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The game colouring number of powers of forests

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Abstract

We prove that the game colouring number of the m-th power of a forest of maximum degree $\Delta \geq 3$ is bounded from above by

$$\frac{(\Delta - 1)^m - 1}{\Delta - 1} + 2^m + 1,$$

which improves the best known bound by an asymptotic factor of 2.

Key words: game colouring number, marking game, activation strategy, graph power, forest, game chromatic number

MSC 2010: 05C15, 91A43, 05C05

1 Introduction

Graph colouring considers the problem to assign colours to the vertices of a given graph in such a way that adjacent vertices receive distinct colours. Classical graph colouring can be regarded as a one-player game, where the single player has the goal to colour every vertex in such a way that she uses a minimum number of colours. Competitive graph colouring considers the situation that there is a second player, too, who has the goal to increase the number of colours used. In a *maker-breaker graph colouring game*, the players are usually called Alice, who tries to minimize the number of colours, and Bob, who tries to maximize the number of colours. In the basic variant of such a game, popularised to the graph theory community by Bodlaender [5], the players move alternately. In each move they colour exactly one uncoloured vertex of the given graph the vertices of which are initially uncoloured. Alice begins. The most important parameter considered concerning this game is the so-called *game chromatic number*, which is the smallest number of colours that is suffcient to colour every vertex in case both players use optimal strategies.

The maximum game chromatic number for graphs from many interesting classes of graphs has been examined by many authors. The first class of graphs the maximum game chromatic number of which was determined were forests. The result is contained in the initial paper of Faigle et al. [8]. In order to prove that 4 is an upper bound for the game chromatic number of a tree, Faigle et al. used a so-called *activation strategy* for Alice. This type of strategy was named and generalized by Kierstead [9] and modified and used by many authors to obtain upper bounds for the game chromatic number of other classes of more complex graphs (some references can be found in the survey paper of Bartnicki et al. [4], some more, more recent references concerning graph colouring games can be taken from [2, 11]). A remarkable fact about the activation strategy is that it does not depend on the colours assigned to neighbours of uncoloured vertices but only on the number of coloured neighbours of uncoloured vertices. This fact motivated Zhu [12] to introduce the following *maker-breaker marking game* defining a graph parameter that is simultanously an upper bound for the game chromatic number and a competitive version of the colouring number named by Erdős and Hajnal [6]. The following rules of this marking game are very similar to those of Bodlaender's colouring game.

Alice and Bob alternately mark vertices of a given graph $G = (V, E)$ until every vertex is marked. The way they choose the vertices to be marked creates a linear ordering $\leq$ on the set V, where the smallest element is the first vertex that was marked and the largest the last vertex marked. The *back degree* $bd_{\leq}(v)$ of a vertex v with respect to the ordering $\leq$ is defined as the number of previously marked neighbours of v, i.e.

$$bd_{\leq}(v) := |\{w \in V \mid vw \in E, w \leq v\}|.$$

The *score* $sc(G, \leq)$ of G with respect to the linear ordering $\leq$ is defined by

$$sc(G, \leq) := 1 + \max_{v \in V} bd_{\leq}(v).$$

Alice's goal is to minimize the score, Bob tries to maximize it. Let $\leq^*$ be a linear ordering in case both players play according to optimal strategies. Then the *game colouring number* $\mathrm{col}_g(G)$ of G is defined as

$$\mathrm{col}_g(G) := sc(G, \leq^*).$$

For a non-empty class $\mathcal{C}$ of graphs we define

$$\mathrm{col}_g(\mathcal{C}) := \sup_{G \in \mathcal{C}} \mathrm{col}_g(G).$$

An application of game colouring number with regard to the graph packing problem was given by Kierstead and Kostochka [10].

In this paper we consider the game colouring number of the class of powers of forests.

We only consider finite, simple, and loopless graphs. The m-th *power* G^m of a graph $G = (V, E)$ is defined as the graph (V, E_m) with

$$E_m = \{vw \in \binom{V}{2} \mid 1 \leq \mathrm{dist}_G(v, w) \leq m\},$$

where the *distance* $\mathrm{dist}_G(v, w)$ denotes, as usual, the number of edges on a shortest path from v to w in G. In particular, we have $G^0 = (V, \emptyset)$ and $G^1 = G$. The *square* of G is the 2nd power G^2.

In order to examine the marking game on the power F^m of a forest F we will often argue with the forest F itself, which has the same vertex set as F^m. The vertex sets are identified in a canonical manner. It is useful to define a *k-neighbour* of a vertex v as a vertex w with $\mathrm{dist}_F(v, w) = k$. A *$k_\leq$-neighbour* of a vertex v is an ℓ-neighbour for some $1 \leq \ell \leq k$. Hence adjacency in F^m corresponds to $m_\leq$-neighbourhood in F.

Esperet and Zhu [7] and Yang [11] determined upper bounds for the game colouring number of squares of graphs depending on the maximum degree of the original graph. The first author and Theuser [3] generalized a global bound for graphs from the paper of Esperet and Zhu [7] from squares to arbitrary powers of graphs and obtained the following upper bound in the special case of powers of forests.

Theorem 1 (A., Theuser (2015)). *Let F be a forest with maximum degree $\Delta \geq 3$. Let $m \in \mathbb{N}$. Then we have*

$$\mathrm{col}_g(F^m) \leq 2\frac{(\Delta - 1)^m - 1}{\Delta - 2} + 2.$$

Here we will prove a bound which is better by factor ≈ 2 for large Δ.

Theorem 2. *Let F be a forest with maximum degree $\Delta \geq 3$. Let $m \in \mathbb{N}$. Then we have*
$$\mathrm{col}_g(F^m) \leq \frac{(\Delta - 1)^m - 1}{\Delta - 2} + 2^m + 1.$$

2 Proof of Theorem 2

In case $m = 1$, Theorem 2 specializes to the result of Faigle et al. [8] that the game colouring number of a forest is at most 4.

Let us give a brief review of the strategy for Alice Faigle et al. essentially used in order to prove this upper bound. Let F be a forest (of maximum degree Δ). During the game, a special set A of vertices, called *active* vertices is updated. At the beginning, $A = \emptyset$. Whenever a player marks the first vertex in a component T (which is a tree) of F, this vertex is activated and becomes root of the *tree of active vertices* of T, which is a rooted tree induced by the vertex set $V(T) \cap A$. We denote the tree of active vertices of the component T by T_A and its root by $r(T)$. In her first move, Alice marks an arbitrary vertex. Whenever Bob marks a vertex v in a component T, let w be the first active vertex on the path from v to $r(T)$ ($v = w$ might be possible). After Bob's move, every vertex on the path from v to $r(T)$ is *activated*, i.e. it becomes member of A. Alice's next move depends on whether $v = w$ or $v \neq w$. Note that $v = w$ if and only if $v = r(T)$ or v was active ($v \in T_A$) at the time v was marked by Bob. Alice uses the following strategy:

Alice's basic activation strategy:

Rule A1 If $v \neq w$ and w is unmarked, then Alice marks w.

Rule B Otherwise, Alice chooses a component tree T_0 that contains an unmarked vertex and, if $r(T_0)$ exists, she marks an unmarked vertex with smallest distance from $r(T_0)$, if $r(T_0)$ does not exist, she marks a vertex in T_0 (which will become $r(T_0)$).

It is easy to see that if Alice uses this strategy, every unmarked vertex has at most two active children. Therefore it has at most three marked (1-)neighbours, hence $\mathrm{col}_g(F) \leq 4$.

With Theuser [3] the first author applied this strategy to the underlying forest F of its m-th power F^m in order to prove Theorem 1. For this purpose we consider the game on F instead of F^m and have to count the maximal number of marked $m_{\leq}$-neighbours an unmarked vertex may have.

In the proof of Theorem 2 we use a modification of the activation strategy in [3]. The main difference is the additional rule A2, which gives us a significant improvement in the upper bound we establish.

Alice's refined activation strategy:

Rule A1 If $v \neq w$ and w is unmarked, then Alice marks w.

Rule A2 If $v \neq w$ and w is marked and there is an unmarked vertex on the path from w to $r(T)$, then Alice marks the first unmarked vertex on this path (i.e. the unmarked vertex on the path that is nearest to w).

Rule B Otherwise, Alice chooses a component tree T_0 that contains an unmarked vertex and, if $r(T_0)$ exists, she marks an unmarked vertex with smallest distance from $r(T_0)$, if $r(T_0)$ does not exist, she marks a vertex in T_0 (which will become $r(T_0)$).

Proof of Theorem 2. Let $m \geq 2$. Alice uses the strategy explained above. In the following arguments we consider the underlying forest F. We will show that at any time in the game after Alice's move the invariant holds that any unmarked vertex of F has at most

$$M_m := \frac{(\Delta - 1)^m - 1}{\Delta - 2} + 2^m - 1.$$

marked $m_{\leq}$-neighbours. Since Bob can increase the number of marked $m_{\leq}$-neighbours of an unmarked vertex in his next by at most one, this means that Alice can force a score of at most $M_m + 2$ in the marking game on the graph F^m.

We prove the validity of the invariant by induction on the number of moves. In Alice's first move the invariant obviously holds. Assume now it holds after some move of Alice. We consider the next pair of moves of Bob and Alice.

We use the same notions, namely the set of active vertices A, active rooted tree T_A with root $r(T)$ as in the description of the special case $m = 1$ above. To be able to argue more precisely we also consider T as rooted tree with root $r(T)$. In this rooted tree, for a vertex x, let $p(x)$ be the predecessor of x and $C(x)$ the set of children of x. For $k \geq 1$ we define the iterates

$$
\begin{aligned}
p^{k+1}(x) &:= p(p^k(x)), \\
C^0(x) &:= \{x\}, \\
C^{k+1}(x) &:= \bigcup_{y \in C^k(x)} C(y).
\end{aligned}
$$

For a vertex x, let $c_1, c_2, c_3, \ldots, c_{\deg(x)-1}$ be the children of x in the order they are activated in the course of the game; then we call c_i the *i-th active child* of x. In the following lemmata we use the notion of i-th active child even before the move after which it is activated.

The rules of the above refined activation strategy imply the following lemmata.

Lemma 3 (Consequence of Rule A1)**.** *At the time Bob marks a vertex in the subtree rooted in the second active child of a vertex x, the vertex x will be marked after Alice's move.*

Proof. We may assume that x is unmarked before Bob's move. By Rule B, Alice will never mark an inactive vertex that is not adjacent to a marked vertex. Therefore, when Bob, for the first time, marks a vertex v_B in the subtree rooted in the second active child c_2 of x, the vertex v_B is the first active vertex in the subtree rooted in c_2. Therefore the first active vertex on the path from v_B to the root $r(T)$ is the vertex x. By Rule A1, the vertex x will be marked in Alice's next move. $\square$

By contraposition, we conclude

Lemma 4. *After Alice's move, for any unmarked vertex u, there is at most one child $c \in C(u)$ of u such that in the rooted subtree of c (including c) there exists at least one marked vertex.* $\square$

Lemma 5 (Consequence of Rule A2)**.** *At the time Bob marks a vertex in the subtree rooted in the k-th active child of a vertex x, $k \geq 3$, the vertices $p(x), \ldots, p^{k-2}(x)$ will be marked after Alice's move.*

Proof. As above, by the rules of the game, Alice will never mark an inactive vertex that is not adjacent to a marked vertex. Therefore the first marked vertex v_i in the i-th child tree of x, $i = 2, \ldots, k$, must be marked by Bob. By Lemma 3, the vertex x will be marked after Alice's move immediately after Bob marked v_2. By induction on i, it follows from Rule A2, that $p^{i-2}(x)$ will be marked after Alice's move immediately after Bob marked v_i, $i = 3, \ldots, k$. $\square$

Let u be an unmarked vertex after Alice's move. By Lemma 4, at most two neighbours of u are marked, one child c_0 and the parent $p(u)$. We will determine

(i) an upper bound for the number of vertices in $V(T_A) \cap \bigcup_{k=1}^{m-1} C^k(u)$, namely

$$2^m - 1,$$

and

(ii) an upper bound for the number of vertices in the ancestor's part of the active tree with distance at most m from u, namely

$$\frac{(\Delta - 1)^m - 1}{\Delta - 2}.$$

Summing these two values obviously gives an upper bound for the number of marked $m_\leq$-neighbours of an unmarked vertex after Alice's move.

Bound in (i): This bound is proved by a series of lemmata. We first introduce a key notion. A *big vertex* is a vertex $z \in V(T_A) \cap \bigcup_{k=1}^{m-1} C^k(u)$ with the property

(V1) either z has $b \geq 3$ active children and was marked by Alice by Rule A1,

(V2) or z has $b \geq 2$ active children and was marked by Alice by Rule A2 or marked by Bob.

A *rabbit* of a big vertex z is an active child c of z which is in case (V1) neither the first nor the second active child of z and in case (V2) not the first active child of z.

Let S_1 resp. S_2 be the set of rabbits of some big vertex of type (V1) resp. type (V2). Let B_2 be the set of big vertices of type (V2). Let D_2 be the set of active vertices in $\bigcup_{k=1}^{m-2} C^k(u)$ with exactly one active child.

Arguing with Rule A2 we can prove

Lemma 6. (a) *Let $z \in C^k(u)$, $1 \leq k \leq m-1$, be a big vertex of type (V1). Then there exist $b-2$ vertices from $V(T_A) \cap \bigcup_{i=1}^{k-1} C^i(u)$ which were marked by Alice by Rule A2 when Bob marked a vertex in the subtree rooted in a rabbit of z.*

(b) *Let $z \in C^k(u)$, $1 \leq k \leq m-1$, be a big vertex of type (V2). Then there exist $b-1$ vertices from $V(T_A) \cap \bigcup_{i=1}^{k-1} C^i(u)$ which were marked by Alice by Rule A2 when Bob marked a vertex in the subtree rooted in a rabbit of z.*

Proof. Alice, by her strategy, will never mark a vertex in a subtree rooted in a rabbit of z unless every vertex on the path from z to $r(T)$ is marked (which does not hold since $u = p^k(z)$ is unmarked). Therefore, every rabbit will be created by Bob. Moreover in case (a), by Lemma 3, z will be marked by Alice before Bob creates the first rabbit. In case (b), again by Lemma 3, z will be marked by Bob or by Alice before Bob creates the first rabbit, otherwise z would not be of type (V2). Since z is marked, the path from z to u is active before the first

6

rabbit is created. Whenever Bob creates a rabbit, Alice marks a vertex on the path from $r(T)$. Since u is unmarked, this vertex, by Lemma 5, indeed must lie on the path from z to u, i.e. the vertex is in $V(T_A) \cap \bigcup_{i=1}^{k-1} C^i(u)$, which proves the lemma. $\square$

The preceding lemma helps us to prove the following key lemma of the proof.

Lemma 7. *There exists an injective mapping $f: S_1 \cup S_2 \longrightarrow B_2 \cup D_2$.*

Proof. The mapping f is defined by Alice's reaction on Bob's moves creating a rabbit of some big vertex: Each such rabbit is mapped to the vertex Alice marks immediately in the next move. By construction and the rules of the game, the mapping is injective. We only have to show that it is well-defined, i.e. that it maps to $B_2 \cup D_2$. According to the proof of Lemma 6 a vertex in the image $f(S_1 \cup S_2)$ lies in the set $\bigcup_{k=1}^{m-2} C^k(u)$. It suffices to show that such a vertex $y \in f(S_1 \cup S_2)$ will never become a big vertex of type (V1). But this follows from the fact that y is marked by Alice by rule A2, implying that y has at most one active child. Therefore y will be in D_2 as long as it has still one active child and become a big vertex of type (V2), i.e. a member of B_2, whenever a second child is activated. $\square$

Now we modify the part T' of the active tree T_A with vertex set $V(T') := V(T_A) \cap \bigcup_{k=1}^{m-1} C^k(u)$ in the following way. For every big vertex z and every rabbit c of z we delete the active subtree rooted in c, move it, and append it as a child of $f(c)$.

Lemma 8. *In the modified tree T_0 of the part T' of the active tree every vertex has at most two children, i.e. T_0 is a binary tree rooted in the unique active child of u.*

Proof. If $f(c) \in D_2$, the vertex $f(c)$ has exactly one active child before the modification and gets exactly another one after the modification. If $f(c) \in B_2$, it has exactly one active non-rabbit child before the modifcation. Since every subtree rooted in a rabbit of $f(c)$ is deleted, after the modification $f(c)$ has exactly two active children. This proves the lemma. $\square$

Lemma 9. *T_0 has the same number of vertices as the original part T' of the active tree and none of the moved vertices lies outside $\bigcup_{k=1}^{m} C^k(u)$.*

Proof. The first assertion follows since we do not delete subtrees, moreover, we move them. The second follows from the fact that we move them to a lower level in the binary tree, but not below the level of the root (the unique active child of u). $\square$

Corollary 10. *T_0 has at most $2^m - 1$ vertices.*

7

Proof. By Lemma 8, T_0 is a binary tree. By the second assertion of Lemma 9, T_0 has height at most $m - 1$. Therefore, at level i we have at most 2^i vertices, $i = 0, \ldots, m - 1$. Summing up, we get at most

$$\sum_{i=0}^{m-1} 2^i = 2^m - 1$$

vertices. $\qquad\square$

Corollary 11. *The number of marked $m_\leq$-neighbours of u in the child trees of u is at most*

$$\left| V(T_A) \cap \bigcup_{k=1}^{m} C^k(u) \right| \leq 2^m - 1. \tag{1}$$

Proof. By the first assertion of Lemma 9 and Corollary 10

$$\left| V(T_A) \cap \bigcup_{k=1}^{m} C^k(u) \right| = |V(T_0)| \leq 2^m - 1.$$

$\qquad\square$

Corollary 11 gives us the desired bound (i).

Bound in (ii): If we consider $p(u)$ and consider the tree T as rooted in u, then in the new "child" subtree rooted in $p(u)$ (which is the tree of foremothers and aunts and so on) there might be at most $(\Delta - 1)^{k-1}$ marked vertices at distance k from u in the new subtree. Therefore the number of marked vertices in the new subtree is at most

$$\sum_{k=0}^{m-1} (\Delta - 1)^k = \frac{(\Delta - 1)^m - 1}{\Delta - 2}. \tag{2}$$

Combining (i) and (ii), i.e. adding the bounds (1) and (2), in total the number of $m_\leq$-neighbours after Alice's move is at most

$$\frac{(\Delta - 1)^m - 1}{\Delta - 2} + 2^m - 1 = M_m.$$

This completes the proof of Theorem 2. $\qquad\square$

3   Open problems

In [3] a lower bound for the game colouring number of the class of m-th powers of forests of maximum degree Δ, based on an observation of Agnarsson and Halldórsson [1], is given, which is $\Omega(\Delta^{\lfloor \frac{m}{2} \rfloor})$. Therefore even the improved bound in Theorem 2 leaves a large asymptotic gap between lower and upper bound.

Problem 12. *Let $\mathcal{F}$ be the class of forests with maximum degree Δ and $m \in \mathbb{N}$. Determine* $\mathrm{col}_g(\{F^m \mid F \in \mathcal{F}\})$.

If $m = 2$ and $\Delta \geq 9$, the gap in Problem 12 was reduced by Esperet and Zhu [7] who proved that

$$\Delta + 1 \leq \mathrm{col}_g\{F^2 \mid F \in \mathcal{F}\} \leq \Delta + 3.$$

It might be that a generalization of the activation strategy can be applied to powers of members of graph classes with some tree decomposition structure.

Problem 13. *Let $\mathcal{T}_k$ be the class of k-trees with maximum degree Δ and $m \in \mathbb{N}$. Determine* $\mathrm{col}_g(\{G^m \mid G \in \mathcal{T}_k\})$.

More generally,

Problem 14. *Let $\mathcal{G}_k$ be the class of k-degenerate graphs with maximum degree Δ and $m \in \mathbb{N}$. Determine* $\mathrm{col}_g(\{G^m \mid G \in \mathcal{G}_k\})$.

Exact values for the game colouring number of powers of special forests are only known for large paths [3].

Problem 15. *Determine the exact values* $\mathrm{col}_g(F^m)$ *for all $m \in \mathbb{N}$ and interesting special forests F.*

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