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Note on sums of binomial coefficients with roots of unity

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Note on sums of binomial coefficients with roots of unity

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We prove the following

Theorem 1. *Let $n \geq 1$ and ζ_n be a primitive n-th root of unity. Then*

$$\sum_{j=1}^{n} \binom{\zeta_n^j}{n} = \frac{1}{(n-1)!}.$$

This statement seems to be very natural and is easy to prove. However, I did not find any reference mentioning this result. Therefore this note contains a proof of Theorem 1.

We denote by $z^{\underline{k}}$ the factorial

$$z(z-1)(z-2)\cdots(z-k+1),$$

where z is a complex number and $k \in \mathbb{N}$. Let $s_{n,k}$ be the number of n-permutations with k cycles. These numbers are called Stirling numbers of first kind. In particular, $s_{n,0} = 0$ for $n > 0$, and $s_{n,n} = 1$. The following Stirling inversion formula is well-known:

Lemma 2. *Let z be a complex number and $n \in \mathbb{N}$. Then*

$$z^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} s_{n,k} z^k.$$

Proof of Theorem 1:

$$\begin{aligned}
\sum_{j=1}^{n} \binom{\zeta_n^j}{n} &= \sum_{j=1}^{n} \frac{\left(\zeta_n^j\right)^{\underline{n}}}{n!} \\
&= \frac{1}{n!} \sum_{j=1}^{n} \sum_{k=0}^{n} (-1)^{n-k} s_{n,k} \left(\zeta_n^j\right)^k \\
&= \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} s_{n,k} \sum_{j=1}^{n} \left(\zeta_n^k\right)^j \\
&= \frac{1}{n!} \left((-1)^n \underbrace{s_{n,0}}_{=0} n + (-1)^0 \underbrace{s_{n,n}}_{=1} n \right) \\
&= \frac{1}{(n-1)!}
\end{aligned}$$

In the second step we use Lemma 2. In the fourth step we use that, if $\zeta_n^k \neq 1$, we have

$$\sum_{j=1}^{n} \left(\zeta_n^k\right)^j = 0.$$

This proves the Theorem.