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Note on a MaxFlow-MinCut Property for Oriented Matroids

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Abstract

We introduce a new maxflow-mincut (MFMC) property for oriented matroids and give necessary and sufficient conditions for a flow lattice of an oriented matroid or more general for an integer lattice to have this property.

1 Introduction

There have been few attempts in the past to introduce a flow theory for oriented matroids as a generalization of flows in digraphs. Hamacher [6] developed an algebraic flow theory for maximal flows and minimal cost flows for regular oriented matroids. The general algebraic framework of Hamacher [6] has two special cases: The first notion of a flow is a so-called max-balanced flow, i. e. an integer or real-valued vector $x \in \mathbb{R}^n$ which satisfy $\max_{i \in D^+} x_e = \max_{x \in D^-} x_e$ for every signed cocircuit D. Hartmann and Schneider [7] generalized some admissibility and decomposition results of Hamacher [6] for the case of max-balanced flows. They presented a polynomial time algorithm that finds a capacity restricted max-balanced flow or certifies that no such flow exists. The second notion of a flow is obtained from the condition for max-balancedness by replacing the max-operator by the sum-operator. Here, an integer or real valued vector x is called a flow if it is orthogonal to every signed cocircuit, i. e. $\sum_{e \in D^+} x_e = \sum_{e \in D^-} x_e$ for all cocircuits D. In regular oriented matroids, the cocircuit orthogonal flows form a vector space of dimension $|E| - \operatorname{rank}(\mathscr{O})$. Hochstättler and Nešetřil [8] and Hochstättler and Nickel [9, 10] revealed that a considerable mass of non-regular oriented matroids has no non-trivial flow in the above sense at all.

Hochstättler and Nešetřil [8] introduced the flow lattice of an oriented matroid as the integer lattice of signed characteristic vectors of signed circuits. Hence, there is a non-trivial flow whenever the oriented matroid is not free. Based on investigations of [8, 9, 10] regarding the flow lattice structure, we consider "max-flow min-cut"-like properties of the lattice. Our main result is that, essentially, an oriented matroid has the oriented max-flow min-cut property if the flow lattice is regular. However, in contrast to the max-flow min-cut property for matroids of Seymour [14] which essentially holds for matroids that are regular, there are large classes of non-regular oriented matroids which have a regular flow lattice and thus satisfy our oriented max-flow min-cut property.

We assume familiarity with matroid and oriented matroid theory and use standard notation from Oxley [13] and Björner et al. [1]. In particular, $\mathcal{C}$, $\mathcal{D}$ denote the families of signed circuits and signed cocircuits resp. of an oriented matroid $\mathscr{O}$ with elements $E = \{1, \dots, n\}$. For some signed subset $X = (X^+, X^-)$ of E we denote by $\vec{X} \in \{0, 1, -1\}^n$ its signed characteristic vector and if $\mathcal{X}$ is some family of signed subsets, we set $\vec{\mathcal{X}} := \{\vec{X} : X \in \mathcal{X}\}$.

In the following we will provide basics of the theories of integer lattices and maximal flows in digraphs. In the second section we introduce our max-flow min-cut property together with a relaxed variant and present some necessary and sufficient conditions for an integer lattice to have these properties.

1.1 Integer Lattices.

For a set of non-zero integer vectors $V := \{v_1, \dots, v_r\} \subset \mathbb{Z}^n$ let

$$\operatorname{lat} V = \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid \lambda_i \in \mathbb{Z} \right\}$$

denote the *integer lattice spanned by* V. Given some $e \in \{1, \dots, n\}$, a capacity function $c : \{1, \dots, n\} \backslash e \to \mathbb{N}$, and an integer lattice $L \subseteq \mathbb{Z}^n$ we define

$$L^c := \{x \in L : 0 \leq x_i \leq c(i) \text{ for all } i \in \{1, \dots, n\} \backslash e\}$$

as the set of *feasible lattice vectors* or *feasible flows.* For a subset $I \subseteq E \backslash e$ let $c(I) := \sum_{i \in I} c(i)$ and for a vector $y \in \mathbb{Z}^n$ we write

$$c^+(y) := \sum_{\substack{1 \leq i \leq n \\ i \neq e \\ y_i \geq 0}} c(i) y_i$$

and call this the *directed capacity of* y.

Integer lattices define an oriented matroid in the following way (see Tutte [15]): let $L := \operatorname{lat} V \subseteq \mathbb{Z}^n$. For $x \in L$ we denote by x^+ resp. x^- the vectors with positive resp. negative components of x and by $\underline{x} := \{i : x_i \neq 0\}$ its support. A vector $x \in L$ is called *elementary* if there is no $y \in L$ such that $\underline{y} \subsetneq \underline{x}$ and $\frac{1}{k} x \notin L$ for all $k > 1$. An elementary vector x is *primitive* if additionally $x_i \in \{0, +1, -1\}$ for all $i \in E$. It can be derived directly from Tutte [15], who proved that the supports of elementary vectors of L yield the family of circuits of a matroid, that

$$\{(\underline{x^+}, \underline{x^-}) : x \text{ is elementary in } L\}$$

is the set of signed circuits of an oriented matroid which we denote by $\mathscr{O}(L)$. We furthermore call L to be *regular* if all elementary vectors are primitive implying that $\mathscr{O}(L)$ is a regular oriented matroid.

1.2 MaxFlow-MinCut

Let $\tilde{G} = (V, \tilde{E})$ be a digraph, $s, t \in V$ and $c : \tilde{E} \to \mathbb{N}$ a non-negative capacity function on $\tilde{E}$. For some $x \in \mathbb{Z}^{|\tilde{E}|}$ and $v \in V$ let

$$\delta_x(v) := \sum_{f \in \delta^+(v)} x_f - \sum_{f \in \delta^-(v)} x_f$$

be the *net flow* of the vertex v with respect to x. An *st-flow* is a vector $x \in \mathbb{R}^{|\tilde{E}|}$ such that all $v \in V \backslash \{s,t\}$ satisfy Kirchhoff's law [11], i. e. $\delta_x(v) = 0$. x is called *feasible* if $0 \leq x_f \leq c(f)$ holds for all $f \in \tilde{E}$. The *value* $\mathrm{val}(x)$ of a feasible st-flow x is $\delta_x(s) = -\delta_x(t)$. An *st-cut* is a partition $S \dot{\cup} T = V$ with $s \in S$ and $t \in T$. The *capacity* $\mathrm{cap}(S,T)$ of an st-cut (S,T) is the capacity sum of the arcs from S to T, i. e.

$$\mathrm{cap}(S,T) := \sum_{(v,w)\in(S,T)} c(v,w).$$

The famous max-flow min-cut theorem of Ford and Fulkerson [4] states that for any digraph $\tilde{G} = (V, \tilde{E})$ and any capacity function $c : \tilde{E} \to \mathbb{N}$ we have

Theorem 1 ([4])**.** $\max\limits_{\substack{x \text{ is a} \\ \text{feasible } st\text{-flow}}} \mathrm{val}(x) = \min\limits_{\substack{(S,T) \text{ is} \\ \text{an } st\text{-cut}}} \mathrm{cap}(S,T).$

Due to the lack of vertices in oriented matroids we need an equivalent statement that only uses the arcs of $\tilde{G}$. This is traditionally achieved by introducing an additional arc $e = (t,s)$ directed from t to s with infinite capacity. If we extend an st-flow x by $x_e := \mathrm{val}(x)$, we get $\delta_x(v) = 0$ for all $v \in V$, i. e. a *circular flow* in $G := (V, \tilde{E} \dot{\cup} e)$. Now let $E := \tilde{E} \dot{\cup} e$ and $\mathcal{C}(G)$ the family of signed circuits of G. We generalize the following characterization of a circular flow in a digraph (Gallai [5]):

$$x \in \mathbb{Z}^n \text{ is a circular flow} \iff x = \sum_{C\in\mathcal{C}(G)} \lambda_C \vec{C} \text{ for some } \lambda_C \in \mathbb{Z}.$$

Definition 2. *Let $\mathscr{O}$ be an oriented matroid on the ground set E with signed circuits $\mathcal{C}$. A vector $x \in \mathcal{F}_{\mathscr{O}} := \mathrm{lat}\,\vec{\mathcal{C}}$ is called a* flow. *Given an element $e \in E$ and a capacity function $c : E \backslash e \to \mathbb{N}$, x is called* feasible *if additionally $0 \leq x_f \leq c(f)$ for all $f \in E \backslash e$, i. e. $x \in \mathcal{F}^c_{\mathscr{O}}$. A feasible flow is called* maximal *if $x(e)$ is maximal.*

A more general variant of Theorem 1 then is

Theorem 3 (Minty [12])**.** *Let $\mathscr{O}$ be a regular oriented matroid with signed circuits $\mathcal{C}$ and signed cocircuits $\mathcal{D}$ on the ground set E with $|E| = n$. Let furthermore $e \in E$ be an arbitrary element and $c : E \backslash e \to \mathbb{N}$ a non-negative integer capacity function. Then*

$$\sup_{x\in\mathcal{F}^c_{\mathscr{O}}} x_e = \inf_{\substack{D\in\mathcal{D} \\ e\in D^-}} c(D^+). \qquad (*)$$

Note that $(*)$ especially holds when e is a loop or a coloop of $\mathscr{O}$ causing both sides to be zero resp. infinity. But it does not hold when $\mathscr{O}$ is not regular.

Example 1. *We discuss $(*)$ for the oriented 4-point line $\mathscr{O}(U_{2,4})$. The signed cocircuits of the equal orientation in terms of sign vectors are $D_1 = {+}{+}{+}0$, $D_2 = {+}{+}0{-}$, $D_3 = {+}0{-}{-}$, and $D_4 = 0{-}{-}{-}$. Since $\mathrm{e}_1 = D_1 - D_2 + D_3$ and by symmetry and selfduality $\mathcal{F}_{\mathscr{O}^*}$ and $\mathcal{F}_{\mathscr{O}}$ are trivial. Hence, the left side of $(*)$ is infinity but the right side is finite.*

2 MaxFlow-MinCut and Oriented Matroids

The lattice from Example 1 is trivial and coincides with the flow lattice of a digraph consisting of directed loops, only. In particular the lattice is regular. This suggests to reformulate $(*)$ in terms of the lattice, only.

Definition 4. *Let $L \subseteq \mathbb{Z}^n$ be an integer lattice and $e \in \{1, \ldots, n\}$. We say L satisfies*

- *the* oriented max-flow min-cut property *with respect to e if*

$$\sup_{x \in L^c} x_e = \inf_{\substack{y \in L^\perp \cap \mathbb{Z}^n \\ y_e < 0}} \frac{c^+(y)}{|y_e|} \tag{MFMC}$$

- *the* relaxed oriented max-flow min-cut property *with respect to e if*

$$\sup_{x \in L^c} x_e = \inf_{\substack{y \in L^\perp \cap \mathbb{Z}^n \\ y_e < 0}} \left\lfloor \frac{c^+(y)}{|y_e|} \right\rfloor \tag{MFMC'}$$

for all capacity functions $c : E \backslash e \to \mathbb{N}$.

If $L = \mathcal{F}_{\mathcal{O}}$ for an oriented matroid $\mathcal{O}$ and L satisfies (MFMC) resp. (MFMC') with respect to $e \in E$ we say that $\mathcal{O}$ has the property with respect to e. If L resp. $\mathcal{O}$ has the property with respect to all e we say that L resp. $\mathcal{O}$ itself has the property.

Note that (MFMC) implies (MFMC'). Our definition is supported by the fact that in general we have a weak duality.

Proposition 5. *Let $L \subseteq \mathbb{Z}^n$ be an integer lattice, $e \in E$, and $c : E \backslash e \to \mathbb{N}$ a capacity function. Then*

$$\sup_{x \in L^c} x_e \leq \inf_{\substack{y \in L^\perp \cap \mathbb{Z}^n \\ y_e < 0}} \left\lfloor \frac{c^+(y)}{|y_e|} \right\rfloor \leq \inf_{\substack{y \in L^\perp \cap \mathbb{Z}^n \\ y_e < 0}} \frac{c^+(y)}{|y_e|}$$

Proof. The right inequality is immediate. Let $x \in L^c$ and $y \in L^\perp \cap \mathbb{Z}^n$ such that $e \in \underline{y}^-$. Then

$$0 = x^\top y = x_e y_e + \sum_{f \in E} x_f y_f \leq x_e y_e + \sum_{f \in y^+} x_f y_f \leq x_e y_e + c^+(y).$$

Hence, $-x_e y_e \leq c^+(y)$ and the claim follows. □

Note that even for very small integer lattices the inequalities in (MFMC) and (MFMC') might be strict:

Example 2. *Let $L := \{\binom{3}{-2}\}^\perp \cap \mathbb{Z}^2$ and $c(1) := 3$ the capacity of the first coordinate. Then $\binom{2}{3}$ is a maximal flow of value 3 but as $L^\perp \cap \mathbb{Z}^n = (3, -2)^\top \mathbb{Z}$, the right hand side of (MFMC') becomes $\lfloor \frac{3 \cdot 3}{2} \rfloor = 4$.*

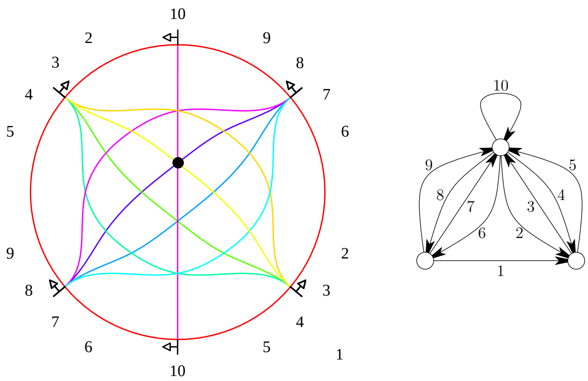
Example 3. *Let $L := \{x \in \mathbb{Z}^2 : x_1 = x_2, x_1 \equiv 0 \pmod{k}\}$, $k > 1$ and $c(1) := k - 1$ the capacity of the first coordinate. Then the maximum value of a vector in L is 0 but the right hand side of (MFMC') yields $k - 1$ showing that the gap can become arbitrarily large.*

But (MFMC) holds for the 4-point line in Example 1 and more general

Proposition 6. *Any regular integer lattice $L \subseteq \mathbb{Z}^n$ satisfies* (MFMC) *with respect to all capacity functions.* □

Hochstättler and Nešetřil [8] and Hochstättler and Nickel [9] determined the flow lattice of oriented matroids that are uniform, have rank 3, or are listed in the catalogue of small oriented matroids of Finschi [3]. A large number of these flow lattices turned out to be regular. We give an example from the catalogue of Finschi [3] which is a prototype of a flow lattice with codimension 2 (c. f. [9]):

Example 4. *The following figure shows the pseudohypersphere configuration of the dual of a non-regular corank 3 oriented matroid together with the graphic representation of $\mathcal{O}(\mathcal{F}_{\mathcal{O}})$. Note that* $\operatorname{rank}(\mathcal{O}) = 7$ *and* $\operatorname{rank}(\mathcal{O}(\mathcal{F}_{\mathcal{O}})) = 3$. *Both matroids have isomorphic flow lattices.*



$$\mathcal{F}_{\mathcal{O}} = \left\{ \begin{array}{l} (1,1,-1,1,-1, \quad 0,0, \quad 0,0,0), \\ (1,0, \quad 0,0, \quad 0,-1,1,-1,1,0) \end{array} \right\}^{\perp} \cap \mathbb{Z}^n$$

Consider e. g. the cocircuit $C = \{1,2,4,5,6,7,9\}$ in the dual representation of $\mathcal{O}$ which is the complement of the emphasized vertex. In $\mathcal{O}$, C is a signed circuit with the sign pattern $\vec{C} = {+}{-}0{+}{+}{+}{+}0{-}0$ which is the sum of circuits of the graphic oriented matroid shown on the right, i. e. $\vec{C} = \vec{C}_{45} + \vec{C}_{67} + \vec{C}_{129}$.

The computational results of Hochstättler and Nickel [9] suggest that fully dimensional lattices dominate the set of flow lattices of non-regular oriented matroids:

Proposition 7. *If* $\dim \operatorname{lin} L = n$ *then both sides of* (MFMC) *and* (MFMC') *become infinity.* □

This includes the case when L is determined by a system of modular equations of the form

$$L = \{x \in \mathbb{Z}^n : x^\top y^{(i)} \equiv 0 \pmod{k_i},\ i = 1, \dots, m\}.$$

Example 5. *Hochstättler and Nickel [9] proved that all uniform oriented matroids $\mathcal{O} = \mathcal{O}(U_{r,n})$ of odd rank r with* $\dim \mathcal{F}_{\mathcal{O}} = n$ *satisfy*

$$\mathcal{F}_{\mathcal{O}} = \{x \in \mathbb{Z}^n : \mathbb{1}^\top x \equiv 0 \pmod 2\}.$$

Apart from fully dimensional lattices, a non-regular flow lattice in general might lead to a gap in (MFMC). For the next results we need the notion of reorientation of a lattice L. In oriented matroids, reorienting some element $f \in E$ leads to a sign reversal of the f-th component of every flow lattice vector. Hence, we say that L' is a *reorientation* of L of it can be obtained from L by reversing the signs of all lattice vectors in some components.

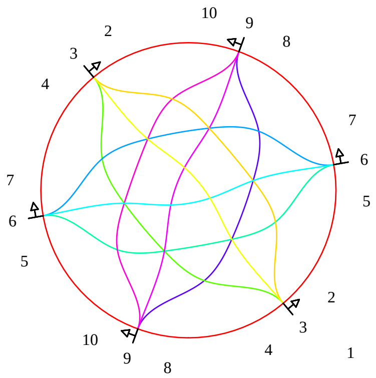
Theorem 8. *Let $L \subset \mathbb{Z}^n$ have an elementary vector x which is not primitive and satisfies $|\underline{x}| > 1$. Then there is a reorientation of L, an element $e \in \{1, \ldots, n\}$, and a capacity function $c : \{1, \ldots, n\} \backslash e \to \mathbb{N}$ such that* (MFMC) *is violated.*

Proof. Let L be reoriented such that x is positive and $f \in \{1, \ldots, n\}$ with $x_f > 1$. We choose an arbitrary $e \in \underline{x} \backslash f$ and set $c(g) := 1$ if $g \in \underline{x} \backslash e$ and 0 otherwise. Since x was chosen to be elementary, we have $L^c = \{0\}$. Given an arbitrary $y \in L^\perp \cap \mathbb{Z}^n$ with $y_e < 0$ we get

$$\frac{c^+(y)}{|y_e|} = \frac{|\underline{y}^+ \cap \underline{x} \backslash e|}{|y_e|} > 0.$$

□

Example 6. *The following is the dual representation of a corank 3 oriented matroid with 10 elements whose flow lattice is characterized by an orthogonality condition and a modular equation. The modular equation causes non-regularity of the lattice and the violation of* (MFMC')*:*



$$\mathcal{F}_{\mathcal{O}} = \left\{ x \in \mathbb{Z}^n : \begin{array}{r} (0,1,0,1,-1,0,-1,1,0,1)x = 0 \text{ and} \\ 2|(0,1,1,0,\quad 1,1,\quad 0,0,1,1)x \end{array} \right\}$$

Choose $e = 5$, $f = 4$, and set $c(f) = 1$ and 0 otherwise. Then $L^c = \{0\}$ and $c^+(y) = 1$. $x := 2(\mathrm{e}_4 + \mathrm{e}_5)$ is an elementary vector as in the proof of Theorem 8.

Example 7. *Non-regularity coupled with non-trivial codimension of a lattice does not necessarily lead to a violation of* (MFMC')*. E. g. consider the lattice $L := \{x \in \mathbb{Z}^3 : (0,1,1)x = 0$ and $2|x_1\}$. This lattice satisfies both properties with respect to all elements, all capacity functions, and all reorientations. An oriented matroid that has this structure is $\mathcal{O}(U_{3,5}) \oplus \mathcal{O}(U_{3,6})$ where $\oplus$ denotes the direct sum and $\mathcal{O}(U_{3,6})$ is a non-neighborly orientation of $U_{3,6}$ (see [9]). However, we are not aware of a connected oriented matroid with a non-regular flow lattice that has non-trivial codimension and satisfies* (MFMC')*.*

We will now derive a sufficient condition for a lattice to satisfy (MFMC'). Let

$$L_1 := \{x \in \mathbb{Z}^n : (z_1, \dots, z_{n-1}, -1)x = 0\}$$

for fixed $z_i \in \mathbb{Z}$, $i \in \{1, \dots, n-1\}$. We set $z := (z_1, \dots, z_{n-1}, -1)^\top$.

Proposition 9. *L_1 satisfies (MFMC) with respect to n for all reorientations and all capacity functions.*

Proof. It is clear that $\mathrm{e}_i + z_i \mathrm{e}_n \in L_1$ for $i = 1, \dots, n-1$. Note that $L_1^\perp \cap \mathbb{Z}^n = z\mathbb{Z}$. We set $y := z$ and

$$x := \sum_{\substack{1 \le i < n \\ z_i > 0}} c(i)(\mathrm{e}_i + z_i \mathrm{e}_n) \in L_1$$

and obtain $x_n = c^+(y)$ as required. □

Note that L_1 does not necessarily satisfy (MFMC) with respect to all $i \in \{1, \dots, n-1\}$. However, Proposition 9 can be used to prove that the next lattice considered at least satisfies (MFMC') with respect to all $i \in \{1, \dots, n\}$.

Let $k \in \mathbb{N}$, $k > 0$, and

$$L_2 := \{x \in \mathbb{Z}^n : x^\top z = 0\}$$

for some $z \in \{0, \pm 1, \pm k\}^n$.

Proposition 10. *L_2 satisfies (MFMC') for all e, all reorientations, and all capacity functions.*

Proof. We wlog. assume that $e = n$. By Proposition 9, the fact that for all $x \in L$ we have that $0 = x^\top z = x^\top(-z)$, and since the case $z_e = 0$ is trivial, we may assume that $z_n = -k$. Let $E_q := \{i \in E \backslash n : z_i = q\}$ for $q \in \{0, 1, -1, k, -k\}$. We set $y := z$ and choose a vector $x \in L_2^c$ such that x_n is maximal and $x_i = 0$ for $i \in E_{-1} \cup E_{-k}$. Note that we must have $x_n = -k \sum_{i \in E_k} x_i - \sum_{i \in E_1} x_i$. We show that $x_n = \left\lfloor \frac{c^+(y)}{k} \right\rfloor$. For assume to the contrary that

$$c^+(y) - x_n = kc(E_k) - k \sum_{i \in E_k} x_i + c(E_1) - \sum_{i \in E_1} x_i \ge k.$$

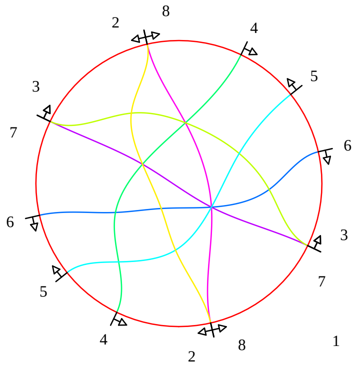
If $x_i < c(i)$ for some $i \in E_k$ then $x + \mathrm{e}_i + \mathrm{e}_n \in L_2^c$ contradicting the choice of x. But then $c(E_1) - \sum_{i \in E_1} x_e \ge k$ and we may choose $k_i \in \mathbb{N}$ such that $k_i \le c(i) - x_i$ and $\sum_{i \in E_1} k_i = k$, and therefore,

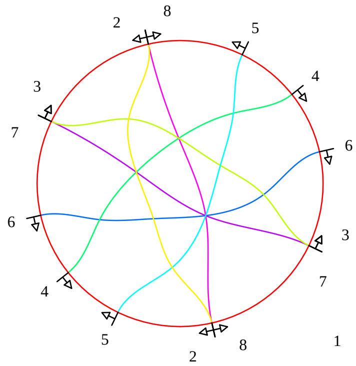
$$x + \sum_{i \in E_1} k_i \mathrm{e}_i + \mathrm{e}_n \in L_2^c$$

also contradicting the choice of x. □

By Hochstättler and Nickel [9], there are oriented matroids that satisfy (MFMC') but not (MFMC). We give two examples with the same underlying matroid the first of which does not satisfy (MFMC) and also demonstrates that (MFMC) and (MFMC') depend on the orientation of a matroid:

Example 8. *The following figure shows the dual representations of two reorientation classes of a rank 5 matroid. The second class is obtained from the first via a so-called* triangular switch *with respect to the triangle formed by* $\{1,4,5\}$*. By Proposition 7, the second satisfies* (MFMC) *but the first does not for* $e = 7$*.*





$\{x \in \mathbb{Z}^8 : (1,-1,-1,1,1,1,2,0)x = 0\}$ $\{x \in \mathbb{Z}^8 : 2 \mid (1,1,1,1,1,1,0,0)x\}$.

Note that this demonstrates that (MFMC) *is not a matroid property and furthermore, since even the first candidate satisfies* (MFMC) *if we reorient* $I = \{2,3\}$*, it is not reorientation invariant. The same holds for* (MFMC') *since a reorientation of* $\{1,2,3,4\}$ *in Example 6 leads to an oriented matroid that satisfies* (MFMC')*.*

3 Final Remarks and Open Questions

We do not know whether our min-max result is a "good characterization" in the sense of Edmonds [2], neither in the realizable case, nor when the oriented matroid is given by a pair of oracles (circuit and cocircuit). The problem of the complexity of minimizing the capacity of a vector in $\mathcal{F}_{\mathcal{O}}^{\perp}$ gives rise to the following problems:

- Characterize the oriented matroids with a regular flow lattice! Note that regularity of the flow lattice is not a property of the underlying matroid.
- Is it possible to recognize oriented matroids with regular flow lattice in polynomial time (in the realizable case or with respect to a suitable oracle)?

These questions address the problem of recognizing oriented matroids where (MFMC) holds. A positive answer to one of the following questions would establish the membership of our MaxFlow-Problem in $\mathcal{NP} \cap \text{co-}\mathcal{NP}$.

- Is it possible to check membership in $\mathcal{F}_{\mathcal{O}}^{\perp}$ in polynomial time?
- Is it possible to find a basis of $\mathcal{F}_{\mathcal{O}}$ in polynomial time?

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