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Abstract

We generalize the methods of Esperet and Zhu [6] providing an upper bound for the game colouring number of squares of graphs to obtain upper bounds for the game colouring number of m-th powers of graphs, $m \geq 3$, which rely on the maximum degree and the game colouring number of the underlying graph. Furthermore, we improve these bounds in case the underlying graph is a forest.

Key words: game colouring number, marking game, graph power, game chromatic number, forest

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1 Introduction

The game colouring number of a graph, introduced by Zhu [14], is a common tool to bound a game-theoretic graph colouring parameter, the game chromatic number, which was introduced by Bodlaender [3], the idea of the underlying game dates back to Steven J. Brams [8]. Though, the concept of game colouring number does not make use of the notion of colours, but is defined simply by the following marking game.

We are given a finite graph $G = (V, E)$. At the beginning every vertex $v \in V$ is unmarked. Two players, Alice and Bob, alternately mark vertices, one vertex in a turn, with Alice beginning. The game ends if every vertex is marked. The players thereby create a linear ordering L of the vertices.

The *back degree* $bd_L(v)$ of a vertex v (with respect to L) is the number of neighbours of v that precede v in L, i.e. the number of previously marked neighbours of v in the game. The *score* $sc(G, L)$ (with respect to L) is the maximum back degree over all vertices. Alice tries to minimize the score, Bob tries to maximize the score. The smallest score Alice can achieve with some strategy against every strategy of Bob is called *game score* $sc_g(G)$ of the graph G, i.e. the game score is the score obtained if both players use optimal strategies.

The *game colouring number* $\mathrm{col}_g(G)$ of G is defined as

$$\mathrm{col}_g(G) = 1 + sc_g(G).$$

For a nonempty class $\mathcal{C}$ of graphs we further define

$$\mathrm{col}_g(\mathcal{C}) = \sup_{G \in \mathcal{C}} \mathrm{col}_g(G).$$

The motivation to consider the value $sc_g(G)+1$ instead of $sc_g(G)$ comes from the observation that $\mathrm{col}_g(G)$ is an upper bound for the game chromatic number, cf. [14]. This is analog to the non-game case: the chromatic number $\chi(G)$ is bounded above by the colouring number $\mathrm{col}(G)$ of a graph G.

The *colouring number* of G, named by Erdős and Hajnal [5], is defined as

$$\mathrm{col}(G) = 1 + \min_{L \in \mathcal{L}} sc(G, L) = 1 + \min_{L \in \mathcal{L}} \max_{v \in V} bd_L(v),$$

where minimization ranges over the set $\mathcal{L}$ of all linear orderings of vertices, hence describing a situation without malicious adversary Bob.

Observation 1. *For any graph G,*

$$\mathrm{col}_g(G) \geq \mathrm{col}(G) \geq \chi(G).$$

All graphs considered in this paper are finite, simple, and loopless.

The m-th *power* of a graph $G = (V, E)$, denoted by G^m, is a graph (V, E_m) with $vw \in E_m$ if and only if $1 \leq \mathrm{dist}_G(v, w) \leq m$. In particular, $G^0 = (V, \emptyset)$ and $G^1 = G$. The 2nd power of a graph is also called *square* of a graph.

Esperet and Zhu [6] obtained the following upper bound for the game colouring number of the square of a graph G, which uses the game colouring number of G and the maximum degree of G.

Theorem 2 ((Esperet and Zhu (2009)))**.** *Let G be a graph with maximum degree Δ. Then*

$$\mathrm{col}_g(G^2) \leq (\mathrm{col}_g(G) - 1)(2\Delta - \mathrm{col}_g(G) + 1) + 1.$$

Using structural properties of specific classes of graphs, they improved the above result in some special cases as follows.

Theorem 3 ((Esperet and Zhu (2009))). *Let G be a graph with maximum degree Δ.*

(i) If G is a forest with $\Delta \geq 9$, then $\mathrm{col}_g(G^2) \leq \Delta + 3$.

(ii) If G is outerplanar, then $\mathrm{col}_g(G^2) \leq 2\Delta + 14$.

(iii) If G is planar, then $\mathrm{col}_g(G^2) \leq 23\Delta + 75$.

Theorem 3 (iii) was improved by Yang [13] in an asymptotic way.

Theorem 4 ((Yang (2009))). *There is a constant C such that, for any planar graph G with maximum degree Δ,*

$$\mathrm{col}_g(G^2) \leq 5\Delta + C.$$

In Section 3 we extend the methods of Esperet and Zhu and obtain a generalization of Theorem 2 to arbitrary powers of a graph. The upper bound is improved in the case of forests in Section 4. The exceptional trivial case of paths is solved completely for large paths. Apart from this result, the tightness of the bounds is a widely open problem, as discussed in Section 5.

2 Notation

In order to examine the marking game on the power G^m of a graph G we will often argue with the graph G itself, which has the same vertex set as G^m. The vertex sets are identified in a canonical manner. For the purpose of that reasoning we also introduce the following notions.

By the *distance* $\mathrm{dist}_G(v, w)$ of two vertices v and w we denote the number of edges on a shortest path between v and w in the original graph G. A *k-neighbour* of a vertex v is a vertex w with distance $\mathrm{dist}_G(v, w) = k$. A *$k_{\leq}$-neighbour* of a vertex v is a vertex w with distance $\mathrm{dist}_G(v, w) = \ell$ and $1 \leq \ell \leq k$.

We denote the path (resp. cycle) with n vertices by P_n (resp. C_n).

3 Upper bounds for powers of graphs

Using an inductive refinement of the methods of Esperet and Zhu [6] we prove the following

Theorem 5. *Let G be a graph with maximum degree $\Delta \geq 3$ and $m \in \mathbb{N}\backslash\{0\}$.*

(a) If $col_g(G) \in \{\Delta, \Delta+1\}$, then

$$\mathrm{col}_g(G^m) \leq \mathrm{col}_g(G) + \Delta(\mathrm{col}_g(G)-1)\frac{(\Delta-1)^{m-1}-1}{\Delta-2}.$$

(b) If $col_g(G) = \Delta - 1$, then

$$\mathrm{col}_g(G^m) \leq 1 - 2m + \Delta\frac{(\Delta-1)^m-1}{\Delta-2}.$$

(c) If $col_g(G) \leq \Delta - 2$, then

$$\mathrm{col}_g(G^m) \leq \underbrace{1 + \Delta\frac{(\Delta-1)^m-1}{\Delta-2} - (\Delta - \mathrm{col}_g(G)+1)\frac{(\Delta-\mathrm{col}_g(G))^m-1}{\Delta-\mathrm{col}_g(G)-1}}_{=:B(\Delta,\mathrm{col}_g(G),m)}.$$

Theorem 5 generalizes the result of Esperet and Zhu [6] (Theorem 2).

Remark 6. *Note that when $m = 2$ the bound in part (c) in Theorem 5 reduces to the bound of Theorem 2, since*

$$\begin{aligned} B(\Delta, \mathrm{col}_g(G), 2) &= 1 + \Delta^2 - (\Delta - \mathrm{col}_g(G) + 1)^2 \\ &= (\mathrm{col}_g(G) - 1)(2\Delta - \mathrm{col}_g(G) + 1) + 1. \end{aligned}$$

Proof of Theorem 5. For $m = 1$ the assertions are trivial.

Let $m \geq 2$. We let $k = \mathrm{col}_g(G) - 1$ and $D = \Delta - 1$. Alice uses the winning strategy to obtain a score of k, when playing the marking game on G. We prove that, if she uses the same strategy for the graph G^m, the score will be at most the upper bound given in the theorem minus one. Moreover, we prove that at any time of the game, any unmarked vertex has at most such many marked neighbours in G^m as this score.

Let v be an unmarked vertex at a certain time of the game. Let M_n resp. U_n be the number of marked resp. unmarked n-neighbours of v in G. We have to give an upper bound for

$$1 + \sum_{n=1}^{m} M_n.$$

We have

$$M_1 \leq k, \qquad U_1 \leq D + 1 - M_1.$$

Since in the graph G, for $2 \leq n \leq m$, any marked vertex at distance $n-1$ from v has at most $D = \Delta - 1$ marked neighbours at distance n from v and any unmarked vertex at distance $n-1$ from v has at most $k = \mathrm{col}_g(G) - 1$ marked neighbours at distance n from v, we have the recursion

$$M_n \leq M_{n-1} D + U_{n-1} k,$$

and, since the number of vertices which have positive distance n from v in G is bounded by $(D+1)D^{n-1}$, we have the recursion

$$U_n \leq (D+1)D^{n-1} - M_n.$$

Using the two previous recursions, we obtain the following:

$$M_n \leq M_{n-1}(D-k) + (D+1)D^{n-2}k. \tag{1}$$

Therefore, for $n \geq 2$, if $\mathrm{col}_g(G) \leq \Delta$, we have

$$\begin{aligned} M_n &\leq (D-k)^{n-1} M_1 + \sum_{i=0}^{n-2} (D+1) D^i k (D-k)^{n-2-i} \\ &\leq (D-k)^{n-1} k + \sum_{i=0}^{n-2} (D+1) D^i k (D-k)^{n-2-i} \\ &=: X_n. \end{aligned}$$

If $k < D$, i.e. if $\mathrm{col}_g(G) \leq \Delta - 1$, using the identity

$$x^{n-1} - y^{n-1} = (x-y) \sum_{i=0}^{n-2} x^i y^{n-2-i} \tag{2}$$

we have

$$\begin{aligned} X_n &\overset{(2)}{=} (D-k)^{n-1} k + (D+1) k \frac{D^{n-1} - (D-k)^{n-1}}{D - (D-k)} \\ &= (D-k)^{n-1} k + (D+1)(D^{n-1} - (D-k)^{n-1}) \\ &= (k - D - 1)(D-k)^{n-1} + (D+1) D^{n-1}. \end{aligned} \tag{3}$$

We remark that, even in case $n = 1$, we have

$$M_1 \leq k = (k - D - 1)(D-k)^{n-1} + (D+1) D^{n-1}.$$

Assuming $k < D$, by (3) we have

$$\begin{aligned}
\mathrm{col}_g(G^m) &\leq 1 + \sum_{n=1}^{m} M_n \\
&\leq 1 + \sum_{n=1}^{m} X_n \\
&= 1 + \sum_{n=0}^{m-1} ((k-D-1)(D-k)^n + (D+1)D^n) \\
&=: B_m.
\end{aligned}$$

Then, if $k < D-1$, i.e. if $\mathrm{col}_g(G) \leq \Delta - 2$, we have

$$\mathrm{col}_g(G^m) \leq B_m = 1 + (D+1)\frac{D^m - 1}{D-1} - (D+1-k)\frac{(D-k)^m - 1}{D-k-1},$$

which proves (c).

On the other hand, if $k = D-1$, i.e. if $\mathrm{col}_g(G) = \Delta - 1$, we have

$$\begin{aligned}
\mathrm{col}_g(G^m) &\leq B_m \\
&= 1 + \sum_{n=0}^{m-1} \left((\underbrace{k-D-1}_{=-2})(\underbrace{D-k}_{=1})^n + (D+1)D^n \right) \\
&= 1 - 2m + (D+1)\frac{D^m - 1}{D-1},
\end{aligned}$$

which proves (b).

If $k = D$, i.e. if $\mathrm{col}_g(G) = \Delta$, for $n \geq 2$, we have

$$M_n \leq X_n = (D+1)D^{n-2}k.$$

In case $k = D+1$, i.e. if $\mathrm{col}_g(G) = \Delta + 1$, for $n \geq 2$, from (1) we also have

$$\begin{aligned}
M_n &\leq (D+1)D^{n-2}k - M_{n-1} \\
&\leq (D+1)D^{n-2}k.
\end{aligned}$$

Therefore, if $k \in \{D, D+1\}$, i.e. if $\mathrm{col}_g(G) \geq \Delta$, we have

$$\mathrm{col}_g(G^m) \leq 1 + k + \sum_{n=2}^{m} M_n$$

$$\begin{aligned} &\leq 1 + k + (D+1)k \sum_{n=0}^{m-2} D^n \\ &= 1 + k + (D+1)k \frac{D^{m-1} - 1}{D-1}, \end{aligned}$$

which proves (a). □

4 Forests

In case the graph of the game is a power of a forest, Theorem 5 can be improved in the following way.

Theorem 7. *Let F be a forest with maximum degree $\Delta \geq 3$. Let $m \in \mathbb{N}$. Then we have*

$$\mathrm{col}_g(F^m) \leq \frac{2(\Delta-1)^m - 2}{\Delta - 2} + 2.$$

Proof. For $m = 0$, $\mathrm{col}_g(F^m) \leq 1$. For $m = 1$, $\mathrm{col}_g(F^m) \leq 4$, by a result of Faigle et al. [7]. Therefore the assertion holds for $m \in \{0, 1\}$.

Let $m \geq 2$. We will describe a strategy for Alice, so that Alice wins the marking game on the graph F^m with score

$$sc := \frac{2(\Delta-1)^m - 2}{\Delta - 2} + 1.$$

This strategy is a generalization of the standard activation strategy for forests (cf. [7, 9, 14]).

For the description of the strategy we use the forest F and consider the vertices of the power F^m as vertices of the underlying forest F. Each vertex that is the first vertex marked in a component T of F is called the *root* of the component tree T and denoted by $r(T)$. In her first move, Alice marks an arbitrary vertex. After that, Alice always marks a certain vertex (to be specified later in Rule A and B) in the same component Bob has marked a vertex in his previous move except if there is not any vertex left in the component. In the latter case Alice proceeds according to Rule B.

All vertices on paths in a component T between marked vertices and $r(T)$ are called *active* vertices. Whenever Bob marks a vertex v, Alice's answer depends on the position of v relative to the subtree T_A induced by the paths between active vertices before Bob's move. If $v \notin V(T_A)$, then let w be the first vertex in $V(T_A)$ on the path from v to $r(T)$.

As long as there are still unmarked vertices, Alice marks according the following rules.

Rule A If $v \notin V(T_A)$ and w is unmarked, then Alice marks w.

Rule B Otherwise, Alice chooses a tree T_0 that contains an unmarked vertex and, if $r(T_0)$ exists, she marks an unmarked vertex with smallest distance from $r(T_0)$, if $r(T_0)$ does not exist, she marks a vertex in T_0 (which will become $r(T_0)$).

We will show that at any time in the game after Alice's move any unmarked vertex has at most $sc - 1$ marked $m_{\leq}$-neighbours.

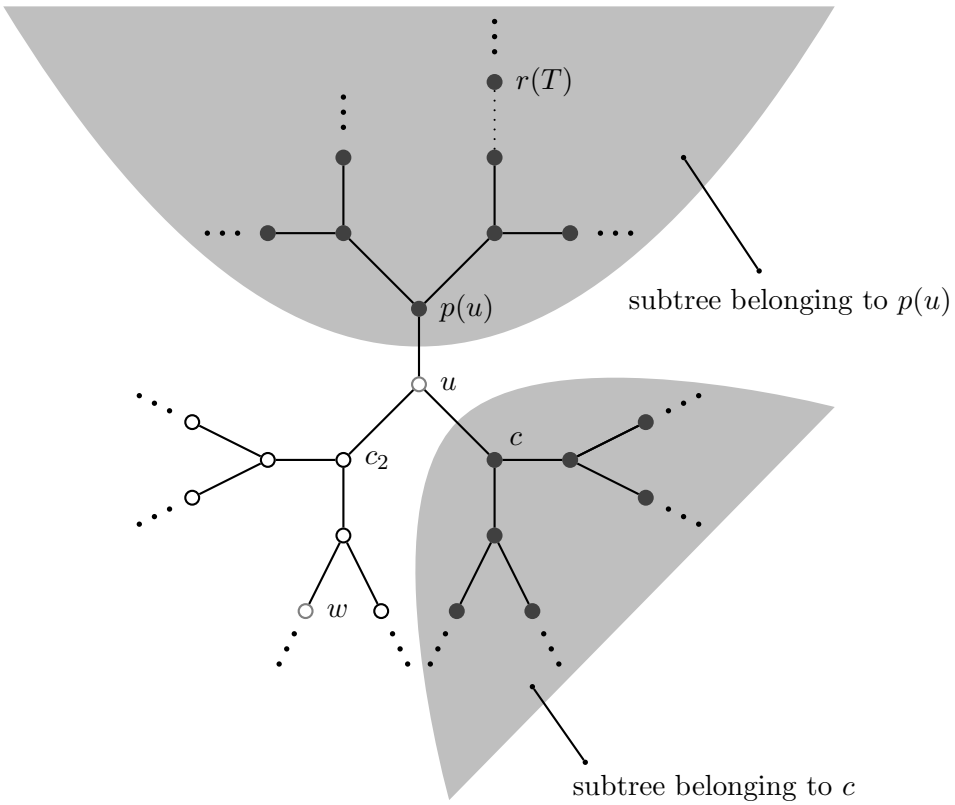

Figure 1: Strategy in the proof of Theorem 7: Immediately after Bob colours the vertex w in the uncoloured child tree belonging to c_2, Alice will colour the uncoloured vertex u

After Alice's first move this is certainly true. Assume it is true after

Alice's N-th move. After Bob's next move any unmarked vertex has at most sc marked $m_{\leq}$-neighbours. We consider the situation after Alice's $(N+1)$st move. Let u be an unmarked vertex in a certain component T of F. We consider T as rooted tree with root $r(T)$ and denote by $p(x)$ resp. $C(x)$ the predecessor resp. the set of children of a vertex x.

The rules of the above activation strategy imply the following lemma, the idea of which was implicitly used already in [7].

Lemma 8. *After Alice's move, for any unmarked vertex u, there is at most one child $c \in C(u)$ of u such that in the rooted subtree of c (including c) there are marked vertices.*

Proof. See [7, 9, 14]. □

See Fig. 1 for a typical situation of the game.

Using Lemma 8, there are at most

$$\sum_{k=0}^{m-1} (\Delta-1)^k = \frac{(\Delta-1)^m - 1}{\Delta - 2} \tag{4}$$

marked vertices which are $m_{\leq}$-neighbours of u in the child trees below u. On the other hand, if we consider $p(u)$ and consider the tree T as rooted in u, then in the new "child" tree rooted in $p(u)$ (which is the tree of foremothers and aunts and so on) there might be, analogously, at most

$$\frac{(\Delta-1)^m - 1}{\Delta - 2}$$

marked $m_{\leq}$-neighbours of u. In total we have at most

$$2\frac{(\Delta-1)^m - 1}{\Delta - 2} = sc - 1$$

marked $m_{\leq}$-neighbours of u, which proves the theorem. □

We remark that the precondition $\Delta \geq 3$ was used in the proof of Theorem 7 when applying the geometric series in (4). In the following, we consider some graphs with $\Delta = 2$.

Proposition 9. *Let $m, n \in \mathbb{N}$ with $n \geq 4m$. Then*

$$\mathrm{col}_g(P_n^m) = 2m + 1.$$

Proof. The case $m = 0$ is trivial. Let $m > 0$. The upper bound is obvious since the number of $m_{\leq}$-neighbours of a vertex in a path is at most $2m$. To prove the lower bound we describe a strategy of Bob to force a score of $2m$ in the marking game on the m-th power of the path P_n. To simplify the strategy we consider the game played on the underlying path P_n.

If $n \geq 4m + 1$, in his first $2m$ moves, Bob ensures that the first m and the last m vertices of the path are marked, leaving at least one unmarked interior vertex. The last interior vertex to be marked will have $2m$ marked $m_{\leq}$-neighbours.

Consider the case $n = 4m$. In his first $2m - 2$ moves, Bob ensures that the first $m - 1$ and the last $m - 1$ vertices of the path are marked. Let u_l resp. u_r be the m-th vertex counted from the left resp. the right end of the path. After Alice's next move there are still three unmarked vertices. In case at least one of u_l, u_r is marked, Bob marks the other one (if any, otherwise an arbitrary vertex) and leaves two interior unmarked vertices. The last vertex of them to be marked will have $2m$ marked $m_{\leq}$-neighbours in the P_n. In case both of u_l and u_r are unmarked, there is an unmarked interior vertex v. Either v has already $2m$ marked $m_{\leq}$-neighbours or $\mathrm{dist}_{P_n}(u_l, v) \leq m$ or $\mathrm{dist}_{P_n}(u_r, v) \leq m$, but not both. Assume w.l.o.g. $\mathrm{dist}_{P_n}(u_l, v) \leq m$. Then Bob marks u_l and v has $2m$ marked $m_{\leq}$-neighbours. Therefore $\mathrm{col}_g(P_n^m) \geq 2m + 1$. □

Proposition 9 is best possible in the following way.

Proposition 10. *Let $m, n \in \mathbb{N}$ with $n \leq 4m - 1$. Then*

$$\mathrm{col}_g(P_n^m) \leq 2m.$$

Proof. We describe a strategy of Alice on P_n that ensures that, at any time in the game, every unmarked vertex of P_n has at most $2m - 1$ marked $m_{\leq}$-neighbours. In her first move, Alice marks a central vertex v such that $P_n \setminus v$ decomposes into a left path L and a right path R, both containing at most $2m - 1$ vertices. Whenever Bob marks a vertex in L (resp. R), Alice marks the unmarked vertex in L (resp. R) with smallest distance to v.

Consider an unmarked vertex w in L at a certain time in the game. We will argue that to the left of w there are at most $m - 1$ marked vertices. Indeed, if there were at least m marked vertices to the left of w, then Alice would have marked at least $m - 1$ vertices in L to the right of w. But then the number of vertices in L would be at least $1 + m + m - 1 = 2m$ which contradicts $|V(L)| \leq 2m - 1$. The similar reasoning holds for R. Therefore at any time there are at most $m - 1$ marked vertices to the exterior of an unmarked vertex and at most m marked $m_{\leq}$-neighbours to the interior. □

Open problem 11. *For all values* $(m,n) \in \mathbb{N}^2$, $n \leq 4m-1$, *determine the exact value of* $\mathrm{col}_g(P_n^m)$.

Some partial results for Open Problem 11 can be found in [11].

We remark that the same idea as in the proof of Proposition 9 can be used to prove the following.

Proposition 12. *Let* $m,n \in \mathbb{N}$ *with* $n \geq 4m$. *Then*

$$\mathrm{col}_g(C_n^m) = 2m+1.$$

Proof. Since P_n^m is a spanning subgraph of C_n^m, when Bob uses his strategy from the proof of Proposition 9 to achieve a score of $2m$ in the marking game on P_n^m for the marking game on C_n^m, he will achieve a score of at least $2m$ because the additional edges might only increase the score but not decrease. Therefore $\mathrm{col}_g(C_n^m) \geq \mathrm{col}_g(P_n^m) \geq 2m+1$. On the other hand, obviously $\mathrm{col}_g(C_n^m) \leq 2m+1$. □

Open problem 13. *For all values* $(m,n) \in \mathbb{N}^2$, $n \leq 4m-1$, *determine the exact value of* $\mathrm{col}_g(C_n^m)$.

5 Remarks on tightness

From Theorem 5 we obtain immediately the following

Corollary 14. *Let* $m \in \mathbb{N} \setminus \{0\}$. *For a graph* G *from the class* $\mathcal{G}(\Delta,c)$ *of graphs with maximum degree* Δ *and* $\mathrm{col}_g(G) = c$ *we have*

$$\mathrm{col}_g(G^m) = O(\Delta^{m-1}c).$$

A lower bound was given by Agnarsson and Halldórsson [1]. Their construction of a complete $(\Delta-1)$-ary tree of height $\lfloor \frac{m}{2} \rfloor$ used to prove the tightness of their Theorem 3.1 shows

Theorem 15 ((Agnarsson and Halldórsson (2003)))**.** *There is a graph* G *with maximum degree* Δ *and*

$$\chi(G^m) = \Omega\left(\Delta^{\lfloor \frac{m}{2} \rfloor}\right).$$

Since, by Observation 1, $\mathrm{col}_g(G^m) \geq \mathrm{col}(G^m) \geq \chi(G^m)$, Theorem 15 implies for the class $\mathcal{G}(\Delta)$ of graphs with maximum degree Δ

$$\mathrm{col}_g(\{G^m \mid G \in \mathcal{G}(\Delta)\}) = \Omega\left(\Delta^{\lfloor \frac{m}{2} \rfloor}\right).$$

We cannot say anything better about the tightness of the bounds in Theorem 5 resp. Theorem 7 for the class of m-th powers of graphs with fixed maximum degree resp. for the class of m-th powers of forests with fixed maximum degree and $m \geq 3$. As already mentioned, Esperet and Zhu [6] improved the bound of Theorem 7 in the case $m = 2$ for forests with large maximum degree to the value $\Delta + 3$ (cf. Theorem 3 (i)).

It seems to be very hard to obtain reasonable tightness results for the game colouring number in the general cases of powers of graphs and even of forests considered in this paper. Similar large gaps between upper bounds and examples which provide lower bounds for the game colouring number of a class of graphs are known for the class of planar graphs which were studied in the last two decades using different sophisticated methods, cf. [2, 4, 9, 10, 12, 14, 15, 16].

Open problem 16. *For the class $\mathcal{G}(\Delta)$ of graphs with maximum degree Δ, what is the asymptotic behaviour of*

$$\mathrm{col}_g(\{G^m \mid G \in \mathcal{G}(\Delta)\})$$

when $\Delta \longrightarrow \infty$?

Open problem 17. *For the class $\mathcal{F}(\Delta)$ of forests with maximum degree Δ, what is the asymptotic behaviour of*

$$\mathrm{col}_g(\{G^m \mid G \in \mathcal{F}(\Delta)\})$$

when $\Delta \longrightarrow \infty$?

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