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A Semi-strong Perfect Digraph Theorem

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A Semi-strong Perfect Digraph Theorem

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Abstract

Reed showed that, if two graphs are P_4-isomorphic, then either both are perfect or none of them is. In this note we will derive an analogous result for perfect digraphs.

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1 Introduction and Notation

Perfect digraphs have been introduced by Andres and Hochstättler [1] as the class of digraphs where the clique number equals the dichromatic number for every induced subdigraph. Reed [7] showed that, if two graphs are P_4-isomorphic, then either both are perfect or none of them is. In this note we will derive an analogous result for perfect digraphs.

We start with some definitions. For basic terminology we refer to Bang-Jensen and Gutin [2]. For the rest of the paper, we only consider digraphs without loops. Let $D = (V, A)$ be a digraph. The *symmetric part* $S(D)$ of $D = (V, A)$ is the digraph (V, A_2) where A_2 is the union of all pairs of antiparallel arcs of D, the *oriented part* $O(D)$ of D is the digraph (V, A_1) where $A_1 = A \setminus A_2$.

A proper k-coloring of D is an assignment $c : V \to \{1, \dots, k\}$ such that for all $1 \le i \le k$ the digraph induced by $c^{-1}(\{i\})$ is acyclic. The *dichromatic number* $\chi(D)$ of D is the smallest nonnegative integer k such that D admits a proper k-coloring. A *clique* in a digraph D is a subdigraph in which for any two distinct vertices v and w both arcs (v, w) and (w, v) exist. The *clique number* $\omega(D)$ of D is the size of the largest clique in $S(D)$. The clique number is an obvious lower bound for the dichromatic number. D is called *perfect* if, for any induced subdigraph H of D, $\chi(H) = \omega(H)$.

An (undirected) graph $G = (V, E)$ can be considered as the symmetric digraph $D_G = (V, A)$ with $A = \{(v, w), (w, v) \mid vw \in E\}$. In the following, we will not distinguish between G and D_G. In this way, the dichromatic number of a graph G is its chromatic number $\chi(G)$, the clique number of G is its usual clique number $\omega(G)$, and G is perfect as a digraph if and only if G is perfect as a graph.

A main result of [1] is the following:

Theorem 1 ([1])**.** *A digraph $D = (V, A)$ is perfect if and only if $S(D)$ is perfect and D does not contain any directed cycle $\vec{C}_n$ with $n \geq 3$ as induced subdigraph.*

Together with the Strong Perfect Graph Theorem (see e.g. [3]) this yields a characterization of perfect digraphs in form of forbidden induced minors. The Weak Perfect Graph Theorem (see [3]), though, does not generalize. The directed 4-cycle $\vec{C}_4$ is not perfect but its complement is perfect, thus perfection is in general not maintained under taking complements.

Two graphs $G = (V, E_1)$ and $H = (V, E_2)$ are P_4*-isomorphic*, if any set $\{a, b, c, d\} \subseteq V$ induces a chordless path, i.e. a P_4, in G if and only if it induces a P_4 in H.

Theorem 2 (Semi-strong Perfect Graph Theorem [7])**.** *If G and H are P_4-isomorphic, then*

$$G \text{ is perfect} \iff H \text{ is perfect.}$$

The graphs without an induced P_4 are the cographs [5]. Thus any pair of cographs with the same number of vertices is P_4-isomorphic. In order to generalize Theorem 2 to digraphs we consider the class of directed cographs [6], which are characterized by a set $\mathcal{F}$ of eight forbidden induced minors. Since the class of directed cographs is invariant under taking complements and perfect digraphs are not, it is clear that isomorphism with respect to $\mathcal{F}$ will not yield the right notion of isomorphism for our purposes. It turns out that restricting to five of these minors yields the desired result.

2 P^4C-isomorphic digraphs

The five forbidden induced minors from [6] we need are the symmetric path P_4, the directed 3-cycle $\vec{C}_3$, the directed path $\vec{P}_3$ and the two possible augmentations $\vec{P}_3^+$ and $\vec{P}_3^-$ of the $\vec{P}_3$ with one antiparallel edge (see Figure 1).

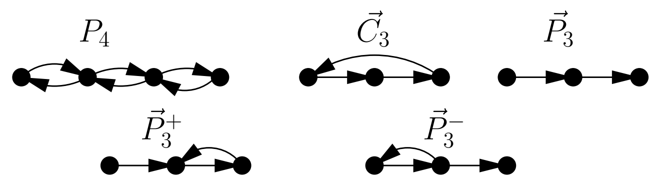

Figure 1: The five induced subdigraphs considered

Definition 3. *Let $D = (V, A)$ and $D' = (V, A')$ be two digraphs on the same vertex set. Then D and D' are said to be P^4C-*isomorphic *if and only if*

1. *any set $\{a, b, c, d\} \subseteq V$ induces a P_4 in $S(D)$ if and only if it induces a P_4 in $S(D')$,*
2. *any set $\{a, b, c\} \subseteq V$ induces a $\vec{C}_3$ in D if and only if it induces a $\vec{C}_3$ in D',*
3. *any set $\{a, b, c\} \subseteq V$ induces a $\vec{P}_3$ with midpoint b in D if and only if it induces a $\vec{P}_3$ with midpoint b in D' and*
4. *any set $\{a, b, c\} \subseteq V$ induces a $\vec{P}_3^+$ or a $\vec{P}_3^-$ in either case with midpoint b in D if and only if it induces one of them with midpoint b in D'.*

Note that the P_4 in case 1 is not necessarily induced in D, resp. in D'.

Lemma 4. *If D and D' are P^4C-isomorphic, then D contains an induced directed cycle of length $k \geq 3$ if and only if the same is true for D'.*

Proof. By symmetry it suffices to prove that, if $\{v_0, \dots v_{k-1}\}$ induces a directed cycle $\vec{C}_k$ in D, then the same holds for D'. The assertion is clear if $k = 3$, thus assume $k \geq 4$. We may, furthermore, assume that the vertices are traversed in consecutive order in D. Since D and D' are P^4C-isomorphic, each set $\{v_i, v_{i+1}, v_{i+2}\}$ induces a $\vec{P}_3$ with midpoint v_{i+1} in D', where indices are taken modulo k. This yields a directed cycle C on $v_0, \dots v_{k-1}$, possibly with opposite orientation wrt. D. In that case we relabel the vertices such that the label coincides with the direction of traversal. We claim the cycle is induced in D', too.

Assume it is not, i.e. C has a chord (v_i, v_j), $j \neq i-1$ in D'. We choose j such that the directed path from v_j to v_i on C is shortest possible. If (v_i, v_j) is an asymmetric arc, then, since $\{v_i, v_j, v_{j+1}\}$ does not induce a $\vec{C}_3$, it must induce a $\vec{P}_3$ with midpoint v_j in D' and hence the same must hold in D, contradicting $\vec{C}_k$ being induced. If we have a pair of antiparallel edges between v_i and v_j, then, similarly, $\{v_i, v_j, v_{j+1}\}$ induces a $\vec{P}_3^+$ or a $\vec{P}_3^-$ with midpoint v_j, also leading to a contradiction. □

Theorem 5. *If D and D' are P^4C-isomorphic then*

$$D \text{ is perfect} \iff D' \text{ is perfect.}$$

Proof. By assumption $S(D)$ and $S(D')$ are P_4-isomorphic, hence using Theorem 2 we find that $S(D)$ is perfect if and only if $S(D')$ is perfect. By Proposition 4, D contains an induced directed cycle of length at least three if and only if the same holds for D'. The assertion thus follows from Theorem 1. □

3 Transitive extensions of cographs

In this section we will analyse the class of digraphs without any of the five subgraphs, which thus are trivially pairwise P^4C-isomorphic.

Since the symmetric part of such a graph is a cograph, we may consider its cotree [5] in canonical form, where the labels alternate between 0 and 1. Since the 1-labeled tree vertices correspond to complete joins, there is no additional room for asymmetric arcs. The 0-labeled vertices correspond to disjoint unions. Assume the connected components in $S(G)$ are $G_1, \ldots G_k$.

Lemma 6. *If there exists an asymmetric arc connecting a vertex v_i in G_i to a vertex v_j in G_j, then G_i and G_j are connected by an orientation of the complete bipartite graph $K_{V(G_i),V(G_j)}$.*

Proof. Since $S(G_i)$ and $S(G_j)$ are connected and by symmetry, it suffices to show that v_i must be connected by an asymmetric arc to all symmetric neighbors of v_j. Let w be such a neighbor. Since there is no symmetric arc from v_i to w and $\{v_i, v_j, w\}$ must neither induce a $\vec{P}_3^-$ nor a $\vec{P}_3^+$, we must have an asymmetric arc between v_i and w. □

Hence, the asymmetric arcs between the components $G_1, \ldots, G_k$ constitute an orientation of a complete ℓ-partite graph for $1 \leq \ell \leq k$. The situation is further complicated by the fact that we must neither create a $\vec{C}_3$ nor a $\vec{P}_3$, where we have to take into account that there may also be asymmetric arcs within the G_i.

We wonder whether this structure is strict enough to make some problems tractable that are $\mathcal{NP}$-complete in general. In particular we would be interested in the complexity of the problem to cover all vertices with a minimum number of vertex disjoint directed paths.

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