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MASTER'S THESIS

Acyclic N-free Colorings of Digraphs and Perfection

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Contents

List of Figures

1 Introduction

Perfect graphs in terms of graph theory are a class of graphs for which every graph G fulfills the following property:

> The *clique number* $\omega(H)$ - the size of the largest complete subgraph of H - equals the *chromatic number* $\chi(H)$ - the minimum number of colors needed to achieve a proper coloring of H - for every induced subgraph H of G.

In this case we consider graph coloring in the *classical* sense, i.e. two adjacent vertices need to be colored differently. This means that perfect graphs are the class of graphs for which the size of the largest clique is the key factor in determining how many colors are needed to color a graph. Moreover this condition does not only hold globally for the whole graph but also locally for every subgraph of a perfect graph. This class of graphs was first introduced by Berge [9;10] in the early 1960's. (cf. [16;13])
Perfect graphs can be considered *perfect* in the way that many algorithmic problems that do not have an efficient solution in general can be solved efficiently for perfect graphs. The problem that led Berge to introducing perfect graphs has to do with the problem of errors in transmissions of information through a communication channel. Shannon [30] discussed the so-called zero error capacity of a channel in a paper published in 1956. Zero error capacity defines the maximal rate at which information can be transmitted through the channel without any possibility of error. This problem can be modeled by a graph G in which the vertex set represents the set of symbols that can be transmitted through the channel and two vertices are adjacent if the symbols they represent cannot be confused for one another. It is then possible to define graphs G^t, $t > 1$, whose vertex sets are all t-tuples of the vertices in G and two vertices in G^t are adjacent if and only if for some coordinate of the t-tuple the corresponding vertices in G are adjacent. So what we are interested in is the size of the largest cliques in G^t. The Shannon capacity of the original graph G is defined as

$$\lim_{n\to\infty} \frac{1}{n} \log \omega(G^n).$$

The following inequalities hold:

$$\omega^n(G) \leq \omega(G^n) \leq \chi(G^n) \leq \chi^n(G)$$

Hence if the graph G is perfect, i.e. especially $\chi(G) = \omega(G)$, the Shannon capacity is the same as the logarithm of this value. In general however the Shannon capacity is difficult to determine. (cf. [13])

Berge [9] conjectured one main property on perfect graphs that became the subject of studies in the field of graph theory for several decades, only being proven in 2002. It was known as the *Strong Perfect Graph Conjecture* and is now referred to as the *Strong Perfect Graph Theorem* after having been proven by Chudnovsky et al. [12]. It states that a graph is perfect if and only if it is *Berge*, i.e. it does not contain cycles of odd length nor complements of cycles of odd length as induced subgraphs.
One important partial result to proving this theorem was already conjectured by Berge [9] as well and is known as the *Weak Perfect Graph Theorem*. It was proven by Lovász [26] in 1971. It states that a graph is perfect if and only if its complement is perfect.
Another main result on perfect graphs states that perfection of graphs is kept under a certain kind of isomorphism of graphs, so-called P_4-isomorphism. In fact if two graphs G_1 and G_2 are P_4-isomorphic, then G_1 is perfect if and only if G_2 is perfect. This property known as the *Semi-strong Perfect Graph Theorem* was conjectured by Chvátal [14] and proved by Reed [28] in 1985.

Once perfection of graphs is established for (undirected) graphs one can consider the same for directed graphs. For the chromatic number we take Neumann-Lara's *dichromatic number* [27]. It corresponds to acyclic coloring of digraphs that forbids monochromatic cycles in a digraph. Andres and Hochstättler [4] used this dichromatic number to introduce *perfect digraphs* as those digraphs D for which the dichromatic number $\chi(H)$ equals the clique number $\omega(H)$ for every induced subdigraph H of D.

Unfortunately it can quite easily be seen that perfection of digraphs does not behave as *perfectly* as perfection of graphs, since for example the directed cycle of size four $\overrightarrow{C_4}$ is not perfect but its complement $\overline{\overrightarrow{C_4}}$ is. So there is no analogon to the Weak Perfect Graph Theorem for digraphs.
Andres and Hochstättler [4] however proved an analogon to the Strong Perfect Graph Theorem. This *Strong Perfect Digraph Theorem* states that a digraph D is perfect if and only if its symmetric part $S(D)$ is perfect and if it does not contain any directed cycle of length greater than three as an induced subdigraph.
It is also possible to formulate an analogon to the Semi-strong Perfect Graph Theorem by modifying the isomorphism that is considered. Andres et al. [3] introduced so-called P^4C-isomorphism and proved a *Semi-strong Perfect Digraph Theorem* which states that for two P^4C-isomorphic digraphs D and D', D is perfect if and only if D' is perfect.

The fact that there is no analogon for the Weak Perfect Graph Theorem suggests that acyclic coloring might not be the best way of coloring digraphs if we want to define perfection for digraphs. We take another family of graphs respectively digraphs into consideration, so-called *cographs* or *directed cographs* respectively. Cographs are graphs that do not have a path of length four - a P_4 - as an induced subgraph (see [17]). Andres et al. [3] used this definition to transfer the Semi-strong Perfect Graph Theorem to digraphs, since for graphs we considered P_4-isomorphism as precondition for this theorem. Directed cographs can be characterized by a set of eight forbidden induced subdigraphs (see [18]). Therefore Andres et al. [3] used five of these eight forbidden induced subdigraphs to define P^4C-isomorphism as precondition for the analogon for digraphs.

In this thesis we examine what happens to the three main properties on perfect (di)graphs if we take the remaining three forbidden subdigraphs into consideration with the aim of finding a possibly better way of coloring digraphs to define perfection.

We will determine that one of these subdigraphs is particularly interesting; a digraph we will call N-structure, that is a digraph with a set of four vertices v_1, v_2, v_3 and v_4 and three arcs (v_1, v_2), (v_3, v_2) and (v_3, v_4). We will use this subdigraph to define a different way of coloring digraphs. We will say a digraph can be (properly) *acyclic N-freely colored* if for every color c the subdigraph induced by the vertices colored with c does not contain an induced N-structure nor a directed cycle as subdigraphs.

As a main result we obtain a *Strong N-perfect Digraph Theorem* similar to the one proven for acyclic coloring: A digraph is N-perfect if and only if its symmetric part is perfect and it contains neither a directed cycle of length greater than three nor an induced N-structure as subdigraphs. We will further modify the definition of P^4C-isomorphism by adding the N-structure to the five considered forbidden subdigraphs and prove a *Semi-strong N-perfect Digraph Theorem*.

Finally we will see that the digraphs that are N-perfect and whose complements are also N-perfect require all eight of the forbidden subdigraphs to formulate a Semi-strong Perfection Theorem, so they form a class that is closely related to the class of directed cographs. Directed cographs are a class of digraphs that has been studied concerning various problems and for which many problems can be efficiently solved (cf. for example [23]). So one might say they are an *easier class* than perfect digraphs. This yields the expectation that maybe N-perfect digraphs are *already* a better way of looking at perfection of digraphs than acyclicly perfect digraphs.

2 Basic Terminology

In this section we want to introduce some basic terminology, notations and results concerning digraphs and graphs which will be used in the following sections.

2.1 Digraphs and Graphs

We start with introducing terminology and notations for digraphs and graphs. Moreover we show how a digraph can be transformed into a graph and vice versa. If not mentioned otherwise the following information is oriented at [7].

Definition 2.1. A *directed graph* or a *digraph* D is a pair $(V(D), A(D))$ consisting of a non-empty finite set $V(D)$ and a finite set $A(D) \subseteq V(D) \times V(D)$. We further define that $A(D)$ may not contain *loops*, i.e. there cannot be an element (v, v), $v \in V(D)$, in $A(D)$. If it is clear from context which digraph D we are talking about, we will write V and A instead of $V(D)$ and $A(D)$. The elements of V are called *vertices* and the elements of A are called *arcs*. Respectively V is called the *vertex set* of D and A is called the *arc set* of D.
The first vertex v_1 of an arc (v_1, v_2) is its *tail* and the second vertex v_2 is its *head*, both vertices are called its *end-vertices*. We will also say that v_1 *dominates* v_2 and that v_2 is *dominated by* v_1 and denote their relation by $v_1 \to v_2$. We say that two vertices v_1 and v_2 are *adjacent* or *connected* if either $(v_1, v_2) \in A$ or $(v_2, v_1) \in A$. A vertex v_1 is *incident* to an arc a if it is its tail or head.

Definition 2.2. For a digraph $D = (V, A)$ and a vertex $v_0 \in V$ we define the sets

$$N_D^+(v_0) = \{v \in V \setminus \{v_0\} \mid (v_0, v) \in A\}, \;\; N_D^-(v_0) = \{v \in V \setminus \{v_0\} \mid (v, v_0) \in A\}.$$

$N_D^+(v_0)$ is called the *out-neighborhood* and $N_D^-(v_0)$ the *in-neighborhood* of v_0. The set $N_D(v_0) = N_D^+(v_0) \cup N_D^-(v_0)$ is called the *neighborhood* of v_0. Respectively the vertices in $N_D^+(v_0)$, $N_D^-(v_0)$ and $N_D(v_0)$ are called *out-neighbors*, *in-neighbors* and *neighbors* of v_0.
We denote by $d_D^+(v_0)$ the number of arcs in D whose tail is v_0 and call this number the *out-degree* of v_0. Analogously we denote by $d_D^-(v_0)$ the number of arcs in D whose head is v_0 and call this number the *in-degree* of v_0. The *degree* of v_0 is $d_D(v_0) = d_D^+(v_0) + d_D^-(v_0)$.

We will often use a graphic visualization of a digraph denoting its vertices by dots and its arcs by arrows connecting the dots. An example is shown in Figure 1.

Note that by definition a digraph cannot contain *parallel arcs*, i.e. there can only be one arc connecting v_1 and v_2 in the same direction. There can however be a

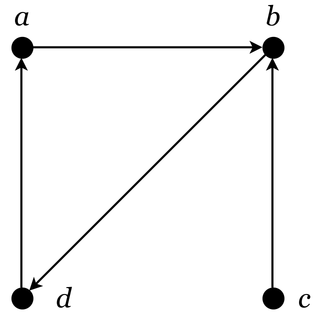


Figure 1: Example of a digraph $D = (V, A)$ with $V = \{a, b, c, d\}$ and $A = \{(a, b), (b, d), (c, b), (d, a)\}$.

digraph $D = (V, A)$ with $(v_1, v_2) \in A$ and $(v_2, v_1) \in A$, in this case we call these arcs a pair of *antiparallel arcs* ([4]). An arc $(v_1, v_2) \in A$ is called a *single arc* if $(v_2, v_1) \notin A$ ([4]). If a digraph D does not contain any pair of antiparallel arcs, D is called an *oriented graph*.

We will sometimes use an arrow with two heads to visualize a pair of antiparallel arcs due to conspicuousness.

Definition 2.3. An *undirected graph* or simply a *graph* G is a pair $(V(G), E(G))$ consisting of a non-empty set $V(G)$ and a finite set $E(G)$, consisting of unordered pairs of elements of $V(G)$. We define that $E(G)$ may not contain *loops*, i.e. an unordered pair of $E(G)$ must contain two different elements of $V(G)$. As for digraphs we will use V and E instead of $V(G)$ and $E(G)$ if there is no chance of misunderstanding. Again the elements of V are called *vertices* and V is called the *vertex set* of G. The elements of E are called *edges* and respectively E is called the *edge set* of G. We will denote edges the same way we denote arcs whereas in this case (v_1, v_2) and (v_2, v_1) denote the same element in E.
If $(v_1, v_2) \in E$ we call v_1 and v_2 *adjacent*.

If we use a graphic visualization of graphs, we again use dots to denote vertices and lines connecting the dots to denote edges.

Definition 2.4. For a graph G and a vertex $v_0 \in V$ the *neighborhood* of v_0 in G is defined as $N_G(v_0) = \{v \in V \setminus \{v_0\} \mid (v_0, v) \in E\}$. The *degree* $d(v_0)$ of a vertex v_0 is the number of edges connected to v_0.

Note that $E(G)$ cannot contain *parallel edges*, i.e. there can only be one edge connecting two specific vertices in G, so obviously we have $d(v_0) = |N_G(v_0)|$.

Definition 2.5. For a graph $G = (V, E)$ a digraph $D = (V, A)$ is called a *biorientation* or a *superorientation* ([4]) of G if D is obtained from G by replacing each

edge $(v_1, v_2) \in E$ by either one of the arcs (v_1, v_2) or (v_2, v_1) in D or the pair of antiparallel arcs (v_1, v_2) and (v_2, v_1). If the digraph D obtained by a biorientation of G is an oriented graph, D is called an *orientation* of G.
The *underlying graph* $UG(D)$ of a digraph D is the unique graph G such that D is a biorientation of G.
The *complete biorientation* $\overleftrightarrow{G}$ of a graph G is a biorientation D of G such that every arc $(v_1, v_2) \in A(D)$ implies $(v_2, v_1) \in A(D)$.
A digraph D is called *symmetric* if $(v_1, v_2) \in A(D)$ implies $(v_2, v_1) \in A(D)$.
The *symmetric part* $S(D)$ of D is the digraph $D_2 = (V, A_2)$ where A_2 is the union of all pairs of antiparallel arcs of D, the *oriented part* $O(D)$ of D is the digraph $D_1 = (V, A_1)$ where $A_1 = A \setminus A_2$ ([3]).

Remark 2.6. Regarding the definitions above an undirected graph $G = (V, E)$ can be identified with the symmetric digraph $D_G = (V, A)$ with $A = \{(v_1, v_2), (v_2, v_1) \mid (v_1, v_2) \in E\}$, therefore we will not distinguish between G and D_G in this thesis ([4]). Most definitions made hereafter for graphs or digraphs can therefore be easily transferred to the other.

Definition 2.7. Two digraphs D_1 and D_2 are *isomorphic* if there exists a bijection $\phi : V(D_1) \to V(D_2)$ such that $(v_1, v_2) \in A(D_1)$ if and only if $(\phi(v_1), \phi(v_2)) \in A(D_2)$ for every ordered pair v_1, v_2 of vertices in $V(D_1)$.

Definition 2.8 ([28])**.** An *endomorphism* of a digraph D is a mapping $f : V(D) \to V(D)$ such that if $v_1 \to v_2$ in $V(D)$ then $f(v_1) \to f(v_2)$ in $f(V(D))$, note that there is no condition for non-neighboring vertices in $V(D)$. An endomorphism is called a *proper endomorphism* if $f(V(D))$ is a proper subset of $V(D)$.

Definition 2.9. A graph G is called *planar* if there exists a mapping $f : G \to \mathbb{R}^2$ with the following properties:

- Every vertex of G is mapped to a point in $\mathbb{R}^2$ and different vertices are mapped to different points and
- every edge $(v_1, v_2) \in E(G)$ is mapped to a not self-intersecting curve $C_{v_1 v_2}$ from $f(v_1)$ to $f(v_2)$ and no two curves corresponding to different edges in G intersect (except for possibly at points corresponding to their end-vertices).

A digraph D is called planar if its underlying graph $UG(D)$ is planar.

This simply means that for every planar (di)graph there exists a visualization of the (di)graph drawn on a plane with no intersecting arcs/ edges.

2.2 Specific Digraphs and Graphs

In this subsection we will define some specific digraphs and graphs. Again if not mentioned otherwise the following information is oriented at [7].

Definition 2.10. A digraph H is called a *subdigraph* or a *minor* of a digraph D if $V(H) \subseteq V(D)$, $A(H) \subseteq A(D)$ and for every arc in $A(H)$ both end-vertices are included in $V(H)$. We will denote a subdigraph by $H \subseteq D$. If $H \neq D$ we call H a *proper subdigraph* of D.
If every arc of $A(D)$ with both end-vertices in $V(H)$ is included in $A(H)$, we say that H is *induced* by $V' = V(H)$ and H is called an *induced subdigraph* of D. We will denote the subdigraph of D induced by V' by $D[V']$.
A set $S \subseteq V$ which induces a subdigraph $H = (S, \varnothing)$ is called a *stable set*, a stable set $K \subseteq V$ is called a *kernel* in a digraph if it is *absorbing*, i.e. for any $v_1 \in V \setminus K$ there is an arc $(v_1, v_2) \in A$ with $v_2 \in K$ ([4]).

Definition 2.11 ([4])**.** The *complement* $\overline{D}$ of a digraph D is the digraph with the same vertex set $V(\overline{D}) = V(D)$ and an arc (v_1, v_2) is in $A(\overline{D})$ if and only if $(v_1, v_2) \notin A(D)$.

Definition 2.12. A *walk* in D is an alternating sequence $W = v_1a_1v_2a_2 \dots v_{k-1}a_{k-1}v_k$ of vertices v_i and arcs a_j from D such that $a_i = (v_i, v_{i+1})$ for every $i \in [1, k-1]$. v_1 is the *initial vertex* of W and v_k is its *terminal vertex*, both vertices are called *end-vertices* of W. We also say that W is a $[v_1, v_k]$-*walk*.
A *path* W is a walk with distinct vertices and a *cycle* is a walk with distinct vertices $v_1, v_2, \dots, v_{k-1}$, $k \geq 3$ and $v_1 = v_k$. A digraph is called *acyclic* if it neither contains a cycle nor a pair of antiparallel arcs.
We will denote a (directed) path by $\overrightarrow{P_n}$ and a (directed) cycle by $\overrightarrow{C_n}$ where n is the number of vertices of the path or cycle and we will call n the *length* of the path or cycle. [1] According to Remark 2.6 walks, paths and cycles are also defined for graphs. To differentiate between digraphs and graphs we will refer to those structures in digraphs as directed walks, directed paths and directed cycles and in graphs as undirected walks, undirected paths and undirected cycles. We will refer to them simply as walks, paths and cycles if it is clear from context whether they are directed or undirected. We will denote an undirected cycle of length n by C_n and an undirected path of length n by P_n.

Definition 2.13 ([4])**.** A *hole* in a graph G is a cycle $C_n \subseteq G$ that is induced in G. An *antihole* of G is an induced subdigraph of G whose complement is a hole in $\overline{G}$. We define the size or length of a hole or an antihole by the number of its vertices.

[1] This definition of length does not correspond to [7], where the length of a path is defined as the number of arcs.

A *disc* is a hole or an antihole that has at least size five ([28]).
For a digraph D, a *filled hole/ antihole* is a subdigraph $H \subseteq D$, so that $S(H)$ is a hole/ antihole.

Definition 2.14. For a graph G that contains an undirected cycle C as a subgraph an edge connecting two vertices $v_1,\ v_2 \in V(C)$ that are not adjacent in C is called a *chord*. A *chordal graph* is an undirected graph G in which every cycle C_n of G, $n \geq 4$, has a chord.
In a digraph D a chord can either be a pair of antiparallel arcs or a single arc.

Definition 2.15. The *union* $D_1 \cup D_2$ of two digraphs D_1 and D_2 is the digraph D such that $V(D) = V(D_1) \cup V(D_2)$ and $(v_1, v_2) \in A(D)$ if $(v_1, v_2) \in A(D_1)$ or $(v_1, v_2) \in A(D_2)$ or both.
The *disjoint union* $D_1 \oplus D_2$ of two vertex-disjoint digraphs D_1 and D_2 is the digraph with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$ ([23]).

Definition 2.16 ([23])**.** The *series composition* $D_1 \otimes D_2$ of two digraphs D_1 and D_2 is the digraph D with vertex set $V(D) = V(D_1) \cup V(D_2)$ and arc set $A(D) = A(D_1) \cup A(D_2) \cup \{(v_1, v_2) \mid (v_1 \in V(D_1), v_2 \in V(D_2)) \text{ or } (v_1 \in V(D_2), v_2 \in V(D_1))\}$.

Union, disjoint union and series composition of more than two digraphs are defined recursively.

Definition 2.17 ([23])**.** The *order composition* $D_1 \oslash D_2 \oslash \cdots \oslash D_k$ of k digraphs $D_1, D_2, \ldots, D_k$ is the digraph D with vertex set $V(D) = V(D_1) \cup V(D_2) \cup \cdots \cup V(D_k)$ and arc set $A(D) = A(D_1) \cup A(D_2) \cup \cdots \cup A(D_k) \cup \{(v_1, v_2) \mid v_1 \in V(D_i), v_2 \in V(D_j), 1 \leq i < j \leq k\}$.

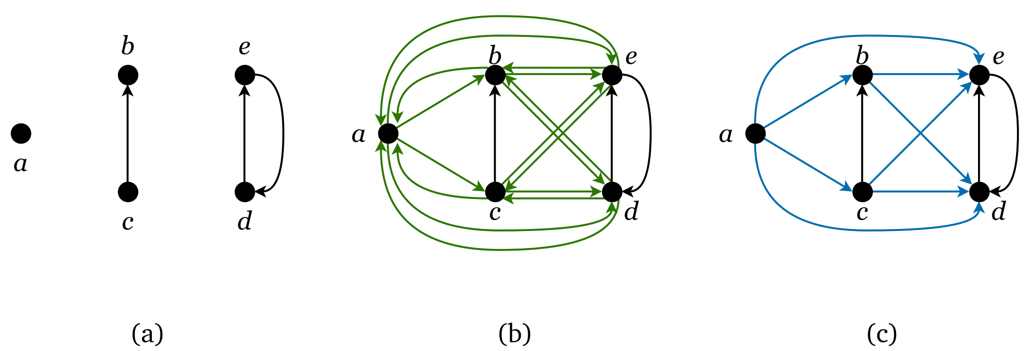


Figure 2: Example of the disjoint union (a), the series composition (b) and the order composition (c) of three vertex-disjoint digraphs; D_1 is the single vertex a, D_2 the directed path of length two and D_3 the directed cycle of length two.

The following definition is needed only for undirected graphs.

Definition 2.18 ([26]). Let G_1 and G_2 be two vertex-disjoint digraphs and $v \in V(G_1)$. By *substituting* G_2 *for* v we mean the construction of a graph G' with $V(G') = (V(G_1) \setminus \{v\}) \cup V(G_2)$ and the edge set of G' consists of the following edges: all edges of the subgraph of G_1 induced by $V(G_1) \setminus \{v\}$, all edges of G_2 and thirdly all vertices of G_1 that are adjacent to v in G_1 are adjacent to all vertices of G_2 in G'.

Definition 2.19. For a digraph D the *line digraph* $L(D)$ is the digraph with vertex set $V(L(D)) = A(D)$ and an arc (v, w) is in $A(L(D))$ if and only if v represents an arc $(\tilde{v_1}, \tilde{v_2})$ in D and w represents an arc $(\tilde{v_2}, \tilde{v_3})$ in D.

2.3 Connectivity of Digraphs and Graphs

In this subsection we focus on connectivity of digraphs and graphs and define some specific graphs and digraphs that are characterized by their connectivity. Again if not mentioned otherwise the following information is oriented at [7].

Definition 2.20. A digraph D is *strongly connected* if for every pair v_1, v_2 of vertices in D, $v_1 \neq v_2$, there exists a $[v_1, v_2]$-path and a $[v_2, v_1]$-path in D.
A digraph D is *connected* if the underlying graph $UG(D)$ is connected, i.e. the complete biorientation of $UG(D)$ is strongly connected.
An undirected graph is *disconnected* if it is not connected.

Definition 2.21. A *strong component* of a digraph D is an induced subdigraph H of D which is strongly connected and for which every subdigraph of D induced by $V(H) \cup v$, $v \notin V(H)$, is not strongly connected.
For an undirected graph G the strong components of the complete biorientation $\overleftrightarrow{G}$ of G are called *components* of G.

Definition 2.22. A graph $G = (V, E)$ is called a *bipartite graph* if there exists a partition of the vertex set $V = V_1 \cup V_2$, $V_1 \cap V_2 = \varnothing$, such that every edge connects a vertex of V_1 to a vertex of V_2.
G is a *complete bipartite graph* if for every pair $v_i \in V_1$ and $v_j \in V_2$ there is an edge $(v_i, v_j) \in E$. We denote a complete bipartite graph by K_{V_1,V_2}.

Definition 2.23. An undirected graph is called a *tree* if it is connected and does not contain a cycle as a subgraph. An *oriented tree* is a superorientation of a tree.
The vertices of a tree with degree one are called *leaves*, all other vertices are called *inner* or *internal vertices*.
Sometimes it is required to consider a kind of ordering of the vertices in a tree. A tree T can be a *rooted tree* with *root* $r \in V(T)$, where r is one specified vertex of the tree that we want to consider superior (or inferior, depending on the interpre-

tation) to all the other vertices. Note that if we have a rooted tree the root is never considered a leaf even if it has degree one. ([21])
In a rooted tree T we call a vertex v_1 the *child* of the vertex v_2, if there is a path $v_1, v_2, \dots, r$ in T. In this case v_2 is called the *parent* of v_1. ([17])
For two vertices v_1, v_2 in a rooted tree T a vertex v_0 is called their *lowest common ancestor* if v_0 is on the unique path from v_1 to r and on the unique path from v_2 to r and if it is the first vertex where those paths intersect. Note that the lowest common ancestor can be one of the vertices or r. ([1])

There are some equivalent definitions for trees. See [21] (Theorem 1.5.1) for the proof of the following lemma.

Lemma 2.24 ([21], Theorem 1.5.1)**.** *If T is a graph with n vertices the following assertions are equivalent:*

1. *T is a tree;*
2. *For any two vertices $v_1, v_2 \in V(T)$ there is a unique path between those vertices;*
3. *T is minimally connected, i.e. T is connected but every subgraph $T' \subseteq T$ with $V(T') = V(T)$ and $E(T') = E(T) \setminus \{e\}$ for any $e \in E(T)$ is not connected;*
4. *T is maximally acyclic, i.e. T does not contain a cycle but adding an edge between any two non-adjacent vertices in T creates a graph that contains a cycle;*
5. *T is connected and has $n-1$ edges.*

Definition 2.25 ([17])**.** In a graph G two vertices v_1, v_2 are called *siblings* if $N_G(v_1) \setminus \{v_2\} = N_G(v_2) \setminus \{v_1\}$. They are *strong siblings* if they are adjacent and *weak siblings* otherwise.

Definition 2.26 ([15;28])**.** A *star-cutset* S in a graph G is a set of vertices such that $V(G) \setminus S$ induces a disconnected graph and some vertex $v \in S$, called the *center*, is adjacent to all vertices in $S \setminus \{v\}$. We call a graph G *unbreakable* if neither G nor its complement $\overline{G}$ contains a star-cutset. If a graph is not unbreakable it is called *fragile*.

Definition 2.27. A *complete* digraph $D = (V, A)$ is a digraph with $(v_1, v_2) \in A$ and $(v_2, v_1) \in A$ for all $v_1,\ v_2 \in V$. We denote a complete digraph with n vertices by K_n.

Definition 2.28 ([4])**.** A subdigraph $H \subseteq D$ is called a *clique* in D if for all $v_1,\ v_2 \in V(H)$ both arcs (v_1, v_2) and (v_2, v_1) exist in H (and D), hence if H is a complete digraph. The *clique number* $\omega(D)$ of D is the size of the largest clique in D.

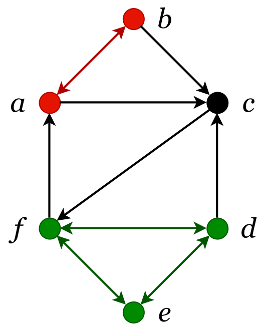


Figure 3: The subdigraph induced by the vertex subset $\{a, b\}$ and the subdigraph induced by $\{d, e, f\}$ are cliques of size two and three in the digraph. As they are no larger cliques in D we have $\omega(D) = 3$.

Observation 2.29 ([4])**.** *For a digraph D we have $\omega(D) = \omega(S(D))$.*

Definition 2.30 ([17])**.** A graph G has the *clique-kernel intersection property* (or *CK-property*) if every clique of G has exactly one vertex in common with every kernel of G.

2.4 Cographs and Directed Cographs

Complement reducible graphs or *cographs* is the name of a family of graphs that occurred independently in different areas of mathematics. An overview on different characterizations and basic properties was given by Corneil et al. [17] in 1981, the following information is oriented at this work. In this subsection we will begin with the definition for undirected graphs and transfer it to digraphs later on.

Definition 2.31. A *complement reducible graph* or *cograph* is a graph constructed recursively as follows:

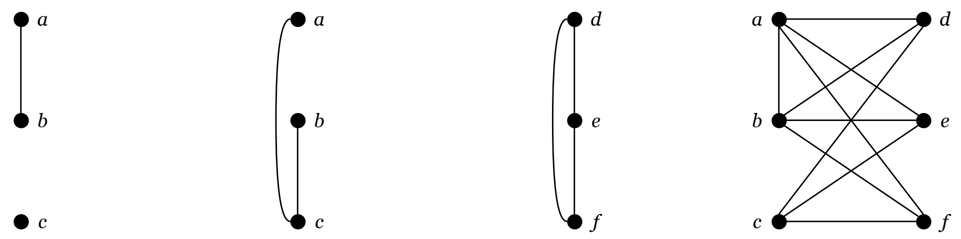
1. A single vertex is a cograph,
2. if $G_1, G_2, \ldots, G_k$ are cographs, then so is their (disjoint) union $G_1 \cup G_2 \cup \cdots \cup G_k$ and
3. if G is a cograph, then so is its complement $\overline{G}$.

Corneil et al. defined a *normalized form of a cograph* to establish a notation of a certain cograph that is unique up to isomorphism.
A connected cograph is in normalized form if it is a single vertex or the complemented union (i.e. the complement of the union) of at least two connected cographs in normalized form. A disconnected cograph is in normalized form if it is the complement of a connected cograph in normalized form.

Figure 4 shows the construction of the cograph of the normalized form

$$\overline{\cup}(\overline{\cup}(\overline{\cup}(a,b),c),\overline{\cup}(d,e,f)),$$

where $\overline{\cup}$ stands for the complemented union (example taken from [19]).
Figure (a) shows the union $\cup(\overline{\cup}(a,b),c)$, Figure (b) shows the complement of Figure (a), Figure (c) shows $\overline{\cup}(d,e,f)$ and Figure (d) the complemented union of Figures (b) and (c), the actual cograph represented by the normalized form.



(a) (b) (c) (d)

Figure 4: Example of a cograph and its construction. (Example taken from [19])

Definition 2.32. A *cotree* for a cograph is a tree representing the normalized form of a cograph in the following way:

1. The leaves of the cotree are the vertices of the cograph,
2. the internal vertices of the cotree represent the complemented unions, i.e. two or more vertices $v_1, v_2, \ldots, v_k$ are children of an internal vertex if there is an expression $\overline{\cup}(v_1, v_2, \ldots, v_k)$ in the normalized form of the cograph where every $v_1, v_2, \ldots, v_k$ is either a vertex in the cograph or a complemented union itself and
3. the root of the cotree represents the outer most complemented union of the normalized form of the cograph.

To establish various properties later on, we label the internal vertices of a cotree in the following way: the root is labeled 1, the children of a vertex with label 1 are labeled 0 and vice versa.

The internal vertices of a cotree can be interpreted in an alternative way: A vertex labeled 1 represents a series composition of all its children and a vertex labeled 0 represents a (disjoint) union of its children. This interpretation corresponds with Definition 2.32, because of the following observations. Firstly a series composition of two single vertices is the same as the complemented union of these two vertices. Secondly all children of vertices labeled 1 have an odd number of complemented unions operating on them. An odd number of complemented unions is again a

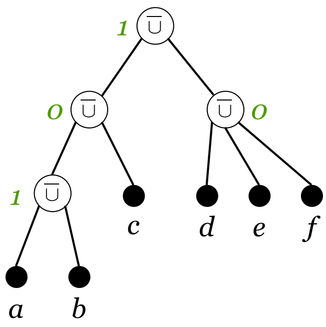


Figure 5: Cotree for the cograph in Figure 4. The green labels are the labels for the internal vertices as described above. (Figure taken from [19])

complemented union. And the children of vertices labeled 0 have an even number of complemented unions operating on them and the complemented union of a complemented union is just a union.

The two proofs of the following lemma clarify how the two interpretations of the construction of a cograph from a cotree lead to the same cograph.

Lemma 2.33. *Let G be a cograph with cotree T. Two vertices $v_1, v_2 \in V(G)$ are adjacent in G if and only if their lowest common ancestor in T is labeled 1.*

Proof. Assume the lowest common ancestor v of two vertices v_1, v_2 in T has label 1. This means that the first operation that operates on both vertices is a series composition, so v_1 and v_2, which were non-adjacent until this operation is executed, get connected by that operation in the construction process of G. All the remaining operations that are executed do not have any effect on the connection of v_1 and v_2 since they are just (disjoint) unions and series compositions with other vertex sets.
Similarly it follows that if v has label 0, the first operation that operates on both vertices is a (disjoint) union, so v_1 and v_2 remain disconnected and all the following operations do not have any effect on that.
If we consider the interpretation of the internal vertices of T being complemented unions the following argumentation leads to the same assertion. Let v be the lowest common ancestor of v_1 and v_2. Both vertices get connected through the complemented union represented by v since this is the first one that effects both of them and they were non-adjacent until this point. If v has label 1, there is an even number of complementations executed after the step of construction represented by v since the root is also labeled 1. Hence in the final graph v_1 and v_2 are again adjacent. If on the other hand v has label 0, there are an odd number of complementations executed after the step of construction represented by v, hence

the complement of the graph constructed in the step of construction represented by v is a subgraph of G and therefore v_1 and v_2 are non-adjacent in G. □

Lemma 2.34 ([17], Lemma 1). *Every induced subgraph of a cograph is a cograph.*

Proof. The assertion is obviously true for a graph with less than three vertices, thus assume the cograph $G = (V, E)$ has at least three vertices.
Any induced subgraph of G can be obtained by removing vertices (and corresponding edges) one by one. Hence it is sufficient to show that the graph obtained by removing a single vertex from a cograph is again a cograph.
Let T be the cotree for G. The subgraph G' induced by $V \setminus \{v\}$ is a cograph if and only if there is a cotree T' representing it. Let v_0 be the parent of v in the cotree. If v_0 has more than two children, T' can be constructed by removing the leaf v from T. If v_0 has exactly two children, v and v', there are two possible cases. Firstly if v' is a leaf, we can construct T' by removing v_0 and v from T and connecting v' to the parent of v_0. Secondly if v' is an internal vertex, we remove v, v' and v_0 from T and connect all children of v' to the parent of v_0.
In all cases T' is a cotree representing the induced subgraph G' thus G' is a cograph. □

The fundamental theorem on cographs gives a variety of equivalent definitions. It was formulated by Corneil et al. [17]. We will only state the equivalent definitions we need to prove the one we will refer to later on. It states that a graph is a cograph if and only if it does not contain a P_4 as an induced subgraph.

Theorem 2.35 ([17], Theorem 2). *Let G be a graph. The following properties are equivalent:*

1. *G is a cograph.*
2. *Any nontrivial induced subgraph of G has at least one pair of siblings.*
3. *Any nontrivial induced subgraph of G has the CK-property.*
4. *G does not contain a P_4 as an induced subgraph.*
5. *If a nontrivial induced subgraph of G is connected, its complement is disconnected.*

Proof. (1) ⇒ (2). Since every induced subgraph of a cograph is again a cograph it suffices to prove the property for any nontrivial cograph. If we consider the corresponding cotree T for the cograph G, we see that any two leaves in T with the same parent are strong siblings in G if the parent is labeled 1 and weak siblings if the parent is labeled 0. Every nontrivial cotree has at least one parent with at

least two leaves, hence the property follows.
(2) ⇒ (3). This is proved by induction on $p = |V(H)|$ for an induced subgraph $H \subseteq G$. Obviously a subgraph with one vertex has the CK-property. Let H' be an induced subgraph with $p+1$ vertices and let v, v' be siblings in H'. We can express the cliques and kernels of H' by the cliques and kernels of the induced subgraph H with $V(H) = V(H') \setminus \{v'\}$ which obviously has p vertices and therefore fulfills the property by induction. If v and v' are strong siblings, any clique of H not containing v remains a clique in H' and any clique containing v becomes a clique with one additional vertex v', since v' is adjacent to the same vertices as v and v itself. Any kernel of H remains a kernel in H' and for every kernel containing v we obtain an additional kernel of the same size in H' if we replace v by v'. Comparing all pairs of kernels and cliques we can differentiate three cases. Firstly they could both be the same as for H hence they have one intersecting vertex by induction. Secondly we could have a clique not containing v that has exactly one intersecting vertex with a *new* kernel - the one it had before with the kernel the new one was constructed of. Thirdly we could have a *new* clique that has exactly one intersecting vertex with a *new* kernel - v' and with an *old* kernel - the one the original clique had with it. If v and v' are weak siblings the assertion follows exactly the same way if we interchange the notions for cliques and kernels.
(3) ⇒ (4). A P_4 does not have the CK-property since the second and third vertex form a clique that does not intersect the kernel formed by the first and fourth vertex. Hence G does not contain a P_4 as an induced subgraph.
(4) ⇒ (5) ([29]). We prove this implication through contraposition. Assume there is a nontrivial induced subgraph of G that is connected and whose complement is also connected. Let X be a vertex set that induces such a subgraph in G and let X be the smallest possible vertex set fulfilling this property. The vertex set $X' = X \setminus \{v_1\}$, $v_1 \in X$, induces a subgraph that is not connected in G or not connected in $\overline{G}$ since X is the smallest vertex set with the assumed property. Without loss of generality assume that it is not connected in G (otherwise change the roles of G and $\overline{G}$).
There is a vertex $v_2 \in X'$ that is adjacent to v_1 in $\overline{G}$ since $\overline{G[X]}$ is connected and $X \geq 2$. Let X'' be the vertex-set of the component of the subgraph induced by X' in G that contains v_2. There are no edges between X'' and $X' \setminus X''$.
Since $G[X]$ is connected there are vertices $v_3 \in X''$ and $v_4 \in X' \setminus X''$ that are both adjacent to v_1 in G. Let S' be the vertex set of all vertices of X'' adjacent to v_1 in $\overline{G}$ and let S'' be the vertex set of all vertices of X'' adjacent to v_1 in G. Since those two sets are complementary vertex subsets of a component of an induced subgraph in G, there are vertices $v_2' \in S'$ and $v_3' \in S''$ that are adjacent in G.
We obtained the edges (v_2', v_3'), (v_3', v_1), (v_1, v_4) in G and the edges (v_1, v_2'), (v_2', v_4),

(v_4, v_3') in $\overline{G}$ (see Figure 6). Hence the subgraph induced by these four vertices is an induced P_4 in G and $\overline{G}$.
(5) ⇒ (1). This is proved by induction on the number of vertices n in G. The statements holds for $n \leq 3$ trivially. Assume the property holds for all $n < p$ and consider a graph G with p vertices. If property (5) holds for G than it also holds for $\overline{G}$. Being a cograph is also kept under complementation by definition. Thus we might examine G or $\overline{G}$ and choose the one that is not connected. All components of this graph are cographs by induction. Hence the graph is a cograph by Definition 2.31. □

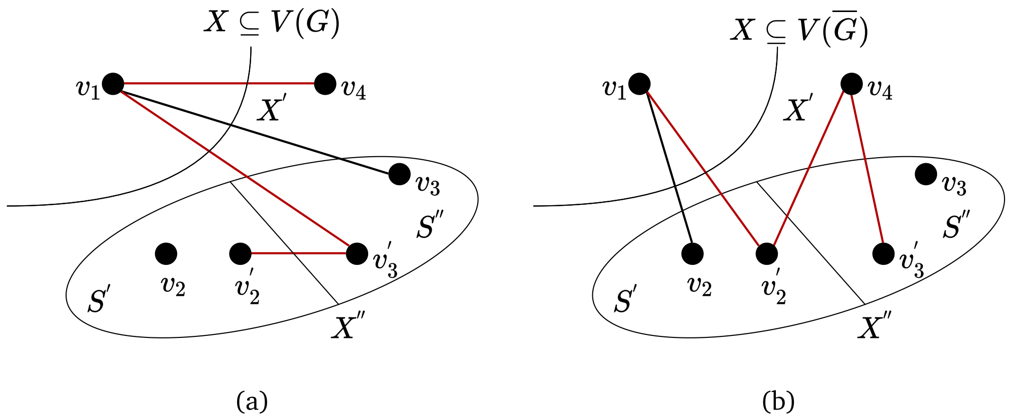


Figure 6: Construction used for the deduction of (4) ⇒ (5) in the proof of Theorem 2.35.

Bechet et al. [8] transferred the idea of cographs to directed graphs. The following information is oriented at Crespelle and Paul [18] and Gurski [23] who introduced algorithms on directed cographs.

Definition 2.36. A *directed cograph* is a digraph constructed recursively as follows:

1. A single vertex is a directed cograph,
2. if $D_1, D_2, \ldots, D_k$ are directed cographs, then so is their disjoint union $D_1 \oplus D_2 \oplus \cdots \oplus D_k$,
3. if $D_1, D_2, \ldots, D_k$ are directed cographs, then so is their series composition $D_1 \otimes D_2 \otimes \cdots \otimes D_k$ and
4. if $D_1, D_2, \ldots, D_k$ are directed cographs, then so is their order composition $D_1 \oslash D_2 \oslash \cdots \oslash D_k$.

In the following we call the three compositions - disjoint union, series composition and order composition of directed graphs - the operations to construct a directed cograph.

Figure 7 shows the construction of the directed cograph $D = ((a \oslash b) \oplus c) \oslash (d \otimes e)$, where a single vertex v stands for the digraph $(\{v\}, \varnothing)$ (example taken from [23]). We see $a \oslash b$ in Figure (a), $(a \oslash b) \oplus c$ in Figure (b), in Figure (c) $d \otimes e$ is added and Figure (d) is the actual directed cograph D.

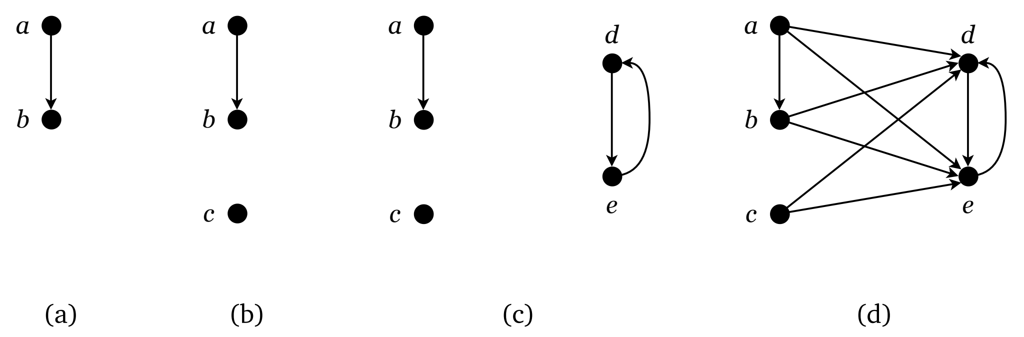


Figure 7: Example of a directed cograph and its construction. (Example taken from [23])

Definition 2.37. The *di-cotree* of a directed cograph is a tree with the following properties:

1. The leaves of the di-cotree are the vertices of the cograph and
2. an inner vertex of the di-cotree represents the operation applied on the children of this vertex.

Note that with this formulation a di-cotree must be interpreted in a specific order, since the order composition is not commutative, to correspond to a unique directed cograph. We can reduce the number of directed cographs represented by the same di-cotree by using the associativity of the operations to transfer a di-cotree into a binary di-cotree in which every inner vertex has exactly two children.

Lemma 2.38 ([23], Lemma 1)**.** *Every di-cotree T for a directed cograph D can be transformed into a binary di-cotree.*

Proof. Let v be a vertex in T with more than two children $v_1, \dots, v_n$. Since the operation $v_1 \circ \dots \circ v_n$ ($\circ$ being one of the three operations) represented by v is associative it can also be written as $(((v_1 \circ v_2) \circ v_3) \circ \dots) \circ v_n$. Hence it can also be represented by a corresponding di-cotree in which every inner vertex has exactly two children. □

Similar as for cotrees we label the internal vertices of a di-cotree in the following way: the root is labeled 1, the children of a vertex with label 1 are labeled 0 and vice versa.

Note that the di-cotree is an undirected graph not a digraph.

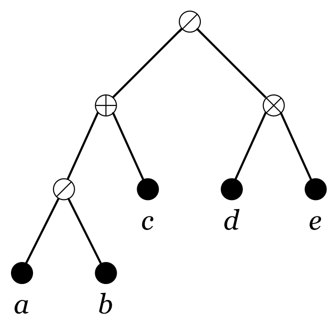


Figure 8: Corresponding di-cotree of the directed cograph in Figure 7. (Figure taken from [23])

Lemma 2.39 ([23], Lemma 2). *For a directed cograph D its complement $\overline{D}$ is a directed cograph.*

Proof. Let D be constructed by a series of operations performed on the directed cographs $D_1, D_2, \ldots, D_n$, $n \in \mathbb{N}$. The series of operations to construct $\overline{D}$ from the directed cographs $\tilde{D_1}, \tilde{D_2}, \ldots, \tilde{D_n}$ can be recursively defined as follows.
If D_i is a single vertex v, than $\tilde{D_i}$ is as well the single vertex v. If $D_i = D_j \oplus D_k$, then $\tilde{D_i} = \tilde{D_j} \otimes \tilde{D_k}$. If $D_i = D_j \otimes D_k$, then $\tilde{D_i} = \tilde{D_j} \oplus \tilde{D_k}$. If $D_i = D_j \oslash D_k$, then $\tilde{D_i} = \tilde{D_k} \oslash \tilde{D_j}$.
Since the disjoint union of two directed cographs creates no arcs between the two directed cographs and the series composition of these two directed cographs includes all arcs between these directed cographs, the two operations are complementary to one another. An order composition of two directed cographs is complementary to its reversion since they create arcs of different directions between the two directed cographs. Hence the defined operations on $\tilde{D_1}, \tilde{D_2}, \ldots, \tilde{D_n}$ construct $\overline{D}$. □

Lemma 2.40 ([23], Lemma 3). *If D is a directed cograph, every induced subdigraph of D is a directed cograph.*

Proof. The proof is similar to the one for undirected cographs. Again the assertion is obviously true for a digraph with less than three vertices. Let D be a directed cograph with at least three vertices, $v \in V(D)$ and let T be the corresponding binary di-cotree. The subdigraph D' induced by $V(D) \setminus \{v\}$ is a directed cograph if and only if there is a di-cotree representing it. We obtain the corresponding di-cotree T' for the subdigraph D' by the following method.
If v is a leaf in T, then T' is obtained by deleting v and the parent of v and

connecting the other child of the parent of v to the parent's parent. T' is obviously a di-cotree, hence D' is a directed cograph.
Any induced subdigraph of D can be obtained by removing vertices one by one, so inductive application of the method above proves the assertion. □

The following theorem gives another characterization of directed cographs. Crespelle and Paul [18] concluded it from a result of [22]. See those papers for further details.

Theorem 2.41 ([18], Theorem 2)**.** *A digraph D is a directed cograph if and only if it does not contain any of the digraphs in Figure 9 as an induced subdigraph.*

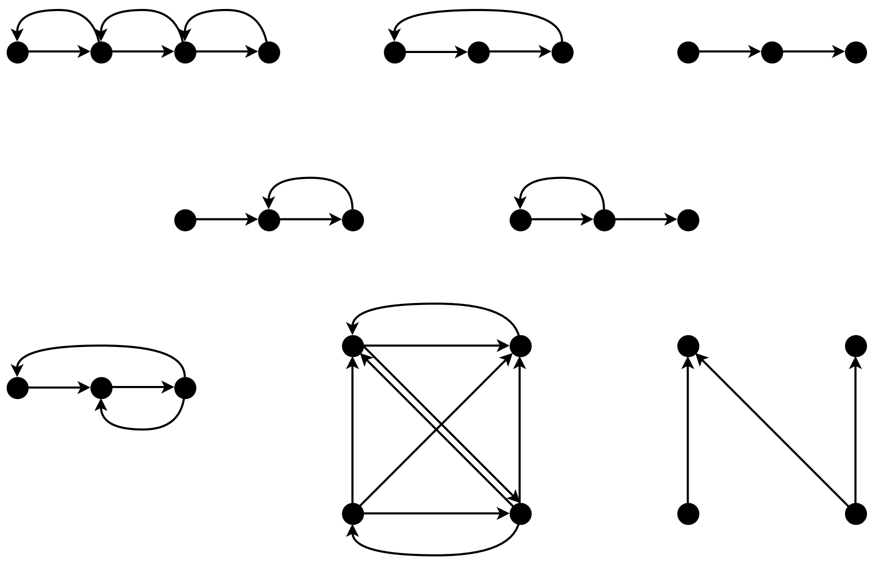


Figure 9: Set of forbidden subdigraphs for directed cographs. (Figure taken from [18])

Observation 2.42. *The set of forbidden subdigraphs for directed cographs is closed under complementation.*

As we will refer to it later on, we will give a name to the forbidden subdigraph in the right bottom corner of Figure 9. Clearly the name descends from the visualization shown in Figure 9.

Definition 2.43. We call a digraph $D = (V, A)$ with $V = \{v_1, v_2, v_3, v_4\}$ and $A = \{(v_1, v_2), (v_3, v_2), (v_3, v_4)\}$ an *N-structure*.

2.5 Hypergraphs

Hypergraphs can be considered as a generalization of graphs with regard to edges containing not only exactly two vertices, but also just one vertex or more than two vertices and multiple edges with the same vertices being allowed. We will use hypergraphs later on to replicate a proof by Lovász [26] for a theorem on perfect graphs (see Section 4.1). Therefore the terminology in this subsection is also taken from the same paper.

Definition 2.44. A *hypergraph* is a pair $H = (V, E)$ consisting of a finite vertex set V and a non-empty finite collection of edges E of non-empty finite sets $E_i \subseteq V$. We allow multiple edges containing exactly the same vertices. If there are $n \in \{1, \dots, |V|\}$ edges with exactly the same vertices, we call n the *multiplicity* of them. The *degree* $\delta(v)$ of a vertex $v \in V$ is the number of edges containing v, we denote by $\delta(H)$ the maximum degree of vertices in H. A *partial hypergraph* of a hypergraph H is a hypergraph $\tilde{H}$ consisting of a subset of edges of H.

Definition 2.45. A set $T \subseteq V$ is called a *transversal* of a hypergraph $H = (V, E)$ if there is at least one vertex from every edge $E_i \in E$ in T. We denote by $\tau(H)$ the size of the smallest transversal T of H.

Definition 2.46. We denote by $\nu(H)$ the maximum number of pairwise disjoint edges of a hypergraph H. If $\nu(\tilde{H}) = \tau(\tilde{H})$ for every partial hypergraph $\tilde{H}$ of H we say that H is *τ-normal*.

There have to be at least as many vertices in a minimal transversal T of a hypergraph H as is the size of the maximum set of pairwise disjoint edges in H (at least one vertex from each edge of this set has to be in T). Hence we always have $\nu(H) \leq \tau(H)$.

Definition 2.47. A hypergraph H has the *Helly property* if any set of edges $\tilde{E} \subseteq E$ (with at least three edges) with $\bigcap_{E_i \in \tilde{E}} E_i = \varnothing$ contains at least two disjoint edges.

A hypergraph that does not have the Helly property cannot be τ-normal, since exists a partial hypergraph with edge set $\tilde{E}$ with $\bigcap_{E_i \in \tilde{E}} E_i = \varnothing$ that has no pairwise disjoint edges but requires a transversal of at least size two. Hence $\nu(\tilde{E}) = 1 < 2 = \tau(\tilde{E})$. So τ-normal hypergraphs always have the Helly property.

There is a way we can transform hypergraphs into undirected graphs and vice versa:
Given a hypergraph $H = ((V(H), E(H))$ we can define its *edge-graph* $G(H) = G$ which is an undirected graph with vertex set $V(G) = E(H)$ and two vertices are adjacent in G if the two corresponding edges in H intersect.
Given an undirected graph G we can construct a hypergraph $H(G) = H$ with the vertices of H corresponding to the maximal cliques (in the sense of these cliques not being part of larger cliques) in G and a vertex v in G creating an edge E_v in H such that E_v equals the set of maximal cliques containing v.

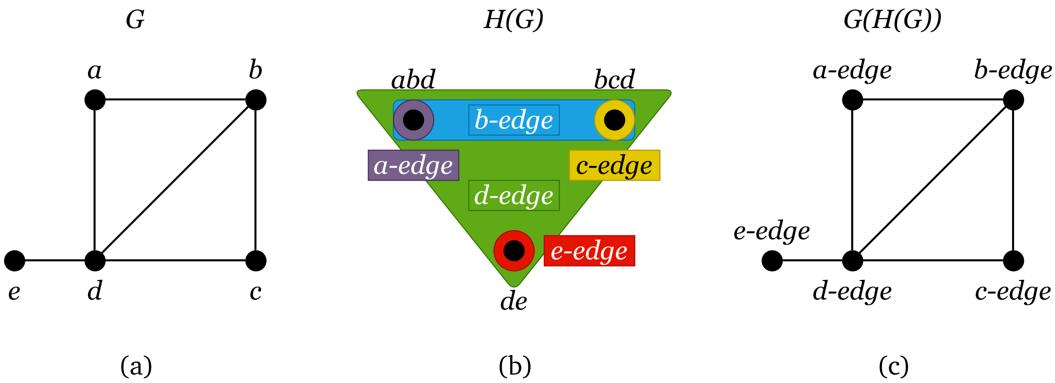


Figure 10: The edges of the hypergraph $H(G)$ with vertices abd, bcd and de are symbolized by different colors (b) and form the vertices of $G(H(G))$, they are adjacent in $G(H(G))$ if and only if the two corresponding edges in $H(G)$ intersect (c).

If a hypergraph H does not have the Helly property there exists a partial hypergraph with edge set $\tilde{E}$ with $\bigcap_{E_i \in \tilde{E}} E_i = \varnothing$ that has no pairwise disjoint edges. So all of its edges intersect pairwise. If there was a graph G with $H = H(G)$, the fact that all edges of $\tilde{E}$ intersect pairwise would correspond to the fact that for any two vertices in the vertex subset of $V(G)$ corresponding to $\tilde{E}$ there is a clique in G (i.e. a vertex in H) containing both vertices. But if these kind of cliques exist for any pair of vertices in this subset, all these vertices are in a single clique in G. This contradicts the maximality of the cliques used to determine the vertices of H. Hence $H(G)$ always has the Helly property.

By construction we have $G(H(G)) = G$ for any undirected graph G. An example can be seen in Figure 10.

3 Coloring

In this section we give a summary of the idea of coloring graphs and digraphs. This will lead us to the definition of perfect graphs and perfect digraphs in the following sections.
In this thesis we will mainly discuss the vertex coloring of graphs and digraphs therefore the term *coloring* will be used instead of *vertex coloring*. For the idea of *edge coloring* see for example [32].

Before we give a formal definition we would like to motivate the idea of coloring a graph by illustrating a famous problem that can be represented by graphs. The history of this problem dates back to 1852 when the British mathematician de Morgan mentioned it in a letter (see [25]). We will see how this idea can be transferred to digraphs and hypergraphs later on.

Imagine you have a map of different states or countries and for reasons of conspicuity you want to color each state with a certain color such that no state has the same color as a bordering state. One question you could ask if presented with this task could be: How many colors do I need at least to find a proper coloring of the map? You may also ask for an instruction on how to start coloring states.
This problem can be modeled by a graph. We set V as the set of states on the map and connect two vertices by an edge if the two corresponding states have a common border.

We will get back to the questions above later after we made some formal definitions concerning coloring of graphs.

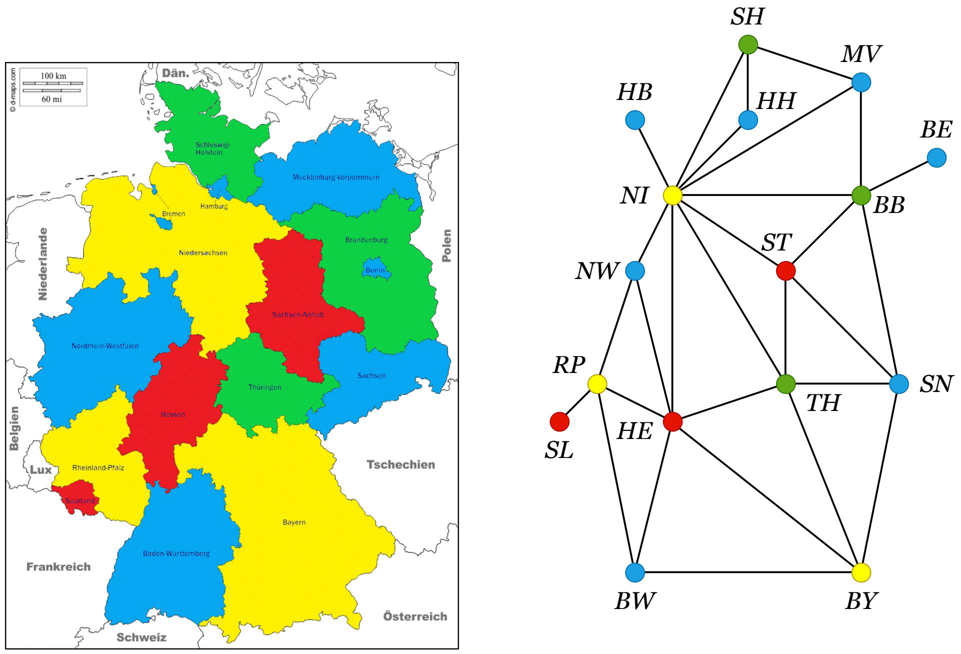
3.1 Coloring of Graphs

The following information is oriented at [21] (Chapter 5).

Definition 3.1. A *coloring* of a graph G is a mapping $f : V(G) \to C$, $C \subseteq \mathbb{Z}$, such that $f(v_1) \neq f(v_2)$ if v_1 and v_2 are adjacent. The elements of C are called *colors*. We call f an *n-coloring* if C has n elements. The set of vertices colored with the same color is called a *color class*.

We will say that a coloring is *proper* or *feasible* if it fulfills the above condition, even though we defined colorings only as proper colorings, to emphasize that we are considering a coloring and not just any mapping on the vertices of a graph.
Formally C is considered a set of integers but we will often use an actual set of colors to illustrate coloring of graphs, especially in graphic visualizations.

Definition 3.2. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer n



(a) (b)

Figure 11: A proper coloring of a map of the German federal states (a) and the corresponding graph (b). (Map taken from [20])

such that there exists an n-coloring of G. We say that G is n-colorable if $\chi(G) \leq n$.

The most famous theorem concerning coloring of graphs answers the question discussed above on how many colors we need to color a map.

Theorem 3.3 (Four Color Theorem, [5;6])**.** *Every planar graph is* 4*-colorable.*

See [5] and [6] for the exact proof of this theorem. We will just give the idea of the proof here oriented at [21] (Notes at the end of Chapter 5).
The proof of this theorem was one of the first to be done with great support of computers, it was criticized and not accepted by some colleagues for using this then unknown method. The idea of the proof is to first show that every maximal planar graph - that is any graph that is planar and that would turn non-planar by adding any further edge between its vertices - must contain at least one of 1482 *unavoidable configurations*. After that it is shown with the assistance of a computer that each of these *unavoidable configurations* can be four-colored. This gives an inductive proof that any maximal planar graph is four-colorable and therefore every planar graph as well.

So we always need at most four colors if we want to color a map since every map can be represented by a planar graph as described above.

We only considered planar graphs so far. In the following we will take a look at some upper bounds for the chromatic number in general.

Proposition 3.4 ([21], Proposition 5.2.1). *For every graph $G = (V, E)$ with $|E| = m$ we obtain*

$$\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}.$$

Proof. Let f be a coloring of G with $n = \chi(G)$ colors. There is at least one edge between every pair of color classes since f uses the minimum number of colors possible. This means that $m \geq \sum_{i=1}^{n-1} i = \frac{1}{2} \cdot n \cdot (n-1)$. Solving this inequation for n yields the assertion. □

We obtain another upper bound by an intuitive algorithm for coloring graphs (cf. [21]). We order the vertices randomly by labeling them $v_1, v_2, \ldots, v_t$, $t = |V|$, and color them one after another with the lowest possible integer from C such that non of the neighbors colored already has the same color. We never need more than $\max\{d(v_i) \mid v_i \in V\} + 1$ colors to color a graph with this method. We can improve this algorithm by considering a helpful ordering of the vertices rather than a random one.

We can also give a lower bound for the chromatic number.

Proposition 3.5. *For a graph G we obtain $\omega(G) \leq \chi(G)$.*

Proof. Let $H \subseteq G$ be a clique of size $k = \omega(G)$. Since every vertex is adjacent to every other vertex in this clique we need exactly k colors to color H. Hence we need at least k colors to color G. □

This simple observation will later lead to the definition of perfect graphs.

3.2 Coloring of Digraphs

Transferring the idea of coloring graphs to digraphs can be realized in different ways. The most intuitive way may be the following: A coloring of a digraph D is a mapping $f : V(D) \to C$ such that f is a proper coloring of the underlying graph $UG(D)$ (see for example [11]). For an oriented graph the following property is often added and the coloring is then called *oriented coloring*: all arcs linking one color class to another have the same direction (see for example [31]).

We will focus on a different definition of coloring in this thesis which is called acyclic coloring of digraphs.

Neumann-Lara [27] used this way of coloring digraphs to define an analogon to the chromatic number for digraphs - the dichromatic number, the following definitions are oriented at his paper.

Definition 3.6. An *acyclic coloring* of a digraph D is a mapping $f : V(D) \to C$, $C \subseteq \mathbb{Z}$ such that for every $c \in C$ the vertices colored by c induce an acyclic subdigraph in D.
For $n = |C|$ we call $f : V(D) \to C$ an *acyclic n-coloring* of D.

Definition 3.7. The *dichromatic number* $\chi(D)$ of a digraph D is the smallest number n such that there exists an acyclic n-coloring of D.

Note that the definition of acyclic coloring of digraphs is compatible with the definition of classic coloring of graphs in the following way: If $f : V(\overleftrightarrow{G}) \to C$ is a proper acyclic coloring of the complete biorientation $\overleftrightarrow{G}$ of a graph G then f induces a proper coloring of G, since a proper acyclic coloring is forbidding cycles of length two in $\overleftrightarrow{G}$ which correspond to edges in G. Moreover the opposite assertion also holds: If $f : V(G) \to C$ is a proper coloring of a graph G then f induces a proper acyclic coloring of the complete biorientation $\overleftrightarrow{G}$ of G, since a proper coloring of G forbids two neighboring vertices of G to be colored the same way which corresponds to cycles of length two not being monochromatic in $\overleftrightarrow{G}$. Obviously every cycle of length greater than two in $\overleftrightarrow{G}$ contains cycles of length two and is colored with at least two different colors.

We therefore obtain $\chi(G) = \chi(\overleftrightarrow{G})$.

Acyclic coloring of digraphs has not been studied as much as classical coloring of (undirected) graphs, but there is still an upper bound known for the dichromatic number. It is given in dependence on the length of cycles in a digraph. See [11] (Theorem 3) for the proof of the following theorem.

Theorem 3.8 ([11], Theorem 3)**.** *Let n and r be integers with $n \geq 2$ and $n \geq r \geq 1$. If a digraph D does not contain a cycle of length r modulo n, then D is n-colorable, hence $\chi(D) \leq n$.*

Just as for (undirected) graphs a lower bound for the dichromatic number of a digraph is given by the clique number.

Lemma 3.9. *For a digraph D we have $\omega(D) \leq \chi(D)$.*

Proof. Let H be a clique of size $\omega(D) = k$ in D. Since H is a complete digraph there is a cycle of length $\omega(D) = k$ in H. To have a proper coloring of H at least one vertex in the cycle has to be colored differently from the others. Assume vertices $v_1, \ldots, v_{k-1}$, are colored with color c_1 and vertex v_k is colored with color c_2.

Since the subdigraph induced by $v_1, \dots, v_{k-1}$ is again a complete digraphs, there is a cycle of length $k-1$ in this subdigraph which requires at least one vertex to be colored with an additional color c_3 to maintain a proper coloring, assume that this vertex is v_{k-1}. The additional color c_3 obviously has to be different from c_1. It also has to be different from c_2 since v_k and v_{k-1} also induce a complete digraph. Inductively it follows that we need $\omega(D) = k$ colors to achieve a proper coloring of H and therefore at least $\omega(D) = k$ colors to achieve a proper coloring of D. □

3.3 Coloring of Hypergraphs

We already know from Section 2.5 that there is a close connection between graphs and hypergraphs by loosely speaking exchanging edges and vertices. Therefore we consider vertex coloring and edge coloring for hypergraphs to be able to transfer assertions made for hypergraphs concerning coloring to undirected graphs and vice versa. We again refer to Lovász [26] for the information presented in this subsection.

Definition 3.10. A *vertex coloring* of a hypergraph $H = (V, E)$ is a mapping $f : V \to C$, $C \subseteq \mathbb{Z}$ such that for every $E_i \in E$ with $|E_i| > 1$ the vertices in E_i are colored with at least two colors $c_1, c_2 \in C$, $c_1 \neq c_2$. The *chromatic number* $\chi(H)$ of a hypergraph H is the minimal number $n = |C|$ such that there is a proper vertex coloring $f : V \to C$ of H.

Definition 3.11. An *edge coloring* of a hypergraph $H = (V, E)$ is a mapping $f : E \to C$, $C \subseteq \mathbb{Z}$ such that if there are two edges E_1 and E_2 with $f(E_1) = f(E_2)$, we have $E_1 \cap E_2 = \varnothing$. The *chromatic index* $\rho(H)$ of a hypergraph H is the minimal number $n = |C|$ such that there is a proper edge coloring $f : E \to C$ of H.

Definition 3.12. A hypergraph H is called *normal* if $\rho(\tilde{H}) = \delta(\tilde{H})$ for every partial hypergraph $\tilde{H}$ of H.

As we need at least as many colors to color the edges of H as there are edges containing one certain vertex of H, we have $\rho(H) \geq \delta(H)$ for every hypergraph H.

If a hypergraph H does not have the Helly property it contains a partial hypergraph $\tilde{H}$ with edge set $\tilde{E}$ with $\bigcap_{E_i \in \tilde{E}} E_i = \varnothing$ that has no pairwise disjoint edges. So every edge in $\tilde{E}$ intersects any other edge in $\tilde{E}$, which requires all edges in $\tilde{E}$ to be colored pairwise differently. Hence $\rho(\tilde{H}) = |\tilde{E}|$. On the other hand every vertex of $\tilde{H}$ has a maximal degree of $|\tilde{E}| - 1$ since $\bigcap_{E_i \in \tilde{E}} E_i = \varnothing$. So we obtain $\delta(\tilde{H}) \leq |\tilde{E}| - 1 < |\tilde{E}| = \rho(\tilde{H})$, hence H is not normal and every normal hypergraph has the Helly property.

4 Perfect Graphs

The term *perfect graph* was established by Berge [9;10] in the early 1960's. He also formulated two famous conjectures on perfect graphs which only got proven significantly later and are now known as the (weak) *Perfect Graph Theorem* and the *Strong Perfect Graph Theorem*. (cf. [16;13])
In this section we will take a closer look at these two theorems and a third one called the *Semi-strong Perfect Graph Theorem*. We will proceed in a chronological order on when these theorems were proven and partially replicate their historic proofs, although, as we will see in Subsection 4.3, we could just deduce the Weak and the Semi-strong Perfect Graph Theorem from the Strong Perfect Graph Theorem nowadays.

Definition 4.1 ([12])**.** A graph G is *Berge* if every hole and antihole induced in G has even length.

Definition 4.2 ([12])**.** A graph G is *perfect* if for every induced subgraph H of G its chromatic number $\chi(H)$ equals its clique number $\omega(H)$. If a graph is not perfect it is called *imperfect*.

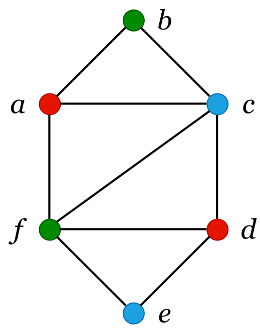


Figure 12: Example of a perfect graph with a proper 3-coloring (red, green and blue). It can easily be seen that for every subgraph H (and the graph itself) we obtain $\chi(H) = \omega(H)$.

We already know from Proposition 3.5 that the clique number of a graph is always a lower bound to the chromatic number. This of course raises the question when it is also an upper bound for the chromatic number and whether graphs fulfilling this criterion form a some-how *uniform class* of graphs. It turns out this is not quite the case. You cannot really tell anything about the properties of a certain graph G_1 just because it fulfills $\chi(G_1) = \omega(G_1)$ and there is another graph G_2 with $\chi(G_1) = \omega(G_1) = \chi(G_2) = \omega(G_2)$, apart from the fact that both graphs contain a clique that determines the colors needed to color the respective graph. The class of graphs however becomes a lot more restricted if we make the property of the chromatic number equaling the clique number *hereditary*, as we just defined it for perfect graphs, because this is a restriction that operates locally on every part of the graph rather than just globally on the whole graph. We will see in the following how the class of perfect graphs can be characterized.

4.1 The Weak Perfect Graph Theorem

Lovász [26] published a proof of one of Berge's conjectures in 1972, which is now known as the (weak) Perfect Graph Theorem. This conjecture states that the complement of a perfect graph is again perfect. In this subsection we will replicate Lovász' proof using hypergraphs and the observation that perfection of graphs is the equivalent to normality respectively τ-normality of hypergraphs (cf. [26]). The following information is oriented at Lovász' paper.

Theorem 4.3 ([26], Theorem 1)**.** *Let G_1 and G_2 be two perfect graphs. Substituting G_2 for a vertex v of G_1 we obtain again a perfect graph.*

Proof. Let G' be the graph we obtain from substituting G_2 for v. We will show that $\chi(G') = \omega(G')$ since the equality follows by the same construction for every induced subgraph of G'. We use induction on $k = \omega(G')$.
For $k = 1$ the assertion is trivial, since every graph with no clique greater than one can be colored with just one color.
Assume $k > 1$ and the assertion is true for $k - 1$. We want to find a stable set T of G' that intersects all cliques with k elements, since then we can color the vertices of the stable set with one color and the remaining vertices with $k-1$ colors by the induction hypothesis and we therefore obtain a k-coloring of G'.
We put $m = \omega(G_1)$, $n = \omega(G_2)$ and p the maximum cardinality of a clique in G_1 containing v. Then we have $k = max\{m, n+p-1\}$ because either a greatest clique of G_1 is also a greatest clique of G' or we obtain a greatest clique of G' by joining the $p - 1$ vertices (all but v) of a greatest clique in G_1 containing v with the n vertices of a greatest clique in G_2.
We denote by K the set of vertices in G_1 having the same color as v considering an m-coloring of G_1. By L we denote a set of pairwise non-adjacent vertices of G_2 that intersect every clique with n vertices of G_2. We can then define $T := L \cup (K \setminus \{v\})$.
T is a stable set. Since on the one hand the vertices of K are pairwise non-adjacent (as they are colored by the same color) and also non-adjacent to any vertex of L (since only vertices originally adjacent to v are adjacent to vertices in G_2 and all vertices of K have the same color as v and are therefore non-adjacent to v) and since on the other hand the vertices of L were chosen pairwise non-adjacent.
Moreover T intersects all cliques of size k in G'. On the one hand if we consider a

clique of size k that intersects G_2 this clique contains a former clique of size n of G_2 and, by definition of L, T contains a vertex of this clique. On the other hand if we consider a clique of size k that does not intersect G_2 it has to be a clique of size m in G_1 and therefore it has to contain a vertex colored the same way as v thus a vertex from $K \setminus \{v\}$. □

Note that recursive implementation of Theorem 4.3 gives us: Substituting perfect graphs for some vertices of a perfect graph we again obtain a perfect graph.

Observation 4.4. *For a hypergraph H and its edge-graph $G(H)$ we have $\chi(G(H)) = \rho(H)$, since the edges of H correspond to the vertices of $G(H)$ in a way that coloring either of them is a proper coloring for the other.*
Furthermore we have $\omega(\overline{G(H)}) = \nu(H)$, since pairwise disjoint edges in H are not adjacent in $G(H)$, so they form a clique in $\overline{G(H)}$.
If H has the Helly property the following holds as well:

- *$\chi(\overline{G(H)}) = \tau(H)$, since the size of a minimal transversal in H equals the number of maximal cliques in $G(H)$ and because of the Helly property this determines the minimal number of colors needed to color $\overline{G(H)}$, and*
- *$\omega(G(H)) = \delta(H)$, since for a vertex v with maximum degree in H all the edges containing this vertex intersect, by construction all corresponding vertices (to these edges) must be pairwise adjacent in $G(H)$, hence they form a clique.*

We therefore obtain the following four equations:

- *$\chi(G) = \rho(H(G))$,*
- *$\omega(G) = \delta(H(G))$,*
- *$\chi(\overline{G}) = \tau(H(G))$ and*
- *$\omega(\overline{G}) = \nu(H(G))$.*

The observations above imply the following theorem:

Theorem 4.5 ([26], Theorem 2)**.** *For a hypergraph H and its edge-graph $G(H)$ the following holds:*

- *H is normal if and only if $G(H)$ is perfect and*
- *H is τ-normal if and only if $\overline{G(H)}$ is perfect.*

For an undirected graph G and the corresponding hypergraph $H(G)$ the following holds:

- *G is perfect if and only if $H(G)$ is normal and*

- *$\overline{G}$ is perfect if and only if $H(G)$ is τ-normal.*

From Theorem 4.3 and Theorem 4.5 we obtain the following lemma:

Lemma 4.6 ([26], Theorem 1'). *Let H be a normal hypergraph. Multiplying some of its edges, the obtained hypergraph is again normal.*

Proof. Since H is normal we know that $G(H)$ is perfect. We can therefore substitute any vertex v of $G(H)$ by a graph with two vertices v_1, v_2 that are adjacent (which is obviously perfect) and obtain a perfect graph according to Theorem 4.3. This substitution is equivalent to doubling the corresponding edge in H. Recursive implementation of this method yields a perfect graph $\widetilde{G(H)}$ and a corresponding hypergraph $\widetilde{H}$ that can be obtained from H by multiplying edges. Since $\widetilde{G(H)}$ is perfect Theorem 4.5 implies that $\widetilde{H}$ is normal. □

We can now prove an analogon of the Perfect Graph Conjecture for hypergraphs.

Theorem 4.7 ([26], Theorem 3). *A hypergraph is τ-normal if and only if it is normal.*

Proof. By Theorem 4.5 it suffices to prove one direction. Thus we will show that if a hypergraph H is normal then $\tau(H) = \nu(H)$, since the assertion for partial hypergraphs follows in the same way.
We will use induction on $n = \tau(H)$, the assertion can be considered true for $n = 0$. We want to find a vertex v such that the partial hypergraph H' consisting of all edges that do not contain v has $\nu(H') < \nu(H)$. Because if we find such a vertex, H' would have a transversal T of size $n-1$ and $T \cup \{v\}$ would be a transversal of size n of H' showing that $\tau(H) \leq n = \nu(H)$.
Assume that there is no such vertex v, thus for every vertex v of H there is a set F_v of n disjoint edges not containing v. Put $H_0 = \bigcup_v F_v$ and consider edges occurring in multiple F_v's with corresponding multiplicity in H_0. H_0 can be constructed from H by removing and multiplying edges, therefore by Lemma 4.6 H_0 is normal. Hence we have $\rho(H_0) = \delta(H_0)$.
By construction H_0 has $n \cdot m$ edges, m being the number of vertices of H. Since there are at most n disjoint edges in H_0 and edges colored the same way have to be disjoint, we have $\rho(H_0) \geq m$.
Since we defined F_v as a set of disjoint edges, a given vertex v is only contained in at most one edge of F_x, $x \neq v$, and in no edge of F_v. We therefore obtain $\delta(H_0) \leq m - 1$.
This yields a contradiction (cf. the observation after Definition 3.12) and we have proven that there is a vertex v with the properties stated above. Hence we have

$\tau(H) \leq n = \nu(H)$ and we always have $\tau(H) \geq \nu(H)$ (cf. the observation after Definition 2.46 so H is τ-normal. □

Theorem 4.8 (Weak Perfect Graph Theorem, [26]). *A graph G is perfect if and only if its complement $\overline{G}$ is perfect.*

Proof. The assertion follows directly from Theorem 4.5 and Theorem 4.7:

G is perfect if and only if $H(G)$ is normal,
$H(G)$ is normal if and only if $H(G)$ is τ-normal,
$H(G)$ is τ-normal if and only if $\overline{G}$ is perfect.

□

4.2 The Semi-strong Perfect Graph Theorem

Reed [28] published the proof of another conjecture concerning perfect graphs in 1987 made by Chvátal [14] a few years earlier. It is now known as the Semi-strong Perfect Graph Theorem. It shows that perfection of graphs is invariant under the premise of a certain kind of isomorphism between graphs, so called P_4-isomorphism. We will replicate the proof of this theorem by referring to [28] for the proof of some lemmas and observations that are essential for the argumentation. The information in this subsection is taken from [28].

Definition 4.9. Two graphs G_1 and G_2 with the same vertex set are *P_4-isomorphic* if the following holds. Every set of four vertices $\{a, b, c, d\}$ induces a path of length four (i.e. a P_4) in G_1 if and only if the same vertex set induces a P_4 in G_2.

Lemma 4.10 (Star-cutset Lemma, [15]). *No minimal imperfect graph has a star-cutset.*

Proof. Let G be a minimal imperfect graph. The following conditions hold for G:

1. every proper induced subgraph of G is $\omega(G)$-colorable and
2. we have $\omega(G[V(G) \setminus S]) = \omega(G)$ for every stable set S in G.

The implication of the second condition can be proved via contraposition. Assume that there is a at least one stable set S' in G such that $\omega(G[V(G) \setminus S']) \neq \omega(G)$. So S' contains exactly one vertex from every largest clique in G. Since no vertices in S' are connected in G they can be colored with one color in G. This implies that the largest cliques in G are not all interconnected in a way that would require more colors than $\omega(G)$. Hence G is perfect.
We will prove that no graph with the properties (1) and (2) has a star-cutset.

Assume that G is a graph that fulfills condition (1) and has a star-cutset $C \subseteq V(G)$, we will prove that G does not fulfill (2).
Since $V(G) \setminus C$ induces a graph that is not connected, we can divide this vertex set into two disjoint sets V_1, V_2, such that there is no edge between any vertex in V_1 and any vertex in V_2. Let G_i, $i = 1, 2$, be the subgraph induced by the vertex set $V_i \cup C$. According to condition (1) there is a proper coloring f_i of G_i with $\omega(G)$ colors. Let $w \in V(C)$ be the center of the star-cutset, so w is adjacent to all other vertices in C. Define the vertex sets $S_i = \{v \mid v \in G_i, f_i(v) = f_i(w)\}$. Each S_i is again a stable set and $S_i \cap C = \{w\}$. According to the construction of the sets S_i the union $S = S_1 \cup S_2$ is also a stable set. Let $Q \subseteq V(G)$ be a clique in $V(G) \setminus S$. Q is fully contained in either $V(G_1) \setminus S_1$ or $V(G_2) \setminus S_2$. Since each of these graphs can be colored with $\omega(G) - 1$ colors, we obtain $|Q| \leq \omega(G) - 1$. Hence condition (2) does not hold. □

See [24] for the proof of the following lemma.

Lemma 4.11 ([24]). *Every graph with at least three vertices that does not contain a disc as an induced subgraph is fragile.*

The proof for the following lemma can be found in [28] (Theorem 2).

Lemma 4.12 ([28], Theorem 2). *If G_1 and G_2 are P_4-isomorphic graphs and if for some disc of size at least six induced by the vertex set S in G_1 the vertex set S induces exactly the same disc in G_2, then one of the following holds:*

1. *$G_1 \cong G_2$, or*
2. *G_2 or $\overline{G_2}$ has a star-cutset, or*
3. *G_2 contains a C_5 as a proper induced subgraph.*

Observation 4.13 ([28], Chapter 3). *All discs except for the ones of size six are P_4-isomorphic only to themselves and their complements.*
C_6 is P_4-isomorphic to itself, its complement $\overline{C_6}$, the graph F from Figure 13 and $\overline{F}$.

The following lemma therefore focuses on a vertex set that induces a C_6 in one graph and an F in a P_4-isomorphic graph, see [28] (Theorem 3) for the proof.

Lemma 4.14 ([28], Theorem 3). *Let G_1 and G_2 be P_4-isomorphic graphs such that G_2 is unbreakable and does not contain an induced C_5. If some vertex set S induces a C_6 in G_2 and an F (see Figure 13) in G_1, then G_2 has a proper endomorphism.*

Lemma 4.15 ([28], Theorem 1). *Let G_1 and G_2 be two P_4-isomorphic graphs such that G_1 is neither G_2 nor $\overline{G_2}$. Then at least one of the following holds:*

1. *G_2 contains a C_5 as a proper induced subgraph, or*

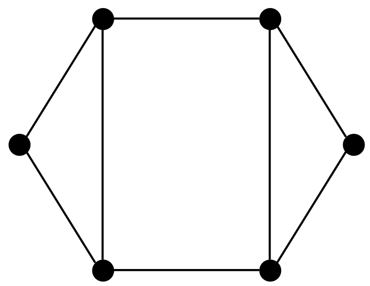


Figure 13: The graph F. (Figure taken from [28])

2. *G_2 or $\overline{G_2}$ contains a star-cutset (i.e. it is fragile), or*

3. *G_2 or $\overline{G_2}$ has a proper endomorphism.*

Proof. Let G_1 and G_2 be two P_4-isomorphic graphs such that G_1 is neither G_2 nor $\overline{G_2}$. If G_1 has less than three vertices we have $G_1 \cong G_2$ or $G_1 \cong \overline{G_2}$, contradicting the assumptions, let therefore G_1 have at least three vertices.
If G_2 has no disc (2) follows by Lemma 4.11.
If G_2 contains a disc of size five, clearly (1) is true since $C_5 \cong \overline{C_5}$.
If G_2 contains a disc of size at least six we have to consider three different cases. Let S be the vertex set of this disc. Denote by S_{G_1} and S_{G_2} the subgraphs induced by S in G_1 and G_2.
If $S_{G_1} \cong S_{G_2}$ Lemma 4.12 can be applied and, since the first condition of this lemma is a contradiction to our assumption, (1) or (2) must be true.
If $S_{\overline{G_1}} \cong S_{G_2}$ we can apply Lemma 4.12 on $\overline{G_1}$ and G_2 (since P_4 is self-complementary) and it follows that (1) or (2) must be true.
If neither $S_{G_1} \cong S_{G_2}$ nor $S_{\overline{G_1}} \cong S_{G_2}$, we can assume by Observation 4.13 that S_{G_1} or $S_{\overline{G_1}}$ is the graph F from Figure 13. In both cases one of the three conditions of the theorem follows from Lemma 4.14 (in the case of $S_{\overline{G_1}}$ being the graph F, we can apply Lemma 4.14 on $\overline{G_1}$ and G_2 because P_4 is self-complementary). □

We will see in the following theorem that this lemma already proves the Semi-strong Perfect Graph Theorem.

Theorem 4.16 (Semi-strong Perfect Graph Theorem, [28]). *If two graphs G_1 and G_2 with the same vertex set are P_4-isomorphic, then G_1 is perfect if and only if G_2 is perfect.*

Proof. Assume the assertion is not true. Thus there are two P_4-isomorphic graphs G_1 and G_2 such that G_1 is perfect and G_2 is not. Let G_2 be minimal imperfect, i.e.

every proper induced subgraph of G_2 is perfect but G_2 is imperfect.
Obviously G_1 is not G_2 and by the Weak Perfect Graph Theorem 4.8 G_1 is not $\overline{G_2}$. We know from Lemma 4.15 that in this case at least one of the three conditions of this lemma must hold for G_2. We will show that a minimal imperfect graph cannot fulfill any of these conditions.
If G_2 has five or less than five vertices it obviously does not contain a proper induced C_5. Assume G_2 has at least six vertices and contains an induced C_5. Since $\omega(C_5) = 2$ and $\chi(C_5) = 3$, C_5 is imperfect and G_2 contains a proper induced subgraph that is imperfect, a contradiction to G_2 being imperfect.
The second condition is excluded by the Star-Cutset Lemma 4.10. Due to the Weak Perfect Graph Theorem 4.8 the Star-Cutset Lemma can also be applied on the complement of an imperfect graph.
Assume that G_2 has a proper endomorphism f and let H be the subgraph of G_2 induced by $f(G_2)$. Due to the minimal imperfection of G_2, H is perfect and therefore $\omega(H)$-colorable. Hence there exists an endomorphism g on H such that $g(H)$ is a clique with $\omega(H)$ vertices where every vertex of $g(H)$ is the image of all vertices with the same color in H. The composition of the two endomorphism g and f is again an endomorphism. So there is an endomorphism $g^* = g \circ f$ such that $g^*(G_2) = (g \circ f)(G_2) = g(f(G_2)) = g(H)$ is a clique with $\omega(f(G_2)) = \omega(H)$ vertices. Therefore G_2 is $\omega(H)$-colorable and since $\omega(H) \leq \omega(G_2)$ also $\omega(G_2)$-colorable, a contradiction. □

4.3 The Strong Perfect Graph Theorem

The proof of another famous conjecture called the *Strong Perfect Graph Conjecture* by Berge was published by Chudnovsky et al. [12] in 2006, the conjecture is now known as the Strong Perfect Graph Theorem.

Theorem 4.17 (Strong Perfect Graph Theorem, [12]). *A graph G is perfect if and only if it is Berge, i.e. G does not contain odd length holes nor odd length antiholes as induced subgraphs.*

Chudnovsky et al. needed roughly 150 pages to prove this theorem, so for obvious reasons we will not replicate their proof in this thesis and therefore refer to [12] for the complete proof and only give a short summary of the idea of the proof oriented at [13] here.

It is easily seen that every perfect graph is Berge since odd length holes and odd length antiholes are not perfect and therefore any graph containing them is not perfect either.
The converse however is difficult to prove. Chudnovsky et al. consider five classes

of graphs and three certain partitions of graphs to prove this inclusion.
Those five classes are bipartite graphs, complements of bipartite graphs, line graphs of bipartite digraphs, complements of line graphs of bipartite graphs and so-called double split graphs. A double split graph G is a graph constructed the following way. Take a bipartite graph H with bipartition (A, B) and define two new vertices s_v and t_v for every vertex $v \in V(H)$. Then set $V(G) = \{s_v, t_v \mid v \in V(H)\}$ and the edges of G are given by:

- $(s_v, t_v) \in E(G)$ if and only if $v \in A$,
- for $v_1 \neq v_2$, $v_1, v_2 \in A$, there are no edges between the vertex sets $\{s_{v_1}, t_{v_1}\}$ and $\{s_{v_2}, t_{v_2}\}$,
- for $v_1 \neq v_2$, $v_1, v_2 \in B$, there are four edges between the vertex sets $\{s_{v_1}, t_{v_1}\}$ and $\{s_{v_2}, t_{v_2}\}$ and
- for $v_1 \in A$ and $v_2 \in B$, there are exactly two edges connecting one vertex of $\{s_{v_1}, t_{v_1}\}$ to one vertex of $\{s_{v_2}, t_{v_2}\}$: if $(v_1, v_2) \in E(H)$ then $(s_{v_1}, s_{v_2}), (t_{v_1}, t_{v_2}) \in E(G)$ and otherwise $(s_{v_1}, t_{v_2}), (t_{v_1}, s_{v_2}) \in E(G)$.

All these five classes of graphs are perfect.

The three partitions considered are 2-joins, M-joins and even skew partitions.
A 2-join in G is a partition of the vertex set (X_1, X_2) such that there exist disjoint non-empty sets $A_i, B_i \subseteq X_i$, $i \in \{1, 2\}$ with:

- every vertex of A_1 is adjacent to every vertex of A_2 and every vertex of B_1 is adjacent to every vertex of B_2, and there are no further edges between X_1 and X_2,
- for $i \in \{1, 2\}$, every component of $G[X_i]$ intersects with both A_i and B_i and
- for $i \in \{1, 2\}$, if $|A_i| = |B_i| = 1$ and $G[X_i]$ is a path joining the vertices of A_i and B_i, it has odd length ≥ 3.

An M-join in G is a partition of the vertex set into six non-empty sets (A, B, C, D, E, F) such that:

- every vertex in A is adjacent to at least one vertex in B and non-adjacent to at least one other vertex in B, and vice versa,
- for the pairs $(C, A), (A, F), (F, B), (B, D)$ every vertex of one set is connected to every vertex of the other set and
- for the pairs $(D, A), (A, E), (E, B), (B, C)$ there are no edges between the two sets.

A skew partition in G is a partition (A, B) of the vertex set such that

- A does not induce a connected subgraph in G and A is not empty and
- B does not induce a connected subgraph in $\overline{G}$ and B is not empty.

A skew partition is even, if every induced path of length at least two, whose beginning and end are in B and all other vertices are in A, has even length and every induced complement of a path (whose complemented path has at least length two), whose beginning and end are in A and all other vertices are in B, has even length.
By referring to some earlier results (see [13] for details) Chudnovsky et al. prove that no minimal imperfect graph has a 2-join or an M-join and furthermore that the complement of a minimal imperfect graphs does not have a 2-join either. Additionally they prove that, assuming a Berge graph G is not perfect but every Berge graph H with $|V(H)| < |V(G)|$ is perfect, G does not have an even skew partition.

So what Chudnovsky et al. proved is that for every Berge graph G exactly one of the following holds:

- G is in one of the five classes,
- G or $\overline{G}$ has a 2-join,
- G has an M-join or
- G has an even skew partition.

With the observations above this proves the theorem.

Remark 4.18 ([28])**.** If G is a graph without odd length holes and without odd length antiholes as induced subgraphs, then its complement $\overline{G}$ does not have an odd length hole or an odd length antihole as induced subgraph either. It is therefore perfect as well. Since $\overline{\overline{G}} = G$ we obtain that:

G is perfect
$\Leftrightarrow$ G does not have an odd length hole or an odd length antihole as an induced subgraph
$\Leftrightarrow$ $\overline{G}$ does not have an odd length hole or an odd length antihole as an induced subgraph
$\Leftrightarrow$ $\overline{G}$ is perfect

So the Strong Perfect Graph Theorem 4.17 implies the (weak) Perfect Graph Theorem 4.8, which also explains the naming as Weak and Strong Perfect Graph Theorems.

Remark 4.19 ([28])**.** If G_1 is a perfect graph and G_2 is P_4-isomorphic to G_1, then G_2 is also a perfect graph according to Theorem 4.16. Since the complement of a P_4 is again a P_4, we obtain that $\overline{G_2}$ is also P_4-isomorphic to G_1 and therefore perfect as well. Since $\overline{\overline{G_2}} = G_2$, Theorem 4.16 implies the Weak Perfect Graph Theorem 4.8 and Chvátal [14] showed that the Strong Perfect Graph Theorem 4.17 implies Theorem 4.16. This means it is *situated between* the two main theorems on perfect graphs and is therefore called the Semi-Strong Perfect Graph Theorem.

5 Perfect Digraphs

As an analogon to Definition 4.2 Andres and Hochstättler introduced perfect digraphs in [4] using Neumann-Lara's definition of the dichromatic number. They also formulated and proved a Strong Perfect Digraph Theorem (see [4]) for characterization of perfect digraphs and a Semi-strong Perfect Digraph Theorem (see [3]), which we will present in this section. However we will see that the Weak Perfect Graph Theorem 4.8 does not have an obvious analogon for digraphs (see [4]).

Definition 5.1 ([4]). A digraph D is called *perfect* if for any induced subdigraph H of D the dichromatic number $\chi(H)$ of H equals the clique number $\omega(H)$ of H.

Note that the definition of perfection for digraphs is compatible with the one of graphs in the following way: A graph G is perfect if and only if its complete biorientation $\overleftrightarrow{G}$ is perfect, since $\chi(G) = \chi(\overleftrightarrow{G})$ as we saw in Section 3.2 and obviously $\omega(G) = \omega(\overleftrightarrow{G})$.

5.1 A Strong Perfect Digraph Theorem

The information in this subsection is taken from [4].

Theorem 5.2 (Strong Perfect Digraph Theorem, [4], Theorem 3). *A digraph $D = (V, A)$ is perfect if and only if the symmetric part $S(D)$ is perfect and D does not contain any directed cycle $\overrightarrow{C_n}$ of length $n \geq 3$ as an induced subdigraph.*

Proof. Assume $S(D)$ is not perfect. Since $S(D)$ can be interpreted as an undirected graph there must exist an induced subgraph $H = (V', E')$ of $S(D)$ with $\omega(H) < \chi(H)$ taking into consideration the definition of perfect graphs 4.2 and the fact that $\omega(H) \leq \chi(H)$ for all H. Since $S(D[V']) = H$ we conclude by Observation 2.29,

$$\omega(D[V']) = \omega(S(D[V'])) = \omega(H) < \chi(H) = \chi(S(D[V'])) \leq \chi(D[V']) \quad ,$$

which implies that D is not perfect.
Assume that D contains a directed cycle $\overrightarrow{C_n}$ of length $n \geq 3$ as an induced subdigraph. On the one hand we have $\omega(\overrightarrow{C_n}) = 1$, since there is no pair of vertices in $\overrightarrow{C_n}$ with a pair of antiparallel arcs between them. On the other hand we have $\chi(\overrightarrow{C_n}) = 2$, since we need at least two different colors to achieve a proper coloring of a cycle. Therefore we have $\omega(\overrightarrow{C_n}) < \chi(\overrightarrow{C_n})$ and since $\overrightarrow{C_n}$ is an induced subdigraph of D, D is not perfect.

Hence if $S(D)$ is not perfect or if D contains an directed cycle $\overrightarrow{C_n}$ of length $n \geq 3$ as an induced subdigraph then D is not perfect.

Now assume that $S(D)$ is perfect but D is not perfect. It suffices to show that D contains a directed cycle $\overrightarrow{C_n}$ of length $n \geq 3$ as an induced subdigraph.
Since D is not perfect there exists an induced subdigraph $H = (V', A')$ of D with $\omega(H) < \chi(H)$. As $S(H) \subseteq S(D)$ is perfect, there is a proper coloring of $S(H) = S(D[V'])$ with $\omega(S(H))$ colors, i.e. by Observation 2.29 with $\omega(H)$ colors. This can however not be a proper coloring of H. Hence there is a (not necessarily induced) monochromatic directed cycle $\overrightarrow{C_n}$ of length $n \geq 3$ in H. Obviously $\overrightarrow{C_n}$ cannot be entirely in $S(H)$. Moreover there cannot be a single arc of $\overrightarrow{C_n}$ that is in $S(H)$ because in this case we would have a monochromatic cycle of length 2 in $S(H)$ which is not possible since the coloring is a proper coloring of $S(H)$. Therefore $\overrightarrow{C_n}$ has to be in $O(H)$.
Let C be a monochromatic directed cycle (of length at least three) in H of minimal length. C cannot have a pair of antiparallel arcs as a chord, since both vertices are colored in the same color contradicting the fact that the coloring is a proper coloring of $S(H)$. Moreover C cannot have a single arc as a chord because in this case there would be a smaller monochromatic cycle in H than C, contradicting the minimality of C. Therefore C is an induced monochromatic directed cycle of length at least three in H and hereby in D.

This suffices to complete the proof because if we assume that D does not contain a cycle of length greater than three and D is not perfect, $S(D)$ cannot be perfect. Because if it was, D would have to have an induced cycle with length greater than three as we have just proven. But this clearly contradicts the assumption we just made. □

Note that this Theorem yields the following assertion.

Remark 5.3 ([4], Remark 4)**.** If D is a perfect digraph, then every proper coloring of $S(D)$ is a proper acyclic coloring of D as well.
The only structure that must be avoided when possibly recoloring the vertices to achieve a proper acyclic coloring of D is a monochromatic directed cycle. Induced monochromatic directed cycles cannot exist in perfect digraphs as just proven in Theorem 5.2. Cycles that have a pair of antiparallel arcs as a chord or as part of the cycle are already colored with at least two colors since the pair of antiparallel edges requires its vertices to be colored differently. If a non-induced cycle has only single arcs as parts of the cycle and as chords, then its vertex set must contain a proper subset that induces a directed cycle of length at least three. This case is therefore also excluded by the Strong Perfect Digraph Theorem.

Corollary 5.4 ([4], Corollary 5). *A digraph D is perfect if and only if it does not contain either of the following structures as an induced subdigraph: a filled odd length hole, a filled odd length antihole and a directed cycle $\overrightarrow{C_n}$ of length $n \geq 3$.*

Proof. Obviously D is not perfect if it contains any of the structures above as an induced subdigraph since they are not perfect.
If D contains none of the structures above as an induced subdigraph then $S(D)$ does not contain an odd length hole nor an odd length antihole as an induced subgraph and therefore $S(D)$ is perfect by Theorem 4.17. Moreover, since directed cycles $\overrightarrow{C_n}$ of length $n \geq 3$ are excluded as induced subdigraphs, Theorem 5.2 implies that D is perfect. □

5.2 A Semi-strong Perfect Digraph Theorem

As seen in Section 4.2 Reed showed that perfection is invariant under P_4-isomorphism. To formulate an analogous result for digraphs Andres et al. [3] introduced what can be considered an analogon to P_4-isomorphism for digraphs; P^4C-isomorphism. We know from Theorem 2.35 that graphs without an induced P_4 form the class of cographs and Theorem 2.41 states that the class of directed cographs can be characterized by a set $\mathcal{F}$ of eight forbidden induced minors. In a way those eight minors correspond to the P_4 when transferring the idea of cographs to digraphs. Forbidding all eight minors cannot yield the right isomorphism to formulate an analogon to the Semi-strong perfect graph Theorem 4.16, since the class of directed cographs is closed under complementation (Lemma 2.39) and perfect digraphs are not (this will be discussed in detail in Section 5.3). Andres et al. therefore restricted the set $\mathcal{F}$ to five induced minors to define P^4C-isomporphism and prove a Semi-strong Perfect Digraph Theorem.

These five forbidden induced minors are the symmetric path P_4, the directed cycle $\overrightarrow{C_3}$, the directed path $\overrightarrow{P_3}$ and the two variations of a directed path $\overrightarrow{P_3}$ with one pair of antiparallel arcs $\overrightarrow{P_3^+}$ and $\overrightarrow{P_3^-}$ (see Figure 14).

The information in this subsection is taken from [3].

Definition 5.5. Two digraphs $D = (V, A)$ and $D' = (V, A')$ on the same vertex set are said to be *P^4C-isomorphic* if and only if all of the following holds:

1. any set $\{a, b, c, d\} \subseteq V$ induces a P_4 in $S(D)$ if and only if it induces a P_4 in $S(D')$,
2. any set $\{a, b, c\} \subseteq V$ induces a $\overrightarrow{C_3}$ in D if and only if it induces a $\overrightarrow{C_3}$ in D',

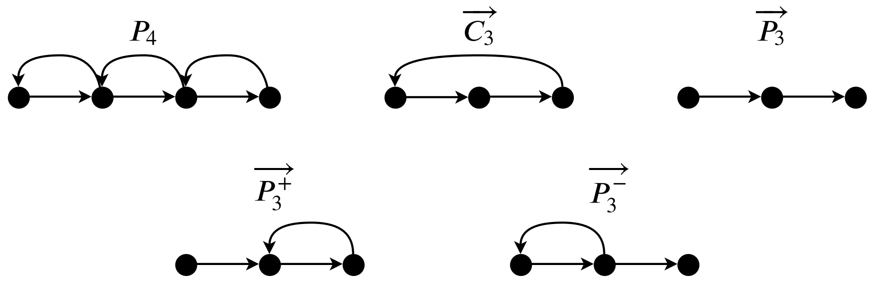


Figure 14: The five forbidden induced minors. (Figure taken from [3])

3. any set $\{a, b, c\} \subseteq V$ induces a $\overrightarrow{P_3}$ with midpoint b in D if and only if it induces a $\overrightarrow{P_3}$ with midpoint b in D' and
4. any set $\{a, b, c\} \subseteq V$ induces a $\overrightarrow{P_3^+}$ or a $\overrightarrow{P_3^-}$ in either case with midpoint b in D if and only if it induces one of them with midpoint b in D'.

Lemma 5.6 ([3], Lemma 4)**.** *Let D and D' be P^4C-isomorphic. D contains an induced directed cycle of length $n \geq 3$ if and only if the same is true for D'.*

Proof. By symmetry it suffices to show one inclusion. Let $\{v_0, \ldots, v_{n-1}\}$ be a vertex set that induces a directed cycle $\overrightarrow{C_n}$ in D.
By definition of P^4C-isomorphism the assertion is true for $n = 3$, thus assume $n \geq 4$ for the following considerations. Without loss of generality we may also assume that the vertices are labeled in consecutive order (in the direction of traversal) in $\overrightarrow{C_n}$.
Each set of vertices $\{v_i, v_{i+1}, v_{i+2}\}$ induces a $\overrightarrow{P_3}$ with midpoint v_{i+1} (indices are taken modulo n) in D' since it induces a $\overrightarrow{P_3}$ in D as part of the cycle $\overrightarrow{C_n}$. As a result we obtain a directed cycle C on $\{v_0, \ldots, v_{n-1}\}$ in D', possibly with opposite direction of the one in D. In this case we relabel the vertices such that the labeling has a consecutive order (in the direction of traversal) again.
It remains to prove that the cycle is induced in D'. Assume it is not, i.e. C has a chord that is either a pair of antiparallel arcs (v_i, v_j) and (v_j, v_i) or a single arc (v_i, v_j) in D', $|i - j| > 1$. We choose j such that the directed path from v_j to v_i on C is the shortest possible.
First consider the case of the chord being a single arc (v_i, v_j). Since $\{v_i, v_j, v_{j+1}\}$ does not induce a $\overrightarrow{C_3}$ in D' it must induce a $\overrightarrow{P_3}$ with midpoint v_j in D' and due to P^4C-isomorphism the same must hold in D, contradicting the assumption that $\overrightarrow{C_n}$ is induced in D.
Now consider the chord being a pair of antiparallel arcs (v_i, v_j) and (v_j, v_i). In this case $\{v_i, v_j, v_{j+1}\}$ must induce a $\overrightarrow{P_3^+}$ or a $\overrightarrow{P_3^-}$ with midpoint v_j in D' and therefore

in D, also contradicting the assumption that $\overrightarrow{C_n}$ is induced in D. □

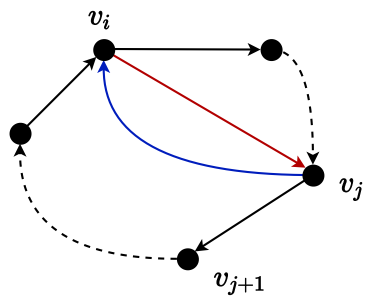


Figure 15: The three vertices induce a directed path of length three with or without one antiparallel arc in the proof of Lemma 5.6 (in this case the directed path from v_i to v_j on the cycle is shorter than the other way around).

With the lemma above we can easily prove an analogon to the Semi-strong Perfect Graph Theorem.

Theorem 5.7 (Semi-strong Perfect Digraph Theorem, [3], Theorem 5)**.** *If D and D' are P^4C-isomorphic then D is perfect if and only if D' is perfect.*

Proof. By the first criterion of Definition 5.5 $S(D)$ and $S(D')$ are P_4-isomorphic and therefore by Theorem 4.16 $S(D)$ is perfect if and only if $S(D')$ is perfect. By Lemma 5.6, D contains an induced cycle of length $n \geq 3$ if and only if the same holds for D'. The assertion now follows from the Strong Perfect Digraph Theorem 5.2. □

Andres et al. [3] conclude their paper on the Semi-strong Perfect Digraph Theorem with an observation on digraphs that do not contain any of the five subdigraphs from Figure 14. These digraphs are trivially P^4C-isomorphic (if they have the same vertex set). Since one of the five forbidden subdigraphs is the P_4 we know from Theorem 2.35 that the symmetric part $S(D)$ of such a digraph D is a cograph. By Definition 2.32 we know that every cograph can be represented by its cotree and we may consider the cotree in normalized form, where vertices labeled 1 represent the complete join of subgraphs and vertices labeled 0 represent the disjoint union of subgraphs. If two cographs in the construction process of $S(D)$ are the children of a 1-labeled vertex in the corresponding cotree they are completely joined in $S(D)$, hence they form one connected component in $S(D)$. Hence it is interesting to see how two cographs in the construction process of the cograph $S(D)$ are connected, that are children of a 0-labeled vertex in the corresponding cotree, or, more generally, how components of $S(D)$ are connected in D. Andres et al. [3] made the following observation.

Lemma 5.8 ([3], Lemma 6)**.** *Let $G_1, \ldots, G_k$ be the connected components of the symmetric part $S(D)$ of a digraph D not containing any of the five subdigraphs from Figure 14 as induced subdigraphs.*
If there exists a single arc in D between a vertex v_i in G_i and a vertex v_j in G_j, $i \neq j$ and $1 \leq i, j \leq k$, then G_i and G_j are connected in D by an orientation of the complete bipartite graph $K_{V(G_i),V(G_j)}$.

Proof. Let w be a neighbor of v_j in $S(D)$, so v_j and w are connected by a pair of antiparallel arcs. We consider the subdigraph induced by v_i, v_j and w in D.
If there is no arc between v_i and w the three vertices induce a $\overrightarrow{P_3^+}$ or a $\overrightarrow{P_3^-}$ in D contradicting the assumption that the five subdigraphs from Figure 14 are not in D. The vertices v_i and w cannot be connected by a pair of antiparallel arcs either since they are in different components of $S(D)$. Hence there has to be a single arc between those vertices.
Inductively we obtain that v_i has to be connected to all vertices of G_j by a single arc.
Conversely it follows from the existence of these single arcs that every vertex of G_j has to be connected to every vertex of G_i by a single arc. Hence we obtain an orientation of the complete bipartite graph $K_{V(G_i),V(G_j)}$ connecting G_i and G_j. □

Obviously the two components G_i and G_j themselves do not have to be an orientation of the complete bipartite graph $K_{V(G_i),V(G_j)}$ since there are arcs connecting vertices within a component.

Note that there may be single arcs between vertices of a component G_i in D. This may exclude certain single arcs between components from existing since we also forbid $\overrightarrow{C_3}$ and $\overrightarrow{P_3}$ from being induced subdigraphs of D.

These observations yield the conclusion that digraphs without any of the five subdigraphs from Figure 14 are pretty *similar* to one another and contain a rather *strict* structure. It may be worth to investigate them further as there are probably certain problems that are tractable for these digraphs but $\mathcal{NP}$-complete in general. (cf. [3])

5.3 Non-existence of a Weak Perfect Digraph Theorem and Strictly Perfect Digraphs

As mentioned above unfortunately there is no analogon to the Weak Perfect Graph Theorem 4.8 for digraphs.

Observation 5.9 ([4])**.** *For a perfect digraph D its complement $\overline{D}$ may not be perfect.*

Take the directed cycle of length four $\overrightarrow{C_4}$ as an example. As mentioned in the proof of the Strong Perfect Digraph Theorem 5.2 we have $\omega(\overrightarrow{C_4}) = 1 < 2 = \chi(\overrightarrow{C_4})$, so obviously $\overrightarrow{C_4}$ is not perfect. However we have $\omega(\overline{\overrightarrow{C_4}}) = 2 = \chi(\overline{\overrightarrow{C_4}})$ (see Figure 16), so the complement of $\overrightarrow{C_4}$ is perfect.

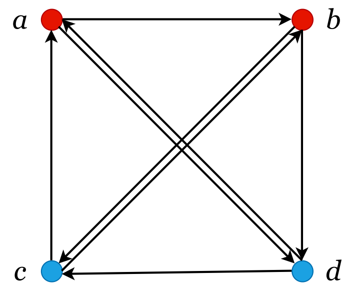


Figure 16: A proper coloring of the complement of the directed cycle of length four with two colors (red and blue), obviously the subdigraphs induced by the vertex sets $\{a, d\}$ and $\{b, c\}$ form cliques of size two.

We can however make some general statements on the structure of the complement of a perfect digraph.

Theorem 5.10 ([4], Theorem 9)**.** *A digraph D is perfect if and only if its complement $\overline{D}$ is a clique-acyclic superorientation of a perfect graph G, i.e. there is no clique in G that is induced by the vertex set of a directed cycle in $O(\overline{D})$.*

Proof. We know from Theorem 5.2 that D is perfect if and only if $S(D)$ is perfect and D does not contain an induced directed cycle of length at least three.

D not containing an induced directed cycle of length at least three is obviously equivalent to $\overline{D}$ not containing an induced antihole of length at least three, since both subdigraphs are complementary and induced by the same vertex set in two complementary digraphs.
We want to prove that $\overline{D}$ not containing an induced antihole of length at least three is equivalent to $\overline{D}$ being clique-acyclic. We prove both directions via contraposition.
Let $\overline{D}$ contain an induced antihole $\overline{\overrightarrow{C_n}}$, $n \geq 3$. The underlying graph $UG(\overline{\overrightarrow{C_n}})$ is the complete graph K_n which obviously defines a clique in $UG(\overline{D})$ and the vertex set of $\overline{\overrightarrow{C_n}}$ forms a directed cycle in $O(D)$. Therefore $\overline{D}$ is not clique-acyclic.
Let on the other hand $\overline{D}$ be not clique-acyclic. So there is a directed cycle $\overrightarrow{C_n}$, $n \geq 3$, in $O(D)$ whose vertex set induces a clique in $UG(\overline{D})$. Let C be such a cycle of minimum length. Assume C has at least length four. Since the vertex set of C induces a clique in $UG(\overline{D})$ there have to be chords between all non-consecutive

vertices of C. If there was a single arc as a chord in C we would obtain a smaller directed cycle fulfilling the same properties contradicting the minimality of C. So each chord has to be a pair of anti-parallel arcs in $\overline{D}$. Therefore $\overline{D}$ contains an induced antihole. If C has length three the assertion follows immediately.

By the Weak Perfect Graph Theorem 4.8 we know that $S(D)$ is perfect if and only if its complement $\overline{S(D)}$ is perfect and therefore if and only if $UG(\overline{D})$ is perfect, since $\overline{S(D)} = UG(\overline{D})$. Therefore $S(D)$ is perfect if and only if $\overline{D}$ is the superorientation of a perfect graph.

Combining the above assertions we obtain the proposed equivalence. □

As we have just seen perfection of digraphs is not closed under complementation. We can however require this by definition.

Definition 5.11. A digraph D is *strictly perfect* if it is perfect and its complement $\overline{D}$ is perfect as well.

An alternative motivation and definition for strictly perfect digraphs is given by Andres in [2].

If we want to formulate a Semi-strong Perfect Digraph Theorem for strictly perfect digraphs we need to forbid six of the subdigraphs from Figure 9 as induced subdigraphs; the five minors we already know from perfect digraphs (see Figure 14) and the directed cycle of length three with one additional arc (subdigraph in the bottom left corner of Figure 9).

We extend our definition of P^4C-isomorphism. We must also extend our criterion for the induced P_4 to all of D and D', because $S(\overline{D})$ and $S(\overline{D'})$ do not have to be P_4-isomorphic just because $S(D)$ and $S(D')$ are P_4-isomorphic. See Figure 17 for an example of digraphs for which P_4-isomorphism of $S(D)$ and $S(D')$ does not require P_4-isomorphism of $S(\overline{D})$ and $S(\overline{D'})$, because a P_4 that is induced in $S(D)$ may not be induced in D. Note that it is however difficult to find examples for such digraphs since many single arcs create a $\overrightarrow{P_3^+}$, a $\overrightarrow{P_3^-}$ or the new minor (directed cycle of length three with one additional arc) as induced subdigraphs in connection with a P_4, which requires them to be induced in both D and D' by definition.

Of course a vertex set that induces a P_4 in D always induces a P_4 in $S(D)$ as well.

Definition 5.12. Two digraphs D and D' on the same vertex set are *P^4S-isomorphic* if and only if all of the following hold:

1. any set $\{a, b, c, d\} \subseteq V$ induces a P_4 in D if and only if it induces a P_4 in D',

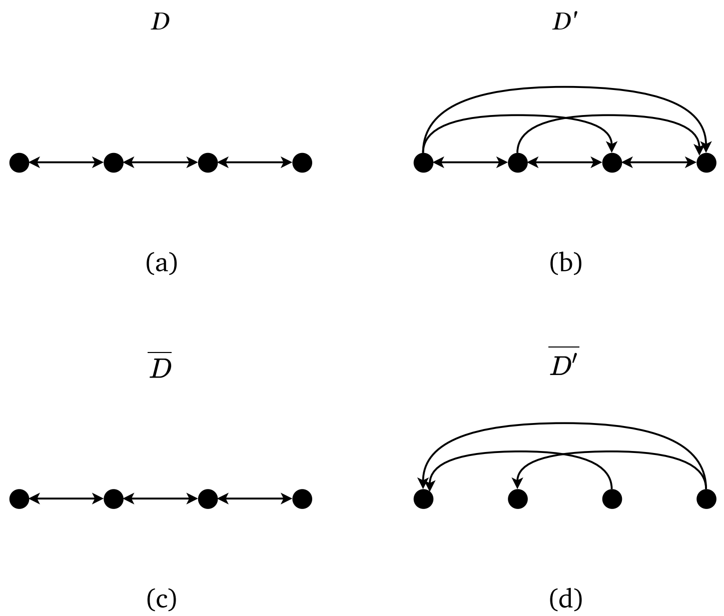


Figure 17: The four vertices induce a P_4 in both $S(D)$ and $S(D')$, but this is not true for the complements.

2. any set $\{a, b, c\} \subseteq V$ induces a $\overrightarrow{C_3}$ in D if and only if it induces a $\overrightarrow{C_3}$ in D',
3. any set $\{a, b, c\} \subseteq V$ induces a $\overrightarrow{P_3}$ with midpoint b in D if and only if it induces a $\overrightarrow{P_3}$ with midpoint b in D',
4. any set $\{a, b, c\} \subseteq V$ induces a $\overrightarrow{P_3^+}$ or a $\overrightarrow{P_3^-}$ in either case with midpoint b in D if and only if it induces one of them with midpoint b in D' and
5. any set $\{a, b, c\} \subseteq V$ induces a directed cycle of length three with one additional arc with midpoint b in D if and only if it induces a directed cycle of length three with one additional arc with midpoint b in D'.

Clearly if two digraphs are P^4S-isomorphic they are also P^4C-isomorphic.

Alternatively we could also include the complement of the N-structure (bottom middle digraph of Figure 9) into our list of forbidden subdigraphs instead of extending the criterion for the P_4 to D and D'. Because in this case P_4-isomorphism of the symmetric part of digraphs extends to digraphs in general. We will use this property later on (cf. Lemma 6.16). But in this section we will just extend our criterion for the P_4 because the main argumentation of the following Semi-strong Perfect Digraph Theorem 5.14 for strictly perfect digraphs does not require the complement of the N-structure as a forbidden minor.

Lemma 5.13. *If D and D' are two P^4S-isomorphic digraphs, then D contains an antihole $\overline{\overrightarrow{C_n}}$ of length $n \geq 3$ as an induced subdigraph if and only if the same is true*

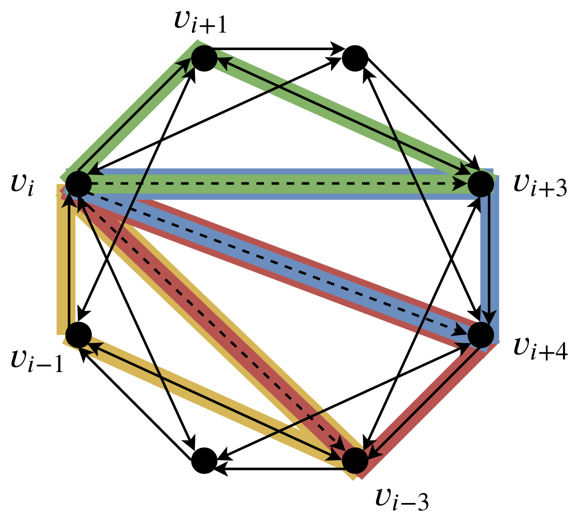


Figure 18: Visualization of the argumentation in the proof of Lemma 5.13 taking the antihole of length eight as an example. The green marking shows the first three vertices we consider and where we assume that there is a single arc (v_i, v_{i+3}). This yields to the consideration of the blue and red markings. Finally we get a contradiction for the three vertices with the yellow marking.

for D'.

Proof. By symmetry it suffices to show one direction. We therefore assume that the vertices $\{v_0, \dots, v_{n-1}\}$ induce an antihole in D. We further assume that the vertices are labeled in consecutive order (in the direction of traversal) of the outer cycle $\overrightarrow{C_n}$.
If $n = 3$ the assertion follows directly from Definition 5.12 since $\overline{\overrightarrow{C_3}} = \overrightarrow{C_3}$.
Thus assume $n \geq 4$. In D any set of three consecutive vertices $\{v_i, v_{i+1}, v_{i+2}\}$, where we consider the indices modulo n (as for the rest of this proof), induces a directed cycle $\overrightarrow{C_3}$ with one additional arc (v_i, v_{i+2}). This is the digraph in the bottom left corner of Figure 9. Since D and D' are P^4S-isomorphic any of these vertex sets induce the same subdigraph in D' possibly with opposite direction of the cycle. We therefore obtain a cycle of length n with additional pairs of antiparallel arcs between two vertices v_i, v_{i+2}, $i = 0, \dots n-1$ in D'. This proves the assertion for $n = 4$ and $n = 5$. If the outer cycle has the opposite orientation in D', we relabel it so the vertices are labeled in consecutive order (in the direction of traversal) again.
We now assume $n \geq 6$. Consider the set of three vertices $\{v_i, v_{i+1}, v_{i+3}\}$ in D'. We want to prove that there has to be a pair of antiparallel arcs between v_i and v_{i+3}.
We have already proven that there has to be a single arc (v_i, v_{i+1}) between v_i and v_{i+1} and a pair of antiparallel arcs between v_{i+1} and v_{i+3}. If there was no arc be-

tween v_i and v_{i+3} or if there was a single arc (v_{i+3}, v_i) between these vertices, the three vertices would induce subdigraphs from Figure 9. This would require that the same subdigraphs were induced by the same vertex set in D, contradicting the assumption that D contains an antihole induced by $\{v_0, \ldots, v_{n-1}\}$. Hence there has to be either a single arc (v_i, v_{i+3}) or a pair of antiparallel arcs between v_i and v_{i+3}.

Assume there is a single arc (v_i, v_{i+3}) between the vertices. We then consider the set of three vertices $\{v_i, v_{i+3}, v_{i+4}\}$. As we assume there is a single arc (v_i, v_{i+3}) and we know that there is a single arc (v_{i+3}, v_{i+4}) there are three cases in which the three vertices would induce a subdigraph from Figure 9; if there was no arc between v_i and v_{i+4}, if there was a single arc (v_{i+4}, v_i) or a pair of antiparallel arcs between those two vertices. All these cases would require that the three vertices induce the same subdigraph in D, which is again contradicting the assumption that D contains an antihole induced by $\{v_0, \ldots, v_{n-1}\}$. Therefore there would have to be a single arc (v_i, v_{i+4}) in D'.

Exactly the same argumentation leads to the conclusion that there would have to be a single arc (v_i, v_{i+5}) in D' if we now consider the set of the three vertices $\{v_i, v_{i+4}, v_{i+5}\}$ and so on until we obtain that there would have to be a single arc (v_i, v_{i-3}) in D'.

We then consider the set of the three vertices $\{v_i, v_{i-3}, v_{i-1}\}$. We know that there is a single arc (v_{i-1}, v_i) and a pair of antiparallel arcs between v_{i-1} and v_{i-3} and following the assumption we obtain that there has to be a single arc (v_i, v_{i-3}) in D'. Hence those three vertices induce a directed cycle of length three with one additional arc in D' which is one of the digraphs in Figure 9. Since D and D' are P^4S-isomorphic the same subdigraph must be induced by these vertices in D, contradicting the assumption that D contains an antihole induced by $\{v_0, \ldots, v_{n-1}\}$. Hence the assumption that there is a single arc (v_i, v_{i+3}) is wrong and there has to be a pair of antiparallel arcs between v_i and v_{i+3}.

Similarly we can conclude that there has to be a pair of antiparallel arcs between v_i and v_{i+4} if we consider the vertex set $\{v_i, v_{i+3}, v_{i+4}\}$. If there was no arc between v_i and v_{i+4} or if there was a single arc (v_{i+4}, v_i) the three vertices would induce a subdigraph from Figure 9, again contradicting the assumption that D contains an antihole induced by $\{v_0, \ldots, v_{n-1}\}$. If there was a single arc (v_i, v_{i+4}) the same argumentation as above leads to a contradiction, hence there has to be a pair of antiparallel arcs between v_i and v_{i+4}.

Inductively we obtain that there has to be a pair of antiparallel arcs between v_i and v_{i+j}, $j \in \{4, \ldots, i-3\}$.

Hence the vertices $\{v_0, \ldots, v_{n-1}\}$ induce an antihole in D'. □

With Lemma 5.13 we can easily formulate a Semi-strong Perfect Digraph Theorem for strictly perfect digraphs.

Theorem 5.14. *If D and D' are two P^4S-isomorphic digraphs, then D is a strictly perfect digraph if and only if D' is a strictly perfect digraph.*

Proof. From the Semi-Strong Perfect Digraph Theorem 5.7 we know that if D and D' are two P^4S-isomorphic digraphs, then D is a perfect digraph if and only if D' is a perfect digraph, since P^4S-isomorphism induces P^4C-isomorphism.
It remains to show that if D and D' are two P^4S-isomorphic digraphs, then $\overline{D}$ is a perfect digraph if and only if $\overline{D'}$ is a perfect digraph.
$S(\overline{D})$ and $S(\overline{D'})$ are P_4-isomorphic as D and D' are P_4-isomorphic and a P_4 is self-complementary. So by the Semi-Strong Perfect Graph Theorem 4.16 $S(\overline{D})$ is perfect if and only if $S(\overline{D'})$ is perfect.
By Lemma 5.13 D contains an antihole of length at least three as an induced subdigraph if and only if the same is true for D'. Hence $\overline{D}$ contains an induced cycle of length at least three if and only if the same holds for $\overline{D'}$.
Hence the assertion follows from the Strong Perfect Digraph Theorem 5.2. □

6 N-free Coloring and N-perfection of Digraphs

As we have seen in Section 5 perfection of digraphs does not behave quite as *perfectly* as perfection for graphs does, most notably it is not closed under complementation. This suggests that maybe acyclic coloring is not the best way of defining coloring for (perfect) digraphs. As we saw in Section 5.2 we need to forbid five subdigraphs to obtain an analogon to the Semi-strong Perfect Graph Theorem for digraphs. An intuitive way of modifying the definition of coloring digraphs could be in a way so that we obtain a Semi-strong Perfect Digraph Theorem that requires more than these five subdigraphs.
We have already discussed that we obtain strictly perfect digraphs if we include the directed cycle of length three with one additional arc (bottom left subdigraph of Figure 9) in our list of forbidden digraphs but this does not modify the way we color digraphs. We therefore consider the remaining two subdigraphs; the N-structure and its complement.
To obtain a Semi-strong perfect Digraph Theorem that requires the five subdigraphs from Figure 14 and the N-structure as forbidden induced subdigraph we can modify the definition of acylic coloring in the following way.

Definition 6.1. An *acyclic N-free coloring* of a digraph D is a mapping $f : V(D) \to C$, $C \subseteq \mathbb{Z}$, such that the following properties hold:

1. for every $c \in C$ the vertices colored by c induce an acyclic subdigraph and
2. there is no monochromatic induced N-structure in D.

For $m = |C|$ we call f an *acyclic N-free m-coloring* of D.
Analogously to Definition 3.7 we call $\chi_N(D) = m_0$ the *N-dichromatic number* of D where m_0 is the smallest number for which there is an acyclic N-free m_0-coloring of D.

For convenience and as there is no danger of confusion we will refer to acyclic N-free coloring just as *N-free coloring*.

Note that for the symmetric part $S(D)$ of a digraph D N-free coloring is the same as acyclic coloring since there cannot be an induced N-structure in the symmetric part of a digraph. Therefore the dichromatic number $\chi(S(D))$ and the N-dichromatic number $\chi_N(S(D))$ of the symmetric part of a digraph are the same.

Some digraphs might just require a recoloring when we consider N-free coloring instead of acyclic coloring (see Figure 19 for an example). But there are obviously digraphs for which $\chi(D) < \chi_N(D)$, an example can be found in Figure 20.

There are some interesting observations concerning the existence of induced N-

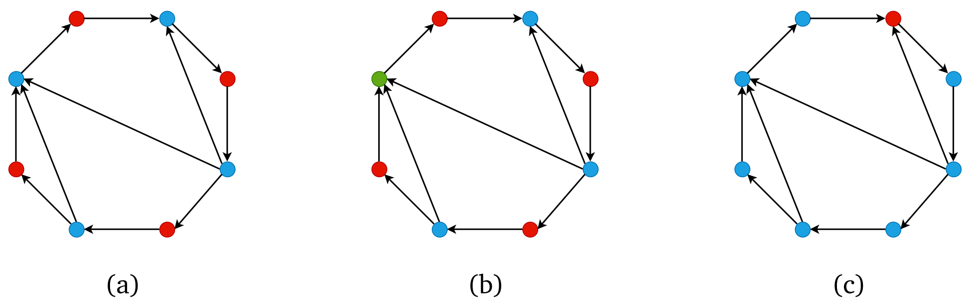


Figure 19: The shown digraph only requires two colors (red and blue) if acyclic coloring is applied (a), three colors (red, blue and green) seem to be needed if N-free coloring is applied (b) but we can recolor the digraph such that again two colors (red and blue) suffice.

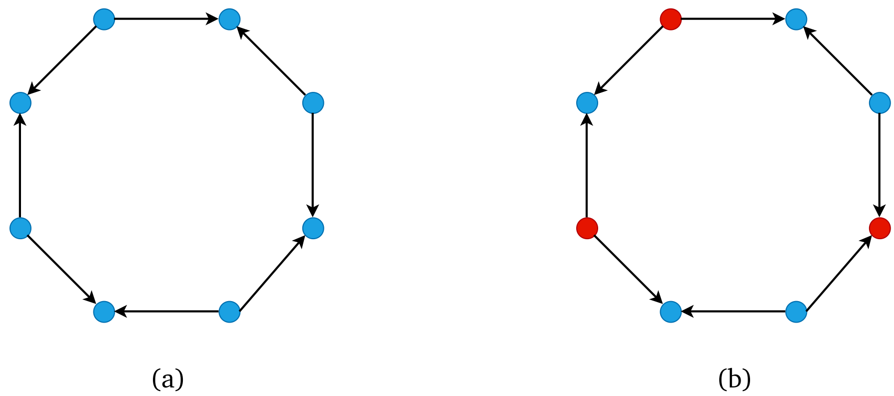


Figure 20: The shown digraph only requires one color (blue) if acyclic coloring is applied (a) and at least two colors (red and blue) if N-free coloring is applied (b), since there are several N-structures induced in the digraph.

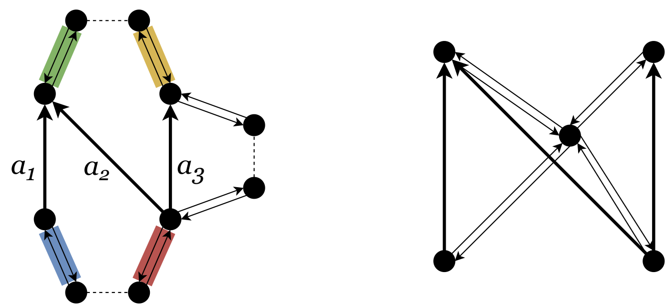
structures in specific digraphs. To begin with we take a look at a class of digraphs we already discussed earlier. In Lemma 5.8 at the end of Section 5.2 we saw that one asymmetric arc between two components of the symmetric part $S(D)$ of a digraph D that does not contain any of the five forbidden subdigraphs from Figure 14 implies the existence of an orientation of the complete bipartite graph between these components. In the following we examine how the remaining three subdigraphs from Figure 9 can be induced subdigraphs of such a digraph D.

Lemma 6.2. *Let D be a digraph not containing any of the five subdigraphs from Figure 14 as an induced subdigraph. An N-structure cannot be induced by four vertices of exactly two different components of $S(D)$.*

Proof. Assume there is a single arc between two components G_i and G_j of $S(D)$. By Lemma 5.8 we know that there has to be an orientation of a complete bipartite graph between those two components.

Assume there is an N-structure (not necessarily induced) containing two vertices of G_i and two vertices of G_j. This N-structure cannot be induced in D since there are only three arcs in an N-structure but the orientation of a complete bipartite graph between those four vertices requires the existence of four arcs between them.
Assume there is an N-structure (not necessarily induced) containing one vertex of G_i and three vertices of G_j (the opposite case follows by symmetry). This N-structure cannot be induced in D since there are at most two arcs connecting the single vertex of G_i to vertices of G_j in an N-structure but the orientation of a complete bipartite graph between those four vertices requires the existence of three arcs between them.
Hence an induced N-structure in D cannot contain vertices from exactly two different components of $S(D)$. □

Note that all other cases (the four vertices being in three, four or just one component of $S(D)$) are possible but there are always further restrictions since there cannot be a $\overrightarrow{P_3^+}$ or a $\overrightarrow{P_3^-}$ induced in D. For example if there is an induced N-structure within one component of $S(D)$ that restricts the structure of such a component due to the non-existence of an induced $\overrightarrow{P_3^+}$ or $\overrightarrow{P_3^-}$ in D as illustrated in Figure 21.



(a) (b)

Figure 21: (a) shows a forbidden induced N-structure within a component in the symmetric part of a digraph without the five forbidden induced subdigraphs, since a_1 forms a forbidden subdigraph with either the green or the blue marked pair of antiparallel arcs as well as a_2 with the green or red one and a_3 with the yellow or red one. (b) shows an allowed induced N-structure within one component of the symmetric part of a digraph that does not contain any of the five forbidden induced subdigraphs.

If D is a digraph not containing any of the five subdigraphs from Figure 14 as an induced subdigraph, the complement of an N-structure (bottom middle subdigraph of Figure 9) can obviously only appear in a single component of $S(D)$ as an induced subdigraph, since there already exists a bidirected path between all four vertices in this structure.

On the other hand there are many possible cases in which a digraph D that does not contain any of the five forbidden subdigraphs from Figure 14 contains a directed cycle of length three with one additional arc (bottom left subdigraph of Figure 9) since this structure is one possibility how one vertex from one component is connected to two vertices from another component (that are neighboring by a pair of antiparallel arcs). Obviously the structure can also appear within a component of $S(D)$.

We now turn to some other digraphs for which we can immediately formulate some observations.

Lemma 6.3. *Let D be a digraph and let every vertex $v \in V(D)$ have either in-degree 0 or out-degree 0. There exists an N-free 2-coloring of D.*

Proof. Let $V_1 \subseteq V(D)$ be the vertex set containing the vertices with in-degree 0 and let $V_2 \subseteq V(D)$ be the vertex set containing all vertices with out-degree 0. There are obviously no cycles in D since we would need at least two vertices with in- and out-degree at least one to form a cycle. Hence D can be colored with one color if we consider acyclic coloring. The vertices of every induced N-structure in D alternate between V_1 and V_2. By coloring all vertices of V_1 with a color c_1 and all vertices of V_2 with a different color c_2 we therefore obtain a proper N-free 2-coloring of D. □

Observation 6.4. *A digraph D with five vertices can contain at most three induced N-structures.*

This observation can easily be verified by adding additional arcs between an existing N-structure induced by a vertex set $\{v_1, v_2, v_3, v_4\}$ in a digraph D and an additional vertex v_5. Obviously it is not possible to add arcs between the vertices of the existing N-structure if we want to maintain the N-structure being induced. Moreover adding arcs between v_5 and the N-structure to create new induced N-structures becomes very limited if we want to maintain the already created N-structures being induced. Trying all possible cases leads to the conclusion that we never obtain more than three induced N-structures between five vertices.
An example can be seen in Figure 22.

Remark 6.5. Any digraph D with five vertices can be N-freely colored with two colors. It always suffices to color two vertices differently from the other three vertices since any induced N-structure in D contains four vertices which then

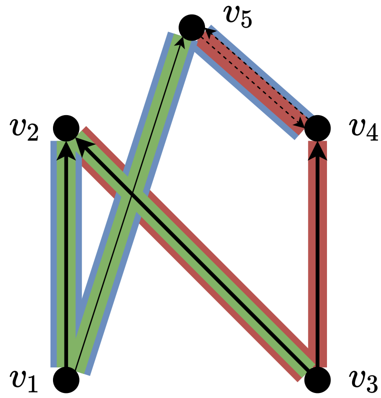


Figure 22: A digraph with five vertices that has the maximum of three induced N-structures. One N-structure is given by the bold arrows, one by the green marked arrows and a third one either by the blue or the red marked arrows depending on which of the dotted arrows is included in the digraph.

cannot all have the same color.

Clearly many more general questions would be interesting to consider for N-structures and N-free coloring, but due to the focus of this thesis we want to turn to the aspect of perfection once again and consider this topic in the sense of N-free coloring. We can obviously give an alternative definition of perfection using N-free coloring instead of acyclic coloring.

Definition 6.6. A digraph D is called *N-perfect* if for any induced subdigraph H of D the N-dichromatic number $\chi_N(H)$ of H equals the clique number $\omega(H)$ of H.

Observation 6.7. *For a digraph D we have $\chi(D) \leq \chi_N(D)$. With Lemma 3.9 we therefore obtain $\omega(D) \leq \chi_N(D)$.*

In the following subsections we will show that the main theorems on perfect digraphs can be transferred to N-perfect digraphs as well.

6.1 A Strong N-perfect Digraph Theorem

We can quite intuitively formulate a Strong Perfect Digraph Theorem for N-perfect digraphs.

Theorem 6.8 (Strong N-perfect Digraph Theorem)**.** *A digraph D is N-perfect if and only if all of the following holds:*

1. *$S(D)$ is N-perfect,*
2. *D does not contain any directed cycle $\overrightarrow{C_n}$ of length $n \geq 3$ as an induced subdigraph and*

3. *D does not contain an N-structure as an induced subdigraph.*

Proof. Assume $S(D)$ is not N-perfect. With Observation 6.7 the argumentation of Theorem 5.2 can be transferred to N-free coloring. There is an induced subdigraph $H = (V', A')$ of $S(D)$ with $\omega(H) < \chi_N(H)$. Taking into consideration Observation 2.29 we obtain:

$$\omega(D[V']) = \omega(S(D[V'])) = \omega(H) < \chi_N(H) = \chi_N(S(D[V'])) \leq \chi_N(D[V'])$$

Therefore D is not N-perfect.
Assume D contains a directed cycle $\overrightarrow{C_n}$ of length $n \geq 3$ as an induced subdigraph, then D is not N-perfect since $\omega(\overrightarrow{C_n}) < \chi(\overrightarrow{C_n}) \leq \chi_N(\overrightarrow{C_n})$.
Assume D contains an N-structure N as an induced subdigraph. D is not N-perfect since $\omega(N) = 1 < 2 = \chi_N(N)$.

Now assume that $S(D)$ is N-perfect but D is not. It suffices to show that D has either a directed cycle of length greater than three or an N-structure as an induced subdigraph.
Since D is not N-perfect there is an induced subdigraph $H \subseteq D$ with $\omega(H) < \chi_N(H)$. As $S(H)$ is perfect there is a proper N-free coloring of $S(H)$ with $\omega(S(H)) = \omega(H)$ colors. This cannot be a proper coloring of H, so there must be a (not necessarily induced) monochromatic cycle of length greater than three or an induced N-structure in H. If there is an induced N-structure in H the assertion follows. Let C be a monochromatic cycle of minimal length greater than three in H. It follows that C has to be induced by the same minimality argument as in Theorem 5.2. Hence the assertion is true.

Just like in Theorem 5.2 this completes the proof because if we assume that D is not N-perfect and has neither a directed cycle of length greater than three nor an N-structure as induced subdigraphs, then $S(D)$ must not be N-perfect. Because if it was, D would have either a directed cycle of length greater than three or an N-structure as induced subdigraphs by what we have just proven. This contradicts the assumption we just made. □

Similar to Remark 5.3 we have actually proven the following.

Remark 6.9. If D is an N-perfect digraph, then any proper N-free coloring of $S(D)$ is also a proper N-free coloring of D.

6.2 A Semi-strong N-perfect Digraph Theorem

To formulate an analogon to the Semi-strong Perfect Graph Theorem 4.16 we once again turn to the eight forbidden subdigraphs in Figure 9. Quite intuitively we include the N-structure into our set of forbidden minors and can then define P^4N-isomorphism. We will see that this suffices to prove a Semi-strong N-perfect Digraph Theorem.

Definition 6.10. Two digraphs $D = (V, A)$ and $D' = (V, A')$ on the same vertex set are said to be P^4N-*isomorphic* if and only if all of the following holds:

1. any set $\{a, b, c, d\} \subseteq V$ induces a P_4 in $S(D)$ if and only if it induces a P_4 in $S(D')$,
2. any set $\{a, b, c\} \subseteq V$ induces a $\overrightarrow{C_3}$ in D if and only if it induces a $\overrightarrow{C_3}$ in D',
3. any set $\{a, b, c\} \subseteq V$ induces a $\overrightarrow{P_3}$ with midpoint b in D if and only if it induces a $\overrightarrow{P_3}$ with midpoint b in D',
4. any set $\{a, b, c\} \subseteq V$ induces a $\overrightarrow{P_3^+}$ or a $\overrightarrow{P_3^-}$ in either case with midpoint b in D if and only if it induces one of them with midpoint b in D' and
5. any set $\{a, b, c, d\} \subseteq V$ induces an N-structure in D if and only if it induces an N-structure in D'.

Clearly if two digraphs are P^4N-isomorphic they are also P^4C-isomorphic.

Theorem 6.11 (Semi-strong N-perfect Digraph Theorem)**.** *If D and D' are two P^4N-isomorphic digraphs, then D is N-perfect if and only if D' is N-perfect.*

Proof. By criterion (1) of Definition 6.10 $S(D)$ and $S(D')$ are P_4-isomorphic and therefore it follows from the Semi-strong Perfect Digraph Theorem 5.2 that $S(D)$ is N-perfect if and only if $S(D')$ is N-perfect, since N-perfection and perfection are the same for the symmetric part of a digraph. Since P^4N-isomorphism implies P^4C-isomorphism, Lemma 5.6 implies that D contains an induced cycle of length $n \geq 3$ if and only if the same is true for D'.
Let $\{v_1, v_2, v_3, v_4\} \subseteq V$ be a set of vertices that induces an N-structure N in D, then by criterion (5) of Definition 6.10 the same vertex set induces an N-structure in D'. By symmetry it follows that D contains an N-structure as an induced subdigraph if and only if D' contains an induced N-structure.
The assertion follows by the Strong N-perfect Digraph Theorem 6.8. □

6.3 Non-existence of a Weak N-perfect Digraph Theorem

Unfortunately there is still no analogon to the Weak Perfect Graph Theorem for N-free colored digraphs.

Observation 6.12. *For an N-perfect digraph D its complement $\overline{D}$ may not be N-perfect.*

Take an N-structure N as an example. We have already seen that N is not N-perfect since $\omega(N) = 1 < 2 = \chi_N(N)$ in the proof of the Strong N-perfect Digraph Theorem 6.8. The complement $\overline{N}$ however is N-perfect, since $\omega(\overline{N}) = 2 = \chi_N(\overline{N})$ (see Figure 23).

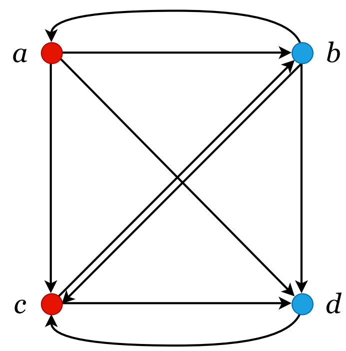


Figure 23: A proper N-free coloring of the complement of an N-structure with two colors (red and blue), clearly the subdigraphs induced by the vertex sets $\{a, b\}$, $\{b, c\}$ and $\{c, d\}$ form cliques of size two.

Similar to Theorem 5.10 we can however formulate some properties for the complement of an N-perfect digraph.

Theorem 6.13. *A digraph D is N-perfect if and only if its complement $\overline{D}$ is a superorientation of a perfect graph G that is clique-acyclic and there does not exist a clique in G which is induced by a vertex set that induces an N-structure in $O(\overline{D})$.*

Proof. We know from the Strong N-perfect Digraph Theorem 6.8 that D is N-perfect if and only if $S(D)$ is N-perfect, D does not contain any directed cycle $\overrightarrow{C_n}$ of length $n \geq 3$ as an induced subdigraph and D does not contain an N-structure as an induced subdigraph. In Theorem 5.10 we have already proven that the first two conditions are equivalent to $\overline{D}$ being a clique-acyclic superorientation of a perfect graph G, since perfection and N-perfection are the same for the symmetric part of a digraph. It therefore remains to show that D not containing an N-structure as an induced subdigraph is equivalent to $\overline{D}$ being the superorientation of a graph not containing a clique that is induced by the vertex set of an induced N-structure in $O(\overline{D})$.

Obviously D not containing an N-structure as an induced subdigraph is equivalent to $\overline{D}$ not containing the complement of an N-structure as an induced subdigraph. We therefore want to show that $\overline{D}$ does not contain the complement of an N-structure as an induced subdigraph if and only if $\overline{D}$ is the superorientation of a graph not containing a clique that is induced by the vertex set of an induced N-structure in $O(\overline{D})$.
Assume that $\overline{D}$ contains the complement of an N-structure $\overline{N}$ as an induced subdigraph. The underlying graph $UG(\overline{N})$ is the complete graph K_4 whose vertices obviously form a clique in $UG(\overline{D})$ and since $\overline{N}$ induces an N-structure in $O(\overline{D})$ we have proved one direction via contraposition.
Assume on the other hand that $\overline{D}$ is the superorientation of a graph that contains a clique that is induced by the vertex set of an induced N-structure N in $O(\overline{D})$. Since N is induced in $O(\overline{D})$ there cannot be a single arc connecting any of the vertices of N in $\overline{D}$ that are not connected within N anyway. To obtain a clique in the underlying graph we therefore have to add three pairs of anti-parallel arcs between the vertices not connected in N. The four vertices then induce the complement of an N-structure in $\overline{D}$ which completes the prove. □

We have actually proven that the only cliques in the underlying graph that have a greater restriction when considering the complement of N-perfect digraphs rather than perfect digraphs are those of size four.

6.4 Strictly N-perfect Digraphs

So far we have not won much from extending the definition of coloring digraphs from acyclic coloring to N-free coloring. However we will see in this subsection that we can obtain a class of digraphs from N-perfect digraphs that is closely related to the class of cographs. This yields the expectation that N-perfect digraphs might be an *easier* class of digraphs than acyclicly perfect digraphs, i.e. certain problems might be traceable for N-perfect digraphs that do not have an efficient solution for acyclicly perfect digraphs. As we still have not achieved closure under complementation for N-perfect digraphs, we define strictly N-perfect digraphs similar to strictly perfect digraphs.

Definition 6.14. A *strictly N-perfect digraph* is a digraph D that is N-perfect and whose complement $\overline{D}$ is N-perfect as well.

If we want to formulate a Semi-strong Perfect Digraph Theorem for strictly N-perfect digraphs we have to forbid all eight digraphs from Figure 9 as induced subdigraphs of D. Interestingly we do not have to extend the criterion for the P_4 to D and D' in this case (cf. Definition 5.12) as Lemma 6.16 shows. We will however do so in the following definition for the sake of consistency within the criteria of the definition.

Definition 6.15. Two digraphs D and D' on the same vertex set are P^4SN-*isomorphic* if and only if all of the following holds:

1. any set $\{a, b, c, d\} \subseteq V$ induces a P_4 in D if and only if it induces a P_4 in D',
2. any set $\{a, b, c\} \subseteq V$ or $\{a, b, c, d\} \subseteq V$ induces one of the subdigraphs of Figure 9 (except for a P_4) in D if and only if it induces the same subdigraph in D' (for an induced $\overrightarrow{P_3^+}$ or an induced $\overrightarrow{P_3^-}$ in D it is also allowed to induce the other subdigraph in D'),
3. for the asymmetric digraphs with three vertices in Figure 9 the midpoint is the same in D and D'.

Lemma 6.16. *Let D and D' fulfill criteria (2) and (3) of Definition 6.15 and let any set $\{a, b, c, d\} \subseteq V$ induce a P_4 in $S(D)$ if and only if it induces a P_4 in $S(D')$. A vertex set induces a P_4 in D if and only if it induces a P_4 in D' as well.*

Proof. According to the premise we know that the corresponding assertion is true for a P_4 induced in $S(D)$ and $S(D')$ respectively. Consider a P_4 that is induced in $S(D)$ but not in D and only has one single arc connecting two vertices of the path. Let $V_1 = \{v_1, v_2, v_3, v_4\}$ be the vertex set that induces the P_4 in $S(D)$ and let (v_i, v_j), $i, j \in \{1, \dots, 4\}$ and $|i - j| > 1$, be a single arc in D. At least two of the vertex sets $\{v_{i\pm1}, v_i, v_j\}$, $\{v_{j\pm1}, v_i, v_j\}$ are subsets of V_1 and at least one of them induces a $\overrightarrow{P_3^+}$ or a $\overrightarrow{P_3^-}$ in D, hence they must induce the same subdigraph in D', maybe with opposite direction of the single arc, by definition. So V_1 induces the same subdigraph in D' as it induces in D, maybe with opposite direction of the single arc.

If we consider a P_4 that is induced in $S(D)$ but not in D and has two single arcs connecting vertices of the path, a similar argumentation holds. Let $V_2 = \{v_1, v_2, v_3, v_4\}$ be the vertex set that induces the P_4 in $S(D)$ and let (v_i, v_j) and (v_k, v_l), $i, j, k, l \in \{1, \dots, 4\}$, $|i - j| > 1$ and $|k - l| > 1$, be two different single arcs in D. If $\{i, j, k, l\} = \{1, 2, 3, 4\}$ the single arcs must connect v_1 to v_3 and v_2 to v_4. Without loss of generality let (v_i, v_j) connect v_1 to v_3 and let (v_k, v_l) connect v_2 to v_4. In this case the vertex sets $\{v_1, v_2, v_4\}$ and $\{v_1, v_3, v_4\}$ induce a $\overrightarrow{P_3^+}$ or a $\overrightarrow{P_3^-}$ in D and must therefore induce the same subdigraph in D', maybe with opposite direction of the single arc. If v_i, v_j, v_k and v_l correspond to only three different vertices, let v_m, $m \in \{2, 3\}$, be the vertex that is not incident to any of the single arcs. The vertex sets $\{v_m, v_i, v_j\}$ and $\{v_m, v_l, v_k\}$ are subsets of V_2 and at least one of them induces a $\overrightarrow{P_3^+}$ or a $\overrightarrow{P_3^-}$ in D, hence they must induce the same subdigraph in D', maybe with opposite direction of the single arc, by definition. So V_2 always induces the same subdigraph in D' as it induces in D, maybe with opposite direction of the single arcs.

If we consider a P_4 that is induced in $S(D)$ but not in D and has three single arcs connecting vertices of the path, we have to consider four different subdigraphs (see Figure 24), all other possible subdigraphs are isomorphic to one of these subdigraphs. Let $V_3 = \{v_1, v_2, v_3, v_4\}$ be the vertex set that induces the P_4 in $S(D)$. The first three subdigraphs have vertex subsets that induce a directed cycle of length three with one additional arc in D (cf. red arcs in Figure 24), hence these vertex subsets have to induce the same subdigraph in D', maybe with opposite direction of the cycle. So V_3 does not induce a P_4 in D'. The fourth subdigraph in Figure 24 is the complement of an N-structure, so V_3 has to induce the complement of an N-structure in D' as well by definition.

Obviously there cannot be more than three single arcs between the vertices of a P_4 in a digraph.

We have shown above that the vertex set of any P_4 that is induced in $S(D)$ but not in D, induces a P_4 in $S(D')$ but not in D'. Obviously a P_4 that is induced in D is induced in $S(D)$ as well, it therefore has to be induced in $S(D')$ by definition. The symmetry of the above argumentation requires that the P_4 is also induced in D'. □

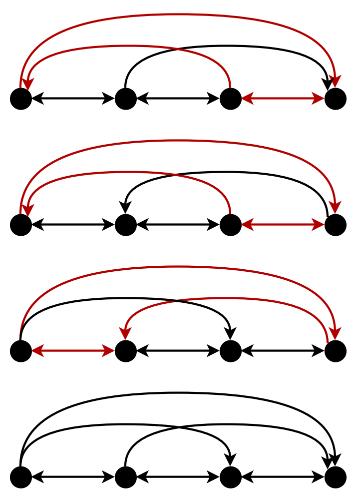


Figure 24: All subdigraphs (up to isomorphism) with a P_4 that is induced in $S(D)$ but not in D with three single arcs connecting the vertices of the path. The red arcs indicate the induced directed cycles of length three with one additional arc.

Due to the fact that the criterion for the P_4 is automatically extended to D and D', P^4SN-isomorphism induces P^4N-isomorphism as well as P^4S-isomorphism and therefore P^4C-isomorphism. The following theorem is an analogon to the Semi-strong N-perfect Digraph Theorem for strictly N-perfect digraphs.

Theorem 6.17. *If D and D' are two P^4SN-isomorphic digraphs, then D is a strictly N-perfect digraph if and only if D' is a strictly N-perfect digraph.*

Proof. From Theorem 6.8 we know that if D and D' are two P^4SN-isomorphic digraphs, then D is an N-perfect digraph if and only if D' is an N-perfect digraph, since P^4SN-isomorphism induces P^4N-isomorphism.
It remains to show that if D and D' are two P^4SN-isomorphic digraphs, then $\overline{D}$ is an N-perfect digraph if and only if $\overline{D'}$ is an N-perfect digraph.
By Theorem 5.14 $S(\overline{D})$ is N-perfect if and only if $S(\overline{D'})$ is N-perfect, since perfection and N-perfection are the same for the symmetric part of a digraph and P^4SN-isomorphism induces P^4S-isomorphism.
By Lemma 5.13 D contains an antihole of length at least three as an induced subdigraph if and only if D' contains an induced antihole of the same length, since P^4SN-isomorphism induces P^4S-isomorphism. Hence $\overline{D}$ contains an induced cycle of length at least three if and only if $\overline{D'}$ contains an induced cycle of the same length.
It remains to show that $\overline{D}$ contains an N-structure as an induced subdigraph if and only if the same is true for $\overline{D'}$. By symmetry it suffices to prove one inclusion. Assume that $\{v_1, v_2, v_3, v_4\}$ induces an N-structure N in $\overline{D}$. Thus these four vertices induce $\overline{N}$ in D. This is the subdigraph in the middle of the bottom line of Figure 9. By Definition 6.15 the same vertex set induces the same subdigraph in D'. Therefore $\{v_1, v_2, v_3, v_4\}$ induces an N-structure in $\overline{D'}$.
Hence the assertion follows from Theorem 6.8. □

The information in this subsection actually states that strictly N-perfect digraphs are a class of digraphs that is closely related to the class of directed cographs, because both can be characterized by the set of eight forbidden minors (see Figure 9).

7 Conclusion

This thesis provides an introduction into the idea of N-free coloring of digraphs. N-free coloring was introduced because acyclic coloring of digraphs does not provide an equivalent of perfection of digraphs as *perfect* as perfection of undirected graphs. Most importantly perfection of digraphs is not closed under complementation. We used the idea of characterizing perfect digraphs through five of the eight forbidden induced subgraphs of cographs, which Andres et al. [3] used to establish a Semi-Strong Perfect Digraph Theorem, to modify the way of coloring digraphs to achieve a *better* perfection of digraphs.
Firstly we examined strictly perfect digraph for which we require closure under complementation by definition. As a result we established that we need to include one additional subdigraph into the list of forbidden minors to prove a Semi-strong Perfect Digraph Theorem for strictly perfect digraphs. This led to the idea of examining the effect of including the two remaining subdigraphs into the list of forbidden minors.

We introduced N-free coloring as a consequence of the inclusion of the N-structure into the list of forbidden minors. We showed that N-perfect digraphs resulting from N-free coloring also fulfill a Strong and a Semi-Strong N-perfect Digraph Theorem. Unfortunately N-perfect digraphs still are not closed under complementation. But we can once again require this by definition. This led us to introduce strictly N-perfect digraphs for which we once again proved a Semi-strong N-perfect Digraph Theorem. It requires all eight subdigraphs. Obviously strictly N-perfect digraphs therefore form a class of digraphs closely related to the class of directed cographs. This class has been studied concerning various problems and many problems can be solved efficiently for it (cf. [23]).

Hence we achieved to introduce a way of coloring digraphs that has the potential to be a better equivalent to classic coloring of graphs if we consider perfection of digraphs. We base this assumption on the fact that N-perfect digraphs are *more closely* related to strictly N-perfect digraphs and therefore to directed cographs than acyclicly perfect digraphs. As we have seen it does not solve the problem of missing closure under complementation. Especially questions concerning the complexity of algorithms on N-perfect digraphs would however be of great interest, as it seems plausible that there are problems that can be solved more efficiently for N-perfect digraphs than for acyclicly perfect digraphs.

They could therefore be suggestions for further research. Moreover general questions related to coloring can be considered for N-free coloring as well. Most of all the minimal number of colors needed to properly N-freely color a planar digraph (as an equivalent to the Four-Color-Theorem) is interesting to consider. But also other questions concerning N-free coloring can be further investigated, like the question of how to find an N-free coloring with a minimal number of colors.

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Selbstständigkeitserklärung

Name: Andrea Möller
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