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Short Note on the Number of Partitions of Wheels and Whirls into Two Trees respectively Two Bases

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Short Note on the Number of Partitions of Wheels and Whirls into Two Trees respectively Two Bases

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Abstract

We prove that the number of partitions of the n-wheel into two trees is $2^n - 2$. Furthermore, this yields that the number of partitions of the n-whirl into two bases is $2^n - 1$.

1 The proof

We start with a crucial of observation.

Proposition 1. *Let $W_n = (V, E)$ denote the n wheel and $E = E_1 \dot\cup E_2$ a partition of the edges into two trees. Let $S \subseteq E$ denote the spikes and $S_1 := S \cap E_1$, $R \subseteq E$ the rim edges and $R_1 := R \cap E_1$. Let $S_1 = \{(cs_{i_1}), (cs_{i_2}), \ldots (cs_{i_k})\}$ in cyclic order. Then*

$$\begin{aligned} R_1 &= R \setminus \{(s_{i_1}, s_{i_1+1}), (s_{i_2}, s_{i_2+1}), \ldots, (s_{i_k}, s_{i_k+1})\} \text{ or} \\ R_1 &= R \setminus \{(s_{i_1}, s_{i_1-1}), (s_{i_2}, s_{i_2-1}), \ldots, (s_{i_k}, s_{i_k-1})\} \end{aligned}$$

where indices are taken modulo n.

Proof. Since $|V| = n + 1$, and E_1 is a set of edges of a spanning tree we must have $|E_1| = n$ and hence $|R| = n - k$. If e is a rim edge adjacent to two spokes from E_2 it must be in E_1, since E_2 has no triangle. Hence, each element from $R \setminus R_1$ is of the form $(s_{i_j}, s_{i_j} + 1)$ or $(s_{i_j}, s_{i_j} - 1)$. Assume that there exists (s_{i_j}, s_{i_j+1}) as well as $(s_{i_\ell}, s_{i_\ell-1})$ in E_2 and $(cs_{i_j+1}), (cs_{i_\ell-1}) \in E_2$. If $j = \ell$ E_2 would contain the cycle $(s_{i_j+1}, s_{i_j})(s_{i_j}, s_{i\_1}), (s_{i_j-1}c)(cs_{i_j+1})$, thus necessarily $j \neq \ell$. We may choose j, ℓ such that (cs_{i_j}) precedes (cs_{i_ℓ}) in S_1. But this contradicts the fact that E_1 induces a connected graph. □

Proposition 2. *Let $W_n = (V, E)$ denote the n wheel and $E = E_1 \dot\cup E_2$ a partition of the edges. Let S, R, S_1, R_1 be as in Proposition 1 and*

$$\begin{aligned} R_1 &= R \setminus \{(s_{i_1}, s_{i_1+1}), (s_{i_2}, s_{i_2+1}), \ldots, (s_{i_k}, s_{i_k+1})\} \text{ or} \\ R_1 &= R \setminus \{(s_{i_1}, s_{i_1-1}), (s_{i_2}, s_{i_2-1}), \ldots, (s_{i_k}, s_{i_k-1})\} \end{aligned}$$

where indices are taken modulo n.

If $\emptyset \neq S_1 \neq S$, then E_1 and E_2 both induce trees.

Proof. First note that if in R_1 the left rim edge is missing at each spoke, the same holds for R_2, vice versa. The same holds if the right rim edge is missing. Hence it suffices to show that E_1 induces a tree. Since $|E_1| = n$ this follows if E_1 is acyclic. The latter is clear, since in each path between two consecutive spokes exactly one edge is missing. The claim follows. □

Theorem 1. *The number of partitions of the edge set of the wheel W_n into two trees is $2^n - 2$.*

Proof. By Propositions 1 and 2 there is a bijection between the oriented proper subsets of S and the trees whose complements are trees as well. We have $2 \cdot (2^n - 2)$ oriented proper subsets of S, and we have counted each partition twice. The claim follows. □

Corollary 1. *The number of partitions of the element set of the n-whirl into two bases is $2^n - 1$.*

Proof. Compared to the wheel we have the additional partition into the spokes and the rim. □