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Immanuel Albrecht:

Well-Scaling Procedure for Deciding Gammoid Class-Membership of Matroids

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Well-Scaling Procedure for Deciding Gammoid Class-Membership of Matroids

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Abstract

We introduce a procedure that solves the decision problem whether a given matroid M is a gammoid. The procedure consists of three pieces: First, we introduce a notion of a valid matroid tableau which captures the current state of knowledge regarding the properties of matroids related to the matroid under consideration. Second, we give a sufficient set of rules that may be used to generate valid matroid tableaux. Third, we introduce a succession of steps that ultimately lead to a decisive tableau starting with any valid tableau. We argue that the decision problem scales well with respect to parallel computation models.

Keywords. matroids, gammoids, directed graphs, decision problem

The *Gammoid Class-Membership Problem* is the following decision problem: Given a matroid $M = (E, \mathcal{I})$, determine whether M is a gammoid or not. It is a well-known fact that the class of gammoids is closed under duality, minors, and direct sums; and that it may not be characterized by a finite number of excluded minors. D. Mayhew even showed that every gammoid is a minor of some excluded minor of the class of gammoids [11], therefore any attempt to solve this

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problem relying solely on excluded minors appears to be futile. We introduce a decision procedure for Gammoid Class-Membership Problems that is guaranteed to ultimately give an answer through exhaustive search, and which is also capable to incorporate knowledge of non-gammoids and strict gammoids in order to give an answer before exhausting the search space in many cases. Furthermore, the derivation steps described in our process may be carried out using massive parallelism, since joining tableaux is a valid derivation.

1 Preliminaries

In this work, we consider *matroids* to be pairs $M = (E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a system of independent subsets of E subject to the usual axioms ([12], Sec. 1.1). The family of bases of M shall be denoted by $\mathcal{B}(M)$, the family of flats of M shall be denoted by $\mathcal{F}(M)$. If $M = (E, \mathcal{I})$ is a matroid and $X \subseteq E$, then the restriction of M to X shall be denoted by $M|X$ ([12], Sec. 1.3). A matroid $N = (E', \mathcal{I}')$ is an *extension* of M, if $E \subseteq E'$ and $\mathcal{I} = \{X \in \mathcal{I}' \mid X \subseteq E\}$ holds. The *dual matroid* of M shall be denoted by M^*. A *modular cut* of M is a set $C \subseteq \mathcal{F}(M)$ that is closed under super-flats and under the intersection of pairs of modular flats. H.H. Crapo showed, that there is a one-to-one correspondence between single-element extensions of a matroid M and its modular cuts [4].

Furthermore, the notion of a *digraph* shall be synonymous with what is described more precisely as *finite simple directed graph* that may have some loops, i.e. a digraph is a pair $D = (V, A)$ where V is a finite set and $A \subseteq V \times V$. All standard notions related to digraphs in this work are in accordance with the definitions found in [1]. A *path* in $D = (V, A)$ is a non-empty and non-repeating sequence $p = p_1 p_2 \ldots p_n$ of vertices $p_i \in V$ such that for each $1 \leq i < n$, $(p_i, p_{i+1}) \in A$. By convention, we shall denote p_n by p_{-1}. Furthermore, the set of vertices traversed by a path p shall be denoted by $|p| = \{p_1, p_2, \ldots, p_n\}$ and the set of all paths in D shall be denoted by $\mathbf{P}(D)$. For $D = (V, A)$ and $S, T \subseteq V$, an S-T-*separator* is a set $X \subseteq V$ such that every path $p \in \mathbf{P}(D)$ from $s \in S$ to $t \in T$ has $|p| \cap V \neq \emptyset$.

Definition 1.1. Let $D = (V, A)$ be a digraph, and $X, Y \subseteq V$. A *routing* from X to Y in D is a family of paths $R \subseteq \mathbf{P}(D)$ such that

(i) for each $x \in X$ there is some $p \in R$ with $p_1 = x$,

(ii) for all $p \in R$ the end vertex $p_{-1} \in Y$, and

(iii) for all $p, q \in R$, either $p = q$ or $|p| \cap |q| = \emptyset$.

We shall write $R\colon X \rightrightarrows Y$ in D as a shorthand for "R is a routing from X to Y in D", and if no confusion is possible, we just write $X \rightrightarrows Y$ instead of R and $R\colon X \rightrightarrows Y$.

Definition 1.2. Let $D = (V, A)$ be a digraph, $E \subseteq V$, and $T \subseteq V$. The *gammoid represented by* (D, T, E) is defined to be the matroid $\Gamma(D, T, E) = (E, \mathcal{I})$ where

$$\mathcal{I} = \{X \subseteq E \mid \text{there is a routing } X \rightrightarrows T \text{ in D}\}.$$

The elements of T are usually called *sinks* in this context, although they are not required to be actual sinks of the digraph D. To avoid confusion, we shall call the elements of T *targets* in this work. A matroid $M' = (E', \mathcal{I}')$ is called *gammoid*, if there is a digraph $D' = (V', A')$ and a set $T' \subseteq V'$ such that $M' = \Gamma(D', T', E')$. A gammoid M is called *strict*, if there is a representation (D, T, E) of M with $D = (V, A)$ where $V = E$.

Definition 1.3. Let $M = (E, \mathcal{I})$ be a matroid. Then M shall be *strongly base-orderable*, if for every pair of bases $B_1, B_2 \in \mathcal{B}(M)$ there is a bijective map $\varphi\colon B_1 \longrightarrow B_2$ such that $(B_1 \backslash X) \cup \varphi[X] \in \mathcal{B}(M)$ holds for all $X \subseteq B_1$.

Lemma 1.4 ([10], Corollary 4.1.4)**.** *Let $M = (E, \mathcal{I})$ be a gammoid. Then M is strongly base-orderable.*

For a proof, see [10].

Definition 1.5. Let $M = (E, \mathcal{I})$ be a matroid. The *α-invariant of M* shall be the map $\alpha_M\colon 2^E \longrightarrow \mathbb{Z}$ that is uniquely characterized by the recurrence relation

$$\alpha_M(X) = |X| - \mathrm{rk}_M(X) - \sum_{F \in \mathcal{F}(M,X)} \alpha_M(F),$$

where $\mathcal{F}(M, X) = \{F \in \mathcal{F}(M) \mid F \subsetneq X\}$.

Theorem 1.6 ([10], Theorems 2.2 and 2.4)**.** *Let $M = (E, \mathcal{I})$ be a matroid. Then M is a strict gammoid if and only if $\alpha_M \geq 0$.*

For a proof, see [10].

Theorem 1.7 ([6], Theorem 13; [2], [3], [5]). *Let $\mathbb{F}_2$ be the two-elementary field, E, C finite sets, and let $\mu \in \mathbb{F}_2^{E\times C}$ be a matrix. Then the linear matroid $M(\mu)$ is a gammoid if and only if there is no minor N of $M(\mu)$ which is isomorphic to $M(K_4)$. The latter is the case if and only if $M(\mu)$ is isomorphic to the polygon matroid of a series-parallel network.*

For proofs of a sufficient set of implications which establish the equivalency stated, refer to [2], [3], and [5].

Theorem 1.8 ([12], Theorem 6.5.4). *Let $M = (E, \mathcal{I})$ be a matroid. Then M is isomorphic to the linear matroid $M(\mu)$ for some matrix $\mu \in \mathbb{F}_2^{E\times C}$ if and only if M has no minor isomorphic to the uniform matroid $U_{2,4} = (E', \mathcal{I}')$, where $E' = \{a, b, c, d\}_{\neq}$ and $\mathcal{I}' = \left\{X \subseteq E' \,\middle|\, |X| \leq 2\right\}$.*

See [12], pp.193f, for a proof.

Definition 1.9. Let $M = (E, \mathcal{I})$ be a matroid, $X \subseteq E$. The restriction $N = M|X$ shall be a *deflate of* M, if $E\backslash X = \{e_1, e_2, \ldots, e_m\}_{\neq}$ can be ordered naturally, such that for all $i \in \{1, 2, \ldots, m\}$ the modular cut

$$C_i = \left\{F \in \mathcal{F}\left(M|\left(X \cup \{e_1, e_2, \ldots, e_{i-1}\}\right)\right) \,\middle|\, e_i \in \mathrm{cl}_M(F)\right\}$$

has precisely one $\subseteq$-minimal element. M shall be called *deflated*, if the only deflate of M is M itself.

Lemma 1.10. *Let $M = (E, \mathcal{I})$ be a matroid, $X \subseteq E$ and let $N = M|X$ be a deflate of M. Then M is a gammoid if and only if N is a gammoid.*

Proof. If M is a gammoid, then N is a gammoid, since the class of gammoids is closed under minors ([10], Sec. 1 and Cor. 4.1.3). Now let N be a gammoid, and let $E\backslash X = \{e_1, e_2, \ldots, e_m\}_{\neq}$ be implicitly ordered with the properties required in Definition 1.9. We proof the statement of this lemma by induction on $|E\backslash X| = m$. The base case $m = 0$ is trivial, the induction step follows from the special case where $E\backslash X = \{e_1\}$. Let $F_1 = \bigcap C_1$ be the unique minimal element of the modular cut C_1. Then M arises from N by adding a new point e_1 to N, which is in general position with respect to the flat F_1. Let (D, T, X) be a representation of N with $D = (V, A)$ and $e_1 \notin V$. Let $D' = (V \cup \{e_1\}, A \cup (\{e_1\} \times F_1))$. It is easy to see that (D', T, E) is a representation of M. $\square$

Theorem 1.11 ([8], Theorem 3). *Let $D = (V, A)$ be a digraph, $E, T \subseteq V$, and $r > 0$ be the cardinality of a minimal E-T-separator in D. There is a set $Z \subseteq V$ with $E \cup T \subseteq Z$ and $|Z| = O(|E| \cdot |T| \cdot r)$ such that for all $X \subseteq E$ and $Y \subseteq T$ there is a minimal X-Y-separator S in D with $S \subseteq Z$.*

For the proof, see [8], where the authors only give the O-behavior of the size of Z in Theorem 1.11, but it is possible to derive the factor hidden in the O-notation by inspecting their proof and the proof of [9] Lemma 4.1. We obtain that $E \cup T \subseteq Z$ and

$$|Z| \leq \binom{r}{1} \cdot \binom{|E|}{1} \cdot \binom{|T|}{1} + |E| + |T| = r \cdot |E| \cdot |T| + |E| + |T|\,.$$

Let (D, T, E) be a representation of M where $|T| = \mathrm{rk}_M(E)$ and $D = (V, A)$. Let $Z \subseteq V$ be a subset of V as in the consequent of Theorem 1.11. Let $D' = (Z, A')$ be the digraph, where for all $x, y \in Z$, there is an arc

$$(x, y) \in A' \quad \iff \quad \exists p \in \mathbf{P}(D; x, y) \colon\; |p| \cap Z = \{x, y\}.$$

Thus there is an arc leaving $y \in Z$ and entering $z \in Z$ in D' if there is a path from y to z in D that never visits another vertex of Z. It is routine to show that (D', T, E) represents the same matroid as (D, T, E). Therefore we obtain:

Corollary 1.12. *Let $M = (E, \mathcal{I})$ be a gammoid. There is a representation (D, T, E) of M where $D = (V, A)$ such that $|T| = \mathrm{rk}_M(E)$ and such that $|V| \leq \mathrm{rk}_M(E)^2 \cdot |E| + \mathrm{rk}_M(E) + |E|$.*

2 Matroid Tableaux

Definition 2.1. A *matroid tableau* is a tuple $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ where

(i) G is a matroid, called the *goal* of $\mathbf{T}$,

(ii) $\mathcal{G}$ is a family of matroids, called the *gammoids* of $\mathbf{T}$,

(iii) $\mathcal{M}$ is a family of matroids, called the *intermediates* of $\mathbf{T}$,

(iv) $\mathcal{X}$ is a family of matroids, called the *excluded matroids* of $\mathbf{T}$, and where

(v) $\simeq$ is an equivalence relation on $\{G' \mid G' \text{ is a minor of } G\} \cup \mathcal{G} \cup \mathcal{M} \cup \mathcal{X}$, called the *equivalence* of $\mathbf{T}$.

Definition 2.2. Let $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ be a matroid tableau. $\mathbf{T}$ shall be *valid*,

(i) if all matroids in $\mathcal{G}$ are indeed gammoids,

(ii) if no matroid in $\mathcal{M}$ is a strict gammoid,

(iii) if all matroids in $\mathcal{X}$ are indeed matroids which are not gammoids, and

(iv) if for every equivalence classes $[M]_\simeq$ of $\simeq$ we have that either $[M]_\simeq$ is fully contained in the class of gammoids or $[M]_\simeq$ does not contain a gammoid.

Definition 2.3. Let $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ be a matroid tableau. $\mathbf{T}$ shall be *decisive*, if $\mathbf{T}$ is valid and if either of the following holds:

(i) There is a matroid $M \in \mathcal{G}$ such that $G \simeq M$.

(ii) There is a matroid $X \in \mathcal{X}$ that is isomorphic to a minor of G.

(iii) For every extension $N = (E', \mathcal{I}')$ of $G = (E, \mathcal{I})$ with

$$|E'| = \mathrm{rk}_G(E)^2 \cdot |E| + \mathrm{rk}_G(E) + |E|$$

there is a matroid $M \in \mathcal{M}$ that is isomorphic to N.

Lemma 2.4. *Let $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ be a decisive matroid tableau. Then G is a gammoid if and only if there is a matroid $M \in \mathcal{G}$ such that $G \simeq M$.*

Proof. Assume that such an $M \in \mathcal{G}$ exists. From Definition 2.2 we obtain that M is a gammoid, and that in this case $G \simeq M$ implies that G is a gammoid, too. Now assume that no $M \in \mathcal{G}$ has the property $G \simeq M$. Since $\mathbf{T}$ is decisive, either case *(ii)* or *(iii)* of Definition 2.3 holds. If case *(ii)* holds, then G cannot be a gammoid since it has a non-gammoid minor, but the class of gammoids is closed under minors. If case *(iii)* holds but not case *(ii)*, then no extension of $G = (E, \mathcal{I})$ with $k = \mathrm{rk}_G(E)^2 \cdot |E| + \mathrm{rk}_G(E) + |E|$ elements is a strict gammoid. Now assume that G is a gammoid, then there is a digraph $D = (V, A)$ with $|V| \leq k$ vertices, such that $G = \Gamma(D, T, E)$ for some $T \subseteq V$ (Corollary 1.12). Let $N' = \Gamma(D, T, V) \oplus (V', \{\emptyset\})$ with $V' \cap V = \emptyset$ and $|V'| + |V| = k$. Clearly, N' is an extension of G on a ground set with k elements, which is also a strict gammoid, a contradiction to the assumption that N' is isomorphic to some $N \in \mathcal{M}$, since $\mathcal{M}$ is a family which consists of matroids that are not strict gammoids. Therefore we may conclude that in case *(iii)* the matroid G is not a gammoid. $\square$

3 Valid Derivations

A *derivation* is an operation on a finite number of input tableaux and possible additional parameters with constraints that produces an output tableau. Furthermore, a derivation is *valid*, if the output tableau is valid for all sets of valid input tableaux and possible additional parameters that satisfy the constraints.

Definition 3.1. Let $\mathbf{T}_i = (G_i, \mathcal{G}_i, \mathcal{M}_i, \mathcal{X}_i, \simeq^{(i)})$ be matroid tableaux for $i \in \{1, 2, \ldots, n\}$. The *joint tableau* shall be the matroid tableaux

$$\bigcup_{i=1}^{n} \mathbf{T}_i = (G_1, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$$

where

$$\mathcal{G} = \bigcup_{i=1}^{n} \mathcal{G}_i, \quad \mathcal{M} = \bigcup_{i=1}^{n} \mathcal{M}_i, \quad \mathcal{X} = \bigcup_{i=1}^{n} \mathcal{X}_i,$$

and where $\simeq$ is the smallest equivalence relation such that $M \simeq^{(i)} N$ implies $M \simeq N$ for all $i \in \{1, 2, \ldots, n\}$. In other words, $\simeq$ is the equivalence relation on the family of matroids $\{G' \mid G' \text{ is a minor of } G\} \cup \mathcal{G} \cup \mathcal{M} \cup \mathcal{X}$ which is generated by the relations $\simeq^{(1)}, \simeq^{(2)}, \ldots, \simeq^{(n)}$.

Lemma 3.2. *The derivation of the joint tableau is valid.*

Proof. Clearly, $\mathcal{G}$, $\mathcal{M}$, and $\mathcal{X}$ inherit their desired properties of Definition 2.2 from the valid input tableaux $\mathbf{T}_i$ where $i \in \{1, 2, \ldots, n\}$. Now let $M \simeq N$ with $M \neq N$. Then there are matroids $M_1, M_2, \ldots, M_k$ and indexes $i_0, i_1, \ldots, i_k \in \{1, 2, \ldots, n\}$ such that there is a chain of $\simeq^{(i)}$-relations

$$M \simeq^{(i_0)} M_1 \simeq^{(i_1)} M_2 \simeq^{(i_2)} \cdots \simeq^{(i_{k-1})} M_k \simeq^{(i_k)} N.$$

The assumption that the input tableaux are valid yields that M is a gammoid if and only if M_1 is a gammoid, if and only if M_2 is a gammoid, and so on. Therefore it follows that M is a gammoid if and only if N is a gammoid, thus $\simeq$ has the desired property of Definition 2.2. Consequently, $\bigcup_{i=1}^{n} \mathbf{T}_i$ is a valid tableau. $\square$

Definition 3.3. Let $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ and $\mathbf{T}' = (G, \mathcal{G}', \mathcal{M}', \mathcal{X}', \simeq')$ be matroid tableaux. We say that $\mathbf{T}$ is a *sub-tableau of* $\mathbf{T}'$ if $\mathcal{G} \subseteq \mathcal{G}'$, $\mathcal{M} \subseteq \mathcal{M}'$, and $\mathcal{X} \subseteq \mathcal{X}'$ holds, and if $M \simeq N$ implies $M \simeq' N$.

Lemma 3.4. *The derivation of a sub-tableau is valid.*

Proof. Clearly $\mathbf{T}$ inherits the properties of Definition 2.2 from the validity of $\mathbf{T}'$. $\square$

Definition 3.5. Let $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ be a matroid tableau. We shall call the tableau $[\mathbf{T}]_{\simeq} = (G, \mathcal{G}', \mathcal{M}, \mathcal{X}', \simeq)$ *expansion tableau of* $\mathbf{T}$ whenever

$$\mathcal{G}' = \bigcup_{M \in \mathcal{G}} [M]_{\simeq} \quad \text{and} \quad \mathcal{X}' = \bigcup_{M \in \mathcal{X}} [M]_{\simeq}.$$

Lemma 3.6. *The derivation of the expansion tableau is valid.*

Proof. If $M' \in \mathcal{G}'$, then there is some $M \in \mathcal{G}$ such that $M \simeq M'$. Since we assume $\mathbf{T}$ to be valid, we may infer that M' is a gammoid if and only if M is a gammoid, and the latter is the case since $M \in \mathcal{G}$. Therefore M' is a gammoid. An analogous argument yields that if $M' \in \mathcal{X}'$, then M' is not a gammoid. $\square$

Definition 3.7. Let $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ be a matroid tableau. We shall call the tableau $[\mathbf{T}]_{\equiv} = (G, \mathcal{G}', \mathcal{M}', \mathcal{X}', \simeq')$ *extended tableau of* $\mathbf{T}$ whenever

$$\mathcal{G}' = \mathcal{G} \cup \{M^* \mid M \in \mathcal{G}\}, \;\; \mathcal{X}' = \mathcal{X} \cup \{M^* \mid M \in \mathcal{X}\}, \;\; \mathcal{M}' = \mathcal{M} \cup \mathcal{X}',$$

and when $\simeq'$ is the smallest equivalence relation that contains the relations $\simeq$ and $\sim$; where $M \sim N$ if and only if N is isomorphic to M or M^*.

Lemma 3.8. *The derivation of the extended tableau is valid.*

Proof. The class of gammoids is closed under duality, therefore a matroid M is a gammoid if and only if M^* is a gammoid. So $\mathcal{G}'$ and $\mathcal{X}'$ inherit their desired properties of Definition 2.2 from the validity of $\mathbf{T}$. If $M \in \mathcal{M}' \backslash \mathcal{M}$, then $M \in \mathcal{X}'$, therefore M cannot be a strict gammoid. $\square$

Definition 3.9. Let $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ be a decisive matroid tableau. The tableau $\mathbf{T}! = (G, \mathcal{G}', \mathcal{M}, \mathcal{X}', \simeq)$ shall be the *conclusion tableau for* $\mathbf{T}$ if either

(i) $\mathcal{G}' = \mathcal{G} \cup \{G' \mid G' \text{ is a minor of } G\}$, $\mathcal{X}' = \mathcal{X}$, and the tableau $\mathbf{T}$ satisfies case (i) of Definition 2.3; or

(ii) $\mathcal{G}' = \mathcal{G}$, $\mathcal{X}' = \mathcal{X} \cup \{G\}$, and $\mathbf{T}$ satisfies case (ii) or (iii) of Definition 2.3.

Corollary 3.10. *The derivation of the conclusion tableau is valid.*

Proof. Easy consequence of Lemma 2.4. □

Definition 3.11. Let $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ be a matroid tableau, let M_1 and M_2 be matroids of the tableau, i.e.

$$\{M_1, M_2\} \subseteq \{G' \mid G' \text{ is a minor of } G\} \cup \mathcal{G} \cup \mathcal{M} \cup \mathcal{X}.$$

Furthermore, let M_1 be a deflate of M_2. The tableau

$$\mathbf{T}(M_1 \simeq M_2) = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq')$$

is called *identified tableau* for $\mathbf{T}$ with respect to M_1 and M_2 if the relation $\simeq'$ is the smallest equivalence relation, such that $M_1 \simeq' M_2$ holds, and such that $M' \simeq N'$ implies $M' \simeq' N'$.

Lemma 3.12. *The derivation of an identified tableau is valid.*

Proof. Follows from Lemma 1.10. □

3.1 Valid Tableaux

Corollary 3.13. *Let $M = (E, \mathcal{I})$ be a matroid with $\alpha_M \geq 0$. Then the matroid tableau $\mathbf{T}$ is valid, where $\mathbf{T} = (M, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ with $\mathcal{G} = \{M, M^*\}$, $\mathcal{M} = \emptyset$, $\mathcal{X} = \emptyset$, and $M \simeq N \Leftrightarrow M = N$.*

Proof. See Theorem 1.6. □

Corollary 3.14. *Let $M = (E, \mathcal{I})$ be a matroid with $\mathrm{rk}_M(X) = 3$, $X \subseteq E$ with $\alpha_M(X) < 0$. Then the matroid tableau $\mathbf{T}$ is valid, where $\mathbf{T} = (M, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ with $\mathcal{G} = \emptyset$, $\mathcal{M} = \emptyset$, $\mathcal{X} = \{M, M^*\}$, and $M \simeq N \Leftrightarrow M = N$.*

Proof. Follows from Theorem 1.6 together with the fact that every gammoid of rank 3 is a strict gammoid ([7], Proposition 4.8). □

Remark 3.15. Let $M = (E, \mathcal{I})$ be a matroid, $X \subseteq E$ with $\alpha_M(X) < 0$. Then the matroid tableau $\mathbf{T}$ is valid, where $\mathbf{T} = (M, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ with $\mathcal{G} = \emptyset$, $\mathcal{M} = \{M\}$, $\mathcal{X} = \emptyset$, and $M \simeq N \Leftrightarrow M = N$.

Corollary 3.16. *Let $M = (E, \mathcal{I})$ be a matroid. If M has no minor isomorphic to $M(K_4)$ and no minor isomorphic to $U_{2,4}$, then the matroid tableau $\mathbf{T}$ is valid, where $\mathbf{T} = (M, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ with $\mathcal{G} = \{M, M^*\}$, $\mathcal{M} = \emptyset$, $\mathcal{X} = \emptyset$, and $M \simeq N \Leftrightarrow M = N$.*

Proof. Direct consequence of Theorems 1.7 and 1.8. □

Corollary 3.17. *Let $M = (E, \mathcal{I})$ be a matroid, $B_1, B_2 \in \mathcal{B}(M)$ be bases of M such that for every bijection $\varphi\colon B_1 \backslash B_2 \longrightarrow B_2 \backslash B_1$ there is a set $X \subseteq B_1 \backslash B_2$ with the property $(B_1 \backslash X) \cup \varphi[X] \notin \mathcal{B}(M)$. Then the matroid tableau $\mathbf{T}$ is valid, where $\mathbf{T} = (M, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$ with $\mathcal{G} = \emptyset$, $\mathcal{M} = \emptyset$, $\mathcal{X} = \{M, M^*\}$, and $M \simeq N \Leftrightarrow M = N$.*

Proof. Direct consequence of Lemma 1.4. □

3.2 Example

Consider the matroid $G = G_{8,4,1} = (E, \mathcal{I})$ where $E = \{1, 2, \ldots, 8\}$ and where $\mathcal{I} = \left\{ X \subseteq E \;\middle|\; |X| \leq 4,\, X \notin \mathcal{H} \right\}$ with

$$\mathcal{H} = \big\{ \{1,3,7,8\}, \{1,5,6,8\}, \{2,3,6,8\}, \{4,5,6,7\}, \{2,4,7,8\} \big\}.$$

Clearly, $\alpha_G(H) = 1$ for all $H \in \mathcal{H}$, and consequently $\alpha_G(E) = 4 - 5 = -1$. The dual matroid $G^* = (E, \mathcal{I}^*)$ has $\mathcal{I}^* = \left\{ X \subseteq E \;\middle|\; |X| \leq 4,\, X \notin \mathcal{H}^* \right\}$ with

$$\mathcal{H}^* = \big\{ \{1,2,3,8\}, \{1,3,5,6\}, \{1,4,5,7\}, \{2,3,4,7\}, \{2,4,5,6\} \big\}.$$

Thus $\alpha_{G^*}(H') = 1$ for all $H' \in \mathcal{H}^*$, and so $\alpha_{G^*}(E) = 4 - 5 = -1$, too. We start with the valid tableaux

$$\mathbf{T}_G = (G, \emptyset, \{G\}, \emptyset, \langle\,\rangle) \text{ and } \mathbf{T}_{G^*} = (G^*, \emptyset, \{G^*\}, \emptyset, \langle\,\rangle)\,,$$

where $\langle . \rangle$ denotes the generated equivalence relation defined on the set of matroids occurring in the respective tableau. We may derive the extended joint tableau

$$\mathbf{T}_1 = [\mathbf{T}_G \cup \mathbf{T}_{G^*}]_{\equiv} = (G, \emptyset, \{G, G^*\}, \emptyset, \langle G \simeq G^* \rangle)\,.$$

Now observe that although G is deflated, G^* is not deflated. We have

$$\begin{aligned} C_8^* &= \left\{ F \in \mathcal{F}\left(G^* | \{1,2,\ldots,7\}\right) \;\middle|\; 8 \in \mathrm{cl}_{G^*}(F) \right\} \\ &= \left\{ F \in \mathcal{F}\left(G^* | \{1,2,\ldots,7\}\right) \;\middle|\; \{1,2,3\} \subseteq F \right\}. \end{aligned}$$

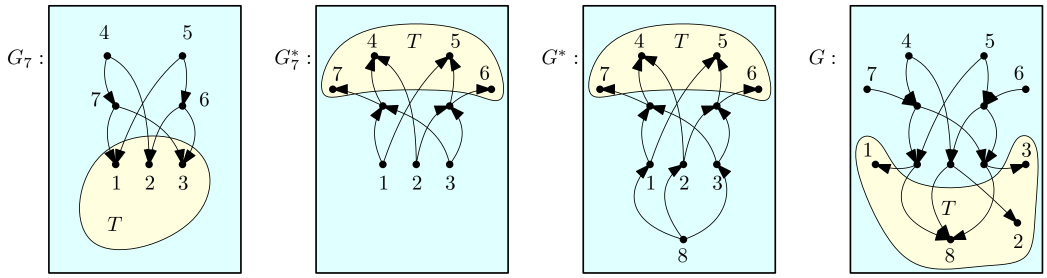

Figure 1: Reconstruction of a representation of $G_{8,4,1}$ from the matroid tableaux in Section 3.2.

Let $G_7^* = G^*|\{1,2,\ldots,7\}$. We have $\alpha_{G_7^*}(\{1,2,\ldots,7\}) = -1$, thus G_7^* is not a strict gammoid, and thus

$$\mathbf{T}_{G_7^*} = (G_7^*, \emptyset, \{G_7^*\}, \emptyset, \langle\,\rangle)$$

is a valid tableau. Since G_7^* is a deflate of G^*, each of them is an induced matroid with respect to the other. Therefore we may identify G^* and G_7^* in the joint tableau

$$\mathbf{T}_2 = \left(\mathbf{T}_1 \cup \mathbf{T}_{G_7^*}\right)(G^* \simeq G_7^*) = (G, \emptyset, \{G, G^*, G_7^*\}, \emptyset, \langle G \simeq G^* \simeq G_7^* \rangle)\,.$$

Now let $G_7 = (G_7^*)^*$, and we have $\alpha_{G_7} \geq 0$. Thus

$$\mathbf{T}_{G_7} = (G_7, \{G_7\}, \emptyset, \emptyset, \langle\,\rangle)$$

is a valid tableau. We now may derive the decisive tableau

$$\mathbf{T}_3 = [\mathbf{T}_2 \cup \mathbf{T}_{G_7}]_{\equiv} = (G, \{G_7\}, \{G, G^*, G_7^*, G_7\}, \emptyset, \langle G \simeq G^* \simeq G_7^* \simeq G_7 \rangle)$$

where case *(i)* of Definition 2.3 holds. Consequently, G is a gammoid.

4 Decision Procedure

We may start the procedure with the valid initial tableau $\mathbf{T} := (G, \emptyset, \emptyset, \emptyset, \langle\,\rangle)$, or any other valid tableau that we may have obtained using some heuristic or intuitive derivation. For instance, if $\mathbf{T}'$ is a tableau obtained for a different goal G', then the joint tableau $\mathbf{T} := (G, \emptyset, \emptyset, \emptyset, \langle\,\rangle) \cup \mathbf{T}'$ may be a better choice to start with, because G and G' may have common extensions and minors (up to isomorphy).

Step 1. If $\mathbf{T}$ is decisive, stop. If the procedure is run in a parallel fashion, you may choose to end a spawned thread here as long as there is another thread that carries on the computation.

Step 2. Choose an intermediate goal $M \in \{G' \mid G' \text{ is a minor of } G\} \cup \mathcal{M}$ such that $M \notin \mathcal{G} \cup \mathcal{X}$, preferably one with $M \simeq G$ which is small both in rank and cardinality. At this point, it is possible to spawn several parallel computations with multiple choices of M. In this case, all subsequent updates of $\mathbf{T}$ shall be considered atomic and synchronized.

Step 3. If $\mathbf{T}_M = (M, \emptyset, \emptyset, \emptyset, \langle\,\rangle) \cup \mathbf{T}$ is decisive, then set $\mathbf{T} := \left[[\mathbf{T} \cup (\mathbf{T}_M!)]_\equiv\right]_\simeq$ and continue with Step 1.

Step 4. Determine whether M has a minor that is isomorphic to $M(K_4)$. If this is the case, then $\mathbf{T}_M = (M, \emptyset, \emptyset, \{M, M^*\}, \langle\,\rangle)$ is valid, we set $\mathbf{T} := \left[[\mathbf{T} \cup \mathbf{T}_M]_\equiv\right]_\simeq$ and then continue with Step 1.

Since $M(K_4) = (M(K_4))^*$, we have that $M(K_4)$ is neither a minor of M nor of M^* when reaching the next step.

Step 5. Determine whether M has a minor that is isomorphic to $U_{2,4}$. If this is not the case, then $\mathbf{T}_M = (M, \{M, M^*\}, \emptyset, \emptyset, \langle\,\rangle)$ is valid, we set $\mathbf{T} := \left[[\mathbf{T} \cup \mathbf{T}_M]_\equiv\right]_\simeq$ and then continue with Step 1.

Since $U_{2,4} = (U_{2,4})^*$, we have that $U_{2,4}$ is neither a minor of M nor of M^* when reaching the next step.

Step 6. If $M \in \mathcal{M}$, continue immediately with Step 7. If $\alpha_M \geq 0$, then $\mathbf{T}_M = (M, \{M, M^*\}, \emptyset, \emptyset, \langle\,\rangle)$ is valid, so we may set $\mathbf{T} := \left[[\mathbf{T} \cup \mathbf{T}_M]_\equiv\right]_\simeq$ and continue with Step 1.

Step 7. If $M^* \in \mathcal{M}$, continue immediately with Step 8. If $\alpha_{M^*} \geq 0$, then $\mathbf{T}_{M^*} = (M^*, \{M, M^*\}, \emptyset, \emptyset, \langle\,\rangle)$ is valid, so we may set $\mathbf{T} := \left[[\mathbf{T} \cup \mathbf{T}_{M^*}]_\equiv\right]_\simeq$ and continue with Step 1.

Step 8. Determine whether M is strongly base-orderable. If this is not the case, then $\mathbf{T}_M = (M, \emptyset, \emptyset, \{M, M^*\}, \langle\,\rangle)$ is valid, we set $\mathbf{T} := \left[[\mathbf{T} \cup \mathbf{T}_M]_\equiv\right]_\simeq$ and then continue with Step 1.

The class of strong base-orderable matroids is closed under duality and minors [5], therefore M^* and all minors of M and M^* are strongly base-orderable upon reaching the next step.

Step 9. Let $M = (E,\mathcal{I})$. Determine whether there is some $X \in \mathcal{I}$ with $|X| = \mathrm{rk}_M(E) - 3$ and some $Y \subseteq E\backslash X$ such that $\alpha_{M.(E\backslash X)}(Y) < 0$. If this is the case, then the tableau $\mathbf{T}_M = (M, \emptyset, \emptyset, \{M, M^*\}, \langle\,\rangle)$ is valid, so we may set $\mathbf{T} := \left[[\mathbf{T} \cup \mathbf{T}_M]_\equiv\right]_\simeq$ and then continue with Step 1.

Step 10. Let $M^* = (E,\mathcal{I}^*)$. Determine whether there is some $X \in \mathcal{I}^*$ with $|X| = \mathrm{rk}_{M^*}(E) - 3$ and some $Y \subseteq E\backslash X$ such that $\alpha_{M^*.(E\backslash X)}(Y) < 0$. If this is the case, then the tableau $\mathbf{T}_{M^*} = (M^*, \emptyset, \emptyset, \{M, M^*\}, \langle\,\rangle)$ is valid, we may set $\mathbf{T} := \left[[\mathbf{T} \cup \mathbf{T}_{M^*}]_\equiv\right]_\simeq$ and then continue with Step 1.

Step 11. Determine whether M is deflated. If not, then find a deflate N of M with a ground set of minimal cardinality, set $\mathbf{T} := \left[\,[(\mathbf{T} \cup \mathbf{T}_N)\,(M \simeq N)]_\equiv\right]_\simeq$ where

$$\mathbf{T}_N = \begin{cases} (N, \{N, N^*\}, \emptyset, \emptyset, \langle\,\rangle) & \text{if } \alpha_N \geq 0, \\ (N, \emptyset, \{N\}, \emptyset, \langle\,\rangle) & \text{otherwise,} \end{cases}$$

and continue with Step 1.

Step 12. Determine whether M^* is deflated. If not, then find a deflate N of M^* with a ground set of minimal cardinality, set $\mathbf{T} := \left[\,[(\mathbf{T} \cup \mathbf{T}_N)\,(M^* \simeq N)]_\equiv\right]_\simeq$ where

$$\mathbf{T}_N = \begin{cases} (N, \{N, N^*\}, \emptyset, \emptyset, \langle\,\rangle) & \text{if } \alpha_N \geq 0, \\ (N, \emptyset, \{N\}, \emptyset, \langle\,\rangle) & \text{otherwise,} \end{cases}$$

and continue with Step 1.

Step 13. Try to find an extension N of M with at most $\mathrm{rk}_G(E)^2 \cdot |E| + \mathrm{rk}_G(E) + |E|$ elements such that N is not isomorphic to any $M' \in \mathcal{G} \cup \mathcal{M} \cup \mathcal{X}$. Set $\mathbf{T} := \left[\,[\mathbf{T} \cup \mathbf{T}_N]_\equiv\right]_\simeq$ where

$$\mathbf{T}_N = \begin{cases} (N, \{N, N^*\}, \emptyset, \emptyset, \langle\,\rangle) & \text{if } \alpha_N \geq 0, \\ (N, \emptyset, \{N\}, \emptyset, \langle\,\rangle) & \text{otherwise,} \end{cases}$$

and continue with Step 1, or repeat this step and add multiple extensions of M. If no such extension of M exists, then set $M := G$ and continue with Step 5.

Clearly, if we continue this process long enough, then Step 13 ensures that the tableau $\mathbf{T}$ will eventually become decisive for G by exhausting all isomorphism classes of extensions of G with at most $\mathrm{rk}_G(E)^2 \cdot |E| + \mathrm{rk}_G(E) + |E|$ elements.

If the procedure is carried out in a parallel fashion, not all spawned threads have to carry out Step 13, as long as it is guaranteed that the step is carried out again and again eventually by some threads.

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