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Orthogonality Axioms for Infinite Oriented Matroids

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Abstract

We introduce the notion of infinite oriented matroids by combining infinite matroids and the orthogonality axioms for finite oriented matroids. Furthermore, we exhibit some basic properties of such oriented matroids and show how a certain class of oriented matroids is characterized by the finite minors of its members.

1 Introduction

Recently, Bruhn et al. (3) presented axiomatic foundations for infinite matroids with duality, based on the work of Higgs (4) and Oxley (5, 6, 8). Duality is also a key element in the definition of a finite oriented matroid via the so called orthogonality axioms. It is thus a natural question to ask whether combining infinite matroids and the orthogonality axioms is a sensible way to define infinite oriented matroids. Therefore, after stating some preliminaries in section 2, in section 3 we start investigating this question by introducing the definition of an infinite oriented matroid in terms of the orthogonality axioms and by deriving some basic properties of such matroids. Based on the work of Bowler and Carmesin (2), we show that a certain class of infinite oriented matroids is characterized by the finite minors of their members. Finally, we take a closer look on the additional properties that oriented matroids add to circuit elimination in the underlying ordinary matroids. This allows us to briefly touch the subject whether the circuit axioms for finite oriented matroids can be reformulated in the infinite setting to obtain a cryptomorphic axiom system for infinite oriented matroids.

2 Preliminaries

2.1 Notation and Terminology

Any notation and terminology regarding matroids and oriented matroids not explained below is taken from Oxley (7) and Björner et al. (1), respectively. In the following, we will always denote a(n) *(oriented) matroid* by M and its finite or infinite *ground set* by E. If $X \subseteq E$, then we will denote its complement $E \setminus X$ by $\overline{X}$. If $X \subseteq E$ and $e \in E$, then we will abbreviate $X \cup \{e\}$ to $X \cup e$ and $X \setminus \{e\}$ to $X \setminus e$. We denote the *power set* of E by 2^E. A *signed subset* X of E is a subset $\underline{X}$ of E together with a partition (X^+, X^-) of $\underline{X}$ where X^+ contains the so called *positive elements* of X and X^- the so called *negative elements* of X. The set $\underline{X}$ is called the *support* of X. If $A \subseteq E$ and X is a signed subset of E, then the *restriction* of X to A is the signed subset $X|_A$ where $X|_A^+ = X^+ \cap A$ and $X|_A^- = X^- \cap A$. If X is a signed subset of E, then we write $X(e) = 1$ if $e \in X^+$ and $X(e) = -1$ if $e \in X^-$. The *separator* of two signed subsets X, Y of E is defined as $\operatorname{sep}(X, Y) = (X^+ \cap Y^-) \cup (X^- \cap Y^+)$. The *opposite* of a signed subset X of E is the signed subset $-X$ where $-X^+ = X^-$ and $-X^- = X^+$. Finally, if X is a signed subset of E and $A \subseteq E$, then we say that the signed subset ${}_{-A}X$ where ${}_{-A}X^+ = (X^+ \setminus A) \cup (X^- \cap A)$ and ${}_{-A}X^- = (X^- \setminus A) \cup (X^+ \cap A)$ is obtained from X by *reorientation* on A.

2.2 Infinite Matroids

We briefly recall one of the cryptomorphic definitions of an (infinite) matroid of Bruhn et al. (3), namely the circuit axioms.

Definition 2.1. A set $\mathcal{C} \subseteq 2^E$ is the set of circuits of a matroid M on a set E if and only if it satisfies the following **circuit axioms**:

(C1) $\emptyset \notin \mathcal{C}$.
(C2) No element of $\mathcal{C}$ is a subset of another.
(C3) Whenever $X \subseteq C \in \mathcal{C}$ and $(C_x \mid x \in X)$ is a family of elements of $\mathcal{C}$ such that $x \in C_y \Leftrightarrow x = y$ for all $x, y \in X$, then for every $f \in C \setminus \left(\bigcup_{x \in X} C_x\right)$ there exists an element $D \in \mathcal{C}$ such that $f \in D \subseteq \left(C \cup \bigcup_{x \in X} C_x\right) \setminus X$.
(CM) Let $\mathcal{I} = \{I \subseteq E \mid C \nsubseteq I \text{ for all } C \in \mathcal{C}\}$. Then, whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{I' \in \mathcal{I} \mid I \subseteq I' \subseteq X\}$ has a maximal element.

If we want to emphasize to which matroid the set of circuits $\mathcal{C}$ belongs, then we will write $\mathcal{C}(M)$ instead of just $\mathcal{C}$.

Since circuit-cocircuit intersections of infinite matroids can be infinite, it is custom to make the following distinction.

Definition 2.2. A matroid is called **tame** if any circuit-cocircuit intersection is finite. Otherwise it is called **wild**.

Like in the finite case, the set of circuits of a contraction of a matroid can be directly characterized as follows.

Proposition 2.3. *Let M be a matroid on a set E and let $X \subseteq E$. Then the set of circuits $\mathcal{C}(M.X)$ of the contraction of M to X is given by*

$$\mathcal{C}(M.X) = \min(\{C \cap X \mid C \in \mathcal{C}(M), C \cap X \neq \emptyset\}).$$

Proof. "$\subseteq$": Let $C' \in \mathcal{C}(M.X)$. Then by (2, Lemma 2.3) there exists a $C \in \mathcal{C}(M)$ such that $C' \subseteq C \subseteq C' \cup \overline{X}$. This implies $C \cap X = C'$. Since $C'' \cap X$ is dependent in $M.X$ for any $C'' \in \mathcal{C}(M)$ by (3, Corollary 3.6) whenever $C'' \cap X \neq \emptyset$, there cannot exist a $C'' \in \mathcal{C}(M)$ such that $\emptyset \neq C'' \cap X \subset C \cap X = C'$. Thus $C \cap X$ is minimal.
"$\supseteq$": Let $C \in \mathcal{C}(M)$ such that $\emptyset \neq C \cap X$ is minimal. Then by (3, Corollary 3.6) $C \cap X$ is dependent in $M.X$ and there exists a $C' \in \mathcal{C}(M.X)$ such that $C' \subseteq C \cap X$. As shown in "$\subseteq$", there exists a $C'' \in \mathcal{C}(M)$ such that $C' = C'' \cap X \subseteq C \cap X$. Since $C \cap X$ is minimal, this implies that $C' = C \cap X$. □

Finally, the following well-known property of circuits and cocircuits of finite matroids carries over to the circuits and cocircuits of infinite matroids.

Lemma 2.4. *Let M be a matroid on E and $C \in \mathcal{C}$. For any two elements $e, f \in C$, there is a cocircuit D of M such that $C \cap D = \{e, f\}$.*

Proof. See (2, Lemma 2.2). □

3 Infinite Oriented Matroids

3.1 Orthogonality Axioms

The following definitions of Björner et al. (1) extend to infinite sets and infinite signed subsets without further modification.

Signed subsets can satisfy an abstract orthogonality property in the following sense.

Definition 3.1. Two signed subsets X, Y are said to be **orthogonal**, denoted by $X \perp Y$, if either $\underline{X} \cap \underline{Y} = \emptyset$, or the restrictions of X and Y to their intersection are neither equal nor opposite, i.e., there are $e, f \in \underline{X} \cap \underline{Y}$ such that $X(e)Y(e) = -X(f)Y(f)$.

Signed subsets are linked to the circuits and cocircuits of a matroid in the following way.

Definition 3.2. Let M be a matroid. A **circuit signature $\mathcal{C}$ of M** assigns to each circuit $\underline{C}$ of M two opposite signed subsets C and $-C$ supported by $\underline{C}$. A circuit signature $\mathcal{C}^*$ of M^* is also called a **cocircuit signature of M**.

Like in the finite case, oriented matroids are matroids whose circuits and cocircuits can be signed in such a way that the corresponding circuit and cocircuit signatures satisfy the orthogonality property from Definition 3.1.

Definition 3.3. Let M be a matroid on a set E. Let $\mathcal{C}$ be a circuit signature and $\mathcal{C}^*$ be a cocircuit signature. Then M is called an **oriented matroid on $\boldsymbol{E}$** if and only if $\mathcal{C}$ and $\mathcal{C}^*$ satisfy one of the following two equivalent conditions:

(O) $\forall C \in \mathcal{C}, U \in \mathcal{C}^* : C \perp U$,
(O') $\forall C \in \mathcal{C}, U \in \mathcal{C}^* : (C^+ \cap U^+) \cup (C^- \cap U^-) \neq \emptyset \Leftrightarrow (C^+ \cap U^-) \cup (C^- \cap U^+) \neq \emptyset$.

In this case we say that M and the pair $\mathcal{C}$ and $\mathcal{C}^*$ satisfy the orthogonality axioms and denote the set of **signed** (or **oriented**) **circuits and cocircuits** by $\mathcal{C}(M)$ and $\mathcal{C}^*(M)$, respectively. We write $\underline{M}$ instead of M if we want to refer to the ordinary matroid M without circuit/cocircuit orientation.

Example 3.4. All finite oriented matroids are oriented matroids in the sense of Definition 3.3.

As one would expect, matroids that are induced by (infinite) directed graphs are oriented matroids, as the following example explains.

Example 3.5. In (2), tame regular matroids are introduced. Such matroids M are signable, i.e. there exists a choice of functions $\sigma_C : C \to \{1, -1\}$ for each circuit C of M and for each cocircuit U a function $\varrho_U : U \to \{1, -1\}$ such that for any circuit C and cocircuit U

$$\sum_{e \in C \cap U} \sigma_C(e) \varrho_U(e) = 0,$$

where the sum is evaluated over $\mathbb{Z}$. Such a signing induces a circuit signature $\mathcal{C}$ of M if we also take the opposite of any signed circuit into consideration. In the same way, we obtain a cocircuit signature $\mathcal{C}^*$ of M. Furthermore, this pair of circuit and cocircuit signatures satisfies the orthogonality axioms 3.3. Any tame regular matroid is thus an oriented matroid. Examples of such oriented matroids are the finite cycle matroid, the algebraic cycle matroid, and the topological cycle matroid of a given graph (cf. (2, Subsection 5.3)).

3.1.1 Minors of Oriented Matroids

The following lemma shows that the class of (infinite) oriented matroids is closed under taking minors.

Lemma 3.6. *Let M be an oriented matroid on a set $E = X \dot\cup F \dot\cup G$ and $N = M/F \backslash G$ be an ordinary minor of M. Then $\mathcal{C}$ and $\mathcal{C}^*$ induce a circuit signature $\mathcal{C}'$ and a cocircuit signature $\mathcal{C}'^*$ on N with the following properties.*

1. *If $\underline{C'}$ is an ordinary circuit of N and C is an oriented circuit of M such that $\underline{C'} \subseteq \underline{C} \subseteq \underline{C'} \cup F$, then the orientations C' and $-C'$ of $\underline{C'}$ are given by $C' = C|_{\underline{C'}}$ and $-C' = C|_{\underline{C'}}$. Dually, if $\underline{U'}$ is an ordinary cocircuit of N and U is an oriented cocircuit of M such that $\underline{U'} \subseteq \underline{U} \subseteq \underline{U'} \cup G$, then the orientations U' and $-U'$ of $\underline{U'}$ are given by $U' = U|_{\underline{U'}}$ and $-U' = U|_{\underline{U'}}$.*

2. *The pair $\mathcal{C}'$ and $\mathcal{C}'^*$ satisfies the orthogonality axioms, i.e. N is an oriented matroid according to Definition 3.3.*

Proof. If $\underline{C'}$ is an ordinary circuit of N, then by (2, Lemma 2.3) there exists an oriented circuit C of M such that $\underline{C'} \subseteq \underline{C} \subseteq \underline{C'} \cup F$. Thus it is possible to orient $\underline{C'}$ as described in (1). This orientation of $\underline{C'}$ does not depend on the choice of C: Assume for a contradiction that there exists another oriented circuit D of M such that $\underline{C'} \subseteq \underline{D} \subseteq \underline{C'} \cup C$ and $C|_{\underline{C'}} \neq D|_{\underline{C'}}, -D|_{\underline{C'}}$. Then there exist $e, f \in \underline{C'}$ such that $C(e) = -D(e)$ and $C(f) = D(f)$. Lemma 2.4 lets us choose a cocircuit $\underline{U'}$ of N such that $\underline{C'} \cap \underline{U'} = \{e, f\}$. Again, by (2, Lemma 2.3), there exists an oriented cocircuit U of M such that $\underline{U'} \subseteq \underline{U} \subseteq \underline{U'} \cup D$. This implies $\underline{C'} \cap \underline{U'} = \underline{C} \cap \underline{U} = \underline{D} \cap \underline{U} = \{e, f\}$ which is not possible since both $C \perp U$ and $D \perp U$ hold. Thus $C|_{\underline{C'}} = D|_{\underline{C'}}$ or $C|_{\underline{C'}} = -D|_{\underline{C'}}$ must hold. The dual statement of (1) follows by applying the same arguments to the circuits of N^*.
To prove (2), let C' be an oriented circuit of N and U' be an oriented cocircuit of N. If C and U are a an oriented circuit and an oriented cocircuit of M such that $C' = C|_{\underline{C}}$ and $U' = U|_{\underline{U'}}$, then it follows from $\underline{C'} \cap \underline{U'} = \underline{C} \cap \underline{U}$ that $C' \perp U'$ holds. □

Given an (infinite) oriented matroid M on a set E and a set $X \subseteq E$, it is thus sensible to speak of the restriction minor $M|X$, the deletion minor $M \backslash \overline{X}$, and the contraction minor $M.X$ or $M/\overline{X}$ of M. Alternatively, minors can be characterized as follows.

Corollary 3.7. *Let M be an oriented matroid on a set E with set of signed circuits $\mathcal{C}(M)$, and let $X \subseteq E$.*

1. *The set $\mathcal{C}'(M|X) = \mathcal{C}'(M \backslash \overline{X}) = \{C \in \mathcal{C}(M) \mid \underline{C} \subseteq X\}$ is the set of circuits of an oriented matroid on X. This oriented matroid is called the* ***restriction of M to X****, denoted by $M|X$, or the minor obtained by* ***deleting $\overline{X}$****, denoted by $M \backslash \overline{X}$.*
2. *The set $\mathcal{C}'(M.X) = \mathcal{C}'(M/\overline{X}) = \min\{C|_X \mid C \in \mathcal{C}(M), \underline{C} \cap X \neq \emptyset\}$ is the set of circuits of an oriented matroid on X. This oriented matroid is called the* ***contraction of M to X****, denoted by $M.X$, or the minor obtained by* ***contracting $\overline{X}$****, denoted by $M/\overline{X}$.*

Proof. The statements follow directly from Lemma 3.6 and Proposition 2.3. □

3.1.2 Properties of Oriented Matroids

Let k be a field. In (2), Bowler and Carmesin showed that a tame matroid M is k-representable or even regular if every finite minor of M is. For tame oriented matroids there exists a similar excluded minors characterization.

Theorem 3.8. *Let M be a tame matroid. Then the following statements are equivalent.*

1. *M is an oriented matroid.*
2. *Every finite minor of M is a (finite) oriented matroid.*

Proof. It is an immediate consequence of Lemma 3.6 that (1) implies (2). Thus it remains to show that (2) implies (1). This can be proven by a compactness argument similar to the one in the proof of the implication "(3) $\Rightarrow$ (1)" of (2, Theorem 4.5). In the following, we will point out the main changes that are necessary to adapt this proof to our context. First, we let $k = \mathbb{F}_3$. Further, we identify the signs 1 and -1 with $1_{\mathbb{F}_3}$ and $-1_{\mathbb{F}_3}$, respectively. Finally, for each (ordinary) circuit o and (ordinary) cocircuit b of M we let

$$C_{o,b} = \left\{c \in (k^*)^H \mid (o \cap b \neq \emptyset) \Rightarrow (\exists e, f \in o \cap b : c(o,e)c(b,e) = -c(o,f)c(b,f))\right\}.$$

We then follow the steps in the proof of (2, Theorem 4.5) up to the point where the existence of a finite minor M' of M, which is an oriented matroid, has been deduced. Next, we have to show that $\bigcap_{(o,b)\in K} C_{o,b} \neq \emptyset$. This is done in our setting by choosing a suitable $c \in (k^*)^H$ in the following way: First, we note that the circuits o' and cocircuits b' of M' that appear in the proof of (2, Theorem 4.5) are oriented circuits and cocircuits in our case. We thus have to choose and fix an orientation for each such circuit and cocircuit. Then, returning to the proof of (2, Theorem 4.5), we let $c(o,e) = o'(e)$ for each $o \in O$ and $e \in o \cap F$, and $c(b,e) = b'(e)$ for each $b \in B$ and $e \in b \cap F$. This implies $c \in \bigcap_{(o,b)\in K} C_{o,b}$ and, finally, $\bigcap_{(o,b)\in \mathcal{C}(M)\times\mathcal{C}(M^*)} C_{o,b} \neq \emptyset$. To conclude the proof, we explain how a circuit and a cocircuit signature for M can be derived from an element $c \in \bigcap_{(o,b)\in \mathcal{C}(M)\times\mathcal{C}(M^*)} C_{o,b}$. Such a c contains exactly one orientation for each circuit and each cocircuit of M. All opposite orientations can thus be obtained by multiplying c by $1_{\mathbb{F}_3}$ componentwise. The pair of circuit/cocircuit signatures on M deduced in this way then satisfies both of the properties (O) and (O'). $\square$

Given an arbitrary circuit signature $\mathcal{C}$ of a matroid, it is not always possible to choose a cocircuit signature $\mathcal{C}^*$ such that $\mathcal{C}$ and $\mathcal{C}^*$ satisfy the properties (O) and (O'). But, if it is possible, then $\mathcal{C}^*$ is uniquely determined.

Proposition 3.9. *Let M be an oriented matroid with cocircuit signature $\mathcal{C}^*$. If $\tilde{\mathcal{C}}^*$ is another cocircuit signature that satisfies (O) or (O'), then $\tilde{\mathcal{C}}^* = \mathcal{C}^*$.*

Proof. Assume for a contradiction that there is a cocircuit $\underline{U}$ which is signed by $\mathcal{C}^*$ and $\tilde{\mathcal{C}}^*$ in different ways. Denote the cocircuit signed by $\mathcal{C}^*$ by U and the cocircuit signed by $\tilde{\mathcal{C}}^*$ by $\tilde{U}$. Then $U \neq \tilde{U}, -\tilde{U}$ and there exist elements $e, f \in \underline{U}$ such that $U(e) = \tilde{U}(e)$ and $U(f) = -\tilde{U}(f)$. By Lemma 2.4, it is possible to pick a circuit C such that $\underline{C} \cap \underline{U} = \{e, f\}$. Then either $C \perp U$ or $C \perp \tilde{U}$ holds, but not both, a contradiction. $\square$

Like in the finite case, when doing circuit elimination in oriented matroids, it is possible to keep elements contained in the intersection of circuits if they have the same sign. This property extends to infinite oriented matroids, where it must be taken into account that circuit elimination is not restricted to the case of eliminating single elements only.

Proposition 3.10. *Let M be an oriented matroid, $C \in \mathcal{C}(M), X \subseteq \underline{C}$, and $(C_x \mid x \in X)$ be a family of elements of $\mathcal{C}(M)$ such that $x \in \underline{C_y} \Leftrightarrow x = y, C_x \neq -C$, and $x \in \operatorname{sep}(C, C_x)$ for all $x, y \in X$. Then for every $f \in \underline{C} \setminus \left(\bigcup_{x\in X} \operatorname{sep}(C, C_x)\right)$ there exists a $D \in \mathcal{C}(M)$ such that $f \in \underline{D}, D(f) = C(f)$, and $\underline{D} \subseteq \left(\underline{C} \cup \bigcup_{x\in X} \underline{C_x}\right) \setminus X$.*

Proof. Let $F = \left(\underline{C} \cup \bigcup_{x\in X} \underline{C_x}\right)$. Since circuits of the minor $\underline{M}|F$ are also circuits of $\underline{M}$, it suffices to show that there exists a circuit $\underline{D} \in \mathcal{C}(\underline{M}|F)$ such that $f \in \underline{D}$ and $\underline{D} \cap X = \emptyset$. To do so, we will first prove that $X \cup f$ is coindependent in $(\underline{M}|F)^*$. Assume for a contradiction that $X \cup f$ contains a cocircuit $\underline{U}$ of $(\underline{M}|F)^*$. Then one of the following cases must hold.

1. $\underline{U} = \{f\}$: This case implies the contradiction $|\underline{C} \cap \underline{U}| = 1$.
2. $\underline{U} \subseteq X$: Similar to the previous case, this would imply the contradiction $|\underline{C_x} \cap \underline{U}| = 1$ for all $x \in \underline{U}$.
3. $\underline{U} = f \cup Y$ where $\emptyset \neq Y \subseteq X$: This case implies $f \in \underline{C_x}$ for all $x \in Y$ since otherwise there would exist an $x \in Y$ such that $|\underline{C_x} \cap \underline{U}| = 1$. By using reorientation we may assume that C is positive. Note that this implies $C(f) = C_x(f) = C(x) = 1$ and $C_x(x) = -1$ for all $x \in Y$. From Lemma 3.6, it follows that there exists an induced orientation U of the cocircuit $\underline{U}$. By replacing U by $-U$ if necessary, we may assume that $U_f = 1$. Since $C \perp U$, there must exist an $x \in Y$ such that $U(x) = -1$. But this contradicts $C_x \perp U$, since $\underline{C_x} \cap \underline{U} = \{f, x\}$.

It is thus possible to extend $X \cup f$ to a cobasis of $(\underline{M}|F)^*$ to obtain a basis $\mathcal{B}$ of $\underline{M}|F$ such that $\mathcal{B} \subseteq F \setminus (X \cup f)$. Finally, the fundamental circuit $\underline{D}$ of f with respect to $\mathcal{B}$ satisfies $f \in \underline{D}, \underline{D} \subseteq \left(\underline{C} \cup \bigcup_{x\in X} \underline{C_x}\right) \setminus X$, and by Lemma 3.6 either $D(f) = C(f)$ or $(-D)(f) = C(f)$ for an induced orientation D of $\underline{D}$. $\square$

Of course, if we restrict the setting of Proposition 3.10 to finite oriented matroids, it is known that among all such circuits D there must be at least one that additionally satisfies $D^+ \subseteq \left(C^+ \cup \bigcup_{x\in X} C_x^+\right) \setminus X$ and $D^- \subseteq \left(C^- \cup \bigcup_{x\in X} C_x^-\right) \setminus X$. Unfortunately, the construction used in Proposition 3.10 to obtain D does not imply this property, as the following example shows.

Example 3.11. If we construct the circuit D as described in the proof of Proposition 3.10, it is indeed possible that it does not satisfy $D^+ \subseteq \left(C^+ \cup \bigcup_{x\in X} C_x^+\right) \setminus X$ and $D^- \subseteq \left(C^- \cup \bigcup_{x\in X} C_x^-\right) \setminus X$. For example, consider the digraph shown in Figure 1 and its associated (finite) oriented matroid M with ground set $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. It

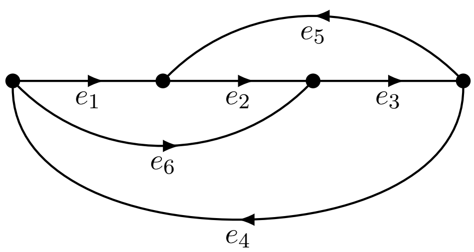

Figure 1: Digraph of Example 3.11

contains the oriented circuits $C = (1, 1, 1, 1, 0, 0)$ and $C_{e_1} = (-1, 0, 1, 0, 1, 1)$. Choosing $f = e_2$ and $X = \{e_1\}$, it is possible to extend $X \cup f = \{e_1, e_2\}$ to a cobasis of M^*, say $\{e_1, e_2, e_3\}$. Then we have to add e_2 to the basis $B = \{e_4, e_5, e_6\}$ to obtain the B-fundamental circuit $\underline{D} = \{e_2, e_4, e_5, e_6\}$. The orientations of this circuit are $D = (0, 1, 0, -1, 1, -1)$ and $-D = (0, -1, 0, 1, -1, 1)$, i.e. both D^- and $-D^-$ are not empty but $\left(C^- \cup C^-_{e_1}\right) \setminus X = \emptyset$.

In the setting of Proposition 3.10, let D be a circuit that does not satisfy $D^+ \subseteq \left(C^+ \cup \bigcup_{x \in X} C^+_x\right) \setminus X$ and $D^- \subseteq \left(C^- \cup \bigcup_{x \in X} C^-_x\right) \setminus X$, and let $e \in \underline{D}$ be one of the elements that make the inclusions fail. The following proposition shows that in this case it is at least possible to find another circuit that does neither include X nor the offending element e.

Proposition 3.12. *Let $M, C, X, (C_x \mid x \in X), f$, and D be as in Proposition 3.10. Denote the set $\left(C^+ \cup \bigcup_{x \in X} C^+_x\right)$ by A and the set $\left(C^- \cup \bigcup_{x \in X} C^-_x\right)$ by B. If there exists an $e \in (D^- \cap (A \setminus B)) \cup (D^+ \cap (B \setminus A))$, then there exists a $D' \in \mathcal{C}(M)$ such that $f \in \underline{D'}$, $D(f) = C(f)$, and $\underline{D'} \subseteq \left(\underline{C} \cup \bigcup_{x \in X} \underline{C_x}\right) \setminus (X \cup e)$.*

Proof. As in the proof of Proposition 3.10, we let $F = \left(\underline{C} \cup \bigcup_{x \in X} \underline{C_x}\right)$, and show that there exists a circuit $\underline{D} \in \mathcal{C}(\underline{M}|F)$ that has the desired properties. From the proof of Proposition 3.10, we know that $X \cup f$ is coindependent in $(\underline{M}|F)^*$. Thus it suffices to prove that $X \cup f$ stays coindependent in $(\underline{M}|F)^*$ if we add e to this set. Assume for a contradiction that $X \cup \{e, f\}$ contains a cocircuit $\underline{U}$ of $(\underline{M}|F)^*$. Then one of the following cases must hold.

1. $\underline{U} = \{e\}$: This case implies the contradiction $|\underline{C} \cap \underline{U}| = 1$.
2. $\underline{U} \subseteq Y \cup e$ where $\emptyset \neq Y \subseteq X$: Similar to the previous case, this would imply the contradiction $|\underline{D} \cap \underline{U}| = 1$.
3. $\underline{U} = \{e, f\}$: This case implies $e \in \underline{C}$. Let U be an induced orientation of the cocircuit $\underline{U}$. Since $C(f) = D(f)$ and $C(e) = -D(e)$, it cannot hold that U is orthogonal to both C and D, a contradiction.
4. $\underline{U} = Y \cup \{e, f\}$ where $\emptyset \neq Y \subseteq X$: Let U be an induced orientation of the cocircuit $\underline{U}$. From $U \perp D$ and $\underline{D} \cap \underline{U} = \{e, f\}$, it follows that $D(f)U(f) = -D(e)U(e)$. Since $C \perp U$, there must exist an $x \in Y \cup e$ such that $C(f)U(f) = -C(x)U(x)$. If $x = e$, then $C(e) = -D(e)$ and $C(f)U(f) = -C(e)U(e) = D(e)U(e) = -D(f)U(f) = -C(f)U(f)$, a contradiction. Thus $x \in Y$ must hold. Since $x \in \underline{C_x} \cap \underline{U} \subseteq \{e, f, x\}$, the circuit C_x includes e, f or both. Because of $C_x \perp U$ we conclude that at least one of the two equalities $C_x(x)U(x) = -C_x(e)U(e)$ and $C_x(x)U(x) = -C_x(f)U(f)$ must hold. The first equality implies $C_x(e) = -D(e)$ and thus $C_x(x)U(x) = D(e)U(e) = -D(f)U(f) = -C(f)U(f) = C(x)U(x) = -C_x(x)U(x)$, a contradiction. The second equality implies $C_x(x)U(x) = -C_x(f)U(f) = -C(f)U(f) = C(x)U(x) = -C_x(x)U(x)$, a contradiction again.

It follows that $X \cup \{e, f\}$ is coindependent in $(\underline{M}|F)^*$. To conclude the proof, we apply the same argument as in the proof of Proposition 3.10. □

The question whether there exists such a circuit D that additionally satisfies $D^+ \subseteq \left(C^+ \cup \bigcup_{x \in X} C^+_x\right) \setminus X$ and $D^- \subseteq \left(C^- \cup \bigcup_{x \in X} C^-_x\right) \setminus X$ in the setting of Proposition 3.10

is motivated by the following question: Do there exist cryptomorphic axiom systems of infinite oriented matroid similar to the ones of the finite case? In the case of the circuit axioms for finite oriented matroids, one has do deal with the problem of extending the (strong) circuit elimination axiom in a sensible way. The following conjecture about (infinite) oriented matroids implicitly states one possible extension of this axiom. Please also note how the circuit elimination axiom (C3) from the definition of (infinite) matroids 2.1 is incorporated into this reformulation.

Conjecture 3.13. Let M be an oriented matroid on a set E. Whenever $C \in \mathcal{C}(M), X \subseteq \underline{C}$, and $(C_x \mid x \in X)$ is a family of elements of $\mathcal{C}(M)$ such that $x \in \underline{C_y} \Leftrightarrow x = y$ and $x \in \operatorname{sep}(C, C_x)$ for all $x, y \in X$, then for every $f \in \underline{C} \setminus \left(\bigcup_{x \in X} \operatorname{sep}(C, C_x)\right)$ there exists a $D \in \mathcal{C}(M)$ such that $f \in \underline{D}, D^+ \subseteq \left(C^+ \cup \bigcup_{x \in X} C_x^+\right) \setminus X$, and $D^- \subseteq \left(C^- \cup \bigcup_{x \in X} C_x^-\right) \setminus X$.

If this conjecture is true, then the existence of a cryptomorphic axiom system for infinite oriented matroids based on an extension of the circuit axioms for finite oriented matroids would follow easily.

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