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Game-perfect Digraphs
— Paths and Cycles

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Abstract

We consider an extension of Bodlaender's graph coloring game [5] which is played on digraphs instead of undirected graphs and in which the first player is allowed to miss a turn. This game defines the A-game chromatic number of a digraph. A digraph D is called A-perfect if for every induced subdigraph H of D, the A-game chromatic number of H is equal to the size of the largest clique in H. We characterize all A-perfect semiorientations of complete graphs with clique number 2, and all A-perfect paths and cycles.
Keywords: game chromatic number, perfect graph, digraph, path, cycle, game-perfectness

1 Introduction

A game-theoretic variant of the chromatic number of a graph is well-studied in the literature of discrete mathematics, based on a work of Bodlaender [5]. Questions of perfectness in this settings are examined in a previous paper [3]. Here, we will focus on a game-theoretic variant of the dichromatic number of a digraph and questions of perfectness in this more general situation.

Bodlaender's game. In 1991, Bodlaender introduced the following graph coloring 2-player game g. It is played on a graph G, which is uncolored at the beginning, and with a finite set C of colors. Alternately, with the first player, Alice, starting, the players color a vertex of G with a color from C, so that adjacent vertices receive distinct colors. The game ends when no move is posssible any move.

This game could be considered as a combinatorial game if the winning rule was that the player that cannot move any more looses. However, Bodlaender, and after him many other contributors, considered a different winning rule, a so called Maker-Breaker winning rule: Alice wins if every vertex is colored at the end of the game, otherwise the second player, Bob, wins. In this context, Alice is the Maker who tries to color the graph properly, and Bob, the Breaker, tries to create a situation in which an uncolored vertex cannot be colored any more (since it is surrounded by vertices of all colors).

The Maker-Breaker winning rule motivates the definition of a game-theoretic analogon of the chromatic number of a graph: The smallest cardinality of a color set C for which Alice has a winning strategy for the game g played on G is called *game chromatic number* $\chi_g(G)$ of G. It is well-defined since if the cardinality of C is at least the number of vertices of G, then Alice will always win.

Previous results. In recent years, the study of the game chromatic number has become popular in the field of discrete mathematics. A lot of work has been done in order to give upper bounds for the game chromatic number of certain types of graphs. Faigle et al. [11] proved that the game chromatic number of a forest is at most 4, a bound which is tight by a result of Bodlaender [5]. Guan and Zhu [12] showed that $\chi_g(O) \leq 7$ for any outerplanar graph O. The upper bound for the game chromatic number of planar graphs of 33 by Kierstead and Trotter [15] was reduced to 30 by Dinski and Zhu [9], to 19 by Zhu [20], to 18 by Kierstead [14] and recently to 17 by Zhu [22]. Zhu [21] and Kierstead [14] also determined upper bounds for the game chromatic number of graphs embeddable in an orientable surface. The game chromatic number of special graphs embeddable in a surface with given girth was bounded by He et al. [13] for planar graphs, by Wang [19] for graphs embeddable in a surface of nonnegative Euler characteristic, and by the author [2] for graphs embeddable in some other surface. The upper bound $\Delta + 1$ for the game chromatic number of line graphs of forests of maximum degree $\Delta \neq 4$ was determined by a series of papers of Cai and Zhu [7], Erdös et al. [10] and the author [1]. Cai and Zhu [7] also considered upper bounds for the game chromatic number of line graphs of k-degenerate graphs.

Game-perfectness. Since the game chromatic number of a graph is a game-theoretic coloring parameter that is obviously bounded below by the *clique number* $\omega(G)$, i.e. the largest size of a complete subgraph of G, we may ask the question for graphs for which $\chi_g(G) = \omega(G)$. Such graphs will be called *g-nice*. A graph G is called *g-perfect* if every induced subgraph of G is g-nice. In the paper [3] it was proved that the g-perfect graphs with clique number 2 are exactly the forests of stars. This does not seem to be a very interesting class of graphs, therefore a modified game was also considered.

The game A has the same rules as the game g, except for one extra rule: Alice is allowed to miss one or several turns, in particular she may miss her first turn. This game defines the *A-game chromatic number* $\chi_A(G)$ of G as the smallest cardinality of a color set C for which Alice has a winning strategy for the game A played on the graph G. A graph G is *A-nice* if $\chi_A(G) = \omega(G)$. G is

A-perfect if every induced subgraph of G is A-nice.

For a graph G, we have

$$\omega(G) \leq \chi(G) \leq \chi_A(G) \leq \chi_g(G), \tag{1}$$

see [1]. Here, $\chi(G)$ denotes the chromatic number of G. $\chi_g(G)$ and $\chi_A(G)$ may really differ, e.g. $\chi_g(C_4) = 3$ but $\chi_A(C_4) = 2$ for the cycle C_4 with 4 vertices since in the game A Alice can use her right to miss the first turn. The cycle C_6 with 6 vertices (which has $\chi_A(C_6) = 2$) is an example of an A-nice graph which is not A-perfect since its induced subgraph P_5 (a path with 5 vertices) has $\chi_A(P_5) = 3$.

By (1), g-perfect graphs are A-perfect, and A-perfect graphs are perfect. Perfect graphs have been introduced by Berge [4] and recently been characterized by the Strong Perfect Graph Theorem [8]. It states that a graph is perfect if, and only if, it does not contain cycles with an odd number of vertices or their complements as induced subgraphs. This was formerly known as Berge's Strong Perfect Graph Conjecture.

The class of A-perfect graphs is much richer than the class of g-perfect graphs. In [3], the following theorem was proved:

Theorem 1. *A graph G with $\omega(G) \leq 2$ is A-perfect if, and only if, every component of G is either a singleton K_1 or a complete bipartite graph $K_{m,n}$ or a complete bipartite graph $K_{m,n} - e$ in which one edge e is missing (for some m, n).*

Another result in [3] is that there is no Weak Perfect Graph Theorem for A- resp. g-perfectness of graphs as an analogon to the famous result of Lovász [16] that a graph is perfect if, and only if, its complement is perfect.

Our contribution. In this paper we will consider a generalization of the game A to directed graphs (digraphs). A *digraph* $D = (V, E)$ consists of a finite vertex set V and an arc set $E \subseteq V \times V$. For simplicity, we will assume that E does not contain loops, i.e. arcs of the form (v, v). A graph G will be considered as a digraph D_G with the same vertex set, where each edge vw is replaced by the pair of arcs (v, w) and (w, v). We do not distinguish between G and D_G in the following. In this way, the game A we define for digraphs will be, in the special case of graphs, the same as the game A defined above.

We consider the following maker-breaker game A which is played by two players, Alice and Bob, on a digraph D and with a finite color set C. At the beginning, the vertices of D are uncolored. The players move alternately, with Alice having the first move. A move of Bob consists in coloring a vertex of D with a color from C. Alice, in her move, can choose whether she misses her turn or also colors a vertex of D with a color from C. When a player colors a vertex v with a color c, the player must obey the rule that none of the in-neighbors of v has been already colored with c. (The coloring of the out-neighbors imposes no restriction.) The game ends when no vertex can be colored any more by this rule. If every vertex is colored at the end of the game, Alice wins. Otherwise Bob wins. So Bob wins if he can achieve a situation in which there is an uncolored vertex v_0 for which there are arcs starting at vertices of every color pointing towards v_0. Note that Bob does not need to create such a situation by himself, he might also force Alice to create it.

We define the A-game chromatic number $\chi_A(D)$ of a digraph D as the smallest cardinality of C, so that Alice has a winning strategy for the game A played on D. Note that this definition coincides with the definition of the A-game chromatic number of an undirected graph. $\chi_A(D)$ is not related to the oriented game chromatic number of Nešetřil and Sopena [17].

Note that the color classes the players produce in the game A are acyclic digraphs. Therefore $\chi_A(D)$ is a competitive version of the dichromatic number $\chi(D)$ of a digraph introduced by Neumann-Lara [18]. The dichromatic number of a digraph D is the smallest number of colors needed, so that a coloring of D exists in which every color class induces an acyclic digraph. In the special case of graphs the dichromatic number becomes the chromatic number. (In the case of graphs, the color classes in an acyclic coloring of Neumann-Lara are independent sets since every pair of adjacent vertices induces a directed 2-cycle.) However, in general for digraphs the determination of the dichromatic number is very difficult. It is even $\mathcal{NP}$-complete to decide whether a digraph has dichromatic number 2, see [6].

We want to extend the notion of A-perfectness to digraphs. Here we have to be careful to explain what the clique number $\omega(D)$ of a digraph D means. We define $\omega(D)$ as the largest size of an induced undirected complete graph in D, in which between every pair of vertices v and w there are the arcs (v, w) and (w, v). Then we call D *A-nice* if $\omega(D) = \chi_A(D)$. D is *A-perfect* if every induced subdigraph of D is A-nice.

In this paper we give a classification of some special classes of A-perfect digraphs with clique number of at most 2. In particular we will characterize the A-perfect digraphs with clique number 1 (Proposition 2) and the A-perfect semiorientations of paths (Theorem 8) and cycles (Theorem 10). A *semiorientation* of a graph G is a digraph with the same vertex set as G in which for every edge vw of G there is either an arc (v, w), which is called *single arc*, or both arcs (v, w) and (w, v). Such a pair of antiparallel arcs will be called an *edge*. We will call a semiorientation of an undirected path simply a *path* and a semiorientation of an undirected cycle simply a *cycle*. The undirected path resp. cycle on n vertices is denoted by P_n resp. C_n.

Our main results imply that the lists of A-perfect paths resp. cycles are finite.

2 Some special A-perfect digraphs

In this section we discuss A-perfect digraphs with clique number 1 and A-perfect semiorientations of complete graphs with clique number 2. An *in-star* is a digraph with a center vertex of in-degree n and n other vertices with in-degree 0 and no further vertices. We observe

Proposition 2. *The A-perfect digraphs with clique number 1 are exactly the digraphs in which one component is an in-star and the other components are isolated vertices.*

Proof. Obviously, an in-star plus isolated vertices is A-perfect: Alice colors the center of the in-star. If a simple digraph D (i.e., a digraph with clique number 1) contains two nontrivial components, it has a subdigraph on 4 vertices with two nonadjacent arcs, which is not A-nice, implying that D is not A-perfect. Now let D be a connected simple digraph (i.e., a digraph with clique number 1) which is not an in-star. We will prove that D is not A-nice (and therefore, not A-perfect).

Since D is not an in-star but connected, D has at least two vertices with in-degree of at least 1. Let v and w be such vertices. A winning strategy for Bob with 1 color is the following. If Alice, in her first move, colors a vertex z with non-zero out-degree, Bob wins, since an out-neighbor of z cannot be colored any more. Otherwise, we may assume w.l.o.g. that w has not been colored by Alice. Then Bob colors an in-neighbor of w, so that w cannot be colored any more. Thus he wins in any case. □

Fig. 1 depicts all semiorientations of the complete graph K_3 with clique number of at most 2. In all figures, a straight line between two vertices v and w represents the two arcs (v, w) and (w, v). A single arc (v, w) is depicted by an arrow directed from v to w.

Theorem 3. *The only A-perfect semiorientations of K_3 with clique number of at most 2 are $K_3^{1,++}$ and $K_3^{1,+-}$.*

Proof. This is a case analysis on the configurations of Fig. 1. □

Theorem 4. *The only A-perfect semiorientation of K_4 with clique number of at most 2 is $\overline{\vec{C}_4}$, the complement of the directed 4-cycle.*

Proof. Let D be a semiorientation of K_4 with vertices v_1, v_2, v_3, v_4. In case D has at most one edge, D contains an orientation of a K_3 which is not A-perfect. If D has two adjacent edges, either D contains a K_3^2 which is not A-perfect or D has clique number at least 3. So we may assume that D contains the edges v_1v_2 and v_3v_4 and no further edges. W.l.o.g. the arc between v_1 and v_3 is directed as (v_1, v_3). Since the subdigraph on the vertices v_1, v_3, v_4 may not be $K_3^{1,--}$ which is not A-perfect, the arc between v_1 and v_4 is directed as (v_4, v_1). With

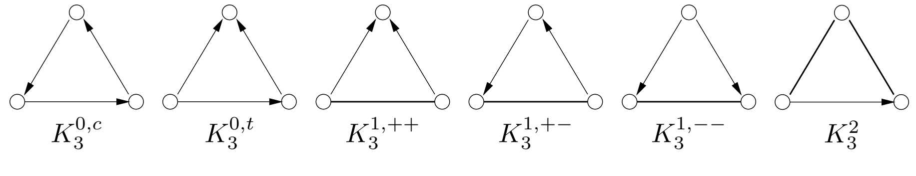


Figure 1: Semiorientations of K_3

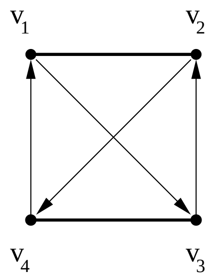


Figure 2: $\overline{\vec{C_4}}$

the same arguments concerning the sets of vertices $\{v_1, v_2, v_4\}$ resp. $\{v_2, v_3, v_4\}$ one finds the orientation of the other arcs that are (v_2, v_4) resp. (v_3, v_2).

A winning strategy for Alice on $\overline{\vec{C_4}}$ is the following: She misses her first turn. Then she colors the vertex which is connected by an edge to the vertex Bob has colored. After her move the coloring is fixed and she will win. Alice also wins on every proper induced subdigraph of $\overline{\vec{C_4}}$ as follows from Theorem 3. □

$\overline{\vec{C_4}}$ is depicted in Fig. 2.

Theorem 5. *There is no semiorientation of K_n, $n \geq 5$, with clique number 2 that is A-perfect.*

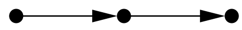
Proof. Obviously, it is sufficient to prove the theorem for $n = 5$. Let D be a semiorientation of K_5 with vertices v_1, v_2, v_3, v_4, v_5. Assume that D is A-perfect and has clique number 2. By Theorem 4, the subdigraph on the vertices v_1, v_2, v_3, v_4 must be the digraph of Fig. 2. Again, by Theorem 4 the subdigraph on the vertices v_1, v_5, v_3, v_4 must be isomorphic to the digraph of Fig. 2, in particular there must be an edge v_1v_5. But then the digraph induced by v_1, v_2, v_5 is either an undirected triangle (which contradicts the precondition that D has clique number 2) or K_3^2 (which is not A-perfect). □

3 Paths

Paths are a simple class of digraphs since every subdigraph of a path is a forest of paths. Therefore we consider the hereditary class $\vec{\mathcal{PF}}$ of forests of paths, i.e. of those digraphs each component of which is a path.

Lemma 6. *If a digraph D contains any of the forbidden configurations $F_{3,1}$, $F_{3,2}$, F_4, $F_{5,1}$, $F_{5,2}$, $F_{7,1}$, $F_{7,2}$, or F_8 depicted in Fig. 3 as induced subdigraph, then D is not A-perfect.*

Proof. It is easy to see that the forbidden configurations have A-game chromatic number 2 if they are simple digraphs, and 3 otherwise, thus they are not A-perfect. Then, by the definition of A-perfectness, D is not A-perfect. □



(a) The forbidden configuration $F_{3,1}$

(b) The forbidden configuration $F_{3,2}$

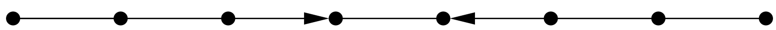
(c) The forbidden configuration F_4

(d) The forbidden configuration $F_{5,1}$

(e) The forbidden configuration $F_{5,2}$

(f) The forbidden configuration $F_{7,1}$

(g) The forbidden configuration $F_{7,2}$



(h) The forbidden configuration F_8

Figure 3: Some forbidden configurations for A-perfectness

Lemma 7. *Let P be a path with $n \geq 10$ vertices. Then P is not A-perfect. Moreover, P contains a forbidden configuration as an induced subdigraph.*

Proof. Assume P is A-perfect. If P contains 3 single arcs, P has an induced $F_{3,1}$, $F_{3,2}$ or F_4. So P has at most 2 single arcs. If there are 2 single arcs then these are adjacent or at distance 1, otherwise P has an induced F_4. Since the length of the path is $n-1 \geq 9$, P contains either an induced $P_5 = F_{5,2}$, which is a forbidden configuration, or P is of the form $v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}$, where $v_1v_2v_3v_4$ and $v_7v_8v_9v_{10}$ are (undirected) P_4's, v_5v_6 is an edge, and between v_4 and v_5 resp. between v_6 and v_7 there are single arcs. If there was an arc (v_5, v_4) or an arc (v_6, v_7), P would contain $F_{5,1}$. So there are arcs (v_4, v_5) and (v_7, v_6). But then P contains $F_{7,2}$, which is a contradiction. $\square$

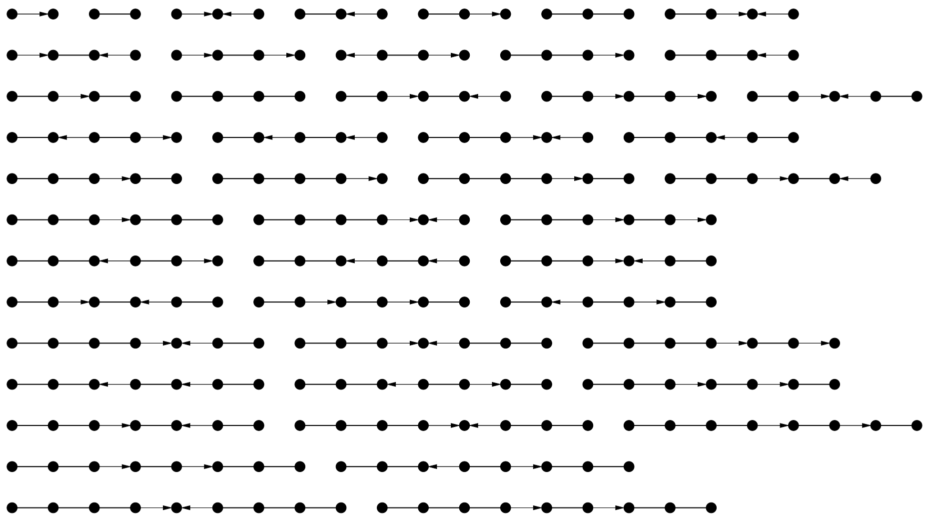


Figure 4: The 47 A-perfect paths

Theorem 8. *Let F be a forest of paths with components $D_1, D_2, \dots, D_k$. Then the following statements are equivalent:*

(a) F is A-perfect.

(b) F does not contain any of the forbidden configurations $F_{3,1}$, $F_{3,2}$, F_4, $F_{5,1}$, $F_{5,2}$, $F_{7,1}$, $F_{7,2}$, or F_8 depicted in Fig. 3 as an induced subdigraph.

(c) Every component of F, except at most one, is either an undirected path P_1, P_2, P_3, or P_4, and the remaining component is one of the 47 configurations depicted in Fig. 4.

In particular, the only A-perfect paths are those depicted in Fig. 4.

Proof. By Lemma 6 we have (a) $\Longrightarrow$ (b).

Consider (b) $\Longrightarrow$ (c). Assume that F does not contain any forbidden configuration. As F_4 is forbidden, every component D_i (with at most one exception, say D_1) is a graph, i.e. an undirected path P_{n_i}. Since $P_5 = F_{5,2}$ is forbidden, $n_i \leq 4$ for all $i \geq 2$. By Lemma 7 the remaining component has at most 9 vertices. The configurations of Fig. 4 are exactly those paths with at most 9 vertices which do not contain any of the forbidden configurations as induced subdigraphs (list all paths with at most 9 vertices and delete all enlargements of forbidden configurations. Note that, as in the proof of Lemma 7, we can restrict ourselves to paths with at most 2 single arcs, and if a path has two single arcs, they are adjacent or at distance 1, otherwise the path would contain a forbidden configuration $F_{3,1}$, $F_{3,2}$ or F_4). Thus F is of the desired form.

Finally we prove (c) $\Longrightarrow$ (a). Assume that F is of the form as in (c). By case analysis or the use of a computer program that calculates the A-game chromatic number by complete game-tree search it is easy to see that the 47 configurations of Fig. 4 are A-nice. Every digraph consisting of an arbitrary component C' which is one of the digraphs of Fig. 4 and some components which are undirected paths P_1, P_2, P_3, or P_4 is A-nice as well, as we shall see. Indeed, a winning strategy for Alice is the following: in her first move she plays on C', after that she always plays in the component on which Bob has played in his last move, in both cases according to her winning strategy for the respective components. Playing on a component possibly includes the use of Alice's right to miss a turn if this is necessary according to her winning strategy for C' or if a component is completely colored. Note that her winning strategies for P_j, $1 \leq j \leq 4$, always allow her to make Bob color the first vertex, therefore the strategy described above is feasible. Since every induced subdigraph of F is also of the type of digraphs described in (c), F is not only A-nice, but A-perfect. □

4 Cycles

Lemma 9. *Let C be a cycle with $n \geq 7$ vertices. Then C is not A-perfect.*

Proof. Assume C is A-perfect. If C has three single arcs, then it contains a forbidden configuration $F_{3,1}$, $F_{3,2}$, or F_4 as induced subdigraph. So C has at most 2 single arcs, and if there are two, then these are adjacent or at distance 1. There are remaining $m \geq n-3 \geq 4$ edges, which form a (forbidden) $P_5 = F_{5,2}$, a contradiction. □

Theorem 10. *Let C be a cycle. C is A-perfect if, and only if, C is one of the 14 configurations of Fig. 6.*

Proof. Proper subdigraphs of cycles are forests of paths. By case analysis or the use of a computer program it is easy to see that among all 22 cycles with at most 6 vertices which do not contain any of the forbidden configurations $F_{3,1}$, $F_{3,2}$, F_4, $F_{5,1}$, or $F_{5,2}$ as induced subdigraphs (see Figs. 5 and 6) there are exactly the 14 configurations of Fig. 6 which are A-nice. Thus, as they do not contain the forbidden configurations, they are A-perfect. By Lemma 9, cycles with more than 6 vertices are not A-perfect. □

5 Open problems

In Fig. 5, 8 forbidden cycles are depicted. These are minimal forbidden configurations, i.e. they do not contain other forbidden configurations as proper induced subdigraphs. Together with the 7 forbidden paths $F_{3,1}$, $F_{3,2}$, $F_{5,1}$, $F_{5,2}$, $F_{7,1}$, $F_{7,2}$, and F_8, and the non-connected forbidden configuration F_4, so far we have found 16 minimal forbidden configurations for A-perfectness of digraphs. There might be many more minimal forbidden configurations. E.g., from the results in [3] we conclude that $K_{2,3}$ is such a minimal forbidden configuration.

The next step in order to complete the list of minimal forbidden configurations for A-perfectness would be to consider forests in general, instead of forests

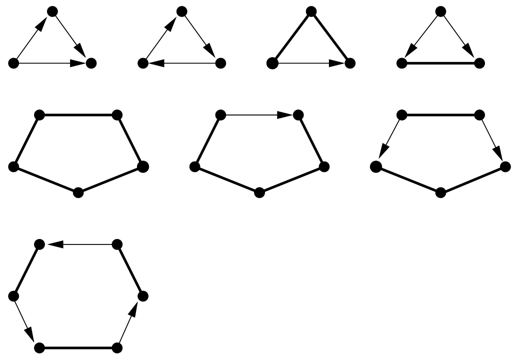


Figure 5: 8 forbidden cycles

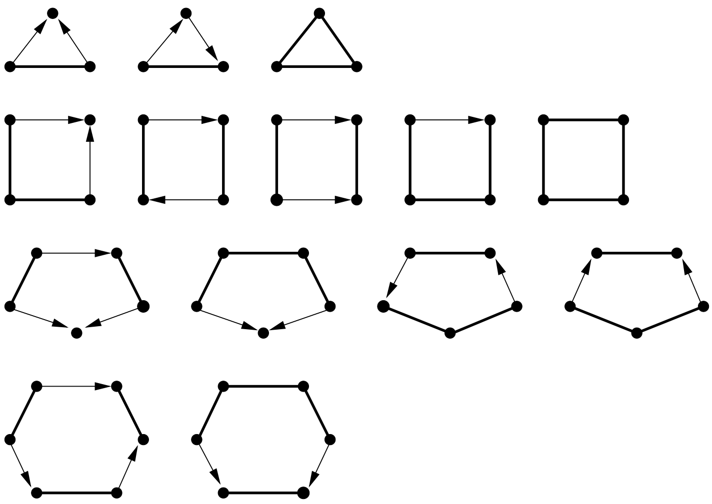


Figure 6: The 14 A-perfect semiorientations of cycles

of paths. By Lemma 7 we have that a tree of diameter $d \geq 9$ is not A-perfect. However, a lot of trees would have to be examined in order to determine the forbidden configurations. Note that the number of A-perfect trees is infinite since, for example, every in-star is A-perfect.

Conjecture 11. *The number of minimal forbidden trees is finite.*

The last step for the classification of A-perfect digraphs with clique number 2 would consist in considering semiorientations of arbitrary graphs. It is clear that every component but one of an A-perfect digraph with clique number 2 must be a bipartite graph of the form as described in Theorem 1. However, the remaining exceptional component will cause a lot of work.

A classification of all A-perfect digraphs (without restriction to the clique number) seems to be a demanding task for the future, as well as a description of A-perfect digraphs by minimal forbidden induced subdigraphs.

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