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Stephan Dominique Andres, Reinhard Börger:

Note on a 1-colouring game on paths and cycles

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Contact: dominique.andres@fernuni-hagen.de,
reinhard.boerger@fernuni-hagen.de

FernUniversität in Hagen
Fakultät für Mathematik und Informatik
Lehrgebiet für Diskrete Mathematik und Optimierung
D – 58084 Hagen

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Note on a 1-colouring game on paths and cycles

Stephan Dominique Andres
Reinhard Börger
Fakultät für Mathematik und Informatik,
Fernuniversität in Hagen, Lützowstr. 125, 58084 Hagen, Germany
dominique.andres@fernuni-hagen.de
reinhard.boerger@fernuni-hagen.de

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Abstract

In the 1-colouring achievement game, two players alternately choose pairwise nonadjacent distinct vertices of a given graph. A player looses if he cannot move any more. We characterize the paths and cycles for which the first player has a winning strategy. This answers an open question of Harary and Tuza.

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1 Introduction

Consider the following *1-colouring achievement game* on a graph $G = (V, E)$. At the beginning of the game, every vertex of the graph is unmarked. Two players move alternately. A move consists in marking an unmarked vertex all of which neighbours are unmarked. The first player who is unable to move looses the game. We call this game G^*.

Let $n \in \mathbb{N}$ and P_n be the path with n vertices, where P_0 denotes the empty graph, and, for $n \geq 3$, let C_n be the cycle with n vertices. In this note we characterize which player has a winning strategy in the games P_n^* resp. C_n^*. This answers an open question raised by Harary and Tuza [4] who examined a similar k-colouring game on paths and cycles for $k \geq 2$.

2 Combinatorial games and nim sum

A *combinatorial game* is a 2-player game, where the players move alternately. The game consists of several *configurations* S_i, $i = 0, 1, \ldots, k$, where S_0 is the *starting configuration*, and each configuration is a set of other configurations, the *options*. The players move alternately, where a move consists in choosing an option (if any) from the actual configuration. The first player's actual configuration is S_0. If a player is unable to move (since the actual configuration is the empty set), the game ends and he looses. We furthermore impose that a combinatorial game is always finite, i.e. the game ends after a finite number of moves. Note that each configuration of a combinatorial game defines a combinatorial game. In the following, we identify the games with their starting configuration.

Let $\mathcal{S}$ be the set of configurations of a combinatorial game with starting configuration S_0. Sprague [6] (and later Grundy [3]) showed that there is a unique mapping $g: \mathcal{S} \longrightarrow \mathbb{N}$ such that

(i) if S' is an option of $S \in \mathcal{S}$, then $g(S') \neq g(S)$, and

(ii) if $g(S) > 0$ for $S \in \mathcal{S}$, then, for any $0 \leq k < g(S)$, the configuration S has an option S' with $g(S') = k$.

The number $g(S_0)$ is called the *Grundy value* of the game. So, if S_0 is the set $\{O_1, \ldots, O_m\}$ of options, then

$$g(S_0) = \text{mex}\{g(O_1), \ldots, g(O_m)\}, \tag{1}$$

where for a finite $M \subseteq \mathbb{N}$ the mex is defined as $\text{mex}M := \min(\mathbb{N} \setminus M)$. The Grundy value describes which player has a winning strategy for the game S: the first player wins if and only if $g(S) > 0$.

The *sum* $S^{(0)} + S^{(1)}$ of two combinatorial games $S^{(0)}$ and $S^{(1)}$ is the game, where in each move a player chooses some $k \in \{0, 1\}$ and plays in $S^{(k)}$ according to the rules of the respective games. Sprague [6] showed that the Grundy value of a sum $S^{(0)} + S^{(1)}$ is the *nim sum*

$$g(S^{(0)} + S^{(1)}) = g(S^{(0)}) \oplus g(S^{(1)}) \tag{2}$$

defined by

$$\left(\sum_{i=0}^{n} \alpha_i 2^i\right) \oplus \left(\sum_{i=0}^{n} \beta_i 2^i\right) := \sum_{i=0}^{n} (\alpha_i + \beta_i \mod 2) 2^i \tag{3}$$

for $n \in \mathbb{N}$, $\alpha_i, \beta_i \in \{0, 1\}$.

The nim sum was already considered by Bouton [2] for nim games, in a generalized version by Moore [5].

3 The 1-colouring achievement game on paths

In this section we discuss the 1-colouring achievement game.

Lemma 3.1. *Let $n \in \mathbb{N}$. Then*

$$g(P_0^*) = 0, \qquad g(P_1^*) = 1,$$

and, for $n \geq 2$, the options of P_n^* are

$$P_{n-2}^*, \text{ and } P_k^* + P_{n-k-3}^* \text{ for any } 0 \leq k \leq n-3.$$

Table 1: The Grundy values of P^*_{17n+k}

n	$k=0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	**0**	1	1	2	0	3	1	1	0	3	3	2	2	4	**0**	5	**2**
1	**2**	3	3	0	1	1	3	0	2	1	1	0	4	5	**2**	7	4
2	**0**	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5
3	**2**	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4
4	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5
5	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4
6	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5
7	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4
8	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5
9	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4
10	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5
11	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4

Proof. In P_n, the first player may either mark a vertex of degree 1, this leads to a path P_{n-2} which can still be marked, or he can mark a vertex of degree 2, this leads to 3 unmarkable vertices in the middle, leaving a path P_k at the left and a path P_{n-k-3} at the right. □

Using Lemma 3.1, (1) and (2), it is possible to calculate the Grundy value of P^*_n recursively via

$$g(P^*_n) = \operatorname{mex}\{g(P^*_{n-2}), g(P^*_k) \oplus g(P^*_{n-k-3}) \mid 0 \le k \le n-3\} \tag{4}$$

for $n \ge 2$. The first 204 values are displayed in Table 1.

Table 1 suggests that the sequence of Grundy values $g(P^*_n)$ is periodic with period 34, except for $0 \le n \le 51$. In fact, this is true.

Lemma 3.2. *For all $N \ge 86$,*

$$g(P^*_N) = g(P^*_{N-34}).$$

Proof. For $86 \le N \le 173$ the lemma is true by the above table. If $N \ge 174$, we have $N-2 \ge 172 \ge 86$, hence $g(P^*_{N-2}) = g(P^*_{N-34-2})$ by the induction hypothesis. Moreover, for $0 \le k \le N-3$ we have $k \ge 86$ or $N-k-3 \ge 86$, hence $g(P^*_k) = g(P^*_{k-34})$ or $g(P^*_{N-k-3}) = g(P^*_{N-k-3-34})$ by the induction hyopthesis. Therefore, $g(P^*_k) \oplus g(P^*_{N-k-3}) = g(P^*_{k-34}) \oplus g(P^*_{N-k-3})$ or $g(P^*_k) \oplus g(P^*_{n-k-3}) = g(P^*_k) \oplus g(P^*_{N-k-3-34})$. Using Lemma 3.1 this means that, for any option of P^*_N, there is an option of P^*_{N-34} with the same Grundy value. Since, for $0 \le k \le N-34-3$, we have that $k \ge 52$ or $N-34-3-k \ge 52$, by a similar argumentation we conclude that, for any option of P^*_{N-34}, there is an option of P^*_N with the same Grundy value. This proves the lemma. □

Now we can prove our main result:

Theorem 3.3. *The second player wins the game P_n^* if and only if*

(i) $n \in \{0, 14, 34\}$ *or*

(ii) $n \equiv c \mod 34$ *with* $c \in \{4, 8, 20, 24, 28\}$.

Proof. The second player wins on P_n if and only if $g(P_n^*) = 0$. Thus the theorem follows from Table 1 and Lemma 3.2. □

Theorem 3.4. *For $n \geq 3$, the first player wins the game C_n^* if and only if*

(i) $n \in \{3, 17, 37\}$ *or*

(ii) $n \equiv c \mod 34$ *with* $c \in \{7, 11, 23, 27, 31\}$.

Proof. In the first move of the game C_n^* the only option is P_{n-3}^*. Therefore the first player wins on C_n if and only if the second player wins on P_{n-3}, thus the theorem follows from Theorem 3.3. □

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Note added

After having completed this paper we came to know that Theorem 3.3 had been already proved by Berlekamp et al. [1], pages 88–90, i.e. even before the problem was announced as an open problem by Harary and Tuza [4], as was remarked in a survey of Tuza [7], see pages 214–215.