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On a base exchange game on bispanning graphs

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On a base exchange game on bispanning graphs

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Abstract

We consider the following maker-breaker game on a *bispanning graph* i.e. a graph that has a partition of the edge set E into two spanning trees E_1 and E_2 . Initially the edges of E_1 are purple and the edges of E_2 blue. Maker and breaker move alternately. In a move of the maker a blue edge is coloured purple. The breaker then has to recolour a different edge blue in such a way that the purple and the blue edges are spanning trees again. The goal of the maker is to exchange all colours, i.e. to make E_1 blue and E_2 purple. We prove that a sufficient but not necessary condition for the breaker to win is that the graph contains a K_4 . Furthermore we characterize the structure of a partition of a wheel into two spanning trees and show that the maker wins on wheels W_n with $n \geq 4$ and provide an example of a graph where, for some partitions, the maker wins, for some others, the breaker wins. We also describe an efficient algorithm for the recognition of bispanning graphs.

Keywords: maker-breaker game, bispanning graph, unique single element exchange, wheel, basis exchange

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1. Introduction

A graph $G = (V, E)$ is a *bispanning graph* if its edge set admits a partition $E = E_1 \dot{\cup} E_2$ into two spanning trees, i.e. such that (V, E_1) and (V, E_2) are

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trees. A *block matroid* is a matroid M on a ground set E such that $E = X_1 \dot{\cup} X_2$ for two bases X_1, X_2 of M . So a bispanning graph is a graphic block matroid. In this paper we often identify a graph G with its corresponding graphic matroid $M(G)$. For basic terminology on graphs resp. matroids we refer to [2] resp. [12].

Consider the following game which is played by two players, a *maker* Alice and a *breaker* Bob, on a bispanning graph $G = (V, E)$ given with a partition $E = E_1 \dot{\cup} E_2$ of the edge set into two spanning trees. During the game, some edges are in the dynamic set P of *purple edges*, the other edges in the dynamic set B of *blue edges*. Initially, $P = E_1$ and $B = E_2$. The players move alternately, the maker begins. A move of the maker consists in colouring a blue edge e purple, i.e. $P \rightarrow P \cup \{e\}$ and $B \rightarrow B \setminus \{e\}$. After that the breaker must colour a purple edge $f \neq e$ blue in such a way that the purple and blue edges each form a spanning tree again. If the maker can enforce that the purple and blue edges are completely exchanged in a finite number of steps, i.e. $P = E_2$ and $B = E_1$, the maker wins. Otherwise, i.e. if the breaker can achieve an infinite sequence of moves without reaching the winning configuration for Alice, the breaker wins. We call this game *base exchange game for bispanning graphs*.

This paper deals with the question: Given a bispanning graph $G = (V, E)$ and a partition of the edge set $E = E_1 \dot{\cup} E_2$ into a purple and a blue spanning tree, which player has a winning strategy for the game described above?

This game is motivated by questions about the connectivity of several matroid base exchange graphs in [14]. It seems to be surprisingly difficult to study the structure of these graphs already in the graphic case [1, 6]. Frequently, a block matroid sits at the heart of the problem.

Our results seem to suggest that the answer to the question of the existence of a winning strategy for the maker might be based solely on the question of connectivity of a certain derived graph, the so-called graph of left unique exchanges.

Graphs of similar types have occurred in several more general contexts on block matroids. Several authors [3, 10, 14] analyze the connectivity of some of these graphs. We will describe the four most important types of such graphs and resume results and conjectures on this connectivity problem in the following.

The graph $\tau_2(M)$ of a block matroid M has as vertices all pairs (B_1, B_2) of disjoint bases $E = B_1 \dot{\cup} B_2$. We have an edge $((B_1, B_2), (B'_1, B'_2))$ if and only if (B'_1, B'_2) arises from (B_1, B_2) by a *symmetric swap*, i.e. for some $e \in B_2$

and $f \in B_1$ we have

$$B'_1 = (B_1 \cup \{e\}) \setminus \{f\} \text{ and } B'_2 = (B_2 \setminus \{e\}) \cup \{f\}.$$

The graph $\tau_1(M)$ of a matroid M has as vertices its bases. We have an edge (B_1, B'_1) if and only if $B'_1 = (B_1 \cup \{e\}) \setminus \{f\}$. Note that identifying B_1 with $(B_1, E \setminus B_1)$, $\tau_2(M)$ maybe considered as induced subgraph of $\tau_1(M)$.

In order to explain the next graphs we recall some definitions from matroid theory. Let M be a matroid on the ground set E . If B is a basis and $e \notin B$, the *fundamental circuit* $C(B, e)$ is the unique circuit (i.e. minimum dependent set) contained in $B \cup e$. If B is a basis and $e \in B$, the *fundamental cocircuit* $D(B, e)$ is the set of all elements f , so that $(B \setminus \{e\}) \cup \{f\}$ is a basis.

The graph $\tau_4(M)$ of *left unique exchanges* has as vertices all pairs (B_1, B_2) of disjoint bases $E = B_1 \dot{\cup} B_2$. We have an edge $((B_1, B_2), (B'_1, B'_2))$ if and only if there exist $e \in B_2, f \in B_1$ such that

$$B'_1 = (B_1 \cup \{e\}) \setminus \{f\} \text{ and } B'_2 = (B_2 \setminus \{e\}) \cup \{f\}$$

and if $C(B_1, e)$ denotes the fundamental circuit and $D(B_2, e)$ the fundamental cocircuit, then

$$C(B_1, e) \cap D(B_2, e) = \{e, f\}. \quad (1)$$

The definition of the graph $\tau_3(M)$ of *unique exchanges* is the same as the definition for $\tau_4(M)$ except that (1) is replaced by

$$C(B_1, e) \cap D(B_2, e) = \{e, f\} \text{ or } D(B_1, f) \cap C(B_2, f) = \{e, f\}. \quad (2)$$

We have

$$E(\tau_4(M)) \subseteq E(\tau_3(M)) \subseteq E(\tau_2(M)) = E(\tau_1(M)[V(\tau_2(M))]) \subseteq E(\tau_1(M)).$$

We list some partial results and remaining open questions concerning the problem of connectivity of the τ_i .

It is clear from the basis axioms of a matroid that

Proposition 1. *For every block matroid M , $\tau_1(M)$ is connected.*

Proposition 2 (Farber, Richter Shank [3]). *For every graphic matroid M , $\tau_2(M)$ is connected.*

Both Propositions 1 and 2 state in particular that $\tau_1(G)$ and $\tau_2(G)$ are connected for every bispanning graph G .

Conjecture 3 (White [14], see also [1]). *For every block matroid M , $\tau_2(M)$ is connected.*

Conjecture 4. *For every regular block matroid M , $\tau_2(M)$ is connected.*

If M^* denotes the dual matroid of a matroid M , then obviously $\tau_2(M) \equiv \tau_2(M^*)$, hence Proposition 2 implies the connectivity of $\tau_2(M)$ for cographic block matroids M . The proof method used in [3] combined with Seymour's result [13], which says that a regular matroid is either graphic, cographic or a very special configuration, might help to find a proof that extends Proposition 2 to regular block matroids.

Not much is known about the structure of the graph $\tau_3(M)$. While $\tau_3(U_2^4)$ consists of 6 isolated vertices, Neil White [14] gives evidence for the following.

Conjecture 5 (White [14]). *For every regular block matroid M , $\tau_3(M)$ is connected.*

However, we do not even know whether $\tau_3(M)$ is connected for (the graphic matroid M of) a bispanning graph. An interesting side result of our paper is that $\tau_4(M)$ is not connected in general, even in the case of (a graphic matroid M of) a bispanning graph. Namely, the proof of Lemma 14 and Theorem 23 imply the following.

Theorem 6. *$\tau_4(M)$ is disconnected for the graphic matroid of K_4 , but connected for all larger wheels.*

On the other hand, McGuinness [10] proves that $\tau_4(M)$ has no isolated vertices if M is regular.

Block matroids and bispanning graphs are a classical subject of research. The first remarkable result on bispanning graphs seems to be the following theorem on the cyclic base order by Farber et al. [3] implying Proposition 2.

Theorem 7 (Wiedemann [15], see also [3, 8]). *Let G be a bispanning graph with a partition (B_1, B_2) into two spanning trees with r edges. Then the edges inside B_1 and B_2 can be ordered to lists \tilde{B}_1 resp. \tilde{B}_2 , so that any r consecutive edges in the cyclic order \tilde{B}_1, \tilde{B}_2 form a spanning tree of G .*

Van den Heuvel and Thomassé [5] brought to our attention that Gabow [4] and Wiedemann [15] conjecture that Theorem 7 can be generalized to block matroids (replacing spanning trees by matroid bases). This conjecture is still open for decades.

Now let us return to our game. White [14] defines the following matroid analogue of the graph game described above: On a block matroid with base pair (X_1, X_2) , the maker chooses $a \in X_1$ and the breaker must choose $b \in X_2$ such that $X_1 - a + b$ and $X_2 - b + a$ are new bases for the next move. If after a finite series of moves X_1 and X_2 are exchanged, the maker wins. We call this game $W(1)$. The graph $\tau_4(M)$ can be regarded as the graph of all moves the maker can enforce in game $W(1)$. In the game corresponding to $\tau_3(M)$, which we call $W(2)$, the maker is allowed to choose $a \in X_1$ or $b \in X_2$. The breaker then must recreate two new bases different from the bases of the previous move.

Our game is the special case of $W(1)$ for graphic matroids. Note that $W(2)$ is the same for graphic and cographic matroids. Our game has more strict rules than $W(2)$. The difference can be seen by the example of the K_4 (see Section 4). Here the breaker has a winning strategy for our game, but not for $W(2)$. Conjecture 5 would imply that for every regular matroid M the maker has a winning strategy for $W(2)$. Similarly, in all examples we know for the game $W(1)$ the maker has a winning strategy starting from (B_1, B_2) if and only if there is a path from (B_1, B_2) to (B_2, B_1) in $\tau_4(M(G))$.

The paper is organized as follows. An algorithm for the recognition of bispanning graphs is given in Section 2. In Section 3 we introduce basic terminology and results for the base exchange game on graphs. We prove that the breaker wins on bispanning graphs that contain a K_4 in Section 4. However, in Section 5 we show that a breaker-win graph does not necessarily contain a K_4 . Section 6 deals with the structure of partitions of the edge set of wheels into spanning trees. These results are needed for Section 7 in which we show that the maker has a winning strategy on wheels that are not the K_4 . In Section 8 we give an example of a bispanning graph in which the maker wins for some partitions, and the breaker for other. Some open questions, in particular on complexity issues, are discussed in Section 9.

2. The recognition of bispanning graphs

In this section we will describe an efficient algorithm for the recognition of bispanning graphs. It may be considered as a specialization of a matroid

intersection algorithm, with a slight difference to the classic one of Lawler [9]. Instead of augmenting a set that is independent in both matroids, we always keep a basis of one matroid and try to increase its rank in the second one using an alternating path. This is similar to the bipartite matching algorithm presented in [7]. Although using matroid terms we could recognize block matroids along the way, in the following we will present the algorithm and its proof in purely graph theoretic terms.

The basic idea of the algorithm is the following. For a given graph $G = (V, E)$ with suitable number of edges, we start with a spanning tree T and consider the complement $C_0 = E \setminus T$. Either the complement is already a spanning tree or, using a search tree, we either find an alternating path that enables us to modify the tree in such a way that we reduce the number of components of C_0 or we find a certificate that proves that G is not a bispanning graph.

Before we formulate the algorithm we need some notation. Let $G = (V, E)$ be a graph and $E' \subseteq E$. The *boundary* $\partial(E')$ of E' in G is the set of all edges whose terminal vertices lie in different components of the graph (V, E') . Let T be a spanning tree of G . The *fundamental cut* $D(T, e)$ for $e \in T$ is the set of all edges of $\partial(T \setminus e)$.

The main iteration of the recognition algorithm is the algorithm described in Fig. 1.

The routine **augment** (f, e_i) does the following. Backtracking the labels starting from e_i until an edge labeled 0+ is encountered, we find a sequence of edges which alternately is labeled with + and -. By the construction of the algorithm every +-labeled edge is in T and every --labeled edge and f are in $C_0 = E \setminus T$. Let S^+ resp. S^- be the +- resp. --labeled edges in the sequence. In the augmentation step we define

$$\begin{aligned} T' &:= (T \cup S^- \cup \{f\}) \setminus S^+ \\ C' &:= (C_0 \cup S^+) \setminus (S^- \cup \{f\}) \end{aligned}$$

This completes the description of the algorithm.

We have to show that after such an augmentation step T' is again a spanning tree and C' has a component less than C_0 . We prove this by induction on the number k of +-labeled edges in the sequence. Let w.l.o.g. $e_1, e_2, e_3, \dots, e_{2k-1}, f$ be the sequence in reverse order, starting with e_1 la-

Input: Graph $G = (V, E)$ with $|E| = 2|V| - 2$, $E = \{e_1, e_2, \dots, e_{2|V|-2}\}$

- (1) Determine a spanning tree T of G
- (2) $C_0 := E \setminus T$, $C^{Rest} := C_0$
- (3) if C_0 is connected, output “ G is bispanning graph”
- (4) Queue $Q := \emptyset$
- (5) label every edge $e \in \partial(C_0)$ with $0+$, add e to Q
- (6) while $Q \neq \emptyset$
 - (6a) remove the first element e_i from Q
 - (6b) if e_i is labeled with $+$:
 - (6b₁) if \exists cycle $C \subseteq C_0$ and $f \in D(T, e_i) \cap C$:
 - augment**(f, e_i) and stop
 - (6b₂) else
 - for every unlabeled $e \in D(T, e_i)$
 - label e with $i-$
 - add e to Q
 - $C^{Rest} := C^{Rest} \setminus \{e\}$
 - (6c) if e_i is labeled with $-$:
 - label every unlabeled $e \in \partial(C^{Rest})$ with $i+$, add e to Q

Output: either (A) a pair of spanning trees (T, C_0)
or (B) a new partition (T', C')
where C' has a component less than C_0
and T' is spanning tree
or (C) a component S of C^{Rest} that has a cycle

Figure 1: An algorithm for bispanning graph recognition

beled $0+$. Let $S_j = \{e_1, \dots, e_j\}$ for $1 \leq j \leq 2k$, where $e_{2k} = f$. Let

$$T_j = (T \cup (S_j \cap (S^- \cup \{f\}))) \setminus (S_j \cap S^+)$$

and

$$C_j = (C_0 \cup (S_j \cap S^+)) \setminus (S_j \cap (S^- \cup \{f\})).$$

In particular, $T_{2k} = T'$, $T_0 = T$ and $C_{2k} = C'$.

Claim 8. (a) For $0 \leq j \leq k$, T_{2j} is a spanning tree.

(b) For $1 \leq j \leq k$, C_{2j-1} has a component less than C_0 .

(c) The number of components of C_{2k-1} and C_{2k} is equal.

Proof. We prove part (a) and (b) of the claim by induction on j .

(a) T_0 is a spanning tree. Now, for $k \geq j > 0$, assume by induction that T_{2j-2} is a spanning tree. It suffices to prove that T_{2j} is acyclic. Assume it is not. Then e_{2j} must close a cycle C with $T_{2j-2} \setminus \{e_{2j-1}\}$. Recall e_{2j} got its $(2j-1)-$ label since $e_{2j} \in D(T, e_{2j-1})$. Hence, $T \setminus \{e_{2j-1}\} \cup \{e_{2j}\}$ is a tree and C must contain another even labeled element e_{2i} . We choose such with smallest possible index. Then $e_{2i} \in D(T, e_{2i-1})$ and hence $C \cap D(T, e_{2i-1}) \neq \emptyset$. Since cuts and circuits meet in an even number of elements, there must be another even labeled element $e_{2\ell}$ in the intersection. This contradicts the fact that, by construction, the sequence can meet every fundamental cut of T in at most one element.

(b) We prove this by induction on j . Since $e_1 \in \partial(C_0)$, $C_1 = C_0 \cup \{e_1\}$ has a component less than C_0 . Now let $j > 1$. By induction, C_{2j-3} has a component less than C_0 . By removing the $-$ -labeled e_{2j-2} , C_{2j-2} has again the same number of components as C_0 . By adding the $+$ -labeled $e_{2j-1} \in \partial C_{2j-2}$, C_{2j} has a component less than C_{2j-2} , i.e. a component less than C_0 .

(c) Since f is chosen in (6b₁) because it is a member of a cycle in C_0 , removing f from C_{2j-1} does not increase the number of components.

□

Claim 8 proves that in case of an augmentation step the algorithm ends with output (B). We will now prove that, if the algorithm terminates with output (C), the subgraph of unlabeled edges contains a subgraph that has too many edges to be a bispanner. We need a lemma and another claim.

Lemma 9. *Let G be a graph and $H = (V, E)$ be an induced subgraph of G with $|E| > 2|V| - 2$. Then G is not a bispanning graph.*

Proof. Assume G is a bispanning graph with disjoint spanning trees T_1 and T_2 . By the pigeon hole principle, $|T_1 \cap E| > |V| - 1$ or $|T_2 \cap E| > |V| - 1$. W.l.o.g. assume the first. But then $T_1 \cap E$ contains a cycle, which is a contradiction. \square

Claim 10. *Assume the algorithm terminates but not in (A) or (B). Let $H = (V'', E'')$ be the subgraph of G induced by the vertex set V'' of some component of C^{Rest} . Then $H_1 = (V'', T \cap E'')$ is connected, thus a spanning tree of H .*

Proof. Assume H_1 is disconnected. Let T' and T'' be two components of H_1 . Since $H_2 = (V'', C^{Rest} \cap E'')$ is connected, there is an edge $e \in C^{Rest} \cap E''$ that connects T' and T'' . On the other hand T' and T'' are connected by a path of T -edges (not lying in H). Let e_i be the first edge on the path from T' to T'' . So at a certain step of the algorithm, we had $e_i \in \partial(C^{Rest})$, therefore e_i was labeled with $+$. When e_i was taken from the queue Q , the algorithm considered the fundamental cut $D(T, e_i)$. Since $e \in D(T, e_i)$ and the algorithm did not produce an augmentation step, e was labeled with $-$. But this is a contradiction, since $e \in C^{Rest}$ and C^{Rest} contains only the unlabeled elements of C_0 . \square

In case the algorithm does not end with output (A) or (B) it ends when the Queue Q is empty. Then $C_0 = E \setminus T$ is not connected and hence must contain a cycle. This also holds for C^{Rest} , since no cycle has been broken in (6b₂). By Claim 10, a component of C^{Rest} with a cycle induces a subgraph $H = (V'', E'')$ of G , such that $(V'', T \cap E'')$ is a spanning tree. Since the number of edges of H is $|E''| \geq |V''| + |T \cap E''| \geq 2|V''| - 1$, H is not a bispanning graph. But H is an induced subgraph of G . By Lemma 9 this implies that G is not a bispanning graph. So the algorithm gives us a certificate (a component with a cycle) for the fact that G is not a bispanning graph.

Proposition 11. *Iterative application of the algorithm in Fig. 1 correctly recognizes a bispanning graph resp. a non-bispanning graph in cubic time.*

Proof. We repeat the algorithm as long as we have a partition into two spanning trees (using the augmented (T', C') to initialize the new (T, C_0)). The correctness of the algorithm follows from Claim 8 and Claim 10 and the preceding discussion.

Since in each iteration the number of components is reduced by one, we have at most $|V|$ iterations. In each iteration every edge is labeled at most once. For each labeled edge we have to determine some boundary, which is possible by depth-first-search in linear time. So the running time is $O(|E|^3) = O(|V|^3)$. \square

3. General results on the base exchange game

For the discussion of the game, for any bispanning graph G , we use an auxiliary graph, the *graph of left unique exchanges* $G^F = \tau_4(M(G))$. Recall that its vertices are all pairs (E_1, E_2) of disjoint spanning trees with $E = E_1 \dot{\cup} E_2$ of the bispanning graph $G = (V, E)$. We have an edge $((E_1, E_2), (E'_1, E'_2))$ if and only if there is an edge $e \in E_2$ such that, if it is coloured by the maker, there is only a single edge $f \in E_1$ the breaker may colour as feasible answer in such a way that $E'_1 = (E_1 \cup \{e\}) \setminus \{f\}$ and $E'_2 = (E_2 \setminus \{e\}) \cup \{f\}$. In this case the move is called *forced*, and *non-forced* otherwise.

The following obvious Proposition is the basis for our further analysis:

Proposition 12. *If G^F is connected, then the maker has a winning strategy for the base exchange game on the graph G for any starting partition into two spanning trees.*

The following proposition that the game is well-defined is the special case of the well-known symmetric base exchange property of matroid theory.

Proposition 13. *Let $G = (V, E)$ be a bispanning graph with a partition $E = P \dot{\cup} B$ of the edge set into two spanning trees. Then*

$$\forall b \in B \exists p \in P : ((P \setminus \{p\}) \cup \{b\}, (B \setminus \{b\}) \cup \{p\}) \text{ is a partition into trees.}$$

Proof. Let $C(P, b)$ denote the fundamental circuit of P and b and $D(B, b)$ the fundamental cocircuit which in the graphic case is the cut induced by the

two components of $B \setminus \{b\}$. Then $|C(P, b) \cap D(B, b)|$ is even, the intersection contains b and

$$\forall p \in C(P, b) \cap D(B, b) : ((P \setminus \{p\}) \cup \{b\}, (B \setminus \{b\}) \cup \{p\})$$

is a partition into trees. □

4. Graphs where the breaker wins

The main purpose of this section is to show that the breaker has a winning strategy if a bispanning graph contains the complete graph K_4 as a subgraph.

Lemma 14. (a) K_4^F has three components.

- (b) If (P, B) is an ordered partition of the edge set of K_4 into two trees, then (P, B) and (B, P) lie in different components of K_4^F .
- (c) If the maker plays the unique non-enforcing move the breaker has a feasible move that does not leave the component.

Proof. (a) Any partition of K_4 into two trees consists of two P_4 s, i.e. paths on four vertices. Let $abcd$ denote such a P_4 . Then the move of the breaker is forced if and only if the maker does not close a C_4 , i.e. plays edge ad . The forced moves (edges ac resp. bd) are indicated by curved lines and the resulting configurations of forced moves are listed in Fig. 2. Hence the component of the purple $abcd$ consists of purple $\{abcd, abdc, bacd, badc\}$ and hence of four ordered partitions. As the number of P_4 s in K_4 is $\frac{1}{2}4! = 12$ the claim follows by symmetry.

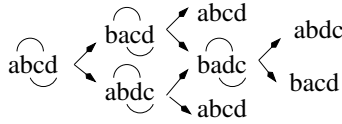


Figure 2: A component of K_4^F

- (b) The above analysis yields that the purple $bdac$ lies in a different component than $abcd$.
- (c) By symmetry, again, it suffices to consider the case that the starting configuration is a purple P_4 $abcd$ and the maker plays ad . Now the breaker recolours bc to blue which yields a purple $badc$ in the same component.

□

Summarizing the last Lemma implies:

Theorem 15. *The breaker has a winning strategy on the K_4 for any starting configuration.*

Theorem 16. *If a bispanning graph G contains a bispanning graph H as an induced subgraph, and the breaker has a winning strategy for H , then the breaker has a winning strategy for G .*

Proof. If a blue edge outside H is recoloured to purple by the maker, then recolouring a purple edge in H would mean that the blue graph in H has two edges more than the purple graph in H , therefore there is a blue cycle. So the answer on recolouring outside H must also be an edge outside H . Therefore the breaker can use his strategy on H , and if the maker plays in the complement, the breaker plays in the complement. In this way the spanning trees of H cannot be exchanged and thus the same holds for the global spanning trees. \square

Corollary 17. *For a bispanning graph that contains a K_4 as subgraph the breaker has a winning strategy.*

5. A breaker-win graph without K_4

By D_6 we denote the 2-sum of two K_4 s “without glueing edge” (see Fig. 3). Clearly, K_4 is a minor but not a subgraph of D_6 .

Theorem 18. *For any partition $E = E_1 \cup E_2$ of the edge set of D_6 into two trees, the breaker has a winning strategy.*

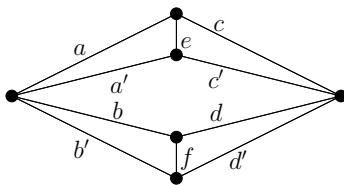


Figure 3: The graph D_6

Proof. We consider two cases.

Case 1: E_1 contains an edge of each of the pairs $\{a, a'\}, \{b, b'\}, \{c, c'\}, \{d, d'\}$ such that these four edges do not form a path. By symmetry, we may assume, that $A := \{a, b, c', d'\} \subseteq E_1$. The proof is based on the fact that $E \setminus A$ is hamiltonian. Note, that any further edge $x \in E \setminus A$ will complement A to a tree such that $E \setminus (A \cup \{x\})$ is a tree as well. Therefore, if $x = E_1 \setminus A$ and the maker recolours $y \in E \setminus (A \cup \{x\})$ to purple, the breaker recolours x to blue. Hence, A will never change its colour and the breaker wins.

Case 2: First we will show that otherwise E_1 must be disjoint from one of the pairs $\{a, a'\}, \{b, b'\}, \{c, c'\}, \{d, d'\}$. Assume not, then E_1 must contain one edge of each pair, which altogether form a path. We may assume, by symmetry, that $\{b, a, c, d'\} \subseteq E_1$ implying $\{f, d, b'\} \in E_2$. As E_2 forms a tree at least one of a', c' must be in E_1 , hence E_1 contains an edge from each pair (namely either $\{a, b, c', d'\}$ or $\{a', b, c, d'\}$) which altogether do not form a path.

Hence we may assume that $\{a, a'\} \subseteq E_2$. Hence $e \in E_1$ and thus w.l.o.g. $c' \in E_2$. So we must not have an E_2 -path in the lower half connecting the two vertices of degree 4. Therefore, E_1 must contain a P_4 in the lower half, w.l.o.g. $\{b', d, f\} \subseteq E_1$, $\{b', d, d'\} \subseteq E_1$ or $\{b, b', d\} \subseteq E_1$ and, again by symmetry, we may assume that E_1 is one from $\{e, c, b', d, f\}$, $\{e, c, b', d, d'\}$ or $\{e, c, b, b', d\}$ (see Fig. 4).

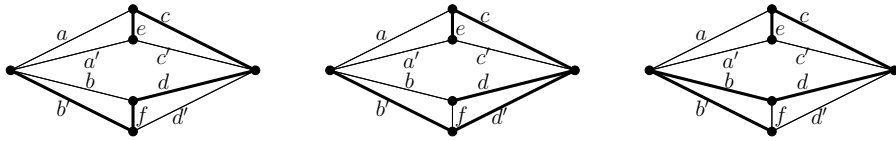


Figure 4: The three possible trees E_1 (fat edges) in Case 2

We will show that the breaker can assure, that $\{e, d, b'\}$ never change their colour. The maker's moves and the answers of the breaker are listed in Table 1.

In any case the breaker reinstalls a partition where E_1 is disjoint from either $\{a, a'\}$ or $\{c, c'\}$ and which contains the vertical edge adjacent to this pair, i.e. e , and two independent edges from the other side, namely

$E_1 := \{e, c, b', d, f\}$				
a	a'	b	c'	d'
c	c	f	c	f

$E_1 := \{e, c, b', d, d'\}$				
a	a'	b	c'	f
c	c	d'	c	d'

$E_1 := \{e, c, b, b', d\}$				
a	a'	c'	d'	f
c	c	c	b	b

Table 1: Fixing b', d, e .

b', d . Hence, by symmetry, the breaker can ensure that $\{e, d, b'\}$ never change their colour and wins.

□

Remark 19. *It can be shown [11] that the graph of left unique exchanges D_6^F has exactly 8 components, four of size 6 corresponding to Case 1 of the above proof and four of size 12 corresponding to Case 2.*

6. On the structure and the number of partitions of a wheel into two trees

We start with a crucial observation.

Proposition 20. (a) *Let $W_n = (V, E)$ denote the n -wheel and $E = E_1 \dot{\cup} E_2$ be a partition of the edges into two trees. Let $S \subseteq E$ denote the spokes and $S_1 := S \cap E_1$, $R \subseteq E$ the rim edges and $R_1 := R \cap E_1$. Let c denote the hub and v_0, \dots, v_{n-1} the outer vertices of W_n and $S_1 = \{cv_{i_1}, cv_{i_2}, \dots, cv_{i_k}\}$ in cyclic clockwise order. Then*

$$R_1 = R \setminus \{v_{i_1}v_{i_1+1}, v_{i_2}v_{i_2+1}, \dots, v_{i_k}v_{i_k+1}\} \text{ or}$$

$$R_1 = R \setminus \{v_{i_1}v_{i_1-1}, v_{i_2}v_{i_2-1}, \dots, v_{i_k}v_{i_k-1}\}$$

where indices are taken modulo n .

(b) *If, on the other hand, S, R, S_1 , and R_1 are as above for some partition $E = E_1 \dot{\cup} E_2$ satisfying $|E_1| = |E_2| = n$, then E_1 and E_2 both induce trees if and only if $\emptyset \neq S_1 \neq S$.*

By Proposition 20, either the purple rim edges follow the purple spokes counter-clockwise or clockwise. In the first case we speak of a *left orientation*, see Fig. 5 left, in the second of a *right orientation*, see Fig. 5 center. Every

purple spoke s_{i_j} that is adjacent to a purple rim edge is called *ending spoke*. There are some special configurations that we will refer to in the next section. If a left orientation has only one purple spoke s_i , the configuration is called *s_i -left path*, see Fig. 5 right. Its complement (i.e. the configuration with only one blue spoke, namely s_i) is called *s_i -left star*. More generally, a left orientation that has only one purple ending spoke s_i (but possibly other purple spokes) is called an *s_i -left half-star*. The complement of an s_i -left half-star S is an s_j -left half-star for some j . We call $s_{j-1} \pmod n$ (which is a purple spoke in S) the *beginning spoke* of S . In all cases, s_i is also called *special spoke*. We use analog notions for right orientations.

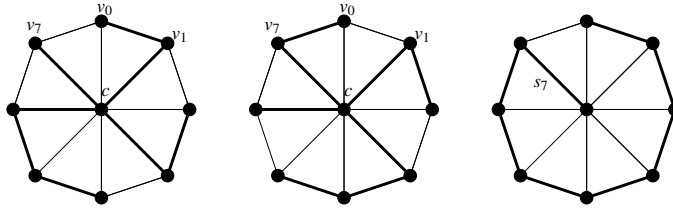


Figure 5: Left and right orientation and a left path

Proof of Proposition 20. (a) Since $|V| = n + 1$, and E_1 is a set of edges of a spanning tree we must have $|E_1| = n$ and hence $|R_1| = n - k$. If e is a rim edge adjacent to two spokes from E_2 it must be in E_1 , since E_2 has no triangle. Hence, each element from $R \setminus R_1$ is of the form $v_{i_j}v_{i_j+1}$ or $v_{i_j}v_{i_j-1}$. Assume that there exists $v_{i_j}v_{i_j+1}$ as well as $v_{i_\ell}v_{i_\ell-1}$ in E_2 and $cv_{i_j+1}, cv_{i_\ell-1} \in E_2$. If $j = \ell$, E_2 would contain the cycle $v_{i_j+1}v_{i_j}, v_{i_j}v_{i_j-1}, v_{i_j-1}c, cv_{i_j+1}$, thus necessarily $j \neq \ell$. We may choose j, ℓ such that cv_{i_j} precedes cv_{i_ℓ} in S_1 . But this contradicts the fact that E_1 induces a connected graph.

(b) First note that if in R_1 the left rim edge is missing at each spoke, the same holds for R_2 , vice versa. The same holds if the right rim edge is missing. Hence it suffices to show that E_1 induces a tree. Since $|E_1| = n$ this follows if E_1 is acyclic. The latter is clear, since in each path between two consecutive spokes of E_1 exactly one rim edge is missing. The claim follows. \square

Theorem 21. *The number of partitions of the edge set of the wheel W_n into two trees is $2^n - 2$.*

Proof. By Proposition 20 there is a bijection between the oriented proper subsets of S and the trees whose complements are trees as well. We have $2 \cdot (2^n - 2)$ oriented proper subsets of S , and we have counted each partition twice. The claim follows. \square

A matroid M is called an n -whirl if it has the same independent sets as an n -wheel (regarded as a graphic matroid) plus one extra independent set, namely the outer cycle of the n -wheel.

Corollary 22. *The number of partitions of the element set of the n -whirl into two bases is $2^n - 1$.*

Proof. Compared to the wheel we have the additional partition into the spokes and the rim. \square

7. The strategy of the maker for wheels

In this section we discover an important class of maker-win graphs, namely the class of wheels. Wheels are the simplest, most natural example for bispanning graphs with a high degree of symmetry.

Theorem 23. *Let $W_n = (V, E)$ be a wheel with $n \geq 4$ and let $E = E_1 \dot{\cup} E_2$ be a partition of the edges into two spanning trees. Let the edges of E_1 be purple and those of E_2 be blue. Then the maker has a strategy in the base exchange game to force an exchange of the colours of E_1 and E_2 .*

Proof. We will prove, using the following two lemmata, that W_n^F is strongly connected for $n \geq 5$. By Observation 27, W_4^F is also strongly connected. Then the theorem follows by Proposition 12. \square

Lemma 24. *If C^1, C^2 are two s_l -left orientations of the wheel W_n , $n \geq 5$ with $S_1^2 \subseteq S_1^1$, where S_1^k denotes the set of purple spokes of C^k , then the maker can enforce the transformation of C^1 into C^2 .*

Proof. We proceed by induction on the number $k = |S_1^1| - |S_1^2|$, the case $k = 0$ being trivial. Let $s_i \in S_1^1 \setminus S_1^2$. We distinguish two cases.

Case 1: C^1 is not a left half-star with beginning spoke s_i .

The maker recolours $v_{i+1}v_i$, making it purple. Since C^1 has at least 2 purple spokes, we may choose $s_j \in S_1^1$ such that $s_m \notin S_1^1$ for $i < m < j$. In order to destroy the cycle $cv_iv_{i+1} \dots v_jc$ and to reinstall a bispanning graph, since in this case not every purple rim edge is contained in the cycle, by Proposition 20 the breaker is forced to colour s_i blue.

Case 2: C^1 is a left half star with beginning spoke s_i .

We may assume that $i = n-1$ and there is some $1 \leq k \leq n-2$ such that the spokes s_0, \dots, s_{k-1} are blue and s_k, \dots, s_{n-1} are purple in C^1 . In case $k = n-2$, the maker recolours s_{n-3} to purple, forcing the breaker to colour the rim edge $v_{n-3}v_{n-2}$ blue. In case $k \geq n-3$, the maker (additionally) recolours s_{n-4} to purple, forcing the breaker to colour the rim edge $v_{n-4}v_{n-3}$ blue. After these up to two preparational steps, the spokes $s_{n-4}, s_{n-3}, s_{n-2}, s_{n-1}$ are purple. Now the maker recolours the rim edge $v_{n-3}v_{n-2}$ to purple, which forces the breaker to colour the spoke s_{n-3} blue. Then the maker recolours the rim edge $v_{n-1}v_0$ to purple, which forces the breaker to colour the spoke s_{n-1} blue. Then the maker recolours the rim edge $v_{n-4}v_{n-3}$ to purple, which forces the breaker to colour the spoke s_{n-4} blue. Possibly we have to invert the preparational steps, i.e. in case $n-3 \geq k \geq n-4$ the maker recolours the spoke s_{n-3} to purple (forcing the breaker to colour $v_{n-3}v_{n-2}$ blue) and in case $k = n-4$ the maker additionally recolours the spoke s_{n-4} to purple (forcing the breaker to colour $v_{n-4}v_{n-3}$ blue). After that the purple spokes are exactly those of $S_1^1 \setminus \{s_{n-1}\}$.

Now, the claim follows by induction. □

Note that the strategy of Case 1 would fail in Case 2, since then the maker's move is non-enforcing.

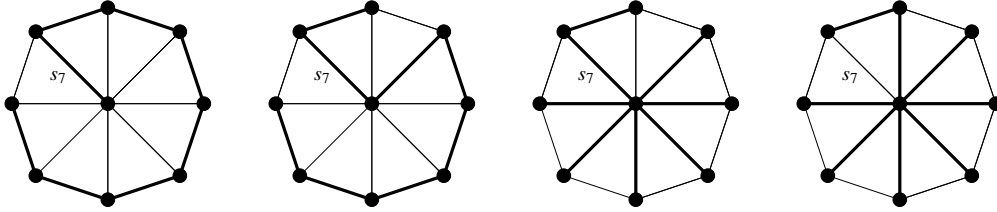


Figure 6: Transforming the s_7 -left path into the s_7 -left star

Lemma 25. *The maker can enforce the transformation of the s_i -left path of the wheel W_n , $n \geq 4$, into the s_i -left star.*

Proof. First the maker recolours the rim edge $v_i v_{i+1}$ to purple, so the breaker is forced to make the rim edge $v_{i-1} v_i$ blue, turning the s_i -left path into the s_i -right path, see Fig. 6 left. Then the maker recolours the spoke s_{i+2} to

purple, so that the breaker is forced to colour the rim edge $v_{i+1}v_{i+2}$ blue, see Fig. 6 center left. Now the maker inductively recolours the spokes s_{i+2+j} , $j = 1, 2, \dots, n-3$ (indices mod n), each move forcing the breaker to recolour the rim edge $v_{i+2+j-1}v_{i+2+j}$ to blue, see Fig. 6 center right. Now, we are left with the s_{i+1} -right star. In order to turn this into the s_i -left star it suffices to recolour s_{i+1} to purple, which forces the breaker to make s_i blue, see Fig. 6 right. Note that the first and last pair of moves requires $n \geq 4$. In case $n = 3$, s_{i+1} and s_{i-1} would be neighboured, and the breakers move is not forced any more. \square

Theorem 26. W_n^F is strongly connected for $n \geq 5$.

Proof. The following chain of arguments is depicted in Figure 7. By Lemma 24 we can transform any left orientation C^1 with spokes S_1^1 where $i \in S_1^1$ and $j \notin S_1^1$ for given $i \neq j$ into the s_i -left path. By Lemma 25 we can transform the s_i -left path into the s_i -left star and by Lemma 24 again from this we reach any left orientation C^2 with spokes S_1^2 and $i \notin S_1^2$, $m \in S_1^2$ for any $m \neq i$. Interchanging the roles of the indices we conclude that we can transform this into C^1 and hence the subgraph of W_n^F induced by the left orientations is strongly connected.

In the proof of Lemma 25 we, furthermore, transformed the s_i -left path into the s_{i+1} -right star and this into the s_i -left star. Since by, symmetry, the right orientations induce a strongly connected graph as well, the claim follows.

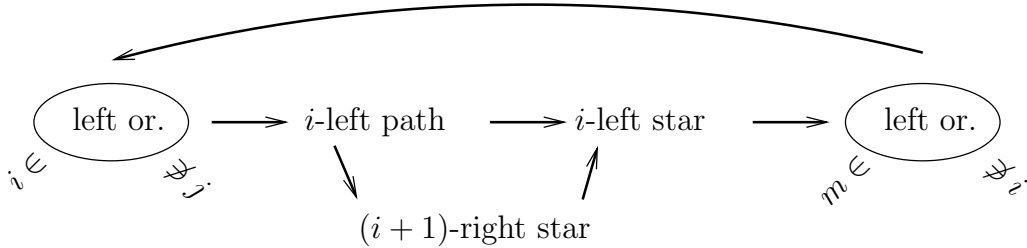


Figure 7: W_n^F is strongly connected

\square

In Case 2 of the proof of Lemma 24 we used the fact that $n \geq 5$. However an explicit computation of W_4^F shows that it is strongly connected as well. For the details we refer to [11].

Observation 27. *The graph W_4^F is strongly connected.*

Proof. For the graph W_4 the graph W_4^F of forced transformations is depicted in Fig. 8. All edges can be used in both directions. \square

In Fig. 8 the vertices that represent left orientations are coloured white, the right orientations are coloured grey. Note that any path from a vertex of degree 3 or 4 to its complement always uses changes of orientation. This might indicate that the changes of orientation we used in Lemma 25 are unavoidable.

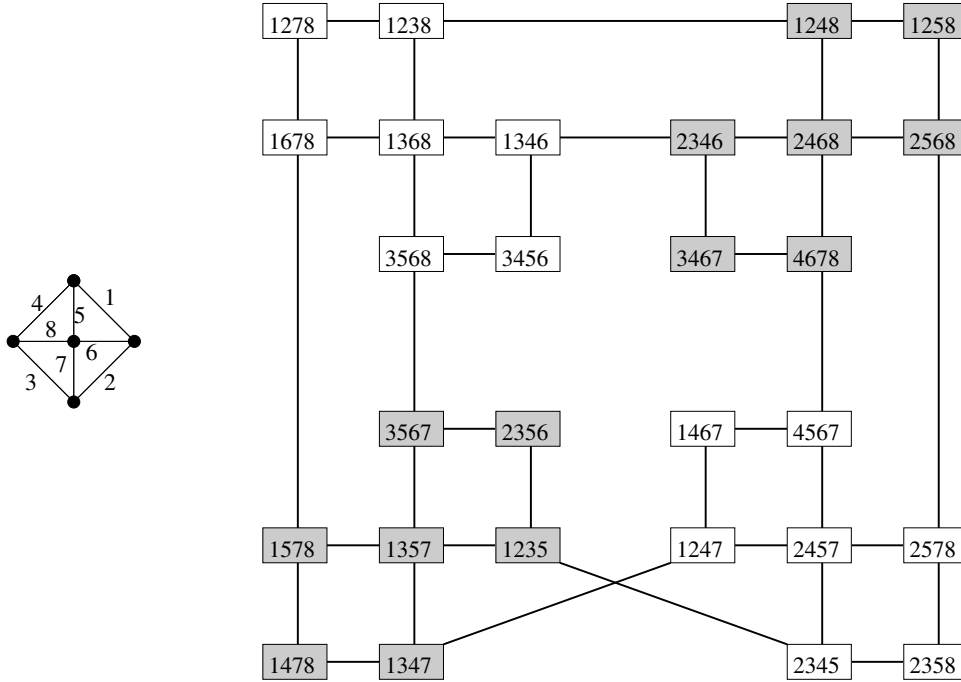


Figure 8: The wheel W_4 and its graph W_4^F of forced transformations

8. A non-homogeneous graph

In this section we will consider the graph $G = K_{3,3} + e$ where e is an additional edge in one of the bipartitions. We will see that for some partitions of G into two trees the breaker has a winning strategy, for others the breaker has a winning strategy. Moreover, we will see that G^F decomposes into two

components, one containing 48 partitions, the other 24 partitions. The maker wins exactly on half of the bispanning graphs of the bigger component.

In order to be able to describe this phenomenon more in detail, we start by identifying the types of bispanning graphs which can occur in G . In Fig. 9 three types and their complements are depicted.

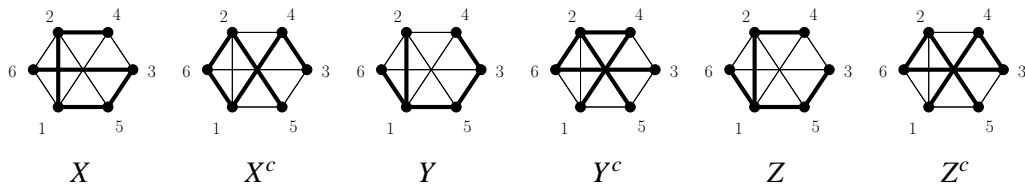


Figure 9: Types X and X^c , Y and Y^c , Z and Z^c

We say two partitions are of the same type if there is an automorphism of G transforming one into the other. It is easy to verify that in each case there are exactly 12 pairs of the same type (and 12 pairs of the complement of these types) since the automorphism group of G is $S_2 \times S_3$, where S_i denotes the permutation group on i elements. In pairs of the type X , Y , and Z the special edge $\{1, 2\}$ is purple, in the complements it is blue. Note that in a pair of type X the purple and the blue edges form a P_6 , in a pair of type Z the purple edges form a generalized star $S_{1,2,2}$ and the blue edges form a P_6 , and in a pair of type Y the purple edges form another generalized star $S_{1,1,3}$ whereas the blue edges form an $S_{1,2,2}$.

Theorem 28. (a) G^F consists of two components A and B , where A contains every partition of type X^c , Y^c , Z , and Z^c and B contains those of types X and Y .

(b) The maker wins if the starting partition is of type Z or Z^c .

(c) The breaker wins if the starting partition is of type X , X^c , Y , or Y^c .

We will prove this theorem by a series of lemmata.

Lemma 29. The bispanning graphs of type X and type Y form a component of G^F . Moreover, the breaker has a strategy never to leave this component if the game is started here.

Proof. In Fig. 10 we depict all possible results of a pair of moves, starting from X (upper row) resp. from Y (lower row). Alice recolours some edge and

in most cases Bob's response is forced (grey edge). In the three non-forced moves we show Bob's possible moves in grey. In all three non-forced moves, if the breaker plays the lower edge $\{1, 5\}$, either a partition of type X or of type Y is created. In the forced moves it can be seen that also only types X or Y are created, they are denoted as X resp. Y with the permutations corresponding to the automorphisms.

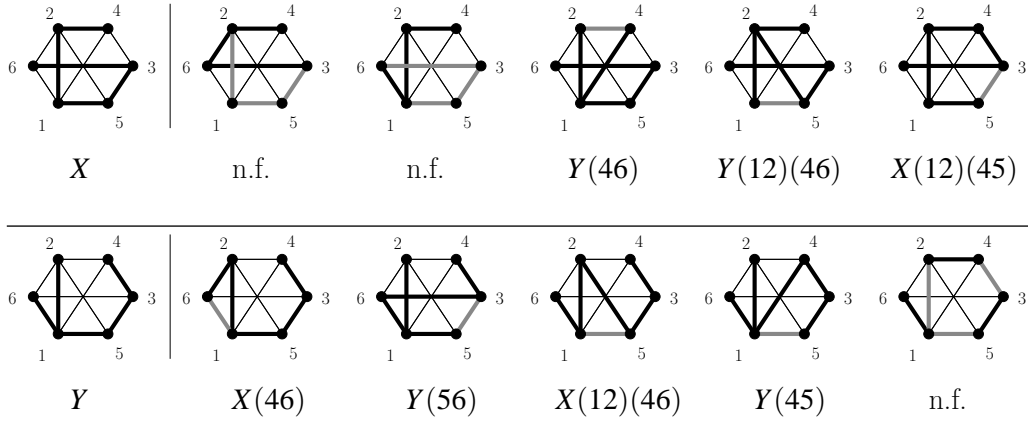


Figure 10: Moves starting with Y

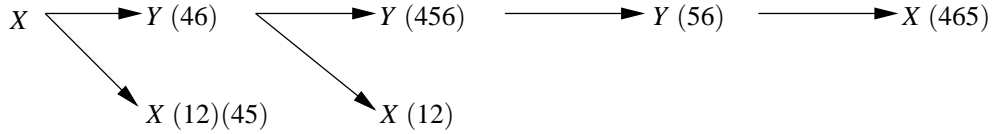


Figure 11: Paths of moves

In Fig. 11 we see paths of moves from X to $Y(4, 6)$ and $X(12)(45)$, $X(12)$ resp. $X(465)$. Since $\{(12)(45), (12), (465)\}$ is a generating set of the automorphism group of G , the partitions of types X and Y form a single component of G^F . \square

Lemma 30. *Types X^c , Y^c , Z and Z^c are in the same component. In particular, each type Z can reach each type Z^c .*

Proof. In Fig. 12 we depict all forced and non-forced (n.f.) moves starting from the bispanning graph X^c , Y^c , Z , Z^c , respectively. The answer of the

breaker in forced moves is the grey edge. It can be seen that there are no forced moves which obtain a partition of type X or Y .

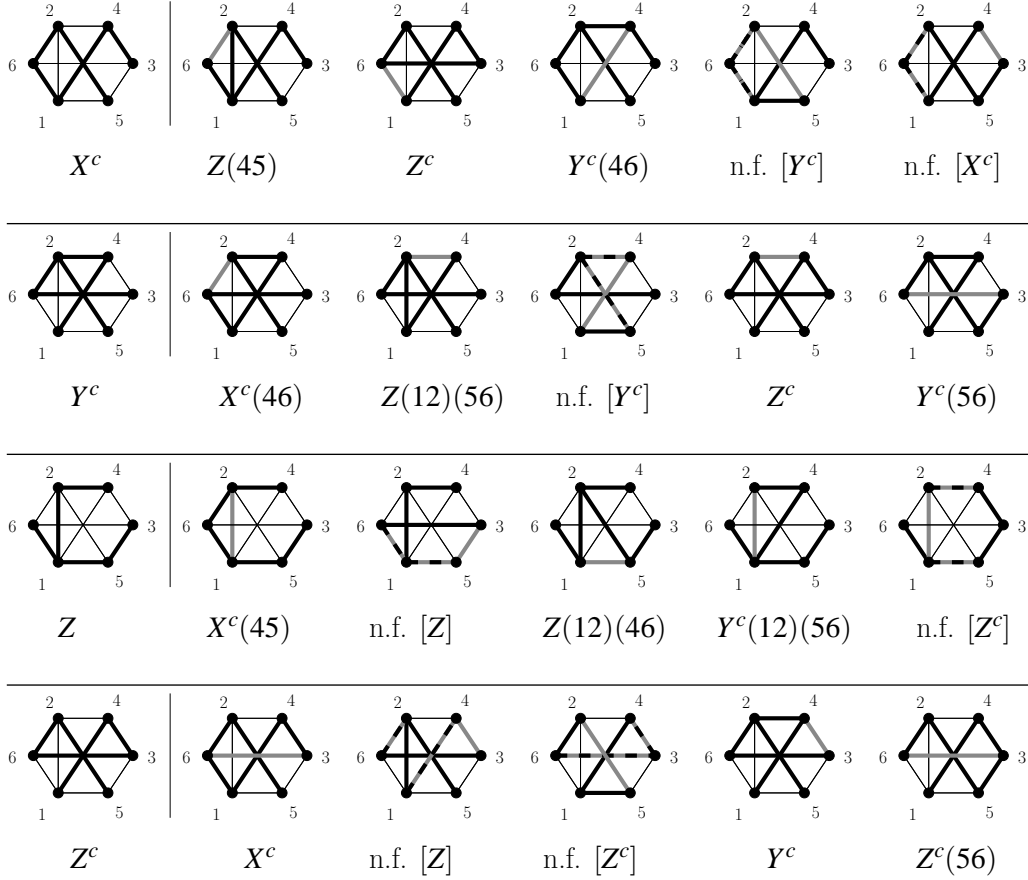


Figure 12: Moves starting with X^c , Y^c , Z , resp. Z^c

Furthermore each permutation of each of the four partitions X^c , Y^c , Z , and Z^c can be reached from any one of them, as is proven by the paths in Fig. 13.

This means that type X^c , Y^c , Z , and Z^c form a component of G^F . \square

Lemma 31. *If the initial partition is of type X^c or Y^c , in the non-forced moves the breaker has a strategy to obtain partitions of types X^c , Y^c , Z or Z^c again (to stay in the same component).*

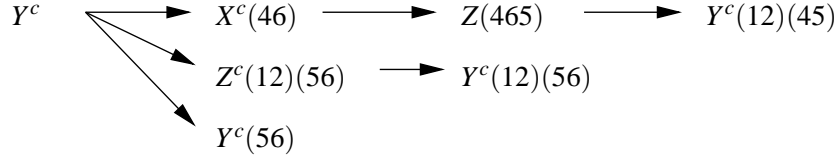


Figure 13: Paths of forced moves

Proof. In the non-forced moves of Fig. 12, if the breaker chooses the grey edge for recolouring (not the grey-black dashed edges), he also obtains a partition of a type which is displayed in brackets. This type is neither X nor Y in any case. \square

This completes the proof of Theorem 28.

9. Concluding remarks

We have seen in the last section that there is a bispanning graph with partitions $E = E_1 \dot{\cup} E_2$ and $E = F_1 \dot{\cup} F_2$ into spanning trees such that the maker wins when the initial partition is (E_1, E_2) but the breaker wins when the initial partition is (F_1, F_2) . However, the following problem is still open.

Problem 32. *Is there a bispanning graph $G = (V, E)$ with partition $E = E_1 \dot{\cup} E_2$ into spanning trees such that the maker wins when the initial partition is (E_1, E_2) but the breaker wins when the initial partition is (E_2, E_1) ?*

In all our examples, if the maker has a winning strategy for a bispanning graph G with initial partition (E_1, E_2) , the partition (E_2, E_1) was in the same component. Note that a positive answer to Problem 32 implies a positive answer to the following

Problem 33. *Is there a bispanning graph $G = (V, E)$ with partition $E = E_1 \dot{\cup} E_2$ into spanning trees such that the maker wins when the initial partition is (E_1, E_2) , but (E_2, E_1) and (E_1, E_2) lie in distinct components of G^F ?*

Finally, we address some questions on the complexity of our problems. Many games are \mathcal{PSPACE} -complete, however there seems to be no obvious reduction to our game.

Problem 34. *Is the problem to decide which player has a winning strategy in the matroid games $W(1)$ resp. $W(2)$ a \mathcal{PSPACE} -complete problem? Is the problem even \mathcal{PSPACE} -complete for bispanning graphs?*

Conjecture 5 would imply a negative answer to the last question in the case of $W(2)$.

Problem 35. *In case the auxiliary graph $\tau_i(M)$ is connected, what is its diameter, for $i = 2, 3, 4$? Is it polynomial in the size of M ?*

Note that in the case of bispanning graphs G , we have seen that the number of vertices of $G^F = \tau_4(M(G))$ can be exponential in the size of G . This is the case for wheels by Proposition 20.

Problem 36. *In case Alice wins the game, how many moves does she need in the worst case? Is there an upper bound on the number of moves which is polynomial in the size of the bispanning graph?*

Note that the answer to Problem 36 might be affirmative even if the diameter of G^F is exponentially large.

Acknowledgements

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