DISKRETE MATHEMATIK UND OPTIMIERUNG

Immanuel Albrecht and Winfried Hochstättler:

Lattice Path Matroids Are 3-Colorable

Technical Report feu-dmo039.15
Contact: immanuel.albrechtfernuni-hagen.de, winfried.hochstaettler@fernuni-hagen.de

FernUniversität in Hagen
Fakultät für Mathematik und Informatik
Lehrgebiet für Diskrete Mathematik und Optimierung
D – 58084 Hagen

2000 Mathematics Subject Classification: 05C15, 05B35, 52C40
Keywords: colorings, lattice path matroids, transversal matroids, oriented matroids

Lattice Path Matroids are 3-Colorable

Immanuel Albrecht Winfried Hochstättler

June 13, 2018

Abstract

We show that every lattice path matroid of rank at least two has a quite simple coline, also known as a positive coline. Therefore every orientation of a lattice path matroid is 3-colorable with respect to the chromatic number of oriented matroids introduced by J. Nešetřil, R. Nickel, and W. Hochstättler.

Keywords. colorings, lattice path matroids, transversal matroids, oriented matroids

Recently, in order to verify the generalization of Hadwiger's Conjecture to oriented matroids for the case of 3-colorability, Goddyn et. al. [3] introduced the class of generalized series parallel (GSP) matroids and asked whether it coincides with the class of oriented matroids without $M(K_4)$-minor. Furthermore, they showed that a minor closed class $\mathcal{C}$ of oriented matroids is a subclass of the GSP-matroids, if every simple matroid in $\mathcal{C}$ contains a flat of codimension 2, i. e. a coline, which is contained in more flats of codimension 1, i. e. copoints, with only one extra element, than in larger copoints. We call such a coline quite simple. They conjectured that every simple gammoid of rank at least 2 has a quite simple

---

I. Albrecht: FernUniversität in Hagen, Fakultät für Mathematik und Informatik, Lehrgebiet für Diskrete Mathematik und Optimierung, D-58084 Hagen, Germany; e-mail: Immanuel.Albrecht@fernuni-hagen.de

W. Hochstättler: FernUniversität in Hagen, Fakultät für Mathematik und Informatik, Lehrgebiet für Diskrete Mathematik und Optimierung, D-58084 Hagen, Germany; e-mail: Winfried.Hochstaettler@fernuni-hagen.de

Mathematics Subject Classification (2010): 05C15, 05B35, 52C40

coline. Gammoids may be characterized as the smallest class of matroids that is closed under minors and under duality, and which contains all transversal matroids – a class of matroids that is not closed under minors nor duals. Bicircular matroids form a minor closed subclass of the transversal matroids, and Goddyn et. al. [3] verified the existence of a quite simple coline in every simple bicircular matroid of rank at least 2.

Another minor closed subclass of the transversal matroids is the class of the lattice path matroids [2]. In this work we show that every simple lattice path matroids of rank at least 2 has a quite simple coline, which implies that orientations of lattice path matroids are GSP, and therefore we obtain the 3-colorability of every orientation of a lattice path matroid.

1 Preliminaries

In this work, we consider *matroids* to be pairs $M = (E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a system of independent subsets of E subject to the usual axioms ([4], Sec. 1.1). Furthermore, *oriented matroids* are considered triples $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$ where E is a finite set, $\mathcal{C}$ is a family of signed circuits and $\mathcal{C}^*$ is a family of signed cocircuits subject to the axioms of oriented matroids ([1], Ch. 3). Every oriented matroid $\mathcal{O}$ has a uniquely determined underlying matroid defined on the ground set E, which we shall denote by $M(\mathcal{O})$.[3]

Definition 1.1 ([3], Definition 4)**.** Let $M = (E, \mathcal{I})$ be a matroid. A flat $X \in \mathcal{F}(M)$ is called *coline of* M, if $\mathrm{rk}_M(X) = \mathrm{rk}_M(E) - 2$. A flat $Y \in \mathcal{F}(M)$ is called *copoint of* M *on* X, if $X \subseteq Y$ and $\mathrm{rk}_M(Y) = \mathrm{rk}_M(E) - 1$. If further $|Y \backslash X| = 1$, we say that Y is a *simple copoint on* X. If otherwise $|Y \backslash X| > 1$, we say that Y is a *multiple copoint on* X[4]. A *quite simple coline*[5] is a coline $X \in \mathcal{F}(M)$, such that there are more simple copoints on X than there are multiple copoints on X.

The following definitions are basically those found in J.E. Bonin and A. deMier's paper *Lattice path matroids: Structural properties* [2].

Definition 1.2. Let $n \in \mathbb{N}$. A *lattice path* of length n is a tuple $(p_i)_{i=1}^n \in \{\mathrm{N}, \mathrm{E}\}^n$. We say that the *i-th step* of $(p_i)_{i=1}^n$ is towards the North if $p_i = \mathrm{N}$, and towards the East if $p_i = \mathrm{E}$.

[3]The underlying matroid is the only notion from oriented matroids that is needed for the comprehension of this work.

[4]In [3] multiple copoints are called *fat copoints*.

[5]In [3] quite simple colines are called *positive colines*.

Definition 1.3. Let $n \in \mathbb{N}$, and let $p = (p_i)_{i=1}^n$ and $q = (q_i)_{i=1}^n$ be lattice paths of length n. We say that p is *south of* q if for all $k \in \{1, 2, \ldots, n\}$,

$$\left|\left\{i \in \mathbb{N}\setminus\{0\} \;\middle|\; i \leq k \text{ and } p_i = \mathrm{N}\right\}\right| \leq \left|\left\{i \in \mathbb{N}\setminus\{0\} \;\middle|\; i \leq k \text{ and } q_i = \mathrm{N}\right\}\right|.$$

We say that p and q have *common endpoints*, if

$$\left|\left\{i \in \mathbb{N}\setminus\{0\} \;\middle|\; i \leq n \text{ and } p_i = \mathrm{N}\right\}\right| = \left|\left\{i \in \mathbb{N}\setminus\{0\} \;\middle|\; i \leq n \text{ and } q_i = \mathrm{N}\right\}\right|$$

holds. We say that the *lattice path* p *is south of* q *with common endpoints*, if p and q have common endpoints and p is south of q. In this case, we write $p \preceq q$.

Definition 1.4. Let $n \in \mathbb{N}$, and let $p, q \in \{\mathrm{E}, \mathrm{N}\}^n$ be lattice paths such that $p \preceq q$. We define the set of *lattice paths between* p *and* q to be

$$\mathrm{P}[p, q] = \left\{r \in \{\mathrm{N}, \mathrm{E}\}^n \;\middle|\; p \preceq r \preceq q\right\}.$$

Definition 1.5. A matroid $M = (E, \mathcal{I})$ is called *strong lattice path matroid*, if its ground set has the property $E = \{1, 2, \ldots, |E|\}$ and if there are lattice paths $p, q \in \{\mathrm{E}, \mathrm{N}\}^{|E|}$ with $p \preceq q$, such that $M = M[p, q]$, where $M[p, q]$ denotes the transversal matroid presented by the family $\mathcal{A}_{[p,q]} = (A_i)_{i=1}^{\mathrm{rk}_M(E)} \subseteq E$ with

$$A_i = \left\{j \in E \;\middle|\; \exists (r_j)_{j=1}^{|E|} \in \mathrm{P}[p, q]\colon\ r_j = \mathrm{N} \text{ and } \left|\{k \in E \mid k \leq j,\, r_k = \mathrm{N}\}\right| = i\right\},$$

i.e. each A_i consists of those $j \in E$, such that there is a lattice path r between p and q such that the j-th step of r is towards the North for the i-th time in total. Furthermore, a matroid $M = (E, \mathcal{I})$ is called *lattice path matroid*, if there is a bijection $\varphi\colon E \longrightarrow \{1, 2, \ldots, |E|\}$ such that $\varphi[M] = \left(\varphi[E], \left\{\varphi[X] \;\middle|\; X \in \mathcal{I}\right\}\right)$ is a strong lattice path matroid.

Example 1.6. (Fig. 1a) Let us consider the two lattice paths $p = (\mathrm{E}, \mathrm{E}, \mathrm{N}, \mathrm{E}, \mathrm{N}, \mathrm{N})$ and $q = (\mathrm{N}, \mathrm{N}, \mathrm{E}, \mathrm{N}, \mathrm{E}, \mathrm{E})$. We have $p \preceq q$ and the strong lattice path matroid $M[p, q]$ is the transversal matroid $M(\mathcal{A})$ presented by the family $\mathcal{A} = (A_i)_{i=1}^3$ of subsets of $\{1, 2, \ldots, 6\}$ where $A_1 = \{1, 2, 3\}$, $A_2 = \{2, 3, 4, 5\}$, and $A_3 = \{4, 5, 6\}$.

Theorem 1.7 ([2], Theorem 2.1)**.** *Let* p, q *be lattice paths of length* n*, such that* $p \preceq q$*. Let* $\mathcal{B} \subseteq 2^{\{1,2,\ldots,n\}}$ *consist of the bases of the strong lattice path matroid* $M = M[p, q]$ *on the ground set* $E = \{1, 2, \ldots, n\}$*. Let*

$$\varphi\colon \mathrm{P}[p, q] \longrightarrow \mathcal{B}, \quad (r_i)_{i=1}^n \mapsto \{j \in \mathbb{N} \mid 1 \leq j \leq n,\, r_j = \mathrm{N}\}.$$

Then φ *is a bijection between the family of lattice paths* $\mathrm{P}[p, q]$ *between* p *and* q *and the family of bases of* M*.*

Proof. Clearly, φ is well-defined: let $r = (r_i)_{i=1}^n \in \mathrm{P}[p,q]$, and let $m = \mathrm{rk}_M(E)$, then there are $j_1 < j_2 < \ldots < j_m$ such that $r_i = \mathrm{N}$ if and only if $i \in \{j_1, j_2, \ldots, j_m\}$. Thus the map

$$\iota_r \colon \varphi(r) \longrightarrow \{1, 2, \ldots, m\},$$

where $\iota_r(i) = k$ for k such that $i = j_k$, witnesses that the set $\varphi(r) \subseteq \{1, 2, \ldots, n\}$ is indeed a transversal of $\mathcal{A}_{[p,q]}$, and therefore a base of $M[p,q]$. It is clear from Definition 1.5 that φ is surjective. It is obvious that if we consider only lattice paths of a fixed given length n, then the indexes of the steps towards the North uniquely determine such a lattice path. Thus φ is also injective. □

Theorem 1.8 ([2], Theorem 3.1)**.** *The class of lattice path matroids is closed under minors, duals and direct sums.*

2 The Western Coline

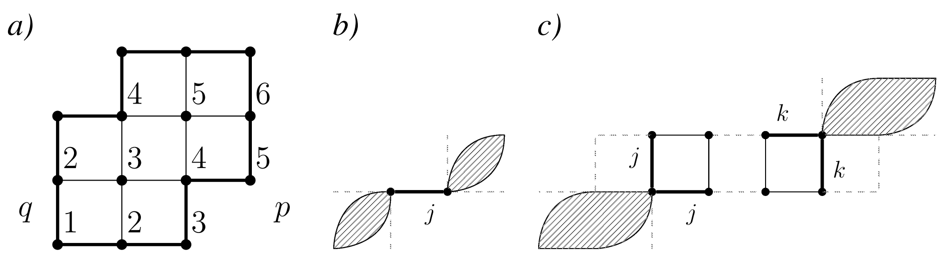


Figure 1: *a)* Lattice paths for Ex. 1.6, *b,c)* situation in Prop.2.1 *(ii)* and *(iii)*.

Proposition 2.1. *Let $p = (p_i)_{i=1}^n$, $q = (q_i)_{i=1}^n$ be lattice paths of length n such that $p \preceq q$. Let $j \in E = \{1, 2, \ldots, n\}$ and $M = M[p,q]$. Then*

(i) $\mathrm{rk}_M(\{1, 2, \ldots, j\}) = |\{i \in \{1, 2, \ldots, j\} \mid q_i = \mathrm{N}\}|$.

(ii) The element j is a loop in M if and only if

$$\left|\left\{i \in \{1, 2, \ldots, j-1\} \;\middle|\; p_i = \mathrm{N}\right\}\right| = \left|\left\{i \in \{1, 2, \ldots, j\} \;\middle|\; q_i = \mathrm{N}\right\}\right|,$$

i.e. the j-th step is forced to go towards East for all $r \in \mathrm{P}[p,q]$ (Fig. 1b).

(iii) For all $k \in E$ with $j < k$, j and k are parallel edges in M if and only if

$$\begin{aligned}\left|\left\{i \in \{1,2,\ldots,j-1\} \;\middle|\; p_i = \mathrm{N}\right\}\right| &= \left|\left\{i \in \{1,2,\ldots,k-1\} \;\middle|\; p_i = \mathrm{N}\right\}\right| \\ &= \left|\left\{i \in \{1,2,\ldots,j\} \;\middle|\; q_i = \mathrm{N}\right\}\right| - 1 \\ &= \left|\left\{i \in \{1,2,\ldots,k\} \;\middle|\; q_i = \mathrm{N}\right\}\right| - 1,\end{aligned}$$

i.e. the j-th and k-th steps of any $r \in \mathrm{P}[p,q]$ are in a common corridor towards the East that is one step wide towards the North (Fig. 1c).

Proof. For every $r \in \mathrm{P}[p,q]$, we have $r \preceq q$, therefore r is south of q, thus for all $k \in E$, $\left|\left\{j \in \{1,2,\ldots,k\} \;\middle|\; r_k = \mathrm{N}\right\}\right| \leq \left|\left\{j \in \{1,2,\ldots,k\} \;\middle|\; q_k = \mathrm{N}\right\}\right|$. Consequently, $\left\{i \in \{1,2,\ldots,j\} \;\middle|\; q_i = \mathrm{N}\right\}$ is a maximal independent subset of $\{1,2,\ldots,j\}$ and so statement *(i)* holds. An element $j \in E$ is a loop in M, if and only if $\mathrm{rk}_M(\{j\}) = 0$, which is the case if and only $\{j\}$ is not independent in M. This is the case if and only if for all bases B of M, $j \notin B$ holds, because every independent set is a subset of a base.The latter holds if and only if for all $(r_i)_{i=1}^n \in \mathrm{P}[p,q]$ the j-th step is towards the East, i.e. $r_j = \mathrm{E}$. This, in turn, is the case if and only if $\left|\left\{i \in \{1,2,\ldots,j-1\} \;\middle|\; p_i = \mathrm{N}\right\}\right| = \left|\left\{i \in \{1,2,\ldots,j\} \;\middle|\; q_i = \mathrm{N}\right\}\right|$. Thus statement *(ii)* holds, too. Let $j,k \in E$ with $j < k$. It is easy to see that if j and k are in a common corridor, then every lattice path $r = (r_i)_{i=1}^n$ of length n with $r_j = r_k = \mathrm{N}$ cannot be between p and q, i.e. $p \preceq r \preceq q$ cannot hold: a lattice path r with $r_j = r_k = \mathrm{N}$ is either below p at $j-1$ or above q at k. Thus $\{j,k\}$ cannot be independent in M. By *(i)*, neither j nor k can be a loop in M, thus j and k must be parallel edges in M. Conversely, let $j < k$ be parallel edges in M. Then j is not a loop in M, so there is a path $r^1 = (r^1_i)_{i=1}^n \in \mathrm{P}[p,q]$ with $r^1_j = \mathrm{N}$ which is minimal with regard to $\preceq$, and then

$$\left|\left\{i \in \{1,2,\ldots,j-1\} \;\middle|\; r^1_i = \mathrm{N}\right\}\right| = \left|\left\{i \in \{1,2,\ldots,j-1\} \;\middle|\; p_i = \mathrm{N}\right\}\right|.$$

Since j and k are parallel edges, $\{j,k\} \not\subseteq B$ for all bases B of M. Therefore there is no $r = (r_i)_{i=1}^n \in \mathrm{P}[p,q]$ such that $r_i = r_k = \mathrm{N}$. This yields the equation

$$\begin{aligned}\left|\left\{i \in \{1,2,\ldots,k\} \;\middle|\; q_i = \mathrm{N}\right\}\right| &= \left|\left\{i \in \{1,2,\ldots,j\} \;\middle|\; r^1_i = \mathrm{N}\right\}\right| \\ &= \left|\left\{i \in \{1,2,\ldots,j-1\} \mid r^1_i = \mathrm{N}\right\}\right| + 1.\end{aligned}$$

Since k is not a loop in M, it follows that

$$\left|\left\{i \in \{1,2,\ldots,j-1\} \;\middle|\; p_i = \mathrm{N}\right\}\right| = \left|\left\{i \in \{1,2,\ldots,j\} \;\middle|\; q_i = \mathrm{N}\right\}\right| - 1.$$

Thus *(iii)* holds. □

Lemma 2.2. *Let $p = (p_i)_{i=1}^n$ and $q = (q_i)_{i=1}^n$ be lattice paths of length n, such that $p \preceq q$, and such that $M = M[p,q]$ is a strong lattice path matroid on $E = \{1,2,\ldots,n\}$ which has no loops. Let $j \in E$ such that $q_j = \mathrm{N}$. Then*

$$\{1,2,\ldots,j-1\} = \mathrm{cl}_M\left(\{1,2,\ldots,j-1\}\right).$$

Furthermore, for all $k \in E$ with $k \geq j$,

$$\mathrm{rk}_M\left(\{1,2,\ldots,j-1\} \cup \{k\}\right) = \mathrm{rk}_M\left(\{1,2,\ldots,j-1\}\right) + 1.$$

Proof. By Proposition 2.1 *(i)*, we have

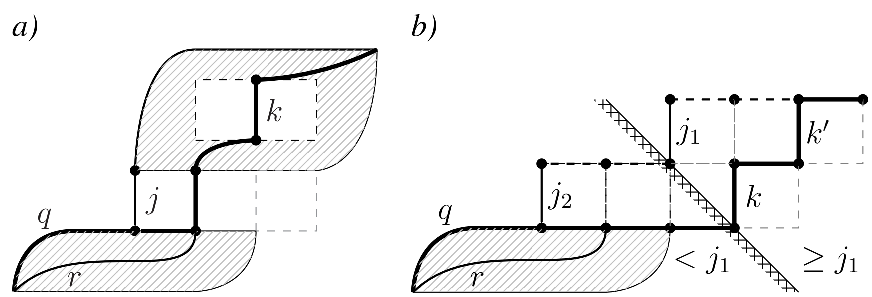
$$\mathrm{rk}_M(\{1,2,\ldots,j-1\}) = \left|\left\{i \in \{1,2,\ldots,j-1\} \;\middle|\; q_i = \mathrm{N}\right\}\right|.$$

Now fix some $k \in E$ with $k \geq j$. Since M has no loop, there is a base B of M with $k \in B$ and thus a lattice path $r = (r_i)_{i=1}^n \in \mathrm{P}[p,q]$ with $r_k = \mathrm{N}$ (Theorem 1.7). We can construct a lattice path $s = (s_i)_{i=1}^n \in \mathrm{P}[p,q]$ that follows q for the first $j-1$ steps, then goes towards the East until it meets r, and then goes on as r does (Fig. 2a). The base $B_s = \{i \in E \mid s_i = \mathrm{N}\}$ that corresponds to the constructed path yields

$$\begin{aligned}\mathrm{rk}_M(\{1,2,\ldots,j-1\} \cup \{k\}) &\geq \left|\left(\{1,2,\ldots,j-1\} \cup \{k\}\right) \cap B_s\right| \\ &= 1 + \left|\left\{i \in \{1,2,\ldots,j-1\} \;\middle|\; q_i = \mathrm{N}\right\}\right| \\ &= 1 + \mathrm{rk}_M(\{1,2,\ldots,j-1\}).\end{aligned}$$

Since rk_M is unit increasing, adding a single element to a set can increase the rank by at most one, thus the inequality in the above formula is indeed an equality.

Figure 2: The lattice paths s in the proof of *a)* Lem. 2.2 and *b)* Thm. 2.3.



This implies that $k \notin \mathrm{cl}_M(\{1, 2, \ldots, j-1\})$. Since k was arbitrarily chosen with $k \geq j$, we obtain $\{1, 2, \ldots, j-1\} = \mathrm{cl}_M(\{1, 2, \ldots, j-1\})$. $\square$

Theorem 2.3. *Let $p = (p_i)_{i=1}^n$, $q = (q_i)_{i=1}^n$ be lattice paths, such that $p \preceq q$ and such that $M = M[p, q] = (E, \mathcal{I})$ has no loop and no parallel edges, and $\mathrm{rk}_M(E) \geq 2$. Let $N_q = \{i \in E \mid q_i = \mathrm{N}\}$, $j_1 = \max N_q$, and $j_2 = \max N_q \backslash \{j_1\}$. Then the following holds*

(i) $\{1, 2, \ldots, j_2 - 1\}$ *is a coline of M, we shall call it the* Western coline of M.

(ii) $\{1, 2, \ldots, j_1 - 1\}$ *is a copoint on the Western coline of M, which is a multiple copoint whenever $j_1 - j_2 \geq 2$.*

(iii) For every $k \geq j_1$ the set $\{1, 2, \ldots, j_2 - 1\} \cup \{k\}$ is a simple copoint on the Western coline of M.

Proof. Lemma 2.2 provides that the set $W = \{1, 2, \ldots, j_2 - 1\}$ as well as the set $X = \{1, 2, \ldots, j_1 - 1\}$ is a flat of M. By construction of j_1 and j_2 we have that $\mathrm{rk}(W) = \mathrm{rk}(E) - 2$ and $\mathrm{rk}(X) = \mathrm{rk}(E) - 1$. Thus W is a coline of M — so *(i)* holds — and X is a copoint of M, which follows from and the construction of j_2 and j_1. Since $|X \backslash W| = |\{j_2, j_2 + 1, \ldots, j_1 - 1\}| = j_1 - j_2$ we obtain statement *(ii)*. Let $k \geq j_1$, and let $X_k = \{1, 2, \ldots, j_2 - 1\} \cup \{k\}$. Lemma 2.2 yields that $\mathrm{rk}(X_k) = \mathrm{rk}(E) - 1$, thus $\mathrm{cl}(X_k)$ is a copoint on the Western coline W. It remains to show that $\mathrm{cl}(X_k) = X_k$, which implies that X_k is indeed a simple copoint on W. We prove this fact by showing that for all $k' \geq j_1$, $\mathrm{rk}(X_k \cup \{k'\}) = \mathrm{rk}(E)$ by constructing a lattice path. Without loss of generality we may assume that $k < k'$. Since M has no loops and no parallel edges, there is a lattice path $r = (r_i)_{i=1}^n \in \mathrm{P}[p, q]$ with $r_k = r_{k'} = \mathrm{N}$. There is a lattice path $s = (s_i)_{i=1}^n \in \mathrm{P}[p, q]$ that follows q for the first $j_2 - 1$ steps, then goes towards the East until it meets r, and then goes on as r does (Fig. 2b). The constructed path s yields that

$$\begin{aligned}
\mathrm{rk}(X_k \cup \{k'\}) &\geq |(W \cup \{k, k'\}) \cap \{i \in E \mid s_i = \mathrm{N}\}| \\
&= 2 + |W \cap \{i \in E \mid q_i = \mathrm{N}\}| \\
&= 2 + \mathrm{rk}(W) = 1 + \mathrm{rk}(X_k) = 1 + \mathrm{rk}(X_{k'}),
\end{aligned}$$

where $X'_k = W \cup \{k'\}$. Thus $k' \notin \mathrm{cl}(X_k)$ and $k \notin \mathrm{cl}(X'_k)$. This completes the proof of statement *(iii)*. $\square$

Theorem 2.4. *Let $M = (E, \mathcal{I})$ be a strong lattice path matroid with $\mathrm{rk}_M(E) \geq 2$ such that $|E| = n$ and such that M has neither a loop nor a pair of parallel edges. Then either the Western coline is quite simple, or the element $n \in E$ is a coloop, and in the latter case there is either another coloop or $\mathrm{rk}_M(E) \geq 3$.*

Proof. If $j_1 \leq n-1$ as defined in Theorem 2.3, $W = \{1, 2, \ldots, j_2 - 1\}$ has at most a single multiple copoint and at least two simple copoints, therefore it is quite simple. Otherwise $j_1 = n$ is a coloop. If there is another coloop e_1, then $\{1, 2, \ldots, n-1\} \backslash \{e_1\}$ is a quite simple coline with two simple copoints. If n is the only coloop, the rank of M is 2, and there is no other coloop, then this would imply that there are parallel edges — a contradiction to the assumption that M is a simple matroid. □

3 Lattice Path Matroids are 3-Colorable

Corollary 3.1. *Every simple lattice path matroid $M = (E, \mathcal{I})$ with $\mathrm{rk}_M(E) \geq 2$ has a quite simple coline.*

Proof. Without loss of generality, we may assume that M is a strong lattice path matroid on $E = \{1, 2, \ldots, n\}$, and we may use j_1 and j_2 as defined in Theorem 2.3. From Theorem 2.4, we obtain the following: If $j_1 < n$, the Western coline is quite simple. Otherwise, if $j_1 = n$, then n is a coloop. If there is another coloop e_1, then $\{1, 2, \ldots, n-1\} \backslash \{e_1\}$ is a quite simple coline. If there is no other coloop, then we have $\mathrm{rk}_M(E) \geq 3$, and the contraction $M' = M|'E \backslash \{n\}$ is a strong lattice path matroid without loops, without parallel edges, and without coloops, such that $\mathrm{rk}_{M'}(E \backslash \{n\}) = \mathrm{rk}_M(E) - 1 \geq 2$. Thus the corresponding $j_1' < n-1$ and the Western coline W' of M' is quite simple in M' (Theorem 2.4). But then $\tilde{W} = W' \cup \{n\}$ is a coline of M, and $\tilde{X}$ is a copoint on $\tilde{W}$ with respect to M if and only if $X' = \tilde{X} \backslash \{n\}$ is a copoint on W' with respect to M'. Since $\left|\tilde{W} \backslash \tilde{X}\right| = |W' \backslash X'|$, we obtain that $\tilde{W}$ is a quite simple coline of M. □

Definition 3.2 ([3], Definition 2)**.** Let $\mathcal{O}$ be an oriented matroid. We say that $\mathcal{O}$ is *generalized series-parallel*, if every non-trivial minor $\mathcal{O}'$ of $\mathcal{O}$ with a simple underlying matroid $M(\mathcal{O}')$ has a $\{0, \pm 1\}$-valued coflow which has exactly one or two nonzero-entries.

Lemma 3.3 ([3], Lemma 5)**.** *If an orientable matroid M has a quite simple coline, then every orientation $\mathcal{O}$ of M has a $\{0, \pm 1\}$-valued coflow which has exactly one or two nonzero-entries.*

For a proof, see [3].

Remark 3.4. A simple matroid of rank 1 has only one element, no circuit and a single cocircuit consisting of the sole element of the matroid; so every rank-1 oriented matroid is generalized series-parallel. Observe that every simple matroid $M = (E, \mathcal{I})$ with $\mathrm{rk}_M(E) = 2$ is a lattice path matroid, as it is isomorphic to the strong lattice path matroid $M[p,q]$ where $p = (p_i)_{i=1}^{|E|}$ with

$$p_i = \begin{cases} \text{E} & \text{if } i < |E| - 2, \\ \text{N} & \text{otherwise,} \end{cases}$$

and where $q = (q_i)_{i=1}^{|E|}$ with

$$q_i = \begin{cases} \text{N} & \text{if } i \leq 2, \\ \text{E} & \text{otherwise.} \end{cases}$$

Therefore Lemma 3.3 and Corollary 3.1 yield that $\mathcal{O}$ has a $\{0, \pm 1\}$-valued coflow which has exactly one or two nonzero-entries. Consequently, every oriented matroid $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$ with $\mathrm{rk}_{M(\mathcal{O})}(E) \leq 2$ is generalized series-parallel.

Corollary 3.5. *All orientations of lattice path matroids are generalized series-parallel.*

Proof. Lemma 3.3, Remark 3.4, Theorem 1.8 and Corollary 3.1. □

Theorem 3.6 ([3], Theorem 3)**.** *Let $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$ be a generalized series-parallel oriented matroid such that $M(\mathcal{O})$ has no loops. Then there is a nowhere-zero coflow $F \in \mathbb{Z}.\mathcal{C}^*$ such that $|F(e)| < 3$ for all $e \in E$. Thus $\chi(\mathcal{O}) \leq 3$.*

For a proof, see [3].

Corollary 3.7. *Let $\mathcal{O}$ be an oriented matroid such that $M(\mathcal{O})$ is a lattice path matroid without loops. Then $\chi(\mathcal{O}) \leq 3$.*

Proof. Theorem 3.6 and Corollary 3.5. □

Acknowledgments. This research was partly supported by a scholarship granted by the FernUniversität in Hagen.

References

[1] A. Björner, M. LasVergnas, B. Sturmfels, N. White, and G. Ziegler. *Oriented Matroids*. Cambridge University Press, second edition, 1999.

[2] Joseph E. Bonin and Anna de Mier. Lattice path matroids: Structural properties. *European Journal of Combinatorics*, 27(5):701 – 738, 2006.

[3] Luis Goddyn, Winfried Hochstättler, and Nancy Ann Neudauer. *Discrete Mathematics*, 339(5):1425–1429, 2016.

[4] James Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.