon - F_\varepsilon)h, \xi\rangle &\le \int_{\mathbb{R}}\int_{\mathbb{R}} |g(s)-g_\varepsilon(s)||h(t-s)|ds|\xi(t)|dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |g(s)-g_\varepsilon(s)||h(t-s)||\xi(t)|dtds \\
&\le \int_{\mathbb{R}} |g(s)-g_\varepsilon(s)|ds \;\; ||h||_{L^2(\mathbb{R})}||\xi||_{L^2(\mathbb{R})}.
\end{aligned} \tag{75}$$

Since $\int_{\mathbb{R}} |g(s)-g_\varepsilon(s)|ds \to 0$ as $\varepsilon \to 0$, we infer that ${F'}_\varepsilon \to F$ in $\mathcal{B}(L^2(\mathbb{R}))$.
Now, the kernel of ${F'}_\varepsilon$ lies in $C_c(\mathbb{R}, C)$. We may again form $\psi = M \rtimes V : C \rtimes_\tau \mathbb{R} \longrightarrow \mathcal{B}(L^2(\mathbb{R}))$, where $M : C \longrightarrow \mathcal{B}(L^2(\mathbb{R}))$ and $V : \mathbb{R} \longrightarrow \mathcal{U}(L^2(\mathbb{R}))$ are given by $M(f)h(t) = f(t)h(t)$ and $V_s h(t) = h(t-s)$ for $h \in L^2(\mathbb{R})$ and $t, s \in \mathbb{R}$ as in Example 5.7. This time, however we cannot infer that the image lies in $\mathcal{K}(L^2(\mathbb{R}))$.
Then the function $\tilde{f}_t^{(\varepsilon)}(r) = g_\varepsilon(r)\chi_\varepsilon(t-r)$ lies in $C_c(\mathbb{R}, C)$. The operator $\psi(\tilde{f}^{(\varepsilon)})$ is ${F'}_\varepsilon$. In particular, we have $\psi(\tilde{f}^{(\varepsilon)}) \to F$ in $\mathcal{B}(L^2(\mathbb{R}))$ as $\varepsilon \to 0$.
As functions $f \in C_c(\mathbb{R}, C)$ satisfy $f_t^*(r) = \overline{f_{t-r}(-r)}$, we infer that the adjoint of $\psi(f)$ is given by

$$\psi(f)^* h(t) = \psi(f^*)h(t) = \int_{\mathbb{R}} \overline{f_{t-s}(-s)}h(t-s)dr. \tag{76}$$

For $f, g \in C_c(\mathbb{R}, C)$, we obtain

$$\psi(f)\psi(g)h(t) = \psi(f * g)h(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(t,s)g(t-s, r-s)h(r-s)drds \tag{77}$$

by the convolution formula.
Now, E is given by the kernel

$$\tilde{e}_t(r) = h(t)h(t-r) = e^{-t/2}\chi(t)e^{-(t-r)/2}\chi(t-r) = e^{-t}e^{r/2}\chi(t)\chi(t-r)$$

where h is the first Laguerre function. We will now calculate the adjoint of F by substituting $s \to -s$, applying Fubini's Theorem and then $t \to t-s$.

$$\begin{aligned}
\int_{\mathbb{R}} Fh(t)\cdot\overline{\xi}(t)dt &= \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{f}_t(s)h(t-s)\overline{\xi}(t)dsdt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{f}_t(-s)h(t+s)\overline{\xi}(t)dsdt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{f}_t(-s)h(t+s)\overline{\xi}(t)dtds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{f}_{t-s}(-s)h(t)\overline{\xi}(t-s)dtds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} h(t)\overline{\tilde{f}_{t-s}(-s)\xi(t-s)}dsdt
\end{aligned} \tag{78}$$

Thus, the adjoint of F is given by the operator with kernel $\tilde{f}_{t-r}(-r)$. Furthermore, we have

$$\begin{aligned}
F^*Fh(t) &= \int_{\mathbb{R}} \tilde{f}_{t-s}(-s)Fh(t-s)ds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{f}_{t-s}(-s)\tilde{f}_{t-s}(r)h(t-s-r)drds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{f}_{t-s}(-s)\tilde{f}_{t-s}(r-s)h(t-r)drds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{f}_{t-s}(-s)\tilde{f}_{t-s}(r-s)ds \;\; h(t-r)dr
\end{aligned} \tag{79}$$

and

$$
\begin{aligned}
FF^*h(t)dt &= \int_{\mathbb{R}} \tilde{f}_t(s)F^*h(t-s)ds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{f}_t(s)\tilde{f}_{t-s-r}(-r)h(t-s-r)drds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{f}_t(s)\tilde{f}_{t-r}(s-r)h(t-r)drds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{f}_t(s)\tilde{f}_{t-r}(-(r-s))ds \;\; h(t-r)dr
\end{aligned}
\tag{80}
$$

where we applied the substitution $r \to r-s$ in both cases. We infer that the kernels of FF^* and F^*F are given by the convolution formulas $\int \tilde{f}_{t-s}(-s)\tilde{f}_{t-s}(r-s)ds$ and $\int \tilde{f}_t(s)\tilde{f}_{t-r}(-(r-s))ds$, respectively. We will now show that $F+F^*-F^*F=0$ and $F^*+F-FF^*=E$. Indeed, we calculate

$$
\begin{aligned}
\int_{\mathbb{R}} \tilde{f}_{t-s}(-s)\tilde{f}_{t-s}(r-s)ds &= \int_{\mathbb{R}} e^{s/2}\chi(-s)\chi(t) \;\; e^{(s-r)/2}\chi(r-s)\chi(t-r)ds \\
&= e^{-r/2}\chi(t)\chi(t-r)\int_{\mathbb{R}} e^{s}\chi(-s)\chi(r-s)ds \\
&= e^{-r/2}\chi(t)\chi(t-r)\begin{cases} 1 & \text{if } r \geq 0 \\ e^r & \text{if } r \leq 0 \end{cases}
\end{aligned}
\tag{81}
$$

and

$$\tilde{f}_t(r) + \tilde{f}_{t-r}(-r) = e^{-r/2}\chi(r)\chi(t-r) + e^{r/2}\chi(-r)\chi(t)$$

If $r \geq 0$, we have $\chi(-r)=0$ and $\chi(t)\chi(t-r)=\chi(r)\chi(t-r)$. That is true, because $\chi(t)\chi(t-r)=\chi(t-r)$ and $\chi(r)=1$ for $r\geq 0$. If $r \leq 0$, we infer $\chi(r)=0$ and $\chi(t)\chi(t-r)=\chi(-r)\chi(t)$, since $t \geq 0$ implies $t \geq r$. We have proved $F^*F+F+F^*=0$.
The second assertion follows from the following calculations:

$$
\begin{aligned}
\int_{\mathbb{R}} \tilde{f}_t(s)\tilde{f}_{t-r}(s-r)ds &= \int_{\mathbb{R}} e^{-s/2}\chi(s)\chi(t-s)\chi(s-r)\chi(t-s) \;\; e^{(r-s)/2}ds \\
&= e^{r/2}\int_{\mathbb{R}} e^{-s}\chi(s)\chi(t-s)\chi(s-r)ds \\
&= e^{r/2}\begin{cases} e^{-r}-e^{-t} & \text{if } t\geq 0 \text{ and } 0\leq r\leq t \\ 1-e^{-t} & \text{if } t \geq 0 \text{ and } r<0 \\ 0 & \text{if } t<0 \end{cases}
\end{aligned}
\tag{82}
$$

$$f_t(r) + f_{t-r}(-r) = \begin{cases} e^{-r/2} & \text{if } t\geq 0 \text{ and } 0\leq r\leq t \\ e^{r/2} & \text{if } t \geq 0 \text{ and } r<0 \\ 0 & \text{if } t<0 \end{cases}$$

$$\tilde{e}_t(r) = \begin{cases} e^{r/2}e^{-t} & \text{if } t\geq 0 \text{ and } 0\leq r\leq t \\ e^{r/2}e^{-t} & \text{if } t \geq 0 \text{ and } r<0 \\ 0 & \text{if } t<0 \end{cases}$$

Thus, $FF^*+F+F^*=E$ is also true.
The following Lemma is being used in [1, Page 150]. We give a proof for the sake of completeness. The proof is motivated by [5, Proposition 7.9.7], [11, the discussion above Corollary 3.20] and by the use of Takai duality in Lemma 5.9.

Lemma 5.27. *The *-homomorphism $\psi = M \rtimes V : C \rtimes_\tau \mathbb{R} \longrightarrow \mathcal{B}(L^2(\mathbb{R}))$ is injective.*

Proof. We will first show that $\ker(M \rtimes V)$ is $\hat{\tau}$-invariant. Indeed, fix $y \in \mathbb{R}$ and $a \in C \rtimes_\tau \mathbb{R}$ with $\psi(a)=0$. There are functions $f^{(n)} \in C_c(\mathbb{R}, C)$ with $f^{(n)} \to a$ as $n \to \infty$ in the universal norm. We have $||\hat{\tau}_y(f^{(n)}) - \hat{\tau}_y(a)||_{\mathrm{u}} = ||f^{(n)}-a||_{\mathrm{u}}$.
Fix $\varepsilon > 0$. For $n \in \mathbb{N}$ big enough, $||f^{(n)}-a||_{\mathrm{u}} \leq \varepsilon$. We infer

$$||\psi(f^{(n)})h||_{L^2(\mathbb{R})} \leq \varepsilon \cdot ||h||_{L^2(\mathbb{R})}$$

for any $h \in L^2(\mathbb{R})$. Set $h_y(t) = e^{-iyt}h(t)$, so that $||h_y||_{L^2(\mathbb{R})} = ||h||_{L^2(\mathbb{R})}$. Furthermore,

$$\begin{aligned} \psi(\hat{\tau}_y(f^{(n)}))h(t) &= \int_{\mathbb{R}} e^{iys} f_t^{(n)}(s)h(t-s)ds \\ &= e^{iyt} \int_{\mathbb{R}} e^{-iy(t-s)} f_t^{(n)}(s)h(t-s)ds \\ &= e^{iyt} \int_{\mathbb{R}} f_t^{(n)}(s)h_y(t-s)ds \\ &= (\psi(f^{(n)})h_y)_{-y}(t). \end{aligned} \tag{83}$$

Taking the square and integrating both sides, we have

$$\begin{aligned} ||\psi(\hat{\tau}_y(f^{(n)}))h||_{L^2(\mathbb{R})} &= ||(\psi(f^{(n)})h_y)_{-y}||_{L^2(\mathbb{R})} \\ &= ||(\psi(f^{(n)})h_y)||_{L^2(\mathbb{R})} \\ &\leq \varepsilon \cdot ||h_y||_{L^2(\mathbb{R})} \\ &= \varepsilon \cdot ||h||_{L^2(\mathbb{R})}. \end{aligned} \tag{84}$$

But then $||\psi(\hat{\tau}_y(a))h||_{L^2(\mathbb{R})} \leq \varepsilon \cdot ||h||_{L^2(\mathbb{R})}$ and thus $\psi(\hat{\tau}_y(a)) = 0$.
We may now apply Proposition 4.29. There is a τ-invariant ideal I of C such that $I \rtimes_\tau \mathbb{R} = \ker(\psi)$. Now if $I \neq 0$, then there is $f \in I$ with $f \neq 0$. As I is τ-invariant, $g = \tau_1(f) \in I$ with $g_t = f_{t-1}$. The limit of f_t for $t \to \infty$ exists by definition, so we have $\lim_{t\to\infty}(f_t - g_t) = 0$ which implies $f - g \in S$. But $f - g \neq 0$, as for $m = \sup_{t\in\mathbb{R}} |f_t|$, the point $t = \inf\{t \in \mathbb{R} : |f_t| \geq m/2\}$, we have $|f_t| = m/2$ and $|g_t| < m/2$.
We infer $I \cap S \neq 0$, so that $I \rtimes_\tau \mathbb{R} \cap S \rtimes_\tau \mathbb{R} \neq 0$, taking for Example the function $(f_t - g_t)\phi(r)$ for $0 \neq \phi \in C_c(\mathbb{R})$ which lies in the intersection. However ψ is injective on $S \rtimes_\tau \mathbb{R} \subseteq C \rtimes_\tau \mathbb{R}$ by Lemma 5.9 which is a contradiction. □

Proof of Lemma 5.26. We have seen that $\psi : C \rtimes_\tau \mathbb{R} \longrightarrow \mathcal{B}(L^2(\mathbb{R}))$ is injective, so ψ is a *-isomorphism onto it's image by Theorem 2.4. Furthermore, $F \in \psi(C \rtimes_\tau \mathbb{R})$ by the above calculations and $E \in \psi(S \rtimes_\tau \mathbb{R})$ by Lemma 5.8. Setting $f = \psi^{-1}(F)$, the formulas $F + F^* - F^*F = 0$ and $F + F^* - FF^* = E$ imply the formulas

$$f + f^* + f^* * f = 0$$

and

$$f + f^* + f * f^* = e.$$

□

Now, let $1 = 1_{\widetilde{CB \rtimes_{\tau\otimes\alpha} \mathbb{R}}}$ denote the unit adjoint to $CB \rtimes_{\tau\otimes\alpha} \mathbb{R}$.
Our next Lemma is based on [1, Page 150].

Lemma 5.28. *If $f \in C \rtimes_\tau \mathbb{R}$ is given by Lemma 5.26, then $f_p, e_p \in CB \rtimes_{\tau\otimes\alpha} \mathbb{R}$ satisfy the following:*

1. *We have $(1 - f_p)^* * (1 - f_p) = 1$ and $(1 - f_p) * (1 - f_p)^* = 1 - e_p$.*
2. *The element $\tilde{\hat{\pi}}(1 - f_p) \in \widetilde{B \rtimes_\alpha \mathbb{R}}$ is a unitary.*
3. *Both, e_p and $1 - e_p$ are projections in $B \tilde{\rtimes}_\alpha \mathbb{R}$.*

Proof. The map $C \rtimes_\tau \mathbb{R} \ni g \mapsto g_p \in CB \rtimes_{\tau\otimes\alpha} \mathbb{R}$ is a *-homomorphism by Lemma 5.25. The identities in Lemma 5.26 now imply that

$$(1 - f_p)^* * (1 - f_p) = 1 - f_p - f_p^* + f_p^* f_p = 1 - (f + f^* - f^* f)_p = 1$$

and

$$(1 - f_p) * (1 - f_p)^* = 1 - f_p - f_p^* + f_p * f_p^* = 1 - (f + f^* - f * f^*)_p = 1 - e_p.$$

The exactness of the Wiener-Hopf extension at $CB \rtimes_{\tau\otimes\alpha} \mathbb{R}$ is equivalent to $SB \rtimes_{\tau\otimes\alpha} \mathbb{R} = \ker(\hat{\pi})$ As $e \in S \rtimes_\tau \mathbb{R}$, $e_p \in SB \rtimes_{\tau\otimes\alpha} \mathbb{R}$ by Lemma 5.25. Thus, $\tilde{\hat{\pi}}(e_p) = 0$ and $\tilde{\hat{\pi}}(1 - f_p) \in \widetilde{B \rtimes_\alpha \mathbb{R}}$ is a unitary. As

$e \in S \rtimes_\alpha \mathbb{R}$ is a projection, we infer $e_p * e_p = (e * e)_p = e_p$ and $e_p^* = (e^*)_p = e_p$. So e_p is a projection, but then the same is true for $1 - e_p$. $\square$

The next Lemma will be used to conclude that δ_1 is surjective, see [1, Pages 147 and 150].

Lemma 5.29. *Set $V = (1 - f_p) \in CB \rtimes_{\tau\otimes\alpha} \mathbb{R}$ and $U = \tilde{\tilde{\pi}}(1 - V) \in B \rtimes_\alpha \mathbb{R}$. We have*

$$\delta_1(U) = -[e_p]_0.$$

Proof. By Lemma 5.28, we may apply Proposition 3.33. The element $\delta_1(U)$ is given by

$$\delta_1(U) = [1 - V^* * V]_0 - [1 - V * V^*]_0 = [1 - 1]_0 - [1 - (1 - e_p)]_0 = [0]_0 - [e_p]_0.$$

$\square$

We want to show that δ_1 is surjective. If the $\phi^{(n)} \in C_c(\mathbb{R})$ converge to the first Laguerre function h in $L^2(\mathbb{R})$, then $e^{(n)}$ given by $e_t^{(n)}(r) = \phi_t^{(n)} \phi_{t-r}^{(n)}$ converges to e in the universal norm of $S \rtimes_\tau \mathbb{R}$ by Lemma 5.6 and $T_{e^{(n)}} \to E$ in $\mathcal{K}(L^2(\mathbb{R}))$. We infer that

$$\omega_E(p) = \gamma(\varphi(p)) = \gamma(p \otimes E) = \lim_{n\to\infty} \gamma(p \otimes T_{e^{(n)}})$$

under the *-homomorphism ω_E from Lemma 5.12. But $\gamma(p \otimes T_{e^{(n)}})_t(r) = \alpha_t(p) e_t^{(n)}(r)$ by Lemma 5.10. We set

$$\gamma(p \otimes T_{e^{(n)}}) = \overline{e}^{(n)}.$$

Since $K_0(\omega_E)$ is an isomorphism and elements of the form $[q]_0$ for projection $q \in M_\infty(A)$ generate $K_0(A)$ by Proposition 3.13, it is enough to show that $\omega_E(p) \sim_h e_p$, because $[e_p]_0$ lies in the image of δ_1.
Now, $e_p = \lim_{n\to\infty} e_p^{(n)}$, but $e_{p,t}^{(n)}(r) = p u_r e_t^{(n)}(r)$.
The following Lemma is contained in [1, Page 150].

Lemma 5.30. *Let $\hat{\varphi} : SA \rtimes_{\tau\otimes\alpha} \mathbb{R} \longrightarrow SA \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R}$ denote the *-isomorphism given by*

$$\hat{\varphi}(g)_t(r) = \alpha_{-t}(f_t(r))$$

from Lemma 5.5. Then $\hat{\varphi}({e_p}^{(n)})_t(r) = u_{-t}^* \hat{\varphi}(\overline{e}^{(n)})_t(r) u_{r-t}$.

Proof. We calculate

$$\hat{\varphi}({e_p}^{(n)})_t(r) = \alpha_{-t}(e_{p,t}^{(n)}(r)) = \alpha_{-t}(p u_r) e_t^{(n)}(r) = u_{-t}^* p u_{-t} \alpha_{-t}(u_r) e_t^{(n)}(r)$$

by formula (70). The cocycle identity (69) implies that $u_{r-t} = u_{-t}\alpha_{-t}(u_r)$, so that

$$\alpha_{-t}(u_r) = u_{-t}^* u_{r-t}.$$

We infer

$$\hat{\varphi}({e_p}^{(n)})_t(r) = u_{-t}^* p u_{r-t} e_t^{(n)}(r) = u_{-t}^* \alpha_{-t}(\alpha_t(p) e_t^{(n)}(r)) u_{r-t} = u_{-t}^* \hat{\varphi}(\overline{e}^{(n)})_t(r) u_{r-t}.$$

$\square$

We now follow [1, Pages 150 and 151], also see [4, Pages 38 and 39].
Given $\lambda \in [0, 1]$, we define

$$L_\lambda(f)_t(r) = u_{-\lambda t}^* f_t(r)$$

and

$$R_\lambda(f)_t(r) = f_t(r) u_{\lambda(r-t)}$$

for $f, g \in C_c(\mathbb{R}, SB) \subseteq SB \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R}$. The idea is that the pair (L, R) is a unitary in a bigger C^*-algebra than $SA \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R}$. It is formally given by $(r, t) \mapsto u_{-\lambda t}\delta_0(t)$ where δ_0 is the Dirac delta function on the point mass 0. The map L is then given by left multiplication and R by right multiplication. We have $L_\lambda(R_\mu(f)) = R_\mu(L_\lambda(f))$ for any $\lambda, \mu \in [0,1]$. Furthermore,

$$(R_\lambda(f)^*)_t(r) = (R_\lambda(f)_{t-r}(-r))^* = (f_{t-r}(-r)u_{\lambda(r-(t-r))})^* = u^*_{-\lambda t}\overline{f_{t-r}(-r)} = L_\lambda(f^*)_t(r) \tag{85}$$

But then $R_\lambda(f)^* = L_\lambda(f^*)$, so we get $L_\lambda(f)^* = R_\lambda(f^*)$ by substituting f^* for f. That implies

$$L_\lambda(R_\lambda(f^*)) = L_\lambda(R_\lambda(f))^*.$$

Also,

$$\begin{aligned}
L_\lambda(R_\lambda(f*g))_t(r) &= u^*_{-\lambda t}(f*g)_t(r)u_{\lambda(r-t)} \\
&= \int_{\mathbb{R}} u^*_{-\lambda t} f_t(s) g_{t-s}(r-s) u_{\lambda(r-t)} ds \\
&= \int_{\mathbb{R}} u^*_{-\lambda t} f_t(s) u_{\lambda(s-t)} u^*_{\lambda(-(t-s))} g_{t-s}(r-s) u_{\lambda(r-s-(t-s))} ds \\
&= \int_{\mathbb{R}} L_\lambda(R_\lambda(f))_t(s) L_\lambda(R\lambda(g))_{t-s}(r-s) ds \\
&= (L_\lambda(R_\lambda(f)) * L_\lambda(R_\lambda(g)))_t(r),
\end{aligned} \tag{86}$$

so $L_\lambda \circ R_\lambda = R_\lambda \circ L_\lambda$ is a *-homomorphism.
Furthmore, L_λ and R_λ are both bounded in $||\cdot||_{L^1}$. Indeed,

$$\begin{aligned}
||L_\lambda(f)||_{L^1} &= \int_{\mathbb{R}} \sup_{t\in\mathbb{R}} ||u^*_{-\lambda t} f_t(s)|| ds \\
&\leq \int_{\mathbb{R}} \sup_{t\in\mathbb{R}} ||u^*_{-\lambda t}|| \cdot ||f_t(s)|| ds \\
&= \int_{\mathbb{R}} \sup_{t\in\mathbb{R}} \cdot ||f_t(s)|| ds \\
&= ||f||_{L^1}
\end{aligned} \tag{87}$$

and

$$||R_\lambda(f)||_{L^1} \leq \int_{\mathbb{R}} \sup_{t\in\mathbb{R}} ||f_t(s)|| \cdot ||u_{\lambda(s-t)}|| ds = ||f||_{L^1},$$

because unitaries have norm 1. We will now also calculate that L_λ and R_λ are bounded in the universal norm. Indeed,

$$(R_\lambda(f) * L_\lambda(g))_t(r) = \int_{\mathbb{R}} f_t(s) u_{\lambda(s-t)} u^*_{-\lambda(t-s)} g_{t-s}(r-s) ds = (f*g)_t(r)$$

and

$$\begin{aligned}
\sup_{||f||_{\mathrm{u}}\leq 1} ||R_\lambda(f)||^2_{\mathrm{u}} &= \sup_{||f||_{\mathrm{u}}\leq 1} ||R_\lambda(f) * R_\lambda(f)^*||_{\mathrm{u}} \\
&= \sup_{||f||_{\mathrm{u}}\leq 1} ||R_\lambda(f) * L_\lambda(f^*)||_{\mathrm{u}} \\
&= \sup_{||f||_{\mathrm{u}}\leq 1} ||f * f^*|| \\
&= 1
\end{aligned} \tag{88}$$

by the C^*-identity. We obtain $||R_\lambda|| \leq 1$ and from $L_\lambda(f)^* = R_\lambda(f^*)$ also $||L_\lambda|| \leq 1$.
Thus, $U_\lambda = L_\lambda \circ R_\lambda : C_c(\mathbb{R}, SB) \longrightarrow C_c(\mathbb{R}, SB)$ is a *-homomorophism that is bounded in $||\cdot||_{\mathrm{u}}$ and extends to a *-homomorphism $U_\lambda : SB \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R} \longrightarrow SB \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R}$ and we may also extend both $L_\lambda, R_\lambda : SB \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R} \longrightarrow SB \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R}$.
The next Lemma is contained in [1, Lemma 3].

Lemma 5.31. *For $a \in SB \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R}$, we have $||U_\lambda(a) - U_\mu(a)||_{\mathrm{u}} \to 0$ as $\lambda \to \mu$. That is, $\lambda \mapsto U_\lambda$ is continuous in the topology of strong convergence.*

Proof. We will in fact show that both, L_λ and R_λ are continuous in the topology of strong convergence starting with L_λ. As $||L_\lambda(a)||_\mathrm{u} \le ||a||_\mathrm{u}$ for any $\lambda \in \mathbb{R}$, the estimate

$$||L_\lambda(a) - L_\mu(a)||_\mathrm{u} \le ||L_\lambda(g) - L_\mu(g)||_\mathrm{u} + 2 \cdot ||a - g||_\mathrm{u}$$

shows that it is enough to consider operators g from a dense subspace of $SB \rtimes_{\tau\otimes\mathrm{Id}} \mathbb{R}$.
We apply Lemma 4.17, to find a function $g \in C_c(\mathbb{R}, SB)$ of the form $g_t(r) = \sum_{i=1}^{n} \phi^{(i)}(r) f_t^{(i)}$ with $f^{(i)} \in SB$ and $\phi^{(i)} \in C_c(\mathbb{R})$ and $||g - a||_\mathrm{u} \le \varepsilon$ for any $\varepsilon > 0$.
Looking at the form of the above g, it is enough to show that we have $||L_\lambda(g) - L_\mu(g)||_\mathrm{u} \to 0$ for any $g \in C_c(\mathbb{R}, SB)$ of the form $g_t(r) = \phi(r) f_t$ for $f \in SB$ and $\phi \in C_c(\mathbb{R})$. We will even proof that $||L_\lambda(g) - L_\mu(g)||_{L^1} \to 0$ which is enough by Proposition 4.14. Now, setting $\tilde{g}_t(r) = \phi(r)\tilde{f}_t$, the equation

$$||g - \tilde{g}||_{L^1} = \int \sup_{t\in\mathbb{R}} ||f_t - \tilde{f}_t||_B \cdot |\phi(r)| dr$$

tells us that we may even suppose $\mathrm{supp}(f)$ to be compact, because $|f_t| \to 0$ as $t \to \infty, -\infty$ by formula (87). Now, our cocycle $t \mapsto u_t$ is uniformly continuous on the compact set $[-1, 0] \cdot \mathrm{supp}(f)$. Thus, we have $\sup_{t\in\mathrm{supp}(f)} ||u_{-\lambda t} - u_{-\mu t}||_B = \sup_{t\in\mathrm{supp}(f)} ||u^*_{-\lambda t} - u^*_{-\mu t}||_B \to 0$ as $\lambda \to \mu$. We infer

$$\begin{aligned} ||L_\lambda(g) - L_\mu(g)||_{L^1} &= \int \sup_{t\in\mathrm{supp}(f)} ||(u^*_{-\lambda t} - u^*_{-\mu t}) f_t||_B \cdot |g(s)| ds \\ &\le \int \sup_{t\in\mathrm{supp}(f)} ||(u^*_{-\lambda t} - u^*_{-\mu t})||_B \cdot ||f_t||_B \cdot |g(s)| ds \\ &\le \sup_{t\in\mathrm{supp}(f)} ||(u^*_{-\lambda t} - u^*_{-\mu t})||_B \cdot \sup_{t\in\mathrm{supp}(f)} ||f_t||_B \int |g(s)| ds. \end{aligned} \tag{89}$$

As the constant $\sup_{t\in\mathrm{supp}(f)} ||f_t||_B \int |g(s)| ds < \infty$, we infer $||L_\lambda(g) - L_\mu(g)||_{L^1} \to 0$ as $\lambda \to \mu$. We then have $||L_\lambda(a) - L_\mu(a)||_\mathrm{u} \to 0$ and by taking adjoints, we infer

$$||R_\lambda(a) - R_\mu(a)||_\mathrm{u} = ||L_\lambda(a^*) - L_\mu(a^*)||_\mathrm{u} \to 0.$$

Our claim now follows from the following estimate:

$$\begin{aligned} ||U_\lambda(a) - U_\mu(a)||_\mathrm{u} &= ||L_\lambda(R_\lambda(a)) - L_\mu(R_\mu(a))||_\mathrm{u} \\ &\le ||L_\lambda(R_\lambda(a)) - L_\mu(R_\lambda(a))||_\mathrm{u} + ||L_\mu(R_\lambda(a)) - L_\mu(R_\mu(a))||_\mathrm{u} \\ &= ||R_\lambda(L_\lambda(a)) - R_\lambda(L_\mu(a))||_\mathrm{u} + ||L_\mu(R_\lambda(a)) - L_\mu(R_\mu(a))||_\mathrm{u} \\ &\le ||L_\lambda(a) - L_\mu(a)||_\mathrm{u} + ||R_\lambda(a) - R_\mu(a)||_\mathrm{u} \end{aligned} \tag{90}$$

□

We can now infer that δ_1 is surjective in the unital case.

Lemma 5.32. *If A is unital, then the index map of the Wiener-Hopf extension*

$$\delta_1 : K_1(A \rtimes_\alpha \mathbb{R}) \longrightarrow K_0(SA \rtimes_{\tau\otimes\alpha} \mathbb{R})$$

is surjective.

Proof. The formula $\hat{\varphi}(e_p{}^{(n)})_t(r) = u^*_{-t}\hat{\varphi}(\bar{e}^{(n)})_t(r) u_{r-t}$ from Lemma 5.30 now becomes

$$\hat{\varphi}(e_p{}^{(n)}) = U_1(\hat{\varphi}(\bar{e}^{(n)})). \tag{91}$$

Given $\lambda \in [0, 1]$, we may take the limit $n \to \infty$ of $U_\lambda(\hat{\varphi}(\bar{e}^{(n)}))$ which evaluates to $U_\lambda(\hat{\varphi}(\omega_E(p)))$, because U_λ is bounded in the universal norm.

Lemma 5.31 now guaranties that $\lambda \mapsto U_\lambda(\hat{\varphi}(\omega_E(p)))$ is a continuous path of projections, because U_λ is a *-homomorphism and thus maps projections to projections. Taking the limit in formula (91), we infer $\hat{\varphi}(e_p) = U_1(\hat{\varphi}(\omega_E(p)))$ and $U_0(\hat{\varphi}(\omega_E(p))) = \hat{\varphi}(\omega_E(p))$.
So we have

$$\hat{\varphi}(\omega_E(p)) \sim_h \hat{\varphi}(e_p).$$

in $SB \rtimes_{\tau\otimes \mathrm{Id}} \mathbb{R}$. As *-isomorphisms respect homotopies, we have

$$\omega_E(p) \sim_h e_p$$

in $SB \rtimes_{\tau\otimes\alpha} \mathbb{R}$. Since the map $[p]_0 - [q]_0 \mapsto [\omega_E(p)]_0 - [\omega_E(q)]_0$ for $p, q \in \mathcal{P}_\infty(A)$ is an isomorphism by Lemma 5.12 and Proposition 3.13, we infer that δ_1 is surjective, because p was chosen to be homotopic to an arbitrary projection in $M_\infty(A)$. ☐

5.6 The non-unital Case

We are left with proving the non-unital case, so we now suppose that A has no unit.
Our argument is based on [1, Pages 152 and 153].

Lemma 5.33. *The index map $\delta_1 : K_1(A \rtimes_\alpha \mathbb{R}) \longrightarrow K_0(SA \rtimes_{\tau\otimes\alpha} \mathbb{R})$ is surjective.*

Proof. We know that

$$0 \longrightarrow SB \xrightarrow{\iota} CB \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

is a short exact sequence of C^*-algebras for any C^*-algebra B. By Proposition 3.26, the sequences

$$0 \longrightarrow SA \xrightarrow{S\iota} S\tilde{B} \xrightarrow{S\pi} S\mathbb{C} \longrightarrow 0$$

and

$$0 \longrightarrow CA \xrightarrow{C\iota} C\tilde{B} \xrightarrow{C\pi} C\mathbb{C} \longrightarrow 0$$

are also short exact sequences. So we obtain the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & SA \rtimes_{\tau\otimes\alpha} \mathbb{R} \arrow[r] \arrow[d] & CA \rtimes_{\tau\otimes\alpha} \mathbb{R} \arrow[r] \arrow[d] & A \rtimes_\alpha \mathbb{R} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S\tilde{A} \rtimes_{\tau\otimes\alpha} \mathbb{R} \arrow[r] \arrow[d] & C\tilde{A} \rtimes_{\tau\otimes\alpha} \mathbb{R} \arrow[r] \arrow[d] & \tilde{A} \rtimes_\alpha \mathbb{R} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S \rtimes_\tau \mathbb{R} \arrow[r] \arrow[d] & C \rtimes_\tau \mathbb{R} \arrow[r] \arrow[d] & C^*(\mathbb{R}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{92}$$

where we applied Proposition 4.22 6 times. The action α on $\tilde{A}$ is given by $\alpha(a + 1_{\tilde{A}}) = \alpha(a) + 1_{\tilde{A}}$. The rows are Wiener-Hopf extensions and the columns come from equivariant short exact sequences. For example, the third column comes from the sequence

$$0 \longrightarrow A \xrightarrow{\iota} \tilde{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

which is equivariant, because

$$\alpha_s(\iota(a)) = \alpha_s(a + 0 \cdot 1_{\tilde{A}}) = \alpha_s(a) + 0 \cdot 1_{\tilde{A}} = \alpha_s(\iota(a))$$

and

$$\mathrm{Id}_s(\pi(a + \gamma \cdot 1_{\tilde{A}})) = \mathrm{Id}_s(\gamma) = \gamma = \pi(\alpha_s(a) + \gamma \cdot 1_{\tilde{A}}) = \pi(\alpha_s(a + \gamma \cdot 1_{\tilde{A}}))$$

for any $a \in A$, $\gamma \in \mathbb{C}$. We now apply the six-term exact sequence 2 times to obtain the following commutative diagram with exact columns:

$$\begin{tikzcd}
 & K_1(S \rtimes_\tau \mathbb{R}) \arrow[d, "\delta_1"] \\
K_1(A \rtimes_\alpha \mathbb{R}) \arrow[r, "\delta_1"] \arrow[d] & K_0(SA \rtimes_{\tau\otimes\alpha} \mathbb{R}) \arrow[d] \\
K_1(\tilde{A} \rtimes_\alpha \mathbb{R}) \arrow[r, "\delta_1"] \arrow[d] & K_0(S\tilde{A} \rtimes_{\tau\otimes\alpha} \mathbb{R}) \arrow[d] \\
K_1(C^*(\mathbb{R})) \arrow[r, "\delta_1"] & K_0(S \rtimes_\tau \mathbb{R})
\end{tikzcd} \tag{93}$$

The δ_1-rows correspond to the index maps of the rows from (92). The columns correspond to the six-term exact sequences of the last and first column from (92). Furthermore, the diagram commutes by Proposition 3.32.

We know that the two bottom δ_1-rows are surjective and we must show that the top δ_1-row is surjective. However, as $S \rtimes_\tau \mathbb{R} \cong \mathcal{K}(L^2(\mathbb{R}))$, we infer from Proposition 3.22 that

$$K_0(S \rtimes_\tau \mathbb{R}) \cong \mathbb{Z}$$

and

$$K_1(S \rtimes_\tau \mathbb{R}) \cong 0.$$

Proposition 4.32 now yields $C^*(\mathbb{R}) \cong C_0(\mathbb{R}) = S\mathbb{C}$ and thus

$$K_1(C^*(\mathbb{R})) \cong K_1(S\mathbb{C}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$$

by Bott periodicity, Theorem 3.29. As a surjective homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}$ needs to map 1 to 1, we infer that the bottom index-map is an isomorphism. Also, $K_1(S \rtimes_\tau \mathbb{R}) \cong 0$ yields that the middle map in the right column is injective. The proof that the top δ_1-row is surjective is thus completed by the next Lemma. □

Lemma 5.34. *Suppose that $G_1, G_2, H_1, H_2, K_1, K_2$ are groups and that we have a commutative diagram of group homomorphisms*

$$\begin{tikzcd}
G_1 \arrow[r, "\delta_1"] \arrow[d, "\varphi_1"] & G_2 \arrow[d, "\varphi_2"] \\
H_1 \arrow[r, "\delta_2"] \arrow[d, "\psi_1"] & H_2 \arrow[d, "\psi_2"] \\
K_1 \arrow[r, "\delta_3"] & K_2
\end{tikzcd} \tag{94}$$

such that $\varphi_i(G_i) = \ker(\psi_i)$ for $i = 1, 2$. Furthermore, assume that δ_2 is surjective, δ_3 an isomorphism and φ_2 injective. Then δ_1 is surjective.

Proof. Let $g_2 \in G_2$. We must find $g_1 \in G_1$ with $\delta_1(g_1) = g_2$. Set $h_2 = \varphi_2(g_2)$. As δ_2 is surjective, there is $h_1 \in H_1$ with

$$\delta_2(h_1) = h_2.$$

Now, h_2 lies in the image of φ_2, so $\psi_2(\delta_2(h_1)) = 0$. The bottom square commutes, so we obtain $\delta_3(\psi_1(h_1)) = 0$. But δ_3 is an isomorphism, so we must necessarily have $\psi_1(h_1) = 0$. Now, because $\varphi_1(G_1) = \ker(\psi_1)$, there is $g_1 \in G_1$ with

$$\varphi_1(g_1) = h_1.$$

We have $\delta_2(\varphi_1(g_1)) = \delta_2(h_1) = h_2$. But the top square commutes, so we have $\varphi_2(\delta_1(g_1)) = h_2$. Now, $\varphi_2(\delta_1(g_1)) = h_2 = \varphi_2(g_2)$, but φ_2 is injective, so the only possibility left is $\delta_1(g_1) = g_2$. □

6 Applications and Further Topics

6.1 The Pimsner-Voiculescu Sequence

We are now concerned with crossed Products $A \rtimes_\alpha \mathbb{Z}$ for a C^*-dynamical system $(A, \mathbb{Z}, \alpha)$. In this situation, $\mathbb{Z}$ carries the discrete topology, so that α is automatically continuous in the strong topology. Furthermore, α is determined by it's value α_1 as $\alpha_n = \alpha_1^n$. We will identify the action α with the single automorphism $\alpha = \alpha_1$.

Theorem 6.1 (Pimsner-Voiculescu Sequence)**.** *Suppose $(A, \mathbb{Z}, \alpha)$ is a C^*-dynamical system. There is a cyclic six-term exact sequence.*

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-\alpha_*} & K_0(A) & \xrightarrow{\iota_*} & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \xleftarrow[\iota_*]{} & K_1(A) & \xleftarrow[1-\alpha_*]{} & K_1(A) \end{array} \tag{95}$$

Theorem 6.1 thirst appeared in [3, Theorem 2.4]. The term 1 refers to $\mathrm{Id}_{K_j(A)} : K_j(A) \longrightarrow K_j(A)$. We have not defined a *-homomorphism $\iota : A \longrightarrow A \rtimes_\alpha \mathbb{Z}$. The term ι_* comes from a short exact sequence

$$0 \longrightarrow I \xrightarrow{\iota} C \xrightarrow{\pi} B \longrightarrow 0$$

with $K_j(I) \cong K_j(A)$ and $K_j(C) \cong K_j(A \rtimes_\alpha \mathbb{Z})$. (In the literature, φ_* is often written instead of both $K_0(\varphi)$ and $K_1(\varphi)$.)

6.2 Sketch of the Proof

The following proof is taken from [8, Sections 10.3 and 10.4]. One first needs to relate crossed products by $\mathbb{R}$ with ones by $\mathbb{Z}$. This is achieved by the following Proposition. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote the circle.

Proposition 6.2. *There is an isomorphism $\hat{\mathbb{Z}} \cong \mathbb{T}$ that is also a homeomorphism.*

Proof. See [10, Examples 3.1.3]. □

The above Proposition allows us to relate the discrete actions by $\mathbb{Z}$ to continuous ones using Takai duality. We can identify $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$. This isomorphism comes from $\mathbb{R} \longrightarrow \mathbb{T}$, $s \mapsto e^{i2\pi s}$ and is a homeomorphism whenever we take the quotient topology on $\mathbb{R}/\mathbb{Z}$ or give it the topology from $\mathbb{T} \subseteq \mathbb{C}$.
The following observation allows to reduce crossed products by the circle to ones by the real line.

Lemma 6.3. *Suppose that B is a C^*-algebra.*

(1) If $\beta : \mathbb{R} \longrightarrow B$ is an action and $\beta(n) = \mathrm{Id}_{\mathrm{Aut}(B)}$ for $n \in \mathbb{Z}$, then there is an action $\overline{\beta} : \mathbb{T} \longrightarrow \mathrm{Aut}(B)$ with $\overline{\beta}(s + \mathbb{Z}) = \beta(s)$.

(2) Conversely, for any action $\beta : \mathbb{T} \longrightarrow \mathrm{Aut}(B)$ there is an action $\tilde{\beta} : \mathbb{R} \longrightarrow \mathrm{Aut}(B)$ such that $\overline{\tilde{\beta}} = \beta$.

We will not further distinguish between β, $\overline{\beta}$ and $\tilde{\beta}$.

Definition 6.4. *If A is a C^*-algebra and $\alpha \in \mathrm{Aut}(A)$, we define the mapping torus of α in A as*

$$M_\alpha(A) = \{f \in C([0,1], A) : f(1) = \alpha(f(0))\}.$$

if $(B, \beta, \mathbb{T})$ is C^*-dynamical system, we can again identify the dual action $\hat{\beta} : \mathbb{Z} \longrightarrow \mathrm{Aut}(B \rtimes_\beta \mathbb{T})$ with an automorphism of $B \rtimes_\beta \mathbb{T}$.

Lemma 6.5. *Suppose that $(B, \beta, \mathbb{T})$ is a C^*-dynamical system. Then*

$$B \rtimes_\beta \mathbb{R} \cong M_{\hat{\beta}}(B \rtimes_\beta \mathbb{T}).$$

Lemma 6.6. *If $(A, \mathbb{Z}, \alpha)$ is a C^*-dynamical system, there is a short exact sequence*

$$0 \longrightarrow SA \xrightarrow{\iota} M_\alpha(A) \xrightarrow{\pi} A \longrightarrow 0.$$

Proof. There is an isomorphism $SA \cong C_0((0,1), A)$. This isomorphism comes from taking any homeomorphism $\sigma : (0,1) \longrightarrow \mathbb{R}$ and then sending $SA \ni f \mapsto f \circ \sigma$. Then ι is given as the inclusion map $f \mapsto f \in M_\alpha(A)$ which is well-defined because $\alpha(f(0)) = \alpha(0) = 0 = \alpha(f(1))$. Now, π is given by evaluation at 0, i.e. $\pi(f) = f(0)$. One then calculates that the sequence above is exact. □

Suppose now that $(A, \mathbb{Z}, \alpha)$ is a C^*-dynamical system. We set $B = A \rtimes_\alpha \mathbb{Z}$. We get an action $\beta = \hat{\alpha}$ of $\mathbb{T}$ on B and $B \rtimes_\beta \mathbb{T} \cong A \otimes_{\max} \mathcal{K}(L^2(\mathbb{Z}))$. Furthermore, $B \rtimes_\beta \mathbb{R} \cong M_{\hat{\beta}}(B \rtimes \mathbb{T})$. The short exact sequence from Lemma 6.6 is now of the form

$$0 \longrightarrow S(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{Z}))) \longrightarrow B \rtimes_\beta \mathbb{R} \longrightarrow A \otimes_{\max} \mathcal{K}(L^2(\mathbb{Z})) \longrightarrow 0.$$

We may apply the cyclic six-term exact sequence of K-theory to obtain the following exact sequence:

$$\begin{tikzcd} K_0(S(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{Z})))) \arrow[r] & K_0(B \rtimes_\beta \mathbb{R}) \arrow[r] & K_0(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{Z}))) \arrow[d] \\ K_1(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{Z}))) \arrow[u] & K_1(B \rtimes_\beta \mathbb{R}) \arrow[l] & K_1(S(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{Z})))) \arrow[l] \end{tikzcd} \tag{96}$$

The Connes-Thom isomorphism gives us $K_j(B \rtimes_\beta \mathbb{R}) \cong K_{|j-1|}(B)$ for $j = 0, 1$. Proposition 3.22 and Bott periodicity, Theorem 3.29 and Proposition 3.28 give us

$$K_j(S(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{Z})))) \cong K_{|j-1|}(A)$$

and

$$K_j(A \otimes_{\max} \mathcal{K}(L^2(\mathbb{Z}))) \cong K_j(A).$$

The sequence now becomes

$$\begin{tikzcd} K_1(A) \arrow[r, "\iota_*"] & K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[r] & K_0(A) \arrow[d] \\ K_1(A) \arrow[u] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[l] & K_0(A) \arrow[l, "\iota_*"] \end{tikzcd}$$

Further analysis then shows that both, the index and exponential map are of the form $1 - \alpha_*$. Rotating the sequence, we get the Pimsner-Voiculescu sequence from Theorem 6.1.

6.3 Some Computations with the Pimsner-Voiculescu Sequence

In what follows, we conduct some simple applications of Theorem 6.1. These are taken from [8, Section 10.11] and [6, Section 11.3].

Example 6.7. *Let α be an automorphism of $\mathcal{K} = \mathcal{K}(H)$ for some seperable Hilbert space H. Then*

$$K_1(\mathcal{K} \rtimes_\alpha \mathbb{Z}) \cong \mathbb{Z}$$

and

$$K_0(\mathcal{K} \rtimes_\alpha \mathbb{Z}) \cong \mathbb{Z}.$$

Proof. We have $K_1(\mathcal{K}) \cong 0$ and $K_0(\mathcal{K}) \cong \mathbb{Z}$ by Proposition 3.22. The Pimsner-Voiculescu sequence becomes

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "1-\alpha_*"] & \mathbb{Z} \arrow[r, "\iota_*"] & K_0(\mathcal{K} \rtimes_\alpha \mathbb{Z}) \arrow[d, "\psi"] \\ K_1(\mathcal{K} \rtimes_\alpha \mathbb{Z}) \arrow[u, "\varphi"] & 0 \arrow[l, "\iota_*"] & 0 \arrow[l, "1-\alpha_*"] \end{tikzcd}$$

with group homomorphisms φ, ψ and we want to show that φ, ι_* are isomorphisms.
The homomorphism $1 - \alpha_* = 1 - K_0(\alpha) : K_0(\mathcal{K}) \longrightarrow K_0(\mathcal{K})$ needs to be 0. We must show that any rank one projection $E \in \mathcal{K}$ gets mapped to 0 under $1 - K_0(\alpha)$. We will show that $E \sim \alpha(E)$ by showing that $\alpha(E)$ is a rank one projection too. If $\alpha(E)$ is not rank one, then $\alpha(E) = P + Q$ for some non-zero projections $P, Q \in \mathcal{K}$. But then $E = \alpha^{-1}(P) + \alpha^{-1}(Q)$, so one of these terms has to be 0, because E is rank-one which is a contradiction. The equivalence now comes from taking a unitary $U \in \mathcal{B}(\mathcal{H})$ that maps the projecting space of E to the one of $\alpha(E)$. Then $V = UE \in \mathcal{K}$ and $V^*V = E$, $VV^* = \alpha(E)$.
Now, φ is surjective, because $\ker(1 - \alpha_*) = \mathbb{Z}$. It is injective, because $\ker(\varphi) = \iota_*(0) = 0$. As $1 - \alpha_* = 0$, ι_* is injective. But $\iota_*(\mathbb{Z}) = \ker(\psi) = K_0(\mathcal{K} \rtimes_\alpha \mathbb{Z})$. Thus, φ and ι_* are isomorphisms. ☐

We will now use the Fourier transformation to calculate the K-theory of the circle.

Example 6.8. *The K-theory of $C(\mathbb{T})$ is $K_j(C(\mathbb{T})) \cong \mathbb{Z}$ for $j = 0, 1$.*

Proof. Apply Theorem 4.32, to get $C(S^1) \cong \mathbb{C} \rtimes_{\mathrm{Id}} \mathbb{Z}$. The claim follows from Example 6.7. ☐

We may extend this Example by using the Pimsner-Voiculescu sequence again to calculate the K-theory of the n-torus $\mathbb{T}^n$, see [15, Example 8.5.2].

Example 6.9. *The K-theory of $C(\mathbb{T}^n)$ is $K_j(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}$.*

Proof. We can write $C(\mathbb{T}^n) \cong C(\mathbb{T}^{n-1}) \rtimes_{\mathrm{Id}} \mathbb{Z}$. indeed,

$$C(\mathbb{T}^{n-1}) \rtimes_{\mathrm{Id}} \mathbb{Z} \cong C(\mathbb{T}^{n-1}) \otimes_{\max} (\mathbb{C} \rtimes_{\mathrm{Id}} \mathbb{Z}) \cong C(\mathbb{T}^{n-1}) \otimes_{\max} C(\mathbb{T})$$

by Proposition 4.23. Furthermore, $C(\mathbb{T}^{n-1}) \otimes_{\max} C(\mathbb{T}) \cong C(\mathbb{T}^n)$ by Proposition 2.20. Now, assume that the statement is true for n. The Pimsner-Voiculescu sequence takes the form

$$\begin{tikzcd}
\mathbb{Z}^{2^{n-1}} \arrow[r, "1-\mathrm{Id}_*"] & \mathbb{Z}^{2^{n-1}} \arrow[r, "\iota_*"] & K_0(C(\mathbb{T}^{n+1})) \arrow[d, "\psi"] \\
K_1(C(\mathbb{T}^{n+1})) \arrow[u, "\varphi"] & \mathbb{Z}^{2^{n-1}} \arrow[l, "\iota_*"] & \mathbb{Z}^{2^{n-1}} \arrow[l, "1-\mathrm{Id}_*"]
\end{tikzcd}$$

But $1 - \mathrm{Id}_* = 0$, so we may extract short exact sequences

$$0 \longrightarrow \mathbb{Z}^{2^{n-1}} \xrightarrow{\iota_*} K_0(C(\mathbb{T}^{n+1})) \xrightarrow{\psi} \mathbb{Z}^{2^{n-1}} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z}^{2^{n-1}} \xrightarrow{\iota_*} K_1(C(\mathbb{T}^{n+1})) \xrightarrow{\varphi} \mathbb{Z}^{2^{n-1}} \longrightarrow 0 \ .$$

Both sequences are split exact. For example the upper one is, because we can find $\lambda(e_k)$ such that $\psi(\lambda(e_k)) = e_k$ for the canonical unit vectors $e_k \in \mathbb{Z}^{2^{n-1}}$. Then λ extends to a homomorphism with $\psi \circ \lambda = \mathrm{Id}_{\mathbb{Z}^{2^{n-1}}}$. We can infer $K_0(C(\mathbb{T}^{n+1}) \cong \mathbb{Z}^{2^{n-1}} \oplus \mathbb{Z}^{2^{n-1}} \cong \mathbb{Z}^{2^n}$, see [6], Excercise 1.1. Also, $K_1(C(\mathbb{T}^{n+1}) \cong \mathbb{Z}^{2^n}$ follows from the same argument. ☐

Given a C^*-algebra A, one can use that the map $(0,1) \ni s \mapsto e^{i2\pi s} \in \mathbb{T}$ is a homeomorphism onto it's image in order to construct a split exact sequence

$$\begin{tikzcd}
0 \arrow[r] & SA \arrow[r] & C(\mathbb{T}, A) \arrow[r] & A \arrow[r] \arrow[l, bend left] & 0
\end{tikzcd}$$

by using that $\mathbb{R}$ is homeomorphic to $(0,1)$. The split arrow comes from the missing point in the image. Then Bott periodicity, Theorem 3.29 and Proposition 3.28 yield the example above using Lemma 3.11. However, with the Pimsner-Voiculescu sequence we did not need to construct a sequence of C^*-algebras.

Example 6.10. *Let ϑ be an irrational number. We can define an action of $\mathbb{Z}$ on $C(\mathbb{T})$ by rotating around the angle $e^{i2\pi\vartheta}$. That is $\alpha(f)(z) = f(e^{-2\pi i\vartheta}z)$. The crossed product $A_\vartheta = C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ is called the irrational rotation algebra or the noncommutative 2-torus. It's K-theory is*

$$K_j(A_\vartheta) \cong \mathbb{Z}^2$$

for $j = 0, 1$.

Proof. We use the Pimsner-Voiculescu sequence again.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "1-\alpha_*"] & \mathbb{Z} \arrow[r, "\iota_*"] & K_0(A_\theta) \arrow[d] \\
K_1(A_\theta) \arrow[u] & \mathbb{Z} \arrow[l, "\iota_*"] & \mathbb{Z} \arrow[l, "1-\alpha_*"]
\end{tikzcd}$$

The term $1 - \alpha_*$ needs to be 0, as we can define a homotopy of automorphisms $\alpha_t(f)(s) = f(e^{i2\pi\theta(t-1)})$, see Proposition 3.21 or compute that $p \sim_h \alpha(p)$. We can again extract split-exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & K_j(A_\theta) \arrow[r] & \mathbb{Z} \arrow[r] \arrow[l, bend left] & 0
\end{tikzcd}$$

for $j = 0, 1$ and conclude that $K_j(A_\theta) \cong \mathbb{Z}^2$. □

It is a striking feature of the Pimsner-Voiculescu sequence that we could compute the K-theory of A_θ in Example 6.10 and of $\mathcal{K} \rtimes_\alpha \mathbb{Z}$ in Example 6.7 without knowing anything about the internal structure of these crossed product C^*-algebras.

6.4 Further Topics

There are many further topics related to K-theory and crossed products. We will only mention a few.
In Example 6.10, we have computed the K-theory of irrational rotation algebras using the Pimsner-Voiculescu sequence. In fact, A_θ is one of the most interesting C^*-algebras and a lot of work has been devoted to determining the structure of A_θ in the past. For example A_θ is simple and thus a building block of all the C^*-algebras, see [11, Proposition 2.56]. One may associate the so-called canonical trace to A_θ. It is defined as

$$\tau(f) = \int \mathbb{T} f(0)(z) d\mu_{\mathbb{T}}(z)$$

for $f \in C_c(\mathbb{Z}, C(\mathbb{T}))$. This functional can be extended to $M_n(A_\theta)$ by computing the usual matrix trace, i.e. $\tau((f_{ij})) = \sum_k \tau(f_{kk})$. If $f, g \in M_n(A_\theta)$, then $\tau(fg) = \tau(gf)$. In particular, τ respects Murray-von Neumann equivalence. One may then conclude that τ induces a homomorphism $K_0(\tau) : K_0(A_\theta) \longrightarrow \mathbb{R}$, see [6, Section 5.2]. In [3], the range of $K_0(\tau)$ is being computed. It is $\mathbb{Z} + \theta \cdot \mathbb{Z} \cong \mathbb{Z}^2$. In [16], a complete classification of the A_θ is derived from that.
Our definition of crossed products is not the only possible one. Given a C^*-dynamical system (A, G, α), one may form the reduced crossed product $A \rtimes_{r,\alpha} G$. The norm is not given by the supremum over all seminorms induced by covariant representations, but by a particular representation. Whenever G is an amenable group, both crossed product definitions coincide. (One may take this as the definition of amenability.) But in general, one can only conclude that there is a surjective *-homomorphism $\pi : A \rtimes_\alpha G \longrightarrow A \rtimes_{r,\alpha} G$. Many properties of universal crossed products carry over to reduced ones, see [11, Section 7.2 and Appendix A].
A related notion that has been subject of research in recent years is the one of C^*-uniqueness. A (discrete) group is called C^*-unique if there is exactly one C^*-norm on the convolution algebra $C_c(G, \mathbb{C})$ with respect to the trivial action on $\mathbb{C}$. If a group G is C^*-unique, it is also amenable, but the converse is not true. Indeed, the group $\mathbb{Z}$ is amenable, but it is not C^*-unique. That is, because the convolution algebra of $\mathbb{Z}$ consists of (finite) formal sums $\sum_{n \in \mathbb{Z}} a(n)\delta_n$ with $a(n) \in \mathbb{C}$. One may identify such a sum with the function $f(z) = \sum_{n \in \mathbb{Z}} a(n) z^n$, $z \in \mathbb{T}$. By taking any infinite closed subset $F \subseteq \mathbb{T}$ and setting $||f||_F = \sup_{z \in F} |f(z)|$, one obtains a C^*-norm on $C_c(\mathbb{Z}, \mathbb{C})$, because polynomials with an infinite 0-set are identically 0. If G is finite, then $C_c(G, \mathbb{C})$ is finite dimensional, so any C^*-norm is complete and the C^*-norm is thus unique. A more interesting example may be found in [17].
Our notion of K-theory as a pair of groups associated to a C^*-algebra may be improved. One may form a bivariant K-theory i.e. a theory that takes two C^*-algebras as input instead of just one. The groups are $KK^0(A, B)$ and $KK^1(A, B)$ and the theory is often referred to as KK-theory or Kasparov-theory. Ordinary K-theory is a special case of KK-theory and there is more structure in the bivariant version than in the univariant one. Furthermore, there are generalizations of the Connes-Thom isomorphism and the Pimsner-Voiculescu sequence. The theory was originally developed in [18] and a survey may be found in [8, Chapter 8].
The K-theory of C^*-algebras is a generalization of K-theory for locally compact Hausdorff spaces. Of course, it has been tried to develop noncommutative analogues of other cohomology theories. There is a generalization of De-Rham cohomology called cyclic cohomology.
Cyclic cohomology is part of a much bigger framework called noncommutative geometry. This theory is a noncommutative analogue of differential geometry. In noncommutative geometry, the C^*-algebra A_θ functions as the prototype of a noncommutative Riemannian manifold. A survey may be found in [19].

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