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MASTER'S THESIS

Neumann-Lara-Flows and the Two-Colour-Conjecture

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1 Introduction

A famous problem in graph theory, answered positively by Appel and Haken in 1976, is the Four-Colour-Conjecture, which states that every simple planar graph is 4-colourable. In my research as part of this master's thesis, I was mainly concerned with a directed version of this conjecture, the so-called Two-Colour-Conjecture, which is still unsolved.

A k-colouring of a digraph according to Victor Neumann-Lara is defined to be a decomposition of the vertex set into k subsets each of them inducing an acyclic subdigraph. The conjecture now simply states that every orientation of a simple planar graph admits a 2-colouring.

Since the publication of the conjecture in 1985, over a long period of time, the only techniques applied to tackle it were based on so-called vertex arboricity numbers of graphs (cf. subsection 5.1). Given a graph G, its vertex arboricity is defined as the minimal number of colours needed to colour the vertices of G without any monochromatic cycles at all. In other words, those techniques are used to find vertex-colourings of a given simple planar graph using at most two colours, which is not only a legal digraph colouring for some orientation of the graph, but for any such orientation at the same time. It can be shown that simple planar graphs admit vertex arboricity at most three, and for a lot of small examples, also two colours seem to suffice. On the other hand, compared to the original task, this notion of colourings obviously makes the problem unnecessarily harder, and in fact, we can indeed construct examples of simple planar graph with vertex arboricity 3. This observation demonstrates that if we want to prove the 2-Colour-Conjecture in its whole generality, we must consider arguments involving special orientation properties of the actual digraph.

The first results improving on the approach mentioned above are pretty recent ones, both showing partial positive results when restricting the digirth of the considered planar digraphs, i.e., the minimal length of a directed cycle contained in it. In 2014, Harutyunyan and Mohar first showed that planar digraphs of digirth at least five admit legal 2-colourings, thereby using elaborate discharging techniques. Unfortunately, it does not seem likely that those techniques can be efficiently used to approach a proof of the conjecture. In contrast to that, in 2017, Li and Mohar found a very elegant and simple way of proving the stronger result that legal 2-colourings of simple planar digraphs exist if directed triangles are forbidden. Their argument deals with dual graphs and uses the concept of so-called Tutte paths introduced and used by Thomassen and Tutte when dealing with the Hamiltonicity of 4-connected planar graphs. These developments are described in the subsection 5.2. As is explained further below, they were one of the central motivations to start the research presented here in a more general dualized context of certain integer flows on 3-edge-connected digraphs.

In this master's thesis, after describing the fundamental notions of graphs, digraphs, matroids and oriented regular matroids (digraphoids) and their basic properties (see section 3), we shortly discuss colourings of graphs and digraphs in general. In the rest of the master's thesis, we mainly focus on colourings of planar digraphs and their relation to certain integers flows in their dual graphs. After presenting the Two-Colour-Conjecture and related notions in the first part of section 4, we introduce the well-studied concept of Nowhere-Zero-flows on graphs and the recently developed corresponding directed notion of Neumann-Lara-Flows

from Hochstättler ([Hoc17]), which both resemble colouring properties of primal objects (planar graphs, digraphs) in terms of edge-assignments on their dual graphs:
Given some usual graph colouring together with some orientation of the respective graph, we can define a tension on its arc set, by enumerating the appearing colours and interpreting them as a potential on the vertices. In the case that the considered graph was planar, the thereby defined tensions are in bijection to certain integral flows on the directed dual graph, which are non-zero on every edge. This gives rise to the notions of Nowhere-Zero-k-Flows on digraphs, i.e., flows ranging in $\{\pm 1, ..., \pm(k-1)\}$. It is easy to see that the concrete orientation used to define the flow is irrelevant for the existence and thus, such flows can be interpreted as a pure graph property. They generalize the notion of graph colourings in the planar case and provide a better understanding of hidden structures in those graphs which are used and needed to find graph colourings. Obviously, if we allow graphs to be non-planar, they can admit arbitrarily large chromatic numbers. On the other hand, after having dualized e.g. the 4-Colour-Theorem to the equivalent dual statement "Every bridgeless planar graph admits a NZ-4-Flow", it becomes much less clear why the flow indices, i.e. the minimal numbers k, for which NZ-k-flows exist, should be unbounded too (given a sufficient edge-connectivity), since for arbitrary graphs, there is no well-defined way of generalizing planar duals. And indeed, as was shown by Jaeger and Seymour respectively, a fixed upper bound for the flow indices of all 2-edge-connected (bridgeless) graphs G indeed exists. Tutte proved that the existence of Nowhere-Zero-Flows is independent of the underlying algebraic structure but only dependent on the order of the abelian group the flows are defined on, which was an essential tool used in further developments. While Jaeger was the first to show the existence of such a bound by proving the existence of Nowhere-Zero-8- or equivalently NZ-$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flows respectively, Seymour improved this result by constructing Nowhere-Zero flows in $\mathbb{Z}_2 \times \mathbb{Z}_3$, i.e., showed an upper bound of 6 for 2-edge-connected graphs. What about lower bounds on the minimal upper bound k? Looking at planar graphs, we know that $k \geq 4$. At first sight, one might be tempted to conjecture that the dualized version of the 4-Colour-Theorem as described above extends to arbitrary bridgeless graphs, i.e. $k = 4$ is the minimal upper bound we were looking for, but unfortunately, there is a well-known 2-edge-connected 3-regular but non-planar graph, called Petersen graph, which does not admit a NZ-4-, but only a NZ-5-flow. As it seems, the Petersen graph is the only substantial obstruction to the existence of a 4-flow, i.e., all examples of bridgeless graphs not admitting a 4-flow found so far admit a Petersen graph as minor. Since this discovery, two very prominent conjectures by Tutte, the 4- and 5-Flow-Conjecture, have been intensively investigated. They state that every bridgeless graph without a Petersen minor admits a NZ-4-flow resp. that Seymour's upper bound of 6 can be improved to 5, i.e., every bridgeless graph admits a NZ-5-flow. Although since then, some weaker partial results have been proven, both conjectures in their full generality remain open. These developments are sketched in more detail in the subsections 4.2 and 5.3.1.
As in the case of graph colourings and Nowhere-Zero-Flows, the study of digraph colourings of planar digraphs is closely related to the investigation of certain integral flows in their dual graphs, so-called Neumann-Lara-Flows as introduced by Hochstättler. Again, NL-flows are defined in such a way that for every planar digraph, given a legal digraph k-colouring, we

can find a NL-k-flow on its dual graph and vice versa. Thus, if we translate the 2-Colour-Conjecture to this dual setting, it asserts that every 3-edge-connected planar digraph admits a NL-2-flow. Obviously, upper bounds for NZ-Flows yield the same bounds for NL-flows. Hochstättler proved that 3-edge-connected digraphs admit at least NL-3-flows. For quite a while, I thus was concerned with the "2-Flow-Conjecture", i.e., whether every 3-edge-connected digraph already admits a NL-2-flow, which generalizes the 2-Colour-Conjecture. The corresponding considerations are captured in section 5.3. Unfortunately, as discovered by Kolja Knauer and Petru Valicov, non-planar digraphs without a 2-flow do indeed exist (they use an elaborate construction which yields a counterexample on >200 vertices). From that point on (cf. section 7), I went on to develop techniques to prove/simplify the 2-Colour-Conjecture in the planar case, ending up with special structural properties of minimal counterexamples to an equivalent strengthening of its original formulation. I believe that some of those properties indeed make the task of finding a legal 2-colouring a lot easier. For some restricted subclasses of simple planar digraphs, the arising properties were already sufficient for 2-colourability, yielding new positive partial results and polynomial-time algorithms, e.g. in the case of simple planar digraphs where all the directed cycles admit the same (counter-/clockwise) orientation. Moreover, section 8 contains some newish concepts which try to find methods of colouring minimal counterexamples even in the general case.
Furthermore, although not all 3-edge-connected digraphs admit an NL-2-flow, intuitively, still many of them do. Therefore, in section 5.3.4, I suggested some restricted/improved conjectures and problems, for which no analogous constructions of counterexamples seem to work, and which therefore are more likely to hold true:

- Does every cubic 3-edge-connected bipartite digraph admit a NL-2-flow?
- Does every cubic cyclically 4-edge-connected digraph admit a NL-2-flow?
- Does every 3-edge-connected digraph without a Petersen minor admit a NL-2-flow?
- Does every 3-edge-connected digraph admitting a NZ-4-flow also admit a NL-2-flow?

Finally, I investigated another approach to overcome the gap between 2 and 3 concerning NL-flow-indices of 3-edge-connected digraphs, which is presented in section 6. For this purpose, following up existing concepts such as the Star Chromatic Number of graphs as defined by Vince and the Circular Chromatic Number of digraphs (Mohar et. al), which provide fractional notions of graph resp. digraph colourings, I defined the notion of the Star NL-Flow Index on digraphs (in the dual setting of flows), intending to find digraphs with fractional NL-flow-indices in between 2 and 3, even if they do not admit an NL-2-flow. I was able to derive some upper bounds for these, e.g. proving the existence of NL-2.5-flows for cyclically 4-edge-connected traceable cubic graphs. The introduction of various new fractional notions of flow indices entails a lot of open questions and interesting problems.
In conclusion, it is important to mention the meaning of flow theories for graph and digraph colourings in terms of matroids and oriented matroids. First of all, most of the notions mentioned above, i.e., Nowhere-Zero, Neumann-Lara-Flows and various fractional flow indices, admit natural extensions to (regular/oriented) matroids which are also part of this

thesis. This is possible mainly because flows, in contrast to vertices and colourings, are very matroidesque concepts, i.e., they are assignments of flow values to the edges of a digraph, whose properties can be formulated in terms of cycles and minimal cuts in the digraph. While the latter can be rediscovered as elements, circuits and cocircuits in the graphic matroid, the graph-theoretic concept of vertices admits no consistent generalization to matroids. Thus, translating colourings of graphs and digraphs to coflows and flows is an important step to make a treatment of these concepts in the wider context of matroid theory possible.

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2 Notation

The following introduces the most basic notations used in this thesis.

$\mathbb{N} = \{1, 2, 3, 4...\}$	The set of natural numbers, 0 excluded
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$	The set of natural numbers, 0 included
$\mathbb{Z} = \{0, 1, -1, 2, -2, ...\}$	The set of integers
$\mathbb{Q} = \{\frac{p}{q} \vert p \in \mathbb{Z}, q \in \mathbb{N}\}$	The set of rationals
$\mathbb{R}$	The real numbers (the reals for short)
$\mathbb{R}_+ = \{x \in \mathbb{R} \vert x \geq 0\}$	The non-negative real numbers.
$\mathbb{F}_{p^n}$	Field with characteristic p and p^n elements
$\mathrm{char}(\mathbb{K})$	Characteristic of the field $\mathbb{K}$
$\mathbb{R}^n$	The set of n-dimensional vectors over $\mathbb{R}$
$\mathbb{R}^{m \times n}$	The set of $m \times n$-matrices over $\mathbb{R}$
$\subseteq$	Subset
$\subset$	Proper subset
$2^M, \mathcal{P}(M)$	Power set of M
$\overline{X} := M \backslash X$	Complement of $X \subseteq M$ within the universal set M (given by context)
$f\vert_{A'}$	Restriction of a mapping $f : A \to B$ to $A' \subseteq A$

$\mathbb{1}_X$	Characteristic mapping of the set X, i.e., $\mathbb{1}_X(x) := \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{if } x \notin X \end{cases}$.
$A \Delta B$	Symmetric difference of A and B
$\dot{\cup}$ or $\sqcup$	Disjoint union
$\|M\|$	Size resp. cardinality of the set M
$\mathring{A}$	The topological interior of $A \subseteq \mathbb{R}^n$
$\overline{A}$	The topological closure of $A \subseteq \mathbb{R}^n$
δA	The topological boundary of $A \subseteq \mathbb{R}^n$
$e_i = (0, ..., 0, \underbrace{1}_{i}, 0, ..., 0)^T \in \mathbb{R}^n$	Standard unit vector with a 1 at position i
$\text{sign}(a) := \begin{cases} 1, & \text{if } a > 0 \\ -1, & \text{if } a < 0 \\ 0 & \text{if } a = 0 \end{cases}$	Sign of the real number a
v_i	Entry of $v \in \mathbb{R}^n$ at the position corresponding to i
$A_{.j}, A_{.j}$	i-th row/j-th column of the matrix A
A_{IJ}	$I \subseteq \{1, ..., m\}, J \subseteq \{1, ..., n\}$. Submatrix of $A \in \mathbb{R}^{m \times n}$ with row indices in I and column indices in J
$\text{sign}(\sigma)$	Sign of the permutation $\sigma \in S_n$, the set of all permutations on $\{1, ..., n\}$.
$\text{supp}(f) = f^{-1}(B \backslash \{0\})$	The support of the mapping $f : A \to B$, where 0 is a neutral element of B given by context.

3 Definitions and Preliminaries

In this preliminary section we want to introduce the most fundamental notions and terminology which will be frequently used in the thesis. They are mainly related to graphs, digraphs and matroids. Due to lack of space, we will omit the proofs in most cases.
The presentation in this section is, if not otherwise mentioned, is oriented at [Mü] and [BGH$^+$ar].

3.1 Graphs

Definition 3.1. A *simple graph* is a pair $G = (V, E)$ consisting of finite sets $V =: V(G)$ and $E(G) := E \subseteq \binom{V}{2} := \{M \subseteq V || M| = 2\}$. An element $v \in V$ is called *vertex or node* of the graph, while we refer to an element $e \in E$ as an *edge*. Two vertices $v_1, v_2 \in V$ are said to be *adjacent*, if $\{v_1, v_2\} \in E$, a vertex $v \in V$ and an edge $e \in E$ are called *incident*, if $v \in e$. If $v, u \in V$ are adjacent, we call v a *neighbour* of u and vice versa. Simple graphs can be equivalently considered as finite, irreflexive, symmetric relations. The "densest" simple graphs are the complete graphs K_V resp. $K_n, n := |V| \in \mathbb{N}$, which admit n vertices out of which each pair is adjacent. The "smallest" graph is the empty graph $(\emptyset, \emptyset)$. In order to avoid possible difficulties in this special case, in the rest of this thesis, we exclude empty graphs in our claims, even if not mentioning it explicitly. The *complement* of a simple graph G, denoted by $\overline{G}$, is defined by $V(\overline{G}) := V(G), E(\overline{G}) := \binom{V}{2} \backslash E(G)$.

More generally, a *graph* is a triple $G = (V, E, \delta)$, where V, E are the finite sets of *vertices* resp. *edges* and $\delta : E \to \binom{V}{1} \cup \binom{V}{2} = \{M \subseteq V || M| \in \{1, 2\}\}$ is the so-called *incidence mapping* of G. For each edge $e \in E$, the vertices $v \in \delta(e)$ are the *end vertices* of e, and v and e are called *incident* in this case. Given some edge $e \in E$, we often abbreviate $\delta(e) = \{u, w\}$ by $e = uw$, which is (in the non-simple case) not to be understood as an equality but much more as a statement about the edge e (i.e., $e = uw, f = uw$ for two distinct edges e, f is possible). Two vertices $v_1, v_2 \in V$ are called *adjacent* if there is an edge $e \in E$ such that $v_1, v_2 \in \delta(e)$. In this context, we may assign the *underlying simple graph* to G, which is defined as (V, E_{simp}), $E_{\text{simp}} := \{\{u, w\} | u \neq w \in V$ are adjacent.$\}$. In contrast to the notion of simple graphs above, graphs in general can admit *loops* and *multiple/parallel edges* between two given vertices: $e \in E$ is called a loop, if $|\delta(e)| = 1$. Two distinct edges $e_1 \neq e_2$ are called parallel, if $|\delta(e_1)| = |\delta(e_2)| = 2$ and $\delta(e_1) = \delta(e_2)$. This means that the simple graphs can be identified as the graphs without loops and multiple parallel edges. Formally, a simple graph $G = (V, E)$ admits the incidence mapping $\delta : E \to \binom{V}{2}, \delta(e) := e$.

A graph $H = (V', E', \delta')$ is a *subgraph* of G, in symbols, $H \subseteq G$, if $V' \subseteq V, E' \subseteq E$, and

$\delta' = \delta|_{E'}$. This especially means that $\delta(e) \subseteq V', \forall e \in E'$.
H is called *induced subgraph*, in symbols $H = G[V']$, if E' contains all the edges $e \in E$ with $\delta(e) \subseteq V'$, i.e., if H can be obtained from G by deleting all vertices out of $V \backslash V'$ and the respective incident edges.
If $V' = V$, i.e., if H arises from G by deleting the edges in $E \backslash E'$, we write $H = G[E]$. We furthermore use the notations $G - X := G[V \backslash X], G - Y := G[E \backslash Y]$ for the processes of deleting a vertex resp. an edge subset $X \subseteq V$ resp. $Y \subseteq E$ from G.
If $H \subseteq G$, we call H a *spanning subgraph*, if $V(H) = V(G)$.
A vertex subset V' of G so that $G[V']$ admits a $K_{V'}$ as simple underlying graph (i.e., every pair $u \neq v \in V'$ is adjacent in G) is called a *clique* in G.
Given two graphs $G_1 = (V_1, E_1, \delta_1), G_2 = (V_2, E_2, \delta_2)$, for the most purposes, they admit the same essential properties if they arise from each other by relabeling. Formally, we say that G_1, G_2 are *isomorphic* (in symbols $G_1 \simeq G_2$) if there are bijections $\sigma : V_1 \to V_2$, $\tau : E_1 \to E_2$, so that $\sigma(\delta_1(e)) = \delta_2(\tau(e)), \forall e \in E_1$. In most cases, unless otherwise specified, we will not distinguish between a graph G and its corresponding equivalence class containing all the graphs being isomorphic to G, if this does not lead to misunderstandings.
Given two simple graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$, we define its *cartesian product*, denoted by $G_1 \Box G_2$, according to $V(G_1 \Box G_2) = V_1 \times V_2$, and $(u_1, u_2) \in V_1 \times V_2$, $(w_1, w_2) \in V_1 \times V_2$ are adjacent, iff $u_1 = w_1$ and $u_2 w_2 \in E_2$ or $u_2 = w_2$ and $u_1 w_1 \in E_1$.
The *union* of two graphs $G_1 = (V_1, E_1, \delta_1), (V_2, E_2, \delta_2)$, such that $\delta_1|_{E_1 \cap E_2} = \delta_2|_{E_1 \cap E_2}$ is defined by $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2, \delta)$, where δ is the common extension $\delta|_{E_1} = \delta_1, \delta_{E_2} = \delta_2$.

Definition 3.2. Given a graph $G = (V, E, \delta)$, a pair $e_1 \neq e_2$ of edges is called *adjacent*, if $\delta(e_1) \cap \delta(e_2) \neq \emptyset$. Otherwise, they are called *independent*. The graph $L(G)$, defined by the vertex set $V(L(G)) := E$ and $E(L(G)) := \{(e_1, e_2, v) | e_1 \neq e_2 \in E : v \in \delta(e_1) \cap \delta(e_2)\} \cup \{(e, e, v) | e \in E : \delta(e) = \{v\}\}$ as well as $\delta'(e_1, e_2, v) = \{e_1, e_2\}$, is called the *line graph* of G. If G is a simple graph, then $L(G)$ is a simple graph where two given edges are adjacent if and only if they are adjacent in G. This is illustrated by the following figure:

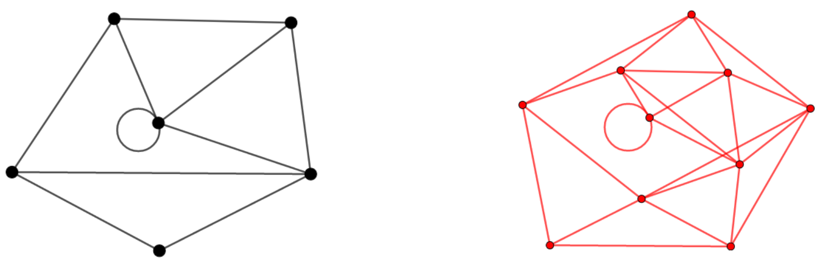


Figure 1: Left: A graph. Right: Its corresponding line graph.

Definition 3.3. Let $G = (V, E, \delta)$ be a graph and $v \in V$. The set

$$N_G(v) := \{w \in V | v, w \text{ adjacent}\}$$

of neighbours of v in G is called the *neighbourhood* of v. Furthermore, we denote the set of incident edges of a vertex $v \in V$ by $E_G(v)$ and the set of incident loops by $L_G(v) \subseteq E_G(v)$. The *degree* $\deg_G(v) := |E_G(v)| + |L_G(v)|$ of a vertex $v \in V$ counts the number of edges incident to v where incident loops have multiplicity two. In the case that G is simple, $\deg_G(v) = |N_G(v)|$ is the number of neighbours of v. If it is clear from the context which graph we refer to, we may also just write $\deg(v)$. The maximum resp. minimum degree of a vertex in a graph G is denoted by $\Delta(G) := \max_{v \in V(G)} \deg_G(v)$ resp. $\delta(G) := \min_{v \in V(G)} \deg_G(v)$.
A vertex of degree 0 is called *isolated*.
For a $k \in \mathbb{N}_0$, we refer to G as being k-regular if every vertex admits degree exactly k. If $k = 3$, we call G *cubic*.

The following is the well-known so-called handshake-lemma, which relates the average vertex-degree in a graph to the number of edges and vertices:

Proposition 3.4. *Let $G = (V, E, \delta)$ be a graph. Then*

$$\sum_{v \in V} \deg_G(v) = 2|E|,$$

i.e., $2\frac{|E|}{|V|}$ is the average degree in G.

Proof. Let $L \subseteq E$ denote the set of loops in E. Then

$$\sum_{v \in V} \deg_G(v) = \sum_{v \in V} (|E_G(v) \backslash L_G(v)| + 2|L_G(v)|) = \sum_{e \in E \backslash L} 2 + 2 \sum_{e \in L} 1 = 2|E \backslash L| + 2|L| = 2|E|.$$

□

Most of the relations between graphs and linear algebra are based on various matrices assigned to a given graph. The most basic matrices of this type are introduced in the following definition:

Definition 3.5. Let $G = (V, E, \delta)$ be a graph and $n := |V|, m := |E|$. Assume we are given orderings $V = \{v_1, ..., v_n\}$ and $E = \{e_1, ..., e_m\}$ of the vertices and edges. Then the matrices $A_G \in \mathbb{R}^{n \times n}, B_G \in \mathbb{R}^{n \times m}$, defined by

$$(A_G)_{i,j} := |\{e \in E | \delta(e) = \{v_i, v_j\}\}| + |\{e \in E | \delta(e) = \{v_i\} = \{v_j\}\}|$$

and

$$(B_G)_{i,j} := \begin{cases} 0, & \text{if } v_i \notin \delta(e_j) \\ 1, & \text{if } e_j \in E_G(v_i) \backslash L_G(v_i) \\ 2, & \text{if } e_j \in L_G(v_i) \end{cases},$$

are called *adjacency matrix* resp. *incidence matrix* of G. We have

$$A_G = B_G B_G^T - \operatorname{diag}(a_1, ..., a_n)$$

with $a_i := 2|L_G(v_i)|, i = 1, ..., n$. Given the incidence matrix, the sum of the elements in each column is 2, while the sum in the ith row is exactly the degree of v_i in G. Double-counting now immediately yields a simple proof of the handshake-lemma.

Definition 3.6. Let $G = (V, E, \delta)$ be a graph. A *walk* in G is an alternating sequence of incident vertices and edges in G, i.e. a sequence $v_1e_1v_2e_2...v_n$, where $\delta(e_i) = \{v_i, v_{i+1}\}$, $i = 1, ..., n-1$. v_1 is called *starting* and v_n *ending vertex* of the walk. The *length* of a walk $W = v_1e_1...v_n$ is defined as the number of edges contained in it $(n-1)$ and denoted by $\ell(W)$. Given a walk in G, we call it a *path*, if all contained vertices are pairwise distinct. A *trail* is defined as a walk without repeated edges. A *closed edge sequence* (also known as cycle, a term which we won't use in order to avoid confusion with the (simple) cycles defined below) in G is a walk which admits the same starting and ending vertex $v_1 = v_n$. A *(simple) cycle* in G is a closed edge sequence such that all the vertices except for the starting and ending vertex are pairwise distinct. A cycle of length 2 is called a *digon*, while we often refer to a cycle of length 3 as a *triangle*. A cycle resp. a path of length $|V|$, i.e., a cycle resp. a path visting every vertex, is called a *Hamiltonian cycle* resp. *Hamiltonian path*. A graph admitting a Hamiltonian cycle is called *Hamiltonian*, a graph admitting a Hamiltonian path *traceable*. Obviously, every Hamiltonian graph is traceable. It is known that deciding Hamiltonicity and traceability of a given graph is an NP-hard problem. One of the most general positive results concerning Hamiltonicity is a Theorem due to Tutte, stating that every 4-connected planar graph (cf. definitions below) is Hamiltonian.
Given a cycle C or a path P in G, we call it *induced*, if the corresponding subgraph of G is an induced subgraph. If to the contrary, C or P admits an edge $e \in E$ not contained in it but connecting a pair of vertices in $V(C)$ resp. $V(P)$, we call it a *chord* of C resp. P.
Finally, we will call a closed edge sequence which is a trail a *closed trail*. It is important to notice that in most cases, the special run order encoded by a sequence as above is irrelevant, i.e., the walk $v_1e_1v_2...v_n$ and the symmetric walk $v_ne_{n-1}v_{n-1}....v_1$ will be often considered equal. Furthermore, we do not distinguish between $v_1e_1...e_{n-1}v_n = v_1$ and the shifted representation $v_ie_i...e_{i-1}v_i$, $i = 1, ..., n-1$ in the case of a closed edge sequence. Moreover, walks, paths, closed edge sequences and cycles can and will be treated as subgraphs of G (i.e., in the above, $\{v_1, ..., v_n\}$ is the vertex set and $\{e_1, ..., e_n\}$ the edge set). All paths and cycles of length $n \in \mathbb{N}$ are isomorphic graphs and often denoted as the *path graph* resp. *cycle graph* P_n resp. C_n.

Definition 3.7. The *girth* of a graph G, denoted by $\mathrm{gir}(G)$ is defined as the length of a shortest cycle in G.

Definition 3.8. Let $G = (V, E, \delta)$ be a graph, and let $e \in E$. *Contracting* e is a process which transforms G into the contraction graph $G/e = (V', E', \delta')$, so that $V' = (V \backslash \delta(e)) \cup \{v_e\}$, where $v_e \notin V$ is an extra vertex representing e, $E' = E \backslash \{e\}$ and

$$\delta'(e') := \begin{cases} \delta(e'), & \text{if } \delta(e') \cap \delta(e) = \emptyset \\ (\delta(e') \backslash \delta(e)) \cup \{v_0\} & \text{if } \delta(e') \cap \delta(e) \neq \emptyset \end{cases}, \forall e' \in E'.$$

The choice of v_e leaves G/e unaffected (up to isomorphisms), and thus, we consider G/e a uniquely defined graph given G. It is easily seen that for any two distinct edges $e_1 \neq e_2 \in E$, we have $(G/e_1)/e_2 = (G/e_2)/e_1$. This "commutativity" of contraction justifies the following definition: If $F = \{e_1, ..., e_k\} \subseteq E$ is any subset, then the graph G/F defined by $(...((G/e_1)/e_2)...)/e_k$ is unique up to isomorphisms and is called the *contraction* of F in G. In such a case, G/F is called a *contraction minor* of G. Generalizing both the notion of contraction minors and of subgraphs, a *minor* of G is defined to be a graph arising from G by a finite sequence of vertex- and edge-deletions or edge-contractions. A *topological minor* of a graph G is defined as a graph H who admits a subdivision (i.e., replacing each edge by a path of arbitrary length) $\tilde{H}$ which is a subgraph of G.

Definition 3.9. Let $G = (V, E, \delta)$ be a graph. We define a binary relation on G by

$$u, v \in V : u \sim_c v :\Longleftrightarrow \text{ There is a walk starting in } u \text{ and ending in } v.$$

It is easy to see that this defines even more an equivalence relation on V. Henceforth, V decomposes to disjoint equivalence classes according to this relation, which we call *connected components* of G. If there is only one such component, there is a walk (and thus, by deleting repeatedly appearing vertices a path) between each given pair of vertices $u \neq w$. In this case, we refer to G as being *(simply) connected*. In the general case, the connected components induce connected subgraphs of G, whose disjoint union is exactly G (there are especially no edges connecting vertices of different components). The number of connected components of a graph G is denoted by $c(G)$.
If G is a graph and $u \neq v \in V(G)$ a pair of vertices, then we denote by $\mathrm{dist}_G(u, v)$ the so-called *distance* of u, v in G, which is the minimal length of a path connecting u and v if they are contained in a common connected component and ∞ otherwise. The *diameter* of G is the maximal distance of a pair of vertices, and $r(G) := \min_{v \in V(G)} \max_{w \in V(G)} \mathrm{dist}_G(u, w)$ is called its *radius*.
An edge $e \in E(G)$ such that deleting e separates a connected component of G, in other words, $c(G - e) > c(G) \Leftrightarrow c(G - e) = c(G) + 1$, is called a *bridge* in G. An edge of a graph is a bridge iff it is not contained in any cycle.

Definition 3.10. Let $G = (V, E, \delta)$ be a graph. G is called *bipartite*, if there is a decomposition of the vertex set $V_1 \dot{\cup} V_2 = V$, the so-called *colour or bipartition classes*, such that $\forall e \in E : \delta(e) \cap V_1, \delta(e) \cap V_2 \neq \emptyset$, i.e. all edges of G are "spanned between" V_1 and V_2.

Obviously, bipartite graphs are loopless. More generally, given any cycle C in a bipartite graph G with bipartition V_1, V_2, its vertices have to be alternately contained in V_1 resp. V_2. Since $V_1 \cap V_2 = \emptyset$, this means that $\ell(C)$ is even. The following now states that this is already a defining property of bipartite graphs:

Theorem 3.11. *Let G be a graph. Then G is bipartite iff it does not contain cycles of odd length.*

Bipartite graphs will be later on identified as the 2-colourable graphs. The "densest" simple bipartite graphs are the *complete bipartite graphs*, which, given some natural numbers $m, n \in \mathbb{N}$ are defined to be the simple graphs admitting bipartition classes of size m resp. n and all possible adjacencies in between them. Since all the m, n-complete graphs are obviously isomorphic to one another, they are often considered equal and denoted by $K_{m,n}$. The following now generalizes the concept of a bipartition to subgraphs:

Definition 3.12. Let $G = (V, E, \delta)$ be a graph, and $V_1, V_2 \subseteq V$ arbitrary vertex subsets. Then $G[V_1, V_2] \subseteq E$ is defined as the set of edges $e \in E$ with $\delta(e) \cap V_1, \delta(e) \cap V_2 \neq \emptyset$ and $\delta(e) \subseteq V_1 \cup V_2$. We thus have $E(G[V']) = G[V', V'], \forall V' \subseteq V$ and $G[V_1, V_2] = \emptyset$ if V_1, V_2 are distinct connected components of G. A *cut* in G is an edge subset of the form $S = G[X, \overline{X}], X \subseteq V$. In that sense, a cut separates the graph G into the halves $G[X]$ and $G[\overline{X}]$, and each path connecting a pair of vertices in X resp. $\overline{X}$ has to use an odd number of edges out of S. Analogously, if C is any cycle in G, when traversing C, one has to switch from X to $\overline{X}$ and vice versa the same number of times and thus, $|S \cap C|$ is even. It can be shown that on the other hand, this property uniquely defines a cut, i.e., $E' \subseteq E$ is a cut if and only if $|E' \cap C|$ is even for all cycles C.
The cuts containing exactly one edge are the bridges of G.
A *minimal* cut S is defined as a non-empty cut which is inclusion-minimal, i.e., there is no non-empty cut $\emptyset \neq S' \subset S$. Given a cut S, a vertex set X, such that $S = G[X, \overline{X}]$, is called a *cut set* of S. It can be easily shown that a cut with an appropriate cut set X is minimal if and only if there are $V_1 \subseteq X, V_2 \subseteq \overline{X}$ so that $S = G[V_1, V_2]$ and $G[V_1], G[V_2]$ are connected graphs. Furthermore, the cuts in a graph G are exactly the cumulated symmetric differences of minimal cuts and at the same time the unions of edge-disjoint minimal cuts. G is connected if and only if every (minimal) cut $G[X, \overline{X}]$ with $\emptyset \neq X \subset V$ is non-empty.

Definition 3.13. Given a $k \in \mathbb{N}$, we refer to a graph $G = (V, E, \delta)$ as being *k-(vertex-) connected*, if $|V(G)| \geq k + 1$ and if each graph $G - V', V' \subseteq V, |V'| \leq k - 1$ arising from G by deleting less than k vertices is simply connected. Consequently, every connected graph on at least two vertices is 1-connected, and k-connectivity for a $k \in \mathbb{N}$ implies k'-connectivity for all $k' = 1, ..., k$. The least natural $k \in \mathbb{N}$ for which G is k-connected is defined as the *(global) connectivity* of G, denoted by $\kappa(G)$.
Analogously, G is said to be *k-edge-connected*, if each graph $G - E', E' \subseteq E, |E'| \leq k - 1$ arising from G by deleting less than k edges is still connected. Again, 1-edge-connectivity is the same as simple connectivity. Furthermore, a simply connected graph G is 2-edge-connected if and only if it is bridgeless. The least $k \in \mathbb{N}$ for which G is k-edge-connected is defined as the *(global) edge-connectivity* of G, denoted by $\kappa'(G)$.
Vertex resp. edge subsets of G whose deletion leaves a disconnected graph are called *separating*. Given a pair $q \neq s \in V$ of vertices, an edge subset resp. a vertex subset not containing q, s in G, denoted by S, is called *q, s- resp. s, q-separating* if in $G - S$, there is no path connecting q and s. With these definitions, we can define the local vertex resp. edge-connectivities of a pair $q \neq s$ of distinct vertices in G by

$$\kappa(G, q, s) := \min\{|S| |S \subseteq V \text{ is } q, s\text{-separating}\}$$

if q and s are not adjacent in G resp.

$$\kappa'(G, q, s) := \min\{|S| |S \subseteq E \text{ is } q, s\text{-separating}\}$$

for arbitrary distinct vertices $q \neq s \in V$. It is obvious that

$$\kappa'(G) = \min\{\kappa'(G, q, s) | q \neq s \in V\}.$$

For the vertex case, we have to be a bit more careful. If there is at least one pair of non-adjacent vertices in G (i.e., G admits no $K_{|V|}$ as a subgraph), then

$$\kappa(G) = \min\{\kappa(G, q, s) | q \neq s \in V, \{q, s\} \notin \delta(E)\}.$$

On the other hand, the complete graph of order n satisfies

$$\kappa(K_n) = n - 1, n = 2, 3, ...$$

Furthermore, given a pair of distinct vertices $q \neq s \in V$, two paths P_1, P_2 connecting q and s are called *internally vertex-disjoint*, if they do not admit a common vertex other than q and s. The maximal number of pairwise internally vertex-disjoint paths between q and s is denoted by $\lambda(G, q, s)$.
Analogously, the maximal number of pairwise edge-disjoint paths connecting q and s is denoted by $\lambda'(G, q, s)$. Additionally,

$$\lambda(G) := \min\{\lambda(G, q, s) | q \neq s \in V\}, \lambda'(G) := \min\{\lambda'(G, q, s) | q \neq s \in V\}.$$

The following introduces a (by now) very prominent graph operation introduced by Lovász in 1974 and since then known as the *Splitting-Off-Operation*. It is a very useful tool whenever we want to locally reduce a graph and preserve the edge-connectivity at the same time:

Definition 3.14 (cf. Lovász [Lov79]). Let $G = (V, E, \delta)$ be a graph and $e_1, e_2 \in E$ with $\delta(e_1) = uv, \delta(e_2) = vw$, where $u, v, w \in V$ are pairwise distinct vertices. The splitting-off of the pair e_1, e_2 now is defined by deleting e_1, e_2 from G and replacing them by an additional edge $e_{1,2} \notin E$ connecting u, w, ending up with a graph $G' = (V, (E\backslash\{e_1, e_2\}) \cup \{e_{1,2}\}, \delta')$ such that $\delta'(e) := \delta(e), \forall E\backslash\{e_1, e_2\}$, $\delta'(e_{1,2}) = \{u, w\}$.

Lovász proved that splitting off a pair of edges at a given vertex is possible while preserving some notion of global connectivity. This was strengthened by Mader (cf. [Mad78]) by the following result:

Theorem 3.15 (Mader). *Let G be a connected graph and $v \in V(G)$ a designated vertex, such that $\deg_G(v) \neq 3$ and assume there are no bridges in G incident to v. Then there is a pair $e_1 \neq e_2$ of edges incident to v, such that for the graph G' arising from G by splitting off e_1, e_2 we have*

$$\lambda_{G'}(q, s) = \lambda_G(q, s), \forall q \neq s \in V(G)\backslash\{v\}.$$

The following statements now finally characterize local and global k-vertex- and k-edge-connectivity, by relating the above quantities to cuts in the actual graph.
According to the following lemma, when looking for smallest separating sets of edges, we may restrict our search to cuts in G:

Lemma 3.16. *Let $G = (V, E, \delta)$ be a graph, $q \neq s \in V$. Then*

$$\kappa'(G, q, s) = \min\{|G[X, \overline{X}]| | \emptyset \neq X \subset V, q \in X, s \notin X\}.$$

Thus, a graph G is k-edge-connected if and only if $|G[X, V\backslash X]| \geq k$, for all $\emptyset \neq X \subset V$, i.e., iff the cuts of size at most $k - 1$ admit trivial cut sets.

There is no analogous characterization of minimal separating sets of vertices. At several points of this thesis, we will deal with cubic graphs, which, due to their vertex degrees, can be at most 3-edge-connected. Still, in many situations, we want to distinguish between different levels of connectivity even within the cubic graphs. Such notions are provided by the following definition:

Definition 3.17. Let G be a graph, $k \in \mathbb{N}$. G is called *internally* or *essentially* k-edge-connected, if every non-trivial cut in G (i.e., the cut sets consist of at least two vertices) admits size at least k, i.e. $|G[X, \overline{X}]| \geq k$, for all $X \subseteq V(G) : |X|, |\overline{X}| \geq 2$.
A weaker notion of "partial" connectivity is the concept of cyclical connectivity: A graph is called *cyclically k-edge-connected*, if every cut $G[X, \overline{X}]$ in G, such that $G[X]$ as well as $G[\overline{X}]$ contains a cycle, admits size at least k. It is easy to show that in the case of cubic 3-edge-connected cubic graphs, essential and cyclical 4-edge-connectivity are equivalent requirements. A proof is given further below in Corollary 3.21 when trees have been introduced.

Theorem 3.18 (Menger–Vertex Version)**.** *Let $G = (V, E, \delta)$ be a graph with $|V| \geq 2$. Then the following holds:*

- *For each pair $q \neq s \in V$ of distinct and non-adjacent vertices, $\lambda(G, q, s) = \kappa(G, q, s)$.*
- *If there is a pair $q \neq s$ of vertices, such that there is at most one edge $e \in E$ with $\delta(e) = \{q, s\}$, then $\lambda(G) = \kappa(G)$.*

The latter especially implies that a given k-connected simple graph admits at least k internally vertex-disjoint paths between any two vertices of distance at least two.

Theorem 3.19 (Menger–Edge Version)**.** *Let $G = (V, E, \delta)$ be a graph with $|V| \geq 2$. Then the following holds:*

- *For each pair $q \neq s \in V$ of distinct vertices, $\lambda'(G, q, s) = \kappa'(G, q, s)$.*
- $\lambda'(G) = \kappa'(G)$.

The latter implies that in any k-edge-connected graph, there are at least k edge-disjoint paths between any given pair of distinct vertices.

The connections between minimal cuts and maximal numbers of paths between two given vertices will reappear in the next subsection in form of the Max-Flow-Min-Cut Theorem when dealing with digraphs.

Definition 3.20. Let $T = (V, E)$ be a simple graph. T is called a *forest* or *acyclic*, if it does not contain any cycles. If additionally, T is connected, then T is called a *tree*. It is easily seen that given some simple graph T, the following are equivalent:

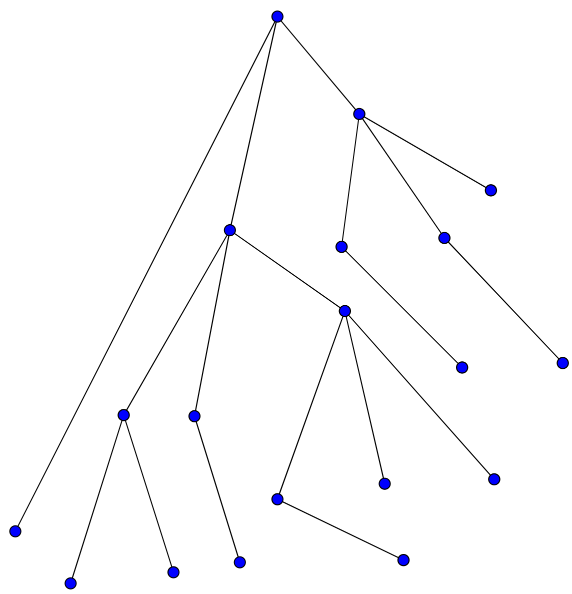
- T is a tree.
- T is a minimal connected graph, i.e., T is connected but $T[E']$ is disconnected for all $E' \subset E$.

- T is a maximal acyclic graph resp. forest, i.e., T is a forest but $T + v_1v_2$ (arising from T by adding a new edge between v_1 and v_2) contains a cycle for each pair of non-adjacent vertices $v_1 \neq v_2 \in V$. In this case, for each edge $e = v_1v_2 \notin E$ (a so-called *chord* of T), $T + v_1v_2$ contains a unique cycle passing through e which is denoted by $C(e, T)$.
- T is connected and $|E| = |V| - 1$.
- T is acyclic and $|E| = |V| - 1$.
- For each pair $v_1 \neq v_2 \in V$, there is a unique path in T connecting v_1, v_2.

From the definition, it becomes clear that forests are exactly the graphs whose connected components are trees. If $V_1, ..., V_k$ are the corresponding vertex sets in a forest $G = (V, E)$, $k := c(G)$, according to the above we have

$$|E| = |E(G[V_1])| + ... + |E(G[V_k])| = (|V_1| - 1) + ... + (|V_k| - 1) = |V| - c(G) \leq |V| - 1.$$

The latter implies (according to the handshake-lemma) that $\frac{2|E|}{|V|} \leq \frac{2|V|-2}{|V|} = 2 - \frac{2}{|V|} < 2$ is the average degree of a forest, i.e., every tree and every forest with a non-trivial component admits (by finiteness even at least two) vertices of degree 1, which are called *leafs*.
Given a graph G, a tree $T \subseteq G$ is called a *tree in* G and a *spanning tree*, if it is a spanning subgraph. It can easily be proven that each connected graph G admits a spanning tree.
Trees (and thus forests) are planar graphs (cf. definition further below). By fixing a special vertex $w \in V(T)$ of a tree T arbitrarily as the so-called *root* and considering a plane embedding of T, the unique connecting paths of w to an arbitrary vertex $v \in V(T)\backslash\{w\}$ can be arranged as "branches" which may arise from each other. By separating two such connecting paths in the cyclic ordering around w, we may distinguish a leftmost and rightmost branch. This is illustrated in the following figure.



Corollary 3.21. *Let G be a 3-edge-connected cubic graph. Then G is essentially 4-edge-connected iff it is cyclically 4-edge-connected.*

Proof. G contains no loops, since such would induce cuts of size 1. Obviously, cyclical 4-edge-connectivity is a consequence of essential 4-edge-connectivity. Assume now contrary to the reverse implication that G is cyclically 4-edge-connected but not essentially 4-edge-connected. By definition, there is a cut $S = G[X, \overline{X}]$ of size 3 so that $|X|, |\overline{X}| \geq 2$. Because of the cyclical connectivity, at least one of the cut sets, say X, contains no cycle, i.e., $G[X]$ is acyclic. By double-counting the pairs $V(G) \times E(G) \ni (v, e) : v \in X \cap \delta(e)$, we conclude

$$3|X| = 2|G[X]| + |S| \leq 2(|X| - 1) + 3 = 2|X| + 1 \Rightarrow 2 \leq |X| \leq 1,$$

and derive the desired contradiction. □

The notion of graphs in the discrete sense has a long history, and probably reaches back to the now famous seven bridges of Königsberg, when Leonhard Euler (1736) resolved the corresponding problem in a negative way, thereby laying the foundations of modern graph theory:
In the city of Königsberg in Prussia (nowadays Kaliningrad), the river Pregel divided the city centre into two islands, which were reachable from each other and the mainland by seven bridges. The historical problem Leonhard Euler dealt with, was to figure out whether one could take a walk through the city, thereby crossing each bridge exactly once, such that in the end, we get back to our starting point. This is illustrated by the following figure:

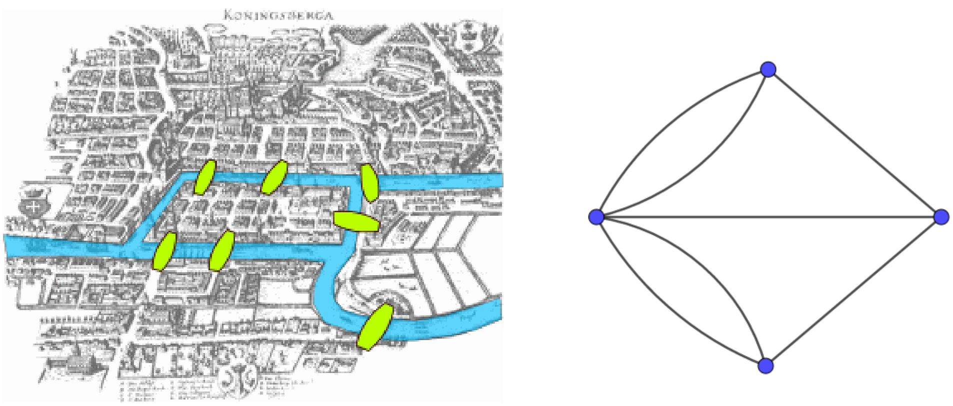


Figure 2: Left: The seven bridges of Königsberg. Right: The corresponding graph.

Euler showed that such a walk cannot exist, by modelling the configuration of islands and bridges as a graph: The vertices correspond to the connected regions of the mainland, and the bridges correspond to different edges connecting them. A walk as required in graph theoretic terms (cf. theorem 3.23 below) now becomes a Eulerian circuit in the corresponding graph, i.e., a closed trail traversing each edge exactly once, which means that it would have to enter and leave every vertex an even number of times. Since two vertices have degree 3 resp. 5, which is odd, this is not possible in the graph above. This is formalized by the following definition and theorem concerning Eulerian circuits:

Definition 3.22. Let $G = (V, E, \delta)$ a graph. A *Eulerian trail* in G is defined as a trail in G visiting each edge in E exactly once. Analogously, a *Eulerian circuit* or *Eulerian cycle* denotes a closed trail in G containing each edge in E exactly once.
We call G *even*, if every vertex admits even degree, and *odd*, if every vertex has odd degree.

Theorem 3.23 (Euler)**.** *Let $G = (V, E, \delta)$ be a (possibly empty) graph. Then the following are equivalent:*

(i) G is even and has at most one non-trivial component (containing at least one edge).

(ii) G admits a Eulerian circuit.

(iii) G admits at most one non-trivial component, which is the union of edge disjoint cycles.

Proof (cf. e.g. [Mü]). (i) $\Rightarrow$ (ii): If G is trivial, i.e., $E(G) = \emptyset$, then the implication holds obviously true, so assume that G admits a unique non-trivial connected component, which thus is also even. Without loss of generality, we may assume that already G itself is connected. Then consider a trail $F = v_1 e_1 ... e_{n-1} v_n$ of maximal length in G. First of all, F has to be a closed trail: Assume by way of contradiction that $v_1 \neq v_n$. Then v_1 and v_n both must have odd degree in $F \subseteq G$, implying (G is even) the existence of an edge $e \in E_G(v_1) \setminus E_F(v_1)$. If v_e is the other end vertex of this edge, then v_e, e, F is a trail in G, contradicting the maximality of F. Therefore F is a closed trail and thus an even subgraph of G. Consequently, also $G - E(F)$ is an even graph. We are done if we can show that this graph does not admit any edges, so assume to the contrary that there was a non-trivial (even) connected component of $G - E(F)$. Every vertex in this component admits degree at least 2 and thus, since forests admit average degree less than 2, there has to be a cycle C in $G - E(F)$. Let $v \in V(C)$ be chosen arbitrarily, by shifting the labeling of F if needed, we may assume that $v = v_1$. Starting in v, by first traversing C in its cyclic order and after that F according to $v_1 e_1 ... v_{n-1} e_{n-1} v_n$ yields a trail in G of length greater than $n - 1$, contradicting the assumed maximality. Thus, the above assumption was false, G indeed admits the Eulerian trail F.

(ii) $\Rightarrow$ (iii): We use induction on the number of edges $|E(G)|$. If this is 0, we are done, so assume for the inductive step that given a fixed $m \geq 1$, every graph admitting a Eulerian trail on at most $m - 1$ edges contains at most one non-trivial component, which is a union of edge-disjoint cycles, and let G be an arbitrary graph with a Eulerian trail F so that $|E(G)| = m$. Obviously, $E(G)$ can only contain one non-trivial component which is traversed by F. Each vertex of this component has at least two distinct incident edges in F and as above, we conclude that the component contains a cycle C. Furthermore, $G - E(C)$ still is an even graph, whose components according to (i) $\Rightarrow$ (ii) admit a Eulerian trail, i.e., are according to the inductive assumption unions of edge-disjoint cycles. Joining these cycles with C now yields a decomposition of $E(G)$ into edge-disjoint cycles, proving the inductive claim and thus the asserted implication.

(iii) $\Rightarrow$ (i): This is obvious. $\square$

Definition 3.24. Let $G = (V, E, \delta)$. A *matching* on G is an edge subset $M \subseteq E$, so that all of its edges are pairwise independent. Obviously, for each matching M in G, we have: $|M| \leq \left\lfloor \frac{|V|}{2} \right\rfloor$. A *perfect matching* is a matching M such that $|M| = \frac{|V|}{2}$, while M is called *almost perfect*, whenever $|M| = \frac{|V| - 1}{2}$. Such a matching is always a *maximum matching*, i.e., it contains a maximal number of edges in G. Again, matchings can equivalently be considered as subgraphs of G.
Moreover, perfect matchings are exactly the 1-regular spanning subgraphs of G. More generally, given some $k \in \mathbb{N}$, a spanning k-regular subgraph of G is referred to as a k*-factor* of G.

In many cases, one is interested in knowing whether a specified graph admits a perfect matching. Since matchings cannot contain loops and since out of a set parallel edges between the same pair of vertices, only one can be contained in a matching at the same time, regarding the existence of certain matchings, we can restrict our analysis to the simple underlying graph of the respective graph.
The following well-known theorem due to Tutte (cf. [Tut47]) gives an equivalent criterion for the existence of perfect matchings in simple graphs using certain inequalities. The assertion of the theorem uses the following notation: Given a graph G, $o(G)$ denotes its number of odd connected components (i.e., containing an odd number of vertices):

Theorem 3.25 (Tutte)**.** *Let $G = (V, E)$ be a simple graph. Then G admits a perfect matching iff $o(G - U) \leq |U|$, for all $U \subseteq V$.*

Proof. The necessity of the condition is seen as follows: Given a perfect matching M and a vertex subset U, we have $|U| \geq |M \cap G[U, V \backslash U]|$, since the end vertices of matching edges in $G[U, V \backslash U]$ in U have to be distinct from one another. On the other hand, no odd component of $G - U$ can be completely covered by matching edges inside it, wherefore there is at least one edge in $G[U, V \backslash U]$ incident to one of its vertices. Thus, $|M \cap G[U, V \backslash U]| \geq o(G - U)$, proving the claimed equality. Although the reverse implication is harder to prove, it is elementary and uses an inductive argument. $\square$

The following is the even more prominent special case when G is a bipartite graph, the so-called Hall Marriage Theorem (cf. [Hal35]):

Theorem 3.26 (Hall Marriage Theorem)**.** *Let G be a simple bipartite graph with bipartition V_1, V_2, where $|V_1| = |V_2|$. Then G admits a perfect matching iff*

$$\forall X \subseteq V_1 : |N_G(X)| \geq |X|.$$

The following is another very old theorem concerning perfect matchings in cubic graphs due to Petersen (cf. [Pet91]), which will play a role when dealing with even subgraphs of those later on in the thesis. Here, we present a slightly strengthened version, allowing to prescribe an edge contained in the perfect matching.

Theorem 3.27 (Petersen)**.** *Let G be a simple cubic and bridgeless graph, and $e \in E(G)$ arbitrary. Then there is a perfect matching M_e in G containing e.*

Proof. Without loss of generality, we may assume that G is simply connected, i.e., 2-edge-connected.
Let u_1, u_2 be the end vertices of e. We have to show that $G - \{u_1, u_2\}$ has a perfect matching, which we do by proving that the Tutte-conditions are satisfied, i.e., for all vertex sets $U \subseteq V(G)$ with $u_1, u_2 \in U$:

$$o(G-U) = o((G-\{u_1,u_2\}) - (U \backslash \{u_1,u_2\})) \leq |U \backslash \{u_1,u_2\}| = |U| - 2.$$

So let U be any vertex set containing u_1, u_2. Let $G_1 = G[V_1], ..., G_k = G[V_k]$ with $k = o(G_U)$ be the odd components of $G - U$ and set $m_i := |G[V_i, V(G) \backslash V_i]|$ for $i = 1, ..., k$ to be the size of the cut corresponding to the ith component. Double-counting now gives that $3|V_i| - m_i = 2|E(G_i)|$, and since G_i is odd, m_i is odd for $i = 1, ..., k$. Furthermore we have $m_i \in \{3, 5, 7, ...\}, i = 1, ..., k$ because G is 2-edge-connected. Using the handshake lemma, we now conclude that:

$$3k \leq \sum_{i=1}^{k} \underbrace{m_i}_{\geq 3} = |\bigcup_{i=1}^{k} |G[V_i, V(G) \backslash V_i]| \leq |G[U, V(G) \backslash U]|$$

$$= \sum_{u \in U} \Big(\deg_G(u) - \deg_{G[U]}(u) \Big) = 3|U| - 2|E(G[U])|.$$

Since $e \in E(G[U])$ we have $k \leq \frac{1}{3}(3|U| - 2) = |U| - \frac{2}{3}$, and so $k \leq |U| - 1$. But $k \equiv \sum_{i=1}^{k} |V_i| \equiv |V(G-U)| = |V(G)| - |U| \equiv |U| \pmod 2$, and therefore we even have $k \leq |U| - 2$. This proves the claim. □

Definition 3.28. A *plane embedding* of a given graph G is a mapping emb from $V(G) \cup E(G)$ into $\mathcal{P}(\mathbb{R}^2)$, so that each vertex in $V(G)$ is represented by a distinct point in the plane and each edge by a not self-intersecting Jordan arc connecting the assigned points of its end vertices (it is a closed Jordan curve if the actual edge is a loop), where no vertex-point is interior to the assigned arc of some edge. G is called *planar*, if it admits a plane embedding such that every two distinct assigned arcs admit no common interior (intersection) point. If G is planar, an appropriate embedding is called *planar embedding* of G or a planarly embedded graph. In this thesis, due to simplicity, we will often not distinguish between a planar graph and one of its plane embeddings if this does not lead to misunderstandings.
Kuratowski's Theorem and *Wagner's Theorem* provide purely combinatorial classifications of planar graphs: According to the much more general and prominent *Robertson-Seymour graph minor structure theorem* on characterizing graph classes by forbidden minors (cf. [NR04]), each graph class which is closed under taking minors (such as the planar graphs) can be defined as the graphs not admitting a certain *finite* set of forbidden minors $\{G_1, ..., G_k\}$. Wagner's Theorem (cf. [Wag37]) makes this explicit for the case of planar graphs and states that a given graph is planar if and only if it does not admit a K_5 resp. a $K_{3,3}$ as minor. The (corresponding) Theorem of Kuratowski (cf. [Kur30]) states that a simple graph is planar if and only if it does not contain a K_5 or $K_{3,3}$ as topological minor.
It is easily seen that plane embeddings of graphs can be equivalently considered as embeddings on the unit sphere without arc-crossings. In such an embedding, by deleting the

images of edges and vertices from the surface or equivalently the plane, i.e., by considering $\mathbb{S}^2 \backslash \mathrm{emb}_{\mathbb{S}^2}(V(G) \cup E(G))$ resp. $\mathbb{R}^2 \backslash \mathrm{emb}(V(G) \cup E(G))$, we obtain different connected components of the sphere surface resp. the plane, which we call *regions* of G. In the plane embedding, we may distinguish between bounded and unbounded regions. There is always a unique unbounded region, called *outer region*.

The union of the arcs assigned to the edges of a simple cycle in a planar graph G induces a closed Jordan curve in an associated plane embedding. This curve divides its complement in $\mathbb{R}^2$ into a bounded (the interior of the cycle) and an unbounded component (the exterior of the cycle). If the interior of a cycle in a plane embedding of G does not contain any further images of vertices or edges, we call the corresponding interior component a *bounded face*, if the exterior contains no further images, we call it an *unbounded face*. In this situation, the cycle itself is called *facial cycle*. Bounded resp. unbounded faces are exactly the bounded resp. unbounded regions without further bridges and vertices contained inside resp. outside of it.

Moreover, given any region, by applying an appropriate stereographic projection to a sphere embedding, we can find a plane embedding of G in which it becomes the unbounded (outer) region.

Given a 2-vertex-connected planar graph, in each plane embedding, interior-minimal and exterior-minimal cycles have to be facial cycles. Moreover, the faces in such an embedding cover the whole plane and can be seen as the remaining connected components after deleting all arcs. In this case, there is a unique unbounded face, which is called the *outer face*. The corresponding cycle of the planar graph is called *outer cycle*. Although 2-connected planar graphs admit "nice" sphere resp. plane embeddings with a decomposition of the graph into facial cycles, still, by e.g. flipping "loose faces" along edges, we may find substantially different plane embeddings, i.e., with non-isomorphic corresponding sphere embeddings in most cases, implying that in general, sphere embeddings of 2-connected planar graphs are not unique.

If we require unique sphere embeddings of planar graphs, we have to step further and consider simple 3-(vertex-)connected planar graphs. Whitney (cf. [BM08]) showed that these graphs (also known as *polyhedral graphs*) admit unique sphere embeddings. In this setting it becomes clear that the set of facial cycles of a 3-connected planar graph is unique and does not depend on the actual graph embedding. Moreover it can be proven that the skeleton graph of every 3-dimensional polyhedron gives rise to a polyhedral graph and on the other hand, every such graph admits a representation as a skeleton graph of a 3D-polyhedron (formally polytope).

A *planar triangulation* is defined to be a 3-connected planar graph in which all faces are triangles. It is also possible to consider planar triangulations without the additional requirement of 3-connectivity, but for our purposes, we will be mostly interested in simple planar (di-)graphs, and a 2-connected planar graph whose facial cycles are all triangles (thereby excluding loops) is 3-connected if and only if it is simple (since cut sets of size 2 come along with parallel edges between the cut vertices).

In some cases, we will furthermore need 4-connected planar triangulations. Again, given some (per definition) 3-connected planar triangulation T, it is 4-connected if and only if it

admits no separating vertex set of size 3. It is not hard to show that furthermore, such three vertices have to be pairwise adjacent and thus form a triangle in T, which is not facial (since in this case, it would not be separating). Thus, T is 4-connected iff it admits no *separating triangle*, which is defined as a triangle which is not facial (i.e., admits vertices in its interior as well as its exterior).
Usually, when dealing with planar triangulations, we consider them as embedded graphs in the plane (if we want to emphasize this fact, we also use the term *plane triangulation*), and denote the outer triangle by $a_1a_2a_3$. a_1, a_2, a_3 are then called *outer vertices*, while we refer to a_1a_2, a_2a_3, a_3a_1 as the *outer edges*.
A planar 2-connected embedded graph with outer cycle C of length $k \in \mathbb{N}, k \geq 3$ so that all the bounded faces are triangles, is called a *k-triangulation* and the corresponding graph *k-triangulated*.

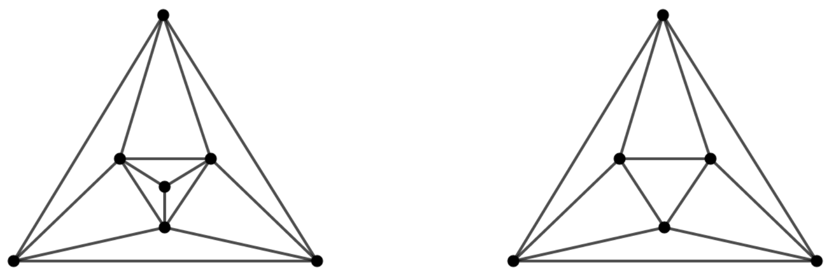


Figure 3: Left: Planar triangulation with separating triangle. Right: 4-connected planar triangulation

The following probably is one of the best-known theorems of planar graph theory (and topology), *Euler's formula*.

Theorem 3.29 (Euler's Formula)**.** *Let G be a connected planarly embedded non-empty graph. Then $|V(G)| - |E(G)| + |\mathcal{F}(G)| = 2$, where $\mathcal{F}(G)$ denotes the set of regions of G.*

Proof. We prove the assertion by induction on the number of edges of G. Assume for the base case that G is a non-empty tree. Then $|E(G)| = |V(G)| - 1, |\mathcal{F}(G)| = 1$, and therefore $|V(G)| - |E(G)| + |\mathcal{F}(G)| = 1 + 1 = 2$ as claimed. For the inductive step, assume that G is a connected planar graph containing a cycle C, $|E(G)| = m \geq 1$ and the claim holds true for all connected planar graphs on at most $m - 1$ edges. Choose some edge $e \in E(C)$. Then e is no bridge and thus, $G - e$ still is a connected planar graph with $m - 1$ edges, for which Euler's formula holds true. Obviously, when stepping from $G - e$ to G, the number of vertices stays the same, the number of edges increases by 1 and, since e is no bridge, adding e separates an existing region of G, thereby increasing the number of regions of G by 1. All in all, the term $|V(G)| - |E(G)| + |\mathcal{F}(G)|$ is unaffected and thus equals 2, proving the induction hypothesis. Finally, the principle of induction yields the claim. $\square$

Corollary 3.30. *Let G be a simple planar graph, $|V(G)| \geq 3$. Then $|E(G)| \leq 3|V(G)| - 6$. If T is a planar triangulation, then $|E(T)| = 3|V(T)| - 6$.*

Proof. We start by proving the inequality.
Assume first that G is additionally 2-connected, i.e., all the regions are faces. We double-count the pairs (e, C) of an edge $e \in E(G)$ and a facial cycle C containing e in G. Since G is simple, $\ell(C) \geq 3$ for all such cycles. This already implies (each edge is obviously contained in at most 2 facial cycles): $2|E(G)| \geq |\{(e, C)|e \in E(G), C$ facial cycle: $e \in E(C)\}| \geq 3|\mathcal{F}(G)|$. Using Euler's formula, we get

$$2 = |V(G)| - |E(G)| + \underbrace{|\mathcal{F}(G)|}_{\leq \frac{2}{3}|E(G)|} \leq |V(G)| - \frac{1}{3}|E(G)|.$$

Rearranging this inequality immediately yields the first claim.
If T is a planar triangulation (and thus 3-connected), then in the above argument, we have $|\ell(C)| = 3$ for all facial cycles C. Thus, all the inequalities formulated there hold with equality, implying the second assertion.
It thus remains to show that $|E(G)| \leq 3|V(G)| - 6$ holds true even for simple planar graphs that are not 2-connected whenever $|V(G)| \geq 3$. We do this, using the positive result for 2-connected simple planar graphs, by induction on the number of vertices of G: If $|V(G)| = 3$, then G is at most a K_3 and thus, $|E(G)| \leq 3 = 3 \cdot 3 - 6$, so assume for the inductive step that $n := |V(G)| \geq 4$ and the claim is true for all simple planar graphs on at most $n - 1$ vertices. Then there are three cases: If G is 2-connected, we are done by the above. Otherwise, it is either disconnected (i.e., it is the union of two vertex-disjoint graphs G_1 and G_2) or connected but admits a cut vertex. In the former case, the inductive hypothesis gives $|E(G)| = |E(G_1)| + |E(G_2)| \leq 3|V(G_1)| + 3|V(G_2)| - 12 \leq 3|V(G)| - 6$, proving the inductive claim. In the latter case, if there is a cut vertex v, such that $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{v\}$, $3 \leq |V(G_1)|, |V(G_2)| \leq |V(G)| - 1$, we deduce

$$|E(G)| = |E(G_1)| + |E(G_2)| \leq 3(|V(G_1)| + |V(G_2)|) - 12 = 3(|V(G)| + 1) - 12 \leq 3|V(G)| - 6.$$

Otherwise, one of the graphs arising from deleting v is trivial, i.e., corresponds to a leaf w. We now conclude $|E(G)| = |E(G - w)| + 1 \leq 3(|V(G)| - 1) - 6 + 1 \leq 3|V(G)| - 6$. Thus, the claim follows in all the three cases, and using the principle of induction, we deduce the first claim in its full generality. □

Definition 3.31. A central definition/graph construction related to planar embeddings is the *dual graph* of a given planarly embedded graph $G = (V, E, \delta)$: Let $\mathcal{F}$ denote the set of regions of G. The *planar dual* or *dual graph* of G is defined as the graph $G^* = (\mathcal{F}, E, \delta^*)$, where $\delta^*(e)$ for an edge $e \in E$ contains the regions bounding e in the sphere embedding of G from the left resp. right (if they are the same, i.e., if e is a bridge in G, then e is a loop in G^*, where $\delta^*(e)$ contains exactly the surrounding region). This graph is equipped with a planar embedding as follows: For each vertex of G^*, place a point inside the corresponding region in the embedded graph G. For every edge $e \in E(G) = E(G^*) = E$ which is no bridge in G, draw a corresponding arc connecting the points inside the two regions incident to e

in G, which crosses only one arc in the plane drawing of G, namely the one corresponding to e. If e was a bridge in G, draw a closed Jordan arc starting and ending in the point of the corresponding region, which crosses $e \in E(G)$ but no other arc in G^*. It is not hard to see that this is possible without producing any intersecting arcs in G^*. This process is illustrated by the following figure:

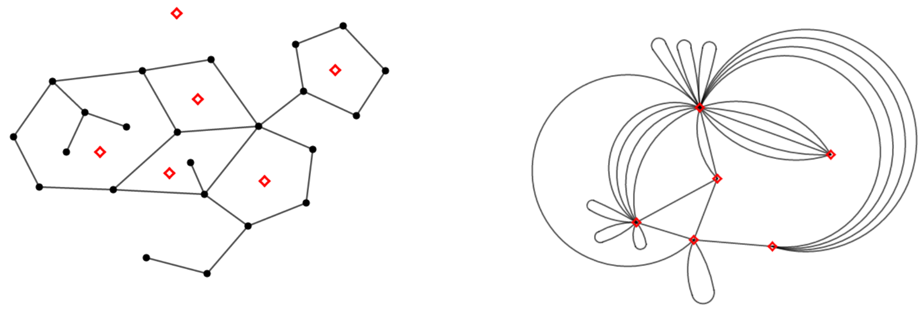


Figure 4: Left: A planarly embedded graph with numerous bridges. Right: A corresponding planarly embedded dual with numerous corresponding loops.

The definition of the dual graph as above only works on planarly embedded graphs. Generally, since even 2-connected planar graphs admit non-isomorphic plane embeddings, their corresponding dual graphs may be non-isomorphic too. Given a planar graph G, we denote by G^* the dual graph of some and any planar embedding of G, thereby knowing that we may possibly be using the same symbol for different graphs. Still, all of the different possible planar duals of a graph admit common interesting properties (described below), which will be essential for our considerations and do not depend on the special underlying plane embedding. Moreover, in most cases, we will be dealing with 3-connected planar graphs, whose duals are (according to the above) unique. In addition to that, it is known that the dual graphs of the simple 3-connected planar graphs are exactly the 3-connected planar graphs again. Analogously, the dual graphs of simple planar graphs are exactly the 3-edge-connected planar graphs.

Why do we even use the term "dual" in the notion of dual graphs? This is because in a given primal-dual planar embedding of a connected graph G and its dual G^*, the regions of G^* contain each exactly one vertex of G. Since each pair of arcs representing the same edge of $E(G) = E(G^*)$ are crossing, this means that constructing the dual graph from the planarly embedded G^* as described above will lead to a graph isomorphic to G again, in other words: For each dual graph G^* of a given planar graph G, G itself can be considered (one of) the dual graphs of G. In the case that G^* is simple and 3-connected, this can be written as $(G^*)^* = G$.

A planar triangulation can be shown to be 4-connected iff its cubic 3-connected dual graph T^* is cyclically resp. internally 4-edge-connected.

The following now contains some important dualities of different objects in a pair of dual planar graphs G, G^* as mentioned above:

Proposition 3.32. *Let G be a connected planar graph and G^* a dual graph of G. Then the following holds ($\mathcal{F}(G)$, $\mathcal{F}(G^*)$ denote the sets of regions in appropriate planar embeddings* *of G and G^*):*

- $E(G^*) = E(G) =: E$.
- *Each vertex $v \in V(G)$ and each region $f \in \mathcal{F}(G)$ uniquely corresponds to a region $f_v \in \mathcal{F}(G^*)$ resp. a vertex $v_f \in V(G^*)$ and vice versa.*
- *$e \in E$ is a loop iff it is a bridge in G^*.*
- *$e \in E$ is a bridge iff it is a loop in G^*.*
- *$E' \subseteq E$ is an even edge subset (a closed trail) in G iff it is the edge set of a cut in G^*.*
- *$E' \subseteq E$ is the edge set of a cut iff it is an even edge subset (a closed trail) in G^*.*
- *$C \subseteq E$ is the edge set of a cycle in G iff it is the edge set of a minimal cut in G^*.*
- *$S \subseteq E$ is the edge set of a minimal cut in G iff it is the edge set of a cycle in G^*.*
- *$E_G(v) \subseteq E, v \in V(G)$ is the set of edges incident to the corresponding region in G^*.*
- *The set of edges incident to a region in G is the set of edges incident to the corresponding vertex in G^*.*
- *$E' \subseteq E$ is the edge set of a spanning tree of G iff $\overline{E'}$ is the edge set of a spanning tree in G^*.*

3.2 Digraphs

The following is to a large extent very analogous (word to word) to corresponding definitions for graphs.

Definition 3.33. A *directed graph* or *digraph* is a triple $D = (V, A, \delta)$, where V and A are finite sets consisting of the *vertices* resp. *arcs, directed edges or edges* of D and $\delta : A \to V \times V$ is an arbitrary mapping, assigning an ordered pair of vertices to each arc of D. In the above setting, we write $V(D) := V, E(D) := A, \delta_D := \delta$. Given $e \in A, \delta(e) = (u, w)$, we denote by $u =: \mathrm{tail}(e)$ the *starting vertex or tail* and by $w =: \mathrm{head}(e)$ the *end vertex or head* of the directed edge e. The digraph D induces a corresponding unoriented graph called *underlying graph of D*, denoted by $U(D)$ which is defined as $(V, E, \delta_{U(D)})$ with $V =: V(D), E =: E(D), \delta_{U(D)}(e) := \{\mathrm{tail}_D(e)\} \cup \{\mathrm{head}_D(e)\}, \forall e \in E$. Because of that, digraphs can equivalently be considered as graphs equipped with a special *orientation* of the edges, i.e., each undirected edge receives an additional direction by specifying a starting and ending vertex. In such a case (especially if we consider different orientations on the same graph), we denote the actual digraph by $\mathcal{O}(G)$, where G is the underlying graph and $\mathcal{O}$ the assignment of directions as described.
Given an edge $e \in E(D)$ of a digraph, we often abbreviate $\delta_D(e) = (u, w)$ with $e = (u, w)$ if this does not lead to misunderstandings due to parallel edges.
We call an edge $e \in E(D)$ of a digraph D a *loop*, if $\mathrm{head}(e) = \mathrm{tail}(e)$. The loops in

a digraph are exactly the orientations of the loops in the underlying graph. Given a pair $e_1 \neq e_2$ of distinct edges which are no loops, we call them *parallel*, if $\delta_D(e_1) = \delta_D(e_2)$, and *antiparallel*, if $\delta_D(e_1) = (u, w), \delta_D(e_2) = (w, u)$, for some $u \neq w \in V(D)$.
Given two vertices $v_1 \neq v_2 \in V(D)$ of a digraph, we call them *adjacent*, if they are adjacent in $U(D)$. We call a vertex $v \in V(D)$ and an edge $e \in E(D)$ incident, if they are incident in $U(D)$. Moreover, they are called *positively* resp. *negatively* incident, if $\text{tail}(e) = v$ resp. $\text{head}(e) = v$.
A digraph $D_1 = (V_1, A_1, \delta_1)$ is a *subdigraph* of $D_2 = (V_2, A_2, \delta_2)$, in symbols, $D_1 \subseteq D_2$, if $V_1 \subseteq V_2, A_1 \subseteq A_2$, and $\delta_1 = \delta_2|_{A_1}$. This especially means that $\delta_2(e) \in V_1 \times V_1$, for all $e \in A_1$.
D_1 is called *induced subdigraph*, in symbols $D_1 = D_2[V_1]$, if A_1 contains all the edges $e \in A_2$ with $\delta(e) \in V_1 \times V_1$, i.e., if D_1 can be obtained from D_2 by deleting all vertices out of $V_2 \setminus V_1$ and the respective incident arcs.
If $V_1 = V_2$, i.e., D_1 arises from D_2 by deleting the edges in $A_2 \backslash A_1$, we write $D_1 = D_2[A_1]$. We furthermore use the notations $D - X := D[V \backslash X], D - Y := D[A \backslash Y]$ for the processes of deleting a vertex set resp. an arc set $X \subseteq V$ resp. $Y \subseteq A$ from a digraph $D = (V, A, \delta)$. If $D_1 \subseteq D_2$, we call D_1 a *spanning* subdigraph, iff $V(D_1) = V(D_2)$.
Given two digraphs $D_1 = (V_1, A_1, \delta_1), D_2 = (V_2, A_2, \delta_2)$, for the most purposes, they admit the same essential properties if they arise from each other by relabeling. Formally, we say that D_1, D_2 are *isomorphic* (in symbols $D_1 \simeq D_2$) if there is a pair of bijections $\sigma : V_1 \to V_2, \tau : A_1 \to A_2$, so that $\delta_1(e) = (u, w) \Leftrightarrow \delta_2(\tau(e)) = (\sigma(u), \sigma(w)), \forall e \in E_1$. In most cases, unless otherwise specified, we will not distinguish between a digraph D and its corresponding equivalence class containing all the digraphs being isomorphic to it, if this does not lead to misunderstandings.
Generally, if not otherwise specified, terms defined for graphs used for a digraph mean that the respective underlying graph has the particular property, e.g. a *planar digraph* is a digraph with planar underlying graph, and a digraph is called *simple* whenever its underlying graph is simple.
When drawing a planar digraph, we usually use a plane embedding of its underlying graph, and replace the (unoriented) arcs by arrows indicating the orientation.

Definition 3.34. Given a digraph $D = (V, E, \delta)$, a pair $e_1 \neq e_2$ of edges is called *consecutive*, if $\text{head}(e_1) = \text{tail}(e_2)$. The graph $L(D)$, defined by

$$V(L(D)) := E, E(L(G)) := \{(e_1, e_2) | e_1 \neq e_2 \in E : \text{consecutive}\}$$

and $\delta'(e_1, e_2) = (e_1, e_2)$, is called the *line digraph* of D.

Definition 3.35. Let $D = (V, A, \delta)$ be a digraph and $v \in V$. The sets of positively resp. negatively incident edges of a vertex v are denoted by $E_D^+(v)$ resp. $E_D^-(v)$. The loops incident to v are the only edges which are positively and negatively incident at the same time, i.e., $E_D^+(v) \cap E_D^-(v) = L_D(v)$. Instead of positively and negatively incident edges, we will also refer to the arcs in $E_D^+(v)$ resp. $E_D^-(v)$ as *outgoing* resp. *incoming*.
As in the case of graphs, we may define the *outdegree* resp. *indegree* of a vertex $v \in V(D)$ as the number of outgoing resp. incoming arcs, i.e., $\deg_D^+(v) := |E_D^+(v)|, \deg_D^-(v) := |E_D^-(v)|$.

We furthermore define $\mathrm{exc}_D(v) := \deg_D^+(v) - \deg_D^-(v)$ and call it the *excess* or *Euler-degree* of D at v. If $G = U(D)$, we have $\deg_G(v) = \deg_D^+(v) + \deg_D^-(v)$ for each $v \in V$.
The sets $N_D^+(v) := \{w \in V | (v, w) \in \delta(A)\}$ and $N_D^-(v) := \{w \in V | (w, v) \in \delta(A)\}$ of neighbours of v in D are (sometimes, especially in rooted trees) called *successors* resp. *predecessors* of v in D. A vertex of degree 0 in $U(D)$ is called *isolated*. A vertex of indegree 0 is called a *source*, while a vertex of outdegree 0 is a *sink* of D.
For a natural $k \in \mathbb{N}_0$, we refer to D as being k-regular resp. cubic, if $U(D)$ is k-regular resp. cubic.

The following is the directed version of the handshake-lemma:

Proposition 3.36. *Let $D = (V, A, \delta)$ be a digraph. Then*

$$\sum_{v \in V} \deg_D^+(v) = \sum_{v \in V} \deg_D^-(v) = |A|,$$

i.e., $\frac{|A|}{|V|}$ is the average in- and outdegree in D. Thus,

$$\sum_{v \in V} \mathrm{exc}_D(v) = 0.$$

Proof. We have

$$\sum_{v \in V} \deg_D^+(v) = \sum_{v \in V} |\{e \in A | \mathrm{tail}(e) = v\}| = |A|,$$

and analogously for the indegrees. This implies the claim. □

Adjacency and incidence matrices can also be defined for digraphs:

Definition 3.37. Let $G = (V, A, \delta)$ be a digraph and $n := |V|, m := |A|$. Assume we are given orderings $V = \{v_1, ..., v_n\}$ and $A = \{e_1, ..., e_m\}$ of the vertices and edges. Then the matrices $A_D \in \mathbb{R}^{n \times n}, B_D \in \mathbb{R}^{n \times m}$,

$$(A_D)_{i,j} := |\{e \in E | \delta(e) = (v_i, v_j)\}|$$

and

$$(B_D)_{i,j} := \begin{cases} 0, & \text{if } v_i \notin \delta_{U(D)}(e_j) \vee e_j \in L_D(v_i) \\ 1, & \text{if } \mathrm{tail}(e_j) = v_i \wedge e_j \notin L_D(v_i) \\ -1, & \text{if } \mathrm{head}(e_j) = v_i \wedge e_j \notin L_D(v_i) \end{cases},$$

are called *adjacency matrix* resp. *incidence matrix* of D. Given the incidence matrix, the sum of the elements in each column is 0, while the sum in the ith row is exactly the excess of v_i in D. Double-counting now immediately gives a simpler proof of the handshake-lemma.

Definition 3.38. All the notions related to sequences in graphs, such as walk, path, trail, closed edge sequence, simple cycle, closed trail, are also defined on a digraph. We consider them as vertex-edge-sequences or equivalently subdigraphs of D, whose corresponding subgraphs or sequences in $U(D)$ have the respective properties as defined for graphs. In the

following, we define their *directed* counterparts in digraphs.
Let $D = (V, A, \delta)$ be a digraph. A *directed walk* in D is a sequence of alternating positively and negatively incident vertices and edges in D, i.e., a sequence $v_1 e_1 v_2 e_2 ... v_n$, where $\delta(e_i) = (v_i, v_{i+1}), i = 1, ..., n-1$. v_1 is called *starting* and v_n *ending vertex* of the walk. The *length* of a directed walk $W = v_1 e_1 ... v_n$ is defined as the number of edges contained in it $(n-1)$ and denoted by $\ell(W)$. Given a directed walk in D, we call it a *directed path*, if all contained vertices are pairwise distinct. A *directed trail* is defined as a directed walk without repeated edges. A *directed closed edge sequence* in D is a walk which admits the same starting and ending vertex $v_1 = v_n$. A *directed cycle* in D is a closed edge sequence such that all the vertices except for the starting and ending vertex are pairwise distinct. A directed cycle of length 2 is also called a *digon*, while we often refer to a directed cycle of length 3 as a *directed triangle*. A directed cycle resp. a directed path of length $|V|$, i.e., a directed cycle resp. a directed path visiting every vertex, is called a *directed Hamiltonian cycle* resp. *directed Hamiltonian path*. A digraph admitting a directed Hamiltonian cycle is called *Hamiltonian*, a digraph admitting a directed Hamiltonian path *traceable*. Obviously, every Hamiltonian digraph is traceable.
Given a cycle C or a path P in D, we call it *induced*, if the corresponding subgraph of $G := U(D)$ is induced. A *chord* of C or P is a *chord* of C resp. P considered as subgraphs of G.
Finally, we will call a directed closed edge sequence which is a trail a *directed closed trail*. In contrary to the graph case, we are not allowed to reverse the order, but still, given a directed closed edge sequence, by shifting the indices according to $v_1 e_1 v_2 ... e_{n-1} v_n = v_1 \longrightarrow v_i e_i v_{i+1} .... e_{i-1} v_i, i = 1, ..., n-1$ we get another directed closed edge sequences, which will be considered equal in most cases. Moreover, directed walks, directed paths, directed closed edge sequences and directed cycles will be equivalently treated as subdigraphs of D.
A digraph is called *acyclic*, if its does not admit any directed cycles.

Definition 3.39. Let $D = (V, A, \delta)$ be a digraph, and let $F \subseteq A$ be an arc set. *Contracting* F is a process which transforms D into the contraction digraph $D/F = (V', E', \delta')$, which is an orientation of $U(D)/F$ so that for each $e \in A \backslash F$, $\delta(e) = (u, w)$ in D implies $\delta'(e) = (\tilde{u}, \tilde{w})$ in D', where $\tilde{u}, \tilde{w}$ represent the contraction vertices of u, w in D/F. In this context, D/F is referred to as a *contraction minor* of D.

Definition 3.40. Let $D = (V, A, \delta)$ be a digraph. D is called *simply connected*, if $U(D)$ is simply connected, and refer to the connected components of $U(D)$ as the connected components of D at the same time. Furthermore, we call D k-vertex- resp. k-edge-connected for a $k \in \mathbb{N}$, iff $U(D)$ is k-(vertex/edge)-connected.
Define a binary relation on D by $u, v \in V : u \sim_c v :\Longleftrightarrow$ There is a directed walk starting in u and ending in v and vice versa.
It is easy to see that even more, this defines an equivalence relation on V. Henceforth, V decomposes to disjoint equivalence classes according to this relation, which we call *strong components* of D. If there is only one such component, there are bidirectional directed walks and consequently bidirectional directed paths between each given pair of vertices $u \neq w$. In this case, we refer to G as being *strongly connected*.

Definition 3.41. Let $D = (V, A, \delta)$ be a digraph, and $V_1, V_2 \subseteq V$ arbitrary vertex subsets. Then $D(V_1, V_2) \subseteq E$ is defined as the set of arcs $e \in A$ starting in V_1 and ending in V_2, i.e. with $\mathrm{tail}(e) \in V_1, \mathrm{head}(e) \in V_2$. We thus have $E(D[V']) = D(V', V'), \forall V' \subseteq V$. An *(oriented) cut* in D is defined as the edge set of a cut in $U(D)$, given $S = U(D)[X, \overline{X}]$, we often abbreviate by $S = (S^+, S^-)$ with $S^+ := D(X, \overline{X}), S^- := D(\overline{X}, X)$. A *directed cut or dicut* in D is defined as an edge set of the form $D(X, \overline{X}), X \subseteq V$ which equals the cut $U(D)[X, \overline{X}]$, in other words, all edges in the cut are directed from X to $\overline{X}$ and $D(\overline{X}, X) = \emptyset$. Given a dicut S, a vertex set X such that $S = D(X, \overline{X})$ is called a *cut set* of S. In that sense, a directed cut separates the digraph D into the halves $D[X]$ and $D[\overline{X}]$, and each directed path connecting a pair of vertices in X resp. $\overline{X}$ has to start in X, while there is no directed path starting in $\overline{X}$ and ending in X. Thus, every digraph D admitting a dicut with nontrivial cut set $X \notin \{\emptyset, V\}$ is not strongly connected. By an inductive argument one can show that also the reverse of this holds true, in other words: A digraph D is strongly connected iff it is simply connected and does not contain dicuts with non-trivial cut sets. A digraph which does admit strongly connected simple components, i.e., which does not admit non-empty dicuts, is called *totally cyclic*.
The (di-)cuts containing exactly one arc are the one-element-sets of the bridges in D.
A *minimal* cut S in D is defined as a cut whose corresponding cut in $U(D)$ is minimal.

Definition 3.42. A digraph $D = (V, A, \delta)$ is said to be k*-arc-connected*, if each digraph $D - E', E' \subseteq E, |E'| \leq k - 1$ arising from D by deleting less than k directed edges is still strongly connected. Again, 1-arc-connectivity is the same as strong connectivity. The least $k \in \mathbb{N}$ for which D is k-arc-connected is defined as the *(global) arc-connectivity* of D, denoted by $\kappa'(D)$.
Arc subsets of D whose deletion leaves a digraph that is not strongly connected are called *separating*. Given a pair $q \neq s \in V$ of vertices, an arc subset S in D is called q, s*-separating* if in $D - S$, there is no directed path starting in q and ending in s. With these definitions, we can define the local edge-connectivity of a pair $q \neq s$ of distinct vertices in D by

$$\kappa'(D, q, s) := \min\{|S| | S \subseteq A \text{ is } q, s\text{-separating}\}.$$

It is obvious that

$$\kappa'(D) = \min\{\kappa'(D, q, s) | q \neq s \in V\}.$$

We denote the maximal number of pairwise edge-disjoint directed paths connecting an ordered pair of distinct vertices q and s (starting in q and ending in s) by $\lambda'(D, q, s)$. Additionally,

$$\lambda'(D) := \min\{\lambda'(D, q, s) | q \neq s \in V\}.$$

The following statements are the content of the arc-version of Menger's Theorem already presented for vertices and edges:

Lemma 3.43. *Let $D = (V, A, \delta)$ be a digraph, $q \neq s \in V$. Then*

$$\kappa'(D, q, s) = \min\{|D(X, \overline{X})| | \emptyset \neq X \subset V, q \in X, s \notin X\}.$$

Thus, a digraph D is k-arc-connected if and only if $\emptyset \neq X \subset V \Rightarrow |D(X, \overline{X})| \geq k$, i.e., if every non-trivial cut admits at least k edges in both directions.

Theorem 3.44 (Menger–Arc Version)**.** *Let $D = (V, A, \delta)$ be a digraph with $|V| \geq 2$. Then the following holds:*

- *For each pair $q \neq s \in V$ of distinct vertices, $\lambda'(D, q, s) = \kappa'(D, q, s)$.*
- *$\lambda'(D) = \kappa'(D)$.*

The latter implies that in any k-arc-connected digraph, there are at least k edge-disjoint directed paths in both directions between any given pair of distinct vertices.

We will derive Menger's theorem for arcs from the more general Max-Flow-Min-Cut Theorem further below.

Definition 3.45. Let D be a digraph, $k \in \mathbb{N}$. D is called *internally* or *essentially* k-edge-connected resp. *cyclically* k-edge-connected, if the same is true for $U(D)$.

Definition 3.46. Let $T = (V, E)$ be a tree. Given a designated vertex v and an orientation $\vec{T} = (V, A)$ of T, $\vec{T}$ is called a *rooted in-tree or in-arborescence in* v, if the unique connection path $P(u, v)$ for each distinct vertex $u \in V \backslash \{v\}$ is directed from u to v. Analogously, $\vec{T}$ is a *rooted out-tree resp. a out-arborescence in* v, if $P(v, u)$ is a directed path starting in v and ending in u, for all $V \ni u \neq v$.

Definition 3.47. Let $D = (V, A, \delta)$ be a digraph. A *Eulerian trail* in D is defined as a directed trail in D visiting each edge in A exactly once. Analogously, a *Eulerian circuit* or *Eulerian cycle* in D denotes a closed directed trail in D containing each edge in A exactly once.
We call D *even*, if $U(D)$ is even and *Eulerian*, if $\operatorname{exc}_D(v) = 0, \forall v \in V(D)$. Every Eulerian digraph is even. On the other hand, from Theorem 3.23 it follows that any even graph admits a Eulerian trail on each non-trivial connected component. Thus, by orienting all edges of such a component forwards with respect to the run order for some Eulerian trail, we end up with a Eulerian orientation on every even graph.
Furthermore, an arc subset $E \subseteq E(D)$ is called *Eulerian* resp. *even*, if in $U(D)[E]$, every vertex admits excess 0 resp. even degree, and *odd*, if every vertex has odd degree.

The following generalizes Euler's Theorem on the existence of Eulerian circuits to digraphs.

Theorem 3.48 (Euler–Digraph Version)**.** *Let $D = (V, A, \delta)$ be a digraph with at most one non-trivial (at least one edge) component. Then the following are equivalent:*

(i) D is Eulerian.

(ii) D admits a Eulerian circuit within its only non-trivial component.

(iii) The unique non-trivial component of D is the union of edge-disjoint directed cycles.

Proof. The proof is completely analogous to the one given for the undirected case and is thus omitted. □

Definition 3.49. As in the case of graphs, we may also define digraphs as duals of planar digraphs, so-called *directed duals*. Assume in the following that $D = (V; A, \delta)$ is a planar digraph equipped with an appropriate planar embedding. If $G = (V, E, \delta)$ is the underlying planar graph of D, we define the *directed dual* of D as the orientation of the graph G^* (defined with respect to the planar embedding of D and G) as follows: By construction, D and G^* admit a common planar embedding, so that the arcs of corresponding edges $e \in E(D) = E(G) = E(G^*)$ cross each other exactly once. We now define the orientation of an arc $e \in E(D^*) = E(G^*)$ uniquely by requiring that when the corresponding arc of D is oriented forwards, the arc of D^* crosses it from the right (tail vertex) to the left side (head vertex). This is illustrated by the figure below. Given a planar digraph D, we denote

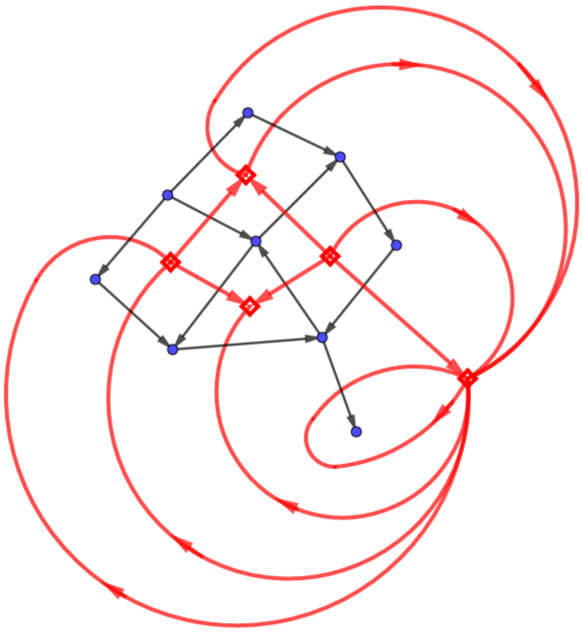


Figure 5: A planarly embedded digraph with its corresponding directed dual.

by D^* the directed dual of some and any planar embedding of G, even if the directed dual digraph is not unique. As in the case of graphs, all of the different possible planar duals of a graph admit common interesting properties (described below), which will be essential for our considerations related to flows and tensions and do not depend on the special corresponding plane embedding. Moreover, in most cases, we will be dealing with 3-connected planar digraphs, whose directed dual is unique.
Furthermore, D (up to reorientation of all the arcs) itself is one of the possible directed duals of any directed dual D^*, justifying the term "dual".

The following contains some important dualities of different objects in a dual pair of digraphs and is analogous to the case of graphs:

Proposition 3.50. *Let D be a simply connected planar digraph and D^* a directed dual of D. Then the following holds:*

- *$E(D^*) = E(D) =: E$.*
- *$e \in E$ is a loop in D iff it is a bridge in D^*.*
- *$e \in E$ is a bridge in D iff it is a loop in D^*.*
- *$E' \subseteq E$ is a Eulerian edge subset (the edge set of a directed closed trail) in D iff it is the edge set of a dicut in D^*.*
- *$E' \subseteq E$ is the edge set of a dicut in D iff it is a Eulerian edge subset (the edge set of a directed closed trail) in D^*.*
- *$C \subseteq E$ is the edge set of a directed cycle in D iff it is the edge set of a minimal dicut in D^*. More generally, if $C = (C^+, C^-)$ is a decomposition of the edges of an oriented cycle C in D into the edges in clockwise resp. counterclockwise direction, then this is also a decomposition of the edges of the corresponding minimal cut in D^* into its edges in "forward" resp. "backward" direction.*
- *$S \subseteq E$ is the edge set of a minimal dicut in D iff it is the edge set of a directed cycle in D^*. More generally, if $S = (S^+, S^-)$ is a decomposition of the edges of a minimal cut S in D into the edges in "backward" resp. "forward" direction, then this is also a decomposition of the edges of the corresponding cycle in D^* into its edges in cw- resp. ccw-direction.*
- *A vertex $v \in V(D)$ is a source or a sink iff the corresponding facial cycle in D^* is directed.*
- *A facial cycle in D is directed iff the corresponding vertex of D^* is a source or a sink.*
- *D is totally cyclic (resp. strongly connected) iff D^* is acyclic and vice versa.*

We now finally turn to one of the major topics of this thesis, namely *flows* and *tensions* on digraphs:

Definition 3.51. Let $D = (V, E, \delta)$ be a digraph and G its underlying graph. Let furthermore $(N, +)$ be some abelian group with neutral element e_N. A group-valued *flow* f on D with respect to N is defined as an assignment $f : E(D) \to N$ on the edges of D, such that f fulfils *Kirchhoff's Law of Flow Conservation*, namely

$$\forall v \in V : f^+(v) := \sum_{e \in E_D^+(v)} f(e) = f^-(v) := \sum_{e \in E_D^-(v)} f(e).$$

This can be equivalently expressed as $B_D f = \mathbf{0}$, where B_D denotes the incidence matrix of the digraph D and $\mathbf{0} \in N^V$ is defined by $\mathbf{0}_v := e_N, \forall v \in V$.
Furthermore, an assignment $f : E(D) \to N$ is called a group-valued *tension* on D, whenever the following (also known as Kirchhoff's mesh rule) holds for any oriented cycle $C = (C^+, C^-)$ decomposed into the edges with forward resp. backward direction when traversing C in some arbitrary cyclical order:

$$\sum_{e \in C^+} f(e) = \sum_{e \in C^-} f(e).$$

The notions of flows and tensions most commonly appear in the context of electrical networks: Given such a network consisting of electrical elements (junctions, vertices) connected by electrical leads (edges), at each electrical element, the sum of currents flowing into a specified junction has to equal the sum of the currents flowing out of it. In the above sense, the current values, given an orientation of the graph corresponding to the electrical network, which represents the current direction at each edge, therefore give rise to a real-valued flow. At the same time, at each junction of the electrical network, we can define a potential p_v. The voltage at each arc $e = (u, w)$ now measures the difference between the potentials at nodes w and u, i.e., $U(e) = p_w - p_u$. It is now easily seen that the voltages on the different arcs of the corresponding digraph satisfy Kirchhoff's mesh rule as stated above, i.e., they define a real-valued tension with respect to this digraph.
Flows on digraphs have many important properties, and grasping all of them would go beyond the scope of this thesis. Still, after mentioning the important role of directed duals for flows and tensions on digraphs, we will shortly sketch a very prominent result, known as the Max-Flow-Min-Cut-Theorem concerning real-valued flows in digraphs, which has many important applications in various fields of discrete mathematics. The Max-Flow-Min-Cut-Theorem deals with the problem that when e.g. wanting to improve the performance of an electricity grid, one has to face specific maximal capacities of the power lines being used, thereby limiting the possible efficiency of such a network. It provides a precise minimum-maximum characterization of the best possible "transport flows" within such a network.

Observation 3.52. *Let $D = (V, A, \delta)$ be a digraph and $(N, +)$ some abelian group with neutral element e_N. Then an assignment $f : E(D) \to N$ is a group-valued flow on D with respect to $(N, +)$ if and only if*

$$\sum_{e \in S^+} f(e) = \sum_{e \in S^-} f(e)$$

for all oriented cuts in D, $S = U(D)[X, \overline{X}], S^+ := D(X, \overline{X}), S^- := D(\overline{X}, X)$.
In the above equations, we may equivalently confine ourselves to the oriented minimal *cuts in D.*

Proof. Obviously, since for a vertex $v \in V$, $S_v := U(D)[v, V \backslash \{v\}]$ with $S_v^+ = E_D^+(v) \backslash L_D(v)$, $S_v^- = E_D^-(v) \backslash L_D(v)$ is a corresponding cut, the above equalities immediately imply

$$f^+(v) - f^-(v) = \sum_{e \in E_D^+(v) \backslash L_D(v)} f(e) - \sum_{e \in E_D^-(v) \backslash L_D(v)} f(e) = e_N,$$

and thus Kirchhoff's Law of flow conservation.
On the other hand, given a flow f and some cut $S = (S^+, S^-)$, $S^+ := D(X, \overline{X})$, $S^- := D(\overline{X}, X)$ with $X \subseteq V$, we deduce

$$\sum_{e \in S^+} f(e) - \sum_{e \in S^-} f(e) = \sum_{e \in A : \text{tail}(e) \in X} f(e) - \sum_{e \in A : \text{head}(e) \in X} f(e) = \sum_{x \in X} f^+(x) - \sum_{x \in X} f^-(x) = e_N.$$

Thus, the first equivalence holds true as claimed. Furthermore, assume that the equalities $\sum_{e \in S^+} f(e) = \sum_{e \in S^-} f(e)$ hold true for all minimal cuts $S_{\min}$, and let $v \in V$ be arbitrarily chosen. If we denote by $X_1, ..., X_r$ the vertex sets of the connected components in $U(D)-v$, according to definition 3.12, we know that $S_i := U(D)[X_i, \overline{X_i}] = U(D)[X_i, \{v\}], i = 1, ..., r$ are minimal cuts in D decomposing $S_v = U(D)[v, V \backslash \{v\}]$. Moreover, regarding the orientations in D, we have $\bigsqcup_{i=1}^r S_i^+ = E_D^+(v) \backslash L_D(v), \bigsqcup_{i=1}^r S_i^- = E_D^-(v) \backslash L_D(v)$. Consequently,

$$f^+(v) - f^-(v) = \sum_{i=1}^{r} \left(\sum_{e \in S_i^+} f(e) - \sum_{e \in S_i^-} f(e) \right) = e_N,$$

and f is indeed a flow. □

This equivalent characterization of flows together with the duality of oriented cuts and cycles now gives rise to the following dualities of flows and tensions concerning planar digraphs:

Proposition 3.53. *Let D be a simply connected planar digraph and D^* a directed dual of D. Then an assignment $f : E(D) \to N$, where $(N, +)$ is an abelian group, is a flow on D iff the corresponding assignment $f : E(D^*) \to N$ on D^* is a tension, and vice versa.*

Proof. This is an immediate consequence of the defining conditions of flows and tensions on digraphs in terms of minimal cuts and cycles and of the duality between minimal cuts in D and cycles in D^* and vice versa. □

The following note introduces a very simple but often appearing kind of flows in digraphs:

Remark 3.54. Let D be a digraph and $E \subseteq E(D)$ a Eulerian arc set in D. Then for any abelian group N and any element $n \in N$, the assignment $n\mathbb{1}_E : E(D) \to N$ admitting value n on the edges in E and e_N elsewhere, is a $(N, +)$-flow on D.
More generally, if $E \subseteq E(D)$ is an even edge subset, then according to definition 3.47, there is an orientation D' of $U(D)$ so that $E \subseteq E(D')$ is Eulerian. Now, the assignment $f_E : E(D) \to \{e_N, -n, n\}$,

$$f_E(e) := \begin{cases} 0, & e \notin E \\ n, & \text{if } e \in E \text{ and } \delta_D(e) = \delta_{D'}(e) \\ -n, & \text{if } e \in E \text{ and } \delta_D(e) \neq \delta_{D'}(e) \end{cases},$$

defines a flow on D with support E, which we will sometimes refer to as the *canonical $\pm n$-flow on E*.

In the rest of this section, we will use N as a symbol for a network, which is not to be confused with the underlying abelian group $(\mathbb{R}, +)$ the flows are defined on.

Definition 3.55. A *network* is defined as a pair $N = (D, c)$, consisting of a digraph $D = (V, A, \delta)$ and a so-called *capacity function* $c : E(D) \to \mathbb{R}_+ \cup \{\infty\}$. $c(e)$ is called *capacity* of the edge e. Given the network N and a pair of distinct vertices $q \neq s$ in D, a *q, s-flow* f in N is defined as a real-valued function $f : E(D) \to \mathbb{R}$ fulfilling the Kirchhoff-Law of flow conservation at each vertex $V(D)\backslash\{q,s\}$ and ranging within the bounds given by the capacity function, i.e., $0 \le f(e) \le c(e)$, for all $e \in E(D)$. If $v \in V$, define

$$f^+(v) := \sum_{e \in E_D^+(v)} f(e), f^-(v) := \sum_{e \in E_D^-(v)} f(e).$$

According to the flow requirement, $f^+(v) = f^-(v), \forall v \neq q, s$. More generally, if $X \subseteq V$, we define $f^+(X) := \sum_{v \in X} f^+(v), f^-(X) := \sum_{v \in X} f^-(v)$ and call $f^+(X) - f^-(X)$ the *excess* $\mathrm{exc}_f(X)$ of X. Due to the flow condition,

$$\mathrm{exc}_f(X) = f^+(X \cap \{q,s\}) - f^-(X \cap \{q,s\}).$$

If $q \in X \not\ni s$, this equals $f^+(q) - f^-(q) =: \mathrm{val}(f)$. The latter is called the *value* of the q,s-flow f. A q,s-flow is called *maximum flow*, if $\mathrm{val}(f) \geq \mathrm{val}(g)$ for all other q,s-flows g, in other words, $\mathrm{maxval}(N;q,s) := \sup\{\mathrm{val}(g)|g$ is a q,s-flow$\}$ equals $\mathrm{val}(f)$.
Obviously, a q,s-flow in N always exists, by considering the basic "zero"-flow which admits flow value 0 at each edge.

In the following, our goal will be to maximize $\mathrm{val}(f)$, where f is a q,s-flow, in other words, to transport as much flow value as possible from q to s while sticking to the given capacity restrictions.

Definition 3.56. Let $N = (D,c)$ be a network with a pair $q \neq s \in V$ of vertices. A q,s-*cut* in N is defined to be an oriented cut K with cut set X and $K^+ = D(X, \overline{X}), q \in X \not\ni s$, and $\mathrm{cap}(K) = \sum_{e \in K^+} c(e) \geq 0$ is its *capacity*.

The following justifies the introduction of q,s-cuts, since they yield upper bounds on the maximal flow values:

Observation 3.57. *If $N = (D,c)$ is a network, $q \neq s \in V$, then for any q,s-flow f and q,s-cut K, $\mathrm{val}(f) \leq \mathrm{cap}(K)$.*

Proof. Let $K = D(X,\overline{X}), q \in X \not\ni s$. Then $\mathrm{val}(f) = f^+(q) - f^-(q) = f^+(X) - f^-(X) = f(D(X,\overline{X})) - \underbrace{f(D(\overline{X},X))}_{\geq 0} \leq \sum_{e \in K^+} \underbrace{f(e)}_{\leq c(e)} \leq \mathrm{cap}(K)$. □

We now provide an answer to the question in which cases maximum flows exist:

Theorem 3.58. *Let $N = (D,c)$ be a network and $q \neq s$. Define* $\mathrm{mincap}(N;q,s) := \min\{\mathrm{cap}(K)|K$ *is q,s-cut* $\}$ *and assume* $\mathrm{mincap}(N;q,s) < \infty$. *Then there is a q,s-maximum flow f, and*

$$\mathrm{maxval}(N;q,s) = \mathrm{val}(f) \leq \mathrm{mincap}(N;q,s).$$

The famous *Max-Flow-Min-Cut-Theorem* due to Ford-Fulkerson (cf. [JF56]) in its original formulation now states that in the above situation, the upper bounds on flow values derived by cuts of minimal capacity are already optimal. In other words:

Theorem 3.59 (Ford and Fulkerson). *Let $N = (D, c)$ be a network, $q \neq s \in V$ with $\mathrm{mincap}(N; q, s) < \infty$. Then there is a maximum-q, s-flow f and a minimum-q, s-cut K, and for each such pair we have*

$$\mathrm{maxval}(N; q, s) = \mathrm{val}(f) = \mathrm{cap}(K) = \mathrm{mincap}(N; q, s).$$

Furthermore, if $c(E(D)) \subseteq \mathbb{N}_0 \cup \{\infty\}$, f can be chosen integer-valued.

Applying this theorem to a certain network now immediately gives the arc version of Menger's Theorem presented above:

Theorem 3.60. *Menger's Theorem for arcs in digraphs (Theorem 3.44) is a consequence of the Ford-Fulkerson Max-Flow-Min-Cut-Theorem.*

Proof. Let D be a digraph and $q \neq s$ a pair of distinct vertices. We have to prove that

$$\lambda'(D, q, s) = \kappa'(G, q, s) = \min\{|D(X, \overline{X})|| \emptyset \neq X \subset V, q \in X, s \notin X\}.$$

We do this by considering the network $N = (D, c)$ with $c(e) := 1, e \in E(D)$. Then, given q and s, the definitions above yield

$$\mathrm{mincap}(N, q, s) = \min\{|D(X, \overline{X})|| \emptyset \neq X \subset V, q \in X, s \notin X\} = \kappa'(G, q, s) < \infty.$$

According to the Max-Flow-Min-Cut Theorem, it thus suffices to prove that in N, $\mathrm{maxval}(N; q, s) = \lambda'(N; q, s)$. First of all, given $k := \lambda'(N; q, s)$ edge-disjoint directed paths $P_1, ..., P_k$ starting in q and ending in v, sending flow value 1 along $E(P_1) \cup ... \cup E(P_k)$ and 0 elsewhere will give rise to a flow of value $k = \lambda'(N; q, s) \leq \mathrm{maxval}(N; q, s)$ on D. On the other hand, according to the Ford-Fulkerson-Theorem, since c is integer-valued, there is a maximum-flow f with $f(e) \in \{0, 1\}, \forall e \in E(D)$. Let D_l for a given $l \in \mathbb{N}_0$ be the digraph arising from D by adding l parallel edges starting in s and ending in q to D. Since f fulfils the flow conservation at each vertex distinct from q and s, in the edge subset $\mathrm{supp}(f) := f^{-1}(\{1\})$ of D each such vertex admits the same number of incoming and outgoing arcs. Since $f^+(q) - f^-(q) = f^-(s) - f^+(s) = \mathrm{val}(f) =: l_0 \in \mathbb{N}_0$, in D_{l_0}, $\mathrm{supp}(f) \cup \delta_{D_l}^{-1}((s, q))$ even is a Eulerian arc set, which according to the above decomposes into edge-disjoint directed cycles. Let now $C_1, ..., C_{l_0}$ be a list of the directed decomposition-cycles containing the l_0 additional edges starting in s and ending in q. Then $C_i - (s, q)$, $i = 1, ..., l_0$ is a list of $l_0 = \mathrm{maxval}(N; q, s)$ edge-disjoint directed paths starting in q and ending in s, i.e., $\mathrm{maxval}(N; q, s) \leq \lambda'(N; q, s)$. This finally implies the local statement in Menger's theorem. The global statement now follows from the local one by taking the maximum over all $q \neq s \in V$ on both sides. □

3.3 Matroids

This subsection introduces a generalization of various notions of independence appearing in different areas of mathematics which may not seem to be related at first sight, in the form of the so-called *matroids*. Finite matroids can be defined in various ways. We will start with the approach of independent and dependent subsets, and introduce this by repeating the following very basic notions and results for vector spaces:

Definition 3.61. Let $(V, +, \cdot)$ be a finitely generated vector space with respect to the field $\mathbb{K}$. Then a finite set $\{v_1, ..., v_r\}$ of vectors in V is called *linearly independent*, if

$$\forall \alpha_1, ..., \alpha_r \in \mathbb{K} : \sum_{i=1}^{r} \alpha_i v_i = 0_{\mathbb{K}} \Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_r = 0_{\mathbb{K}}.$$

Analogously, $\{v_1, ..., v_r\}$ is called *linearly dependent* if there are $\alpha_1, ..., \alpha_r \in \mathbb{K}$, not all of them $0_{\mathbb{K}}$, such that $\sum_{i=1}^{r} \alpha_i v_i = 0_{\mathbb{K}}$.

The following prominent augmentation property of linearly independent vector sets is the foundation of the notion of a *basis* in a vector space (we use one of many possible equivalent formulations of this lemma which is appropriate for the context of matroids):

Theorem 3.62 (Steinitz Exchange Lemma)**.** *Let $(V, +, \cdot)$ be a finitely generated vector space defined over a field $\mathbb{K}$ and $U = \{u_1, ..., u_r\}, W = \{w_1, ..., w_s\}$ with $r < s \in \mathbb{N}_0$ be two linearly independent vector subsets. Then there is a $j \in \{1, ..., s\}$ such that $\{u_1, ..., u_r, w_j\}$ is linearly independent as well.*

This lemma now finally gives rise to the following definition, which abstracts from the above setting of linearly independent vector sets within an underlying algebraic (linear) structure by only using a Steinitz-Exchange-Lemma-like augmentation property as the defining condition for independent subsets of a given finite ground set:

Definition 3.63 (Matroid–Independent sets)**.** A pair $M = (S, \mathcal{I})$ of a finite ground set S and a set $\mathcal{I} \subseteq 2^S$ of subsets of S is called a *matroid*, if the following holds:

($\mathcal{I}$ 1) $\emptyset \in \mathcal{I}$.

($\mathcal{I}$ 2) For $I \subseteq J$, we have $J \in \mathcal{I} \Rightarrow I \in \mathcal{I}$.

($\mathcal{I}$ 3) Given $I, J \in \mathcal{I}$, $|I| < |J|$, there is a $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$.

Given the matroid M, $E(M) := S$ is called its ground set and each $e \in E(M)$ an *element* of M. $\mathcal{I}$ is called the set of *independent sets* in M, any $I \in \mathcal{I}$ is called *independent* while a set $I \in 2^S \setminus \mathcal{I}$ is called *dependent*.

With this definition of matroids as systems of independent sets at hand, the following becomes clear:

Observation 3.64. *Let $(V, +, \cdot)$ be a finitely generated vector space over a field $\mathbb{K}$, $V' = \{v_1, ..., v_r\}$ a finite multiset of vectors in V and denote by $\mathcal{I}$ the indices of linearly independent vector multi-subsets of V'. Then $M[V'] := (\{1, ..., r\}, \mathcal{I})$ is a matroid. Given $m, n \in \mathbb{N}$, we can equivalently consider a matrix $A \in \mathbb{K}^{m \times n}$ and define the matroid $M[A]$ as $M[V']$, where V' is the finite multiset consisting of the column-vectors of A.*

Proof. This is an immediate consequence of the definition above and of the Steinitz exchange lemma. $\square$

As in the case of graphs, we consider matroids as structurally equal if they can be obtained from each other by isomorphisms:

Definition 3.65. Let $M_1 = (S_1, \mathcal{I}_1), M_2 = (S_2, \mathcal{I}_2)$ be two matroids. We call them *isomorphic*, in symbols $M_1 \simeq M_2$, whenever there is a bijection $F : S_1 \to S_2$ such that $\forall I \subseteq S_1 : I \in \mathcal{I}_1 \Leftrightarrow F(I) \in \mathcal{I}_2$.

Definition 3.66. Given a field $\mathbb{K}$, a matroid M is called *representable* over $\mathbb{K}$, whenever there is a matrix $A \in \mathbb{K}^{m \times n}; m, n \in \mathbb{N}$, so that M is isomorphic to $M[A]$, in other words, the notion of independence in M can be understood as linear independence in a vector space over $\mathbb{K}$.
Moreover, M is called *regular*, if it is representable over every field.

If we consider finitely generated vector spaces, the Steinitz exchange lemma gives rise to the notion of bases as maximal linearly independent sets (which are spanning the whole vector space) and the dimension of a vector space.
Analogously, we can define the bases of a matroid as the (inclusion-maximal) independent sets and show that they are all of the same size, called the *rank* of the matroid M.

Definition and Proposition 3.67. Let $M = (S, \mathcal{I})$ be a matroid. A *basis* of M is defined as a inclusion-maximal independent set in $\mathcal{I}$. By $\mathcal{B}$, we denote the set of bases of M. For each pair $B, B' \in \mathcal{B}$, we have $|B| = |B'|$. The common value $|B|, B \in \mathcal{B}$ is called the *rank* of the matroid M, which is denoted by $r(M) \in \mathbb{N}_0$.

Proof. Let $B, B' \in \mathcal{B}$ and assume for contrary that $|B| \neq |B'|$, without loss of generality $|B| < |B'|$. Then according to axiom ($\mathcal{I}$ 3), there is an element $e \in B' \setminus B$ so that $B \cup \{e\}$ is independent. This contradicts the assumed maximality of B and B', proving the claim. □

Strictly speaking, matroids are mathematically not well-defined, since they admit a number of formally different but equivalent ways of definition. If we take e.g. a matroid given by its independent sets $M = (S, \mathcal{I})$, because of ($\mathcal{I}$ 2), M is already uniquely defined by its set $\mathcal{B}$ of bases. On the other hand, the following provides an axiomatic definition of a matroid in terms of its set of bases, which is equivalent to the one involving independent sets. Such equivalences between different approaches and axioms defining matroids are known as *cryptomorphisms*, of which we will present a couple more further below.

Definition and Proposition 3.68. Let $M = (S, \mathcal{I})$ be a matroid with base set $\mathcal{B}$. Then the following holds true:

($\mathcal{B}$ 1) $\mathcal{B} \neq \emptyset$.

($\mathcal{B}$ 2) For each pair $B_1 \neq B_2 \in \mathcal{B}$, and $x \in B_1 \setminus B_2$, there is a $y \in B_2 \setminus B_1$ so that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

The latter can be considered as a generalization of the Steinitz basis exchange theorem from linear algebra. Moreover, given a pair $(M, \mathcal{B})$ satisfying the basis axioms above, the corresponding set $\mathcal{I} := \{I \subseteq S | \exists B \in \mathcal{B} : I \subseteq B\}$ together with S forms a matroid fulfilling axioms ($\mathcal{I}$ 1-$\mathcal{I}$ 3). Thus, this can be seen as a cryptomorphic definition of matroids.

Proof. $\mathcal{B} \neq \emptyset$ is trivial, since $\emptyset \in \mathcal{I}$ and thus, there is a maximal independent set.
Assume that $B_1 \neq B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$. Then according to ($\mathcal{I}$ 2), $B_1 \backslash \{x\}$ is independent and admits less elements than B_2, i.e., according to ($\mathcal{I}$ 3), there is a $y \in B_2 \backslash (B_1 \backslash \{x\}) = B_2 \setminus B_1$ (since $x \notin B_2$), so that $(B_1 \backslash \{x\}) \cup \{y\}$ is independent. Since it admits the same number of elements as B_1 and B_2, according to the above, it has to be a basis.
If on the other hand, $M = (S, \mathcal{B})$ fulfils ($\mathcal{B}$ 1-$\mathcal{B}$ 2), the definition

$$\mathcal{I} := \{I \subseteq S | \exists B \in \mathcal{B} : I \subseteq B\}$$

obviously fulfils $\mathcal{I}$ (1-2), and also axiom 3 can be easily verified by using a simple counting argument. □

Remark 3.69. Given a matroid $M = (S, \mathcal{B})$, we can strengthen ($\mathcal{B}$ 2) by using the following equivalent symmetric version of the basis exchange axiom as stated above:

($\mathcal{B}$ 2') For each pair $B_1 \neq B_2 \in \mathcal{B}$, and $x \in B_1 \setminus B_2$, there is a $y \in B_2 \setminus B_1$ so that $(B_1 \backslash \{x\}) \cup \{y\}$ as well as $(B_2 \backslash \{y\}) \cup \{x\}$ are bases of M at the same time.

Definition 3.70. Let $M_1 = (S_1, \mathcal{I}_1), M_2 = (S_2, \mathcal{I}_2)$ be two matroids with disjoint ground sets $S_1 \cap S_2 = \emptyset$. Then the *matroid union* of M_1 and M_2, denoted by $M_1 \cup M_2$, is the matroid pair $(S, \mathcal{I})$, where $S = S_1 \cup S_2$ and $\mathcal{I} = \{I_1 \cup I_2 | I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$.

It can be easily verified that the above indeed is a valid definition, since the pair $(S, \mathcal{I})$ indeed satisfies the independence axioms ($\mathcal{I}$ 1 – $\mathcal{I}$ 3).
We now finally introduce the so-called graphic matroids or cycle matroids, which are matroids defined on the edge sets of an arbitrary graph. This connection between graphs and matroids will be essential throughout this master's thesis.

Definition and Proposition 3.71. Let $G = (V, E, \delta)$ be a graph. Then the definition $M(G) := (E, \mathcal{I})$, where $\mathcal{I} := \{E' \subseteq E | G[E']$ is acyclic resp. a forest$\}$ gives rise to a matroid, the so-called *graphic matroid* or *cycle matroid* of the graph G. The bases of $M(G)$ are exactly the unions of the edge sets of spanning trees on each connected component of G. Thus, $r(M(G)) = |V(G)| - c(G)$.

Proof. Again, ($\mathcal{I}$ 1) and ($\mathcal{I}$ 2) can be easily verified. Given two acyclic edge subsets $E_1, E_2 \subseteq E$ with $|E_1| < |E_2|$, we know that for each connected component of $G[E_1]$ with n vertices, E_2 can have at most $n-1$ edges connecting two vertices out of it, as well as E_1 (since each connected component of the forest $G[E_1]$ is a tree). Since $|E_1| < |E_2|$, this means that there is at least one $e \in E_2$ connecting different components of $G[E_1]$. Adding this edge to e gives rise to the forest $G[E_1] \cup e$ with edge set $E_1 \cup \{e\}$, proving the independence of the latter. This proves the claim. □

Regarding the most important notions defined on graphs, one might ask whether we can reconstruct e.g. (simple) cycles of a graph only from the information provided by the corresponding graphic matroid. This is indeed the case:

Observation 3.72. *Let G be a graph with edge set E and $M(G)$ its graphic matroid. An edge set $C \subseteq E$ is the edge set of a cycle in G iff it is a minimal dependent set in $M(G)$.*

Proof. Per definition, an edge set in G is dependent iff it contains the edge set of a cycle. $\square$

The above motivates the following definition:

Definition and Proposition 3.73. Let M be a matroid. An inclusion-minimal dependent set in M is called a *circuit*. The set of circuits $\mathcal{C}$ in a matroid M satisfies the following (as in the case of independent sets and bases cryptomorphically defining) *circuit axioms*:

($\mathcal{C}$ 1) $\emptyset \notin \mathcal{C}$.

($\mathcal{C}$ 2) $C_1, C_2 \in \mathcal{C}, C_1 \subseteq C_2 \Rightarrow C_1 = C_2$.

($\mathcal{C}$ 3) For all $C_1 \neq C_2 \in \mathcal{C}$, so that $e \in C_1 \cap C_2$, there is a $C_3 \in \mathcal{C}$ contained in $(C_1 \cup C_2) \backslash \{e\}$.

On the other hand, given any pair $(S, \mathcal{C} \subseteq 2^S)$ satisfying the above axioms, the definitions $\mathcal{I} := \{I \subseteq S | \forall C \in \mathcal{C} : C \nsubseteq I\}$; and $\mathcal{B}$: Inclusion-maximal sets in $\mathcal{I}$; give rise to matroids in the context of independent sets and bases.

Proof. If $\mathcal{C}$ contains the inclusion-minimal dependent sets of M, then ($\mathcal{C}1$–$\mathcal{C}2$) are obvious. Let now $C_1 \neq C_2 \in \mathcal{C}, e \in C_1 \cap C_2$ be arbitrary. We have to prove that $(C_1 \cup C_2) \setminus \{e\}$ is dependent. So assume to the contrary it was independent. Since C_1, C_2 are cycles and $C_1 \neq C_2$, we have that $C_1 \cap C_2$ is independent. Moreover, $C_1 \cup C_2$ is obviously dependent and so, $(C_1 \cup C_2) - e$ is a basis of $M[C_1 \cup C_2]$. Since $C_1 \cap C_2 \ni e$ is independent, there also has to be a basis $B \subseteq C_1 \cup C_2$ containing $C_1 \cap C_2 \ni e$, which thus is of the form $(C_1 \cup C_2) \backslash \{f\}, f \in (C_1 \cup C_2) \backslash (C_1 \cap C_2)$. But then, for an $i \in \{1, 2\}$, we have $f \notin C_i$, implying that the circuit $C_i \subseteq (C_1 \cup C_2) \backslash \{f\}$ is independent, contradiction.
On the other hand, given a set system $\mathcal{C}$ fulfilling ($\mathcal{C}$ 1–$\mathcal{C}$ 3), it is easy to verify ($\mathcal{B}$ 2), where $\mathcal{B}$ denotes the set of inclusion maximal cycle-free subsets of S, while ($\mathcal{B}1$) is obvious. All in all, we derive the claim. $\square$

As in graphs, we may define deletions and contractions of elements in matroids:

Definition 3.74. Let $M = (S, \mathcal{I})$ be a matroid given in the form of independent sets. If $e \in S$, the deletion resp. contraction of e in M is a process resulting in the matroids $M - e, M/e$ defined by

$$E(M-e) = E(M/e) = S \backslash \{e\}, \mathcal{I}_{-e} := \{I \subseteq S \backslash \{e\} | I \in \mathcal{I}\}, \mathcal{I}_{/e} := \{I \subseteq S \backslash \{e\} | I \cup \{e\} \in \mathcal{I}\}.$$

It can be shown that these indeed are well-defined matroids. Moreover, for any pair $e \neq f \in E(M)$, we have $(M - e) - f = (M - f) - e, (M/e)/f = (M/f)/e$. This commutativity allows us to define the deletion resp. contraction of an element set $F \subseteq E(M)$, giving rise to the matroids $M - F, M/F$ by repeatedly deleting resp. contracting the elements of F in some specified but irrelevant order. We also write $M[F] := M - \overline{F}$ for the submatroid of M induced by F. Its independent sets resp. circuits are those of M contained in F.
If G is a graph and $M := M(G)$, then

$$\forall F \subseteq E(G) = E(M) : M[F] = M(G[F]), M - F = M(G - F), M/F = M(G/F).$$

Instead of $M-e, M-F$ for the deletion processes in a matroid M, we may also equivalently write $M\backslash e, M\backslash F$ at some points. A matroid arising from M by finitely many consecutive deletions resp. contractions of elements is called *minor* of M. Regular matroids are closed under taking minors.

Remark 3.75. If M is a matroid and B a basis, then for each $e \notin B$, $B \cup \{e\}$ is a dependent set containing a unique circuit, the so-called *fundamental circuit* of e and B, denoted by $C(e, B)$.

Proof. Because of $|B \cup \{e\}| > |B|$, $B \cup \{e\}$ is dependent. If there were two distinct circuits $C_1, C_2 \subseteq B \cup \{e\}$, then $e \in C_1 \cap C_2$, and according to the circuit axiom ($\mathcal{C}$ 3), there is a circuit $C \subseteq (C_1 \cup C_2)\backslash\{e\} \subseteq B$, contradicting the independence of B. □

The following generalizes the notions of loops and bridges in graphs to matroids:

Definition 3.76. Let M be a matroid and $e \in E(M)$, $\mathcal{B}$ the set of bases. e is called *loop*, if $\{e\}$ is a dependent set, i.e., a circuit of M ($e \notin B, \forall B \in \mathcal{B}$). e is called a *coloop*, if $e \in \bigcap_{B \in \mathcal{B}} B$.

Remark 3.77. If M is a matroid and $X \subseteq E(M)$ contains a loop, then X is dependent. If $X \subseteq E(M)$ is independent and $e \in E(M)$ is a coloop, then $X \cup \{e\}$ is independent.
If $M(G)$ is the graphic matroid of the graph G, then the loops in M are the loops of G and the coloops of M are the bridges of M.

The following, which defines an other cryptomorphism for matroids, is the matroid analogue to even edge subsets in graphs:

Definition and Proposition 3.78. Let $M = (S, \mathcal{C})$ be a matroid given by its set of circuits. A *cycle* in M is defined as a disjoint union of circuits in M ($\emptyset$ included).
This is not to be confused with the notion of cycles in graphs, which correspond to the circuits in the graphic matroid, while the *even* edge subsets of a graph G according to Theorem 3.23 are exactly the cycles in $M(G)$.
A regular matroid can also equivalently be defined (cryptomorphically) by axioms concerning the cycles contained in it. Namely, in this case, the set of cycles in M is closed under symmetric differences, i.e., $C \Delta C'$ is a cycle in M for any pair C, C' of cycles.
On the other hand, given a set system $\mathcal{C}_{\text{cyc}} \subseteq 2^S$ over a ground set S which is closed under symmetric differences, we may define a set system of circuits as the inclusion-minimal non-empty cycles by

$$C \in \mathcal{C} :\Leftrightarrow \emptyset \neq C \in \mathcal{C}_{\text{cyc}} : \forall \emptyset \neq C' \in \mathcal{C}_{\text{cyc}} : C' \subseteq C \Rightarrow C' = C,$$

which indeed defines the set of circuits of a corresponding matroid on S.

Proof. For the latter statement, given the set of cycles $\mathcal{C}_{\text{cyc}}$, we only need to prove that the definition of $\mathcal{C}$ above indeed fulfils the circuit axioms. This is obvious for the axioms ($\mathcal{C}$ 1-2). Furthermore, given any $C_1 \neq C_2 \in \mathcal{C} \subseteq \mathcal{C}_{\text{cyc}}\backslash\{\emptyset\}$ with $e \in C_1 \cap C_2$, according to the above, $\emptyset \neq C_1 \Delta C_2 \in \mathcal{C}_{\text{cyc}}$, and thus, there is a minimal element $C \in \mathcal{C}$ of $\mathcal{C}_{\text{cyc}}$ contained in $C_1 \Delta C_2 \subseteq (C_1 \cup C_2)\backslash\{e\}$. This verifies ($\mathcal{C}$ 3). □

Observation 3.79. *Let M be a matroid given by its set $\mathcal{C}_{\mathrm{cyc}}$ of circuits, $F \subseteq M$. Then the sets of circuits of $M - F, M/F$ are given by*

$$\mathcal{C}_{-F} := \{C \in \mathcal{C} | C \cap F = \emptyset\}$$

and $\mathcal{C}_{/F}$ contains the inclusion-minimal elements of $\{C \backslash F | C \in \mathcal{C}\}$.

Another axiom system for matroids (and the last we will sketch in here) is the one of the *rank function* of a given matroid M, which is defined as the mapping $r_M : 2^{E(M)} \to \mathbb{N}_0$, assigning to each subset $E \subseteq S$ the maximum size of an independent set contained in it (equivalently: the rank of the matroid $M[E]$), the so-called *rank* of the respective set. Obviously, given a rank function r_M of a matroid, we can reconstruct its independent sets via $X \in \mathcal{I} \Leftrightarrow r_M(X) = |X|$ and thus also the bases ($X \in \mathcal{B} \Leftrightarrow r_M(X) = |X| = r(M)$) and the circuits ($X \in \mathcal{C} \Leftrightarrow r_M(C) = r_M(C \setminus e) = |C| - 1, \forall e \in C$). The following provides the axiomatic system defining rank functions of matroids:

Proposition 3.80. *Given a finite ground set S, an assignment $r : 2^S \to \mathbb{N}_0$ is the rank function of a matroid on S iff it fulfils the following rank axioms:*

($\mathcal{R}$ 1) $\forall X \subseteq S : 0 \leq r(X) \leq |X|$.

($\mathcal{R}$ 2) If $X \subseteq Y \subseteq S$, then $r(X) \leq r(Y)$ (monotonicity).

($\mathcal{R}$ 3) $\forall X, Y \subseteq S : r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$ (submodularity).

Remark 3.81.

- If M_1, M_2 are matroids with $E(M_1) \cap E(M_2) = \emptyset$, then

$$r_M(X) = r_M(X \cap E(M_1)) + r_M(X \cap E(M_2)), \forall X \subseteq E(M).$$

- If $M = M(G)$ is the graphic matroid of $G = (V, E, \delta)$, then $r_M(X) = |V(G[X])| - c(X), \forall X \subseteq E$.

It is important to mention that the class of graphic matroids is an integral part of the regular matroids, as will be sketched further below when presenting a structure theorem for regular matroids due to Seymour.

Theorem 3.82. *Let G be a graph and $\mathcal{O}(G)$ some orientation of it with corresponding (directed) incidence matrix B. Then for any field $\mathbb{K}$, $M(G)$ is representable over $\mathbb{K}$ and thus regular. If $\mathrm{char}(\mathbb{K}) \neq 2$, $M[B]$ serves as a representation of $M(G)$, while in the case of $\mathrm{char}(\mathbb{K}) = 2$, $M[B \mod 2]$ does so.*

Finally, the duality between planar graphs and their dual graphs explained in subsection 3.1 can be rediscovered in the context of *bond* matroids of (even non-planar) graphs:

Definition and Proposition 3.83. Let $G = (V, E, \delta)$ be a connected planar graph and G^* one of its planar dual graphs. Then in the graphic matroid $M(G^*)$ with ground set E, the circuits are exactly the minimal cuts in G. Moreover, an edge subset $I \subseteq E$ is independent if and only if deleting I from G does not split a connected component of G, in other words, $G - I$ is connected. The bases of $M(G^*)$ (the edge sets of spanning trees in G^*) are exactly the complements of the edge sets of spanning trees in G.

Proof. According to the definition of a graphic matroid, the circuits of $M(G^*)$ are exactly the edge sets of cycles in G^*, which according to 3.32 are the minimal cuts in G. Moreover, the same result gives that the edge sets of spanning trees in G^* as the bases of $M(G^*)$ are exactly the complements of the spanning trees in G.
An edge subset $I \subseteq E$ is independent in $M(G^*)$ if and only if there is a spanning tree T of G so that $I \subseteq \overline{E(T)}$. If there is such a tree, we have $G - I \supseteq G - \overline{E(T)} = T$, which is connected. On the other hand, if $G - I$ is a connected graph, it admits a spanning tree T which, because of $V(G - I) = V(G)$, is also a spanning tree of G so that $E(T) \subseteq \overline{I} \Leftrightarrow I \subseteq \overline{E(T)}$. $\square$

Matroids on graphs as defined above are not restricted to planar graphs using their duals, but generally give rise to the notion of so-called *cographic matroids*.

Definition and Proposition 3.84. Let $G = (V, E, \delta)$ be a graph. Then the *cographic or bond* matroid of G is denoted by $M(G)^*$ and is defined as $(E, \mathcal{I})$, where the set of independent sets $\mathcal{I}$ in G is given by

$$I \text{ is independent} \Leftrightarrow c(G - I) = c(G).$$

The rank function $r_{M(G)^*}$ of the cographic matroid is given by

$$r_{M(G)^*}(X) = |X| + r_{M(G)}(\overline{X}) - r(M(G)).$$

The set of circuits of $M(G)^*$ consists of the minimal cuts in G, while the bases are the unions of the complements of spanning trees on the connected components of G.

Proof. We show that the given definition of independency with the appropriate rank function $r(X) := \max\{|I| | \mathcal{I} \ni I \subseteq X\}, X \subseteq E$ satisfies the rank axioms ($\mathcal{R}$ 1-$\mathcal{R}$ 3), thereby proving that this indeed defines a matroid. In the proof we may restrict to the case where G is connected, since the independent sets as defined above in the general case are exactly the unions of the independent sets on the cographic matroids of the respective connected components, in other words, if $G_1, ..., G_l$ are the connected components of G, then $M(G)^* = M(G_1)^* \cup ... \cup M(G_l)^*$ is a matroid in the sense of matroid union, and furthermore, if $r_{M(G_i)^*}$ denotes the corresponding rank function of $M(G_i)^*, i = 1, ..., l$, then

$$r_{M(G_i)^*}(Y) = |Y| + r_{M(G_i)}(E(G_i) \backslash Y) - r(M(G_i)), \forall Y \subseteq E(G_i), i = 1, ..., l$$

implies that for each $X \subseteq E(G)$,

$$r_{M(G)^*}(X) = \sum_{i=1}^{l} r_{M(G_i)^*}(X \cap E(G_i))$$

$$
\begin{aligned}
&= \sum_{i=1}^{l} \Big(|X \cap E(G_i)| + r_{M(G_i)}(\overline{X} \cap E(G_i)) - r(M(G_i))\Big) \\
&= |X| + r_{M(G)}(\overline{X}) - r(M(G)),
\end{aligned}
$$

and thus we may restrict to connected graphs when proving the assertion on the rank function of $M(G^*)$.
($\mathcal{R}$ 1) and ($\mathcal{R}$ 2) are immediate consequences of the the above definition of r. For ($\mathcal{R}$ 3), we observe that for a set $I \subseteq E$, $G - I$ is connected iff it contains a spanning tree, which means that there is an edge set T of a spanning tree in G contained in $\overline{I}$. Thus,

$$
\begin{aligned}
r(X) &= \max\{|I| | \mathcal{I} \ni I : \overline{I} \supseteq \overline{X}\} \\
&= |E| - \min\{|\overline{I}| | \mathcal{I} \ni I : \overline{I} \supseteq \overline{X}\} \\
&= |E| - \min\{|T \cup \overline{X}| | T \text{ is the edge set of a spanning tree}\} \\
&= |E| - \min\{|T| + |E| - |X| - |T \backslash X| | T \text{ is the edge set of a spanning tree}\} \\
&= |X| - |V(G)| + c(G) + \max\{|Y| | Y \subseteq \overline{X} \text{ is acyclic}\} \\
&= |X| - |V(G)| + c(G) + |V(G - X)| - c(G - X) \\
&= |X| + r_{M(G)}(\overline{X}) - r(M(G)) \\
&= |X| + c(G) - c(G - X).
\end{aligned}
$$

The asserted submodularity ($\mathcal{R}$ 3) now follows due to ($X, Y \subseteq E$ arbitrary):

$$
\begin{aligned}
&r(X \cup Y) + r(X \cap Y) - r(X) - r(Y) \\
&= \underbrace{|X \cup Y| + |X \cap Y| - |X| - |Y|}_{=0} \\
&+ r_{M(G)}(\overline{X} \cap \overline{Y}) + r_{M(G)}(\overline{X} \cup \overline{Y}) - r_{M(G)}(\overline{X}) - r_{M(G)}(\overline{Y}) \\
&\leq 0,
\end{aligned}
$$

thereby utilizing the submodularity of the rank function $r_{M(G)}$ of $M(G)$.
The statements on circuits and bases of $M(G)^*$ are proven as in the case of planar graphs above. □

Since both the bases of the cographic matroid $M(G)^*$ and the graphic matroid of a planar dual graph $M(G^*)$ (where G is planar and connected) are the complements of edge sets of the spanning trees in G, we have

Remark 3.85. If G is a planar connected graph and G^* one of its dual graphs, then $M(G^*) = M(G)^*$.

The duality between graphic and cographic matroids sketched above (i.e., their bases are each others complements) now admits the following generalization to arbitrary matroids:

Definition and Proposition 3.86. Let M be a matroid given by its set $\mathcal{B} \subseteq 2^{E(M)}$ of bases. Then also $\mathcal{B}^* := \{\overline{B} | B \in \mathcal{B}\}$ is a set system satisfying the basis axioms ($\mathcal{B}$ 1, $\mathcal{B}$ 2'), which

thus defines a matroid over the ground set $E(M)$. This matroid is called the *dual matroid* M^* of M.
A basis of M^* is called *cobasis* of M, while a circuit in M^* is called *cocircuit* of M. A subset $X \subseteq E(M)$ is called *coindependent* with respect to M, if it is an independent subset with respect to M^*.
Given a graph G, the cocircuits of its graphic matroid M are exactly the edge sets of minimal cuts in G.

Proof. It is immediate from the symmetric formulation of ($\mathcal{B}$ 2') that a pair $B_1, B_2 \in \mathcal{B}$ of sets fulfils the condition iff their complements within $\mathcal{B}^*$ do. $\square$

From the above it becomes clear that the duality of matroids indeed gives rise to dual pairings within matroids, i.e., $(M^*)^* = M$ for every matroid M.

Proposition 3.87. *Let $M = (S, \mathcal{B})$ be a matroid, and $e \in E(M), F \subseteq E(M)$. Then*

- *e is a loop of M $\Leftrightarrow$ e is a coloop of M^* and vice versa.*
- *$(M/F)^* = M^* - F, (M - F)^* = M^*/F$.*
- *$r(M^*) = |E(M)| - r(M)$, and $r_{M^*}(X) = |X| + r(\overline{X}) - r(M), \forall X \subseteq E(M)$.*
- *If M is representable over a field $\mathbb{K}$, then also M^* is representable over $\mathbb{K}$. If $M \simeq M[A]$, where $A \in \mathbb{K}^{m \times n}$, without loss of generality, we may assume that $m \leq n$ and $A = (I_m|R), R \in \mathbb{K}^{m \times (n-m)}$ is in standard form ($m = r(M)$). Then $M[A^*]$, $A^* := (-R^\top|I_{n-m})$ is isomorphic to M^*.*

3.4 Oriented (Regular) Matroids

This subsection presents a directed notion of regular matroids, the so-called *digraphoids*, which were introduced by Minty in [Min66], in the same way that digraphs are directed notions of graphs. Since the definitions of digraphs are based on vertices, which do not admit consistent extensions to matroids, when defining digraphoids, our main goal will be to capture orientation properties arising from the interactions of primal (cycles) and dual (cuts) objects in a digraph and transfer them to a more generalized setting. The presentation of the following section is mainly based on a preprint of the book [BGH$^+$ar], Chapter 8.
In order to motivate the notion of digraphoids, we first look at graphs resp. graphic matroids and intersection properties of cycles/circuits and cuts/cocircuits in orientations of those:
Assume that $G = (V, E, \delta)$ is a graph and let $\mathcal{O}(G)$ be some arbitrary but fixed orientation on it. Let $C = (C^+, C^-)$ be an oriented cycle in G resp. $\mathcal{O}(G)$, and denote by $S = (S^+, S^-)$ a decomposition of a minimal (oriented) cut in G resp. $\mathcal{O}(G)$ into its edges in forward- and backward-direction, i.e., $S = G[X, \overline{X}]$ and $S^+ = \mathcal{O}(G)(X, \overline{X}), S^- = \mathcal{O}(G)(\overline{X}, X)$. Let $C = v_1, e_1, ..., e_n, v_{n+1} = v_1$ be some cyclical run order so that

$$C^+ = \{e_i | \delta_{\mathcal{O}}(e_i) = (v_i, v_{i+1})\}, C^- = \{e_i | \delta_{\mathcal{O}}(e_i) = (v_{i+1}, v_i)\}$$

and define for all $e \in E(G)$

$$\operatorname{sign}_C(e) := \begin{cases} 1, & \text{if } e \in C^+ \\ -1, & \text{if } e \in C^- \\ 0, & \text{else.} \end{cases}, \qquad \operatorname{sign}_S(e) = \begin{cases} 1, & \text{if } e \in S^+ \\ -1, & \text{if } e \in S^- \\ 0, & \text{else.} \end{cases}.$$

If we consider the edge-crossings of C and S, i.e. an edge $e \in C \cap S$, we can have the possible configurations $++, +-, -+, --$ of signs in C resp. S. When counting the patterns $\{++, --\}$ and $\{+-, -+\}$ (i.e., $\operatorname{sign}_C(e) = \operatorname{sign}_S(e)$ resp. $\operatorname{sign}_C(e) \neq \operatorname{sign}_S(e)$), both types have to occur the same number of times. This is easily seen as follows: First of all, assume without loss of generality that $C^- = \emptyset$: This can be done without loss of generality, since reversing the orientations of the edges in C^- will change the signs in C and S at the same time, so that the type $++, --$ resp. $+-, -+$ does not change. If now $C = C^+$, then when traversing C in the order $e_1, .., e_n$, we have to alternately cross S starting in X and ending in $\overline{X}$ resp. vice versa, and thus, within $e_1, ..., e_n$, the two types described above appear alternately, proving that indeed both patterns occur the same time.
We now use this observation regarding cycles and minimal cuts in a digraph to define a notion of orientable matroids resp. digraphoids according to Minty as follows:

Definition 3.88 (cf. [Min66])**.** Let M be a matroid. M is called *orientable as a digraphoid* if there are partitions $\vec{C} = (C^+, C^-) \in \mathcal{C}$ of each circuit and $\vec{S} = (S^+, S^-) \in \mathcal{S}$ for each cocircuit in M so that

$$|C^+ \cap S^+| + |C^- \cap S^-| = |C^+ \cap S^-| + |C^- \cap S^+|, \forall C \in \mathcal{C}, S \in \mathcal{S}.$$

The arising orientation structure on M is called *orientation* of M, often denoted by $\omega(M)$. The pair of M and $\omega(M)$ is also referred to as *digraphoid* or *oriented regular matroid*. A justification for the latter expression is given below.

As in the above when defining signs of cycles and cuts, we can equivalently identify an ordered partition $\vec{X} = (X^+, X^-)$ of an element set $X \subseteq E(M)$ with its *characteristic function*

$$\chi_{\vec{X}}(e) := \begin{cases} 1, & \text{if } e \in X^+ \\ -1, & \text{if } e \in X^- \\ 0, & \text{else.} \end{cases}.$$

The the condition for circuits and cocircuits above becomes equivalent to $\chi_{\vec{C}}^\top \chi_{\vec{S}} = 0$, for all $C \in \mathcal{C}, S \in \mathcal{S}$ and thus can be seen as some sort of orthogonality between circuits and their dual objects, cocircuits.
From the above considerations, the following becomes immediately clear:

Observation 3.89. *Let G be a graph and $\mathcal{O}(G)$ some orientation on G. Then the ordered partitions of the cycles and minimal cuts in G as described above give rise to a digraphoid on $M(G)$, i.e., each graphic matroid is orientable as a digraphoid, and each orientation of G admits a unique corresponding orientation on $M(G)$.*

As in digraphs, we have the following notion of directed circuits and cocircuits:

Definition 3.90. Let $\omega(M)$ be some orientation of a matroid M as a digraphoid. Then a signed circuit or cocircuit $X = (X^+, X^-)$ is called *directed*, if $X^- = \emptyset$ (an all-$\oplus$-(co)circuit) or $X^+ = \emptyset$ (an all-$\ominus$-(co)circuit).

The intersection condition for digraphoids above implies that circuits and cocircuits in an orientable matroid M have to intersect in an even number of elements. Furthermore, the duality between circuits and cocircuits of M and M^* and vice versa yields the following:

Remark 3.91. A matroid M is orientable as a digraphoid if and only if its dual M^* is, and in this case, they admit a common orientation, in which circuits in M as the cocircuits in M^* resp. the cocircuits in M as the circuits in M^* admit the same ordered partitions. Thus, also each bond matroid of a graph as the dual of the graphic matroid is orientable as a digraphoid. Given some orientation $\omega(M)$ of M, we denote its corresponding dual orientation of M^* by $\omega(M)^*$.

The following now provides an equivalent characterization of the orientable matroids, drawing links to representability over certain fields and so-called totally unimodular matrices.

Theorem 3.92 (cf. [BGH$^+$ar])**.** *Let M be a matroid. Then the following statements are equivalent:*

(i) M *is orientable as a digraphoid.*

(ii) M *is regular.*

(iii) M *is representable over* $\mathbb{F}_2$ *and some other field* $\mathbb{K}$ *of characteristic* $\mathrm{char}(\mathbb{K}) \neq 2$*.*

(iv) $M \simeq M[A]$*, where* $A \in \mathbb{R}^{m \times n}$ *is a totally unimodular matrix over the real numbers, i.e.,* $\det(A_{IJ}) \in \{-1, 0, 1\}, \forall I \subseteq \{1, ..., m\}, J \subseteq \{1, ..., n\}, |I| = |J|$*.*

As we know that regular matroids and thus digraphoids are closed under taking minors, the following is natural:

Remark 3.93. Let M be a regular matroid equipped with an orientation $\omega(M)$, i.e., a partitioning $C = (C^+, C^-), S = (S^+, S^-)$ of all circuits ($\mathcal{C}$) and cocircuits ($\mathcal{S}$) in M with respect to definition 3.88, and $F \subseteq E(M)$ some element set. Then the regular minors $M - F$ and M/F are orientable as digraphoids according to the following signing of circuits and cocircuits of $M - F$ resp. M/F:

$$C = (C^+, C^-), \forall C \in \mathcal{C}, C \subseteq \overline{F},$$

$$S \backslash F = (S^+ \backslash F, S^- \backslash F), \forall S \in \mathcal{S}, S \backslash F \in \mathcal{S}_{-\mathcal{F}},$$

for $M - F$, and

$$C \backslash F = (C^+ \backslash F, C^- \backslash F), \forall C \in \mathcal{C}, C \backslash F \in \mathcal{C}_{/F},$$

$$S = (S^+, S^-), \forall S \in \mathcal{S}, S \subseteq \overline{F},$$

for M/F.
It is easily seen that these orientations indeed fulfil the conditions from 3.88 and thus define digraphoids on $M-F$ and M/F, denoted by $\omega(M)-F, \omega(M)/F$. Furthermore, it can be shown that each orientation of $M-F, M/F$ can be represented as $\omega'(M)-F, \omega'(M)/F$ where $\omega'(M)$ is some orientation of M.

The following constructive characterization of regular matroids and thus digraphoids due to Seymour ([Sey80]) shows that the class of regular matroids is not much more than graphic and cographic matroids. The matroid R_{10} is a specific regular 10-element matroid which is neither graphic nor cographic, given by the following representation matrix over any field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$:

$$R_{10} = M\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}\right].$$

The following is needed to define the operations Seymour uses for his decomposition theorems:

Definition 3.94. Let $M_1 = (S_1, \mathcal{C}_1^{\mathrm{cyc}}), M_2 = (S_2, \mathcal{C}_2^{\mathrm{cyc}})$ be binary matroids (i.e., representable over $\mathbb{F}_2$) given by their sets of cycles. Then we may define a new matroid $M_1 \Delta M_2$ by $E(M) = S_1 \Delta S_2$ and the set $\mathcal{C}_M^{\mathrm{cyc}} := \{C_1 \Delta C_2 | C_1 \in \mathcal{C}_1^{\mathrm{cyc}}, C_2 \in \mathcal{C}_2^{\mathrm{cyc}}\}$ of cycles. Due to the commutativity of the symmetric difference, this indeed is a matroid again.

Definition 3.95 (cf. [Sey80])**.** Let M_1, M_2 be matroids.

(i) If $E(M_1) \cap E(M_2) = \emptyset, E(M_1), E(M_2) \neq \emptyset$, $M_1 \Delta M_2$ (or equivalently $M_1 \cup M_2$) is called *direct (1-)sum* (or equivalently matroid union of M_1 and M_2).

(ii) If $E(M_1) \cap E(M_2) = \{e\}$, where e is neither a loop nor a coloop of M_1 or M_2, and $|E(M_1)|, |E(M_2)| \geq 3$, then $M_1 \Delta M_2$ is called a *2-sum*.

(iii) If $|E(M_1)|, |E(M_2)| \geq 7; E(M_1) \cap E(M_2) = Z, |Z| = 3$, Z is a circuit of M_1 and M_2, and Z does not contain any cocircuit of M_1 or M_2, then $M_1 \Delta M_2$ is called a *3-sum*.

Theorem 3.96 (Seymour)**.** *Every regular matroid can be built up from graphic, cographic matroids and the R_{10} (or isomorphic matroids) by direct (1-)sums, 2-sums and 3-sums.*

We now turn to a generalization of the notion of flows and tensions on digraphs to digraphoids. For this purpose, we recall that given a digraph D, a tension resp. a flow on D was defined as a group-valued assignment $f: E(D) \to N$, where $(N, +)$ is an abelian group, so that for any oriented cycle $C = (C^+, C^-)$ resp. for any oriented minimal cut $S = (S^+, S^-)$ the following holds:

$$\sum_{e \in C^+} f(e) = \sum_{e \in C^-} f(e) \text{ resp. } \sum_{e \in S^+} f(e) = \sum_{e \in S^-} f(e).$$

This immediately gives rise to the following notion of flows on oriented regular matroids:

Definition 3.97. Let M be a regular matroid equipped with some orientation $\omega(M)$. An assignment $f : E(M) \to N$, where $(N, +)$ is a given abelian group with neutral element e_N, is called a *flow* or more precisely an N-*flow* on $\omega(M)$, if

$$\sum_{e \in S^+} f(e) - \sum_{e \in S^-} f(e) = e_N,$$

for all ordered partitions of cocircuits $\vec{S} = (S^+, S^-)$ in $\omega(M)$, or equivalently $\chi_{\vec{S}}^\top f = e_N$. Analogously, an assignment $f : E(M) \to N$ is called a N-*tension* on $\omega(M)$, if

$$\sum_{e \in C^+} f(e) - \sum_{e \in C^-} f(e) = e_N,$$

for all ordered partitions of circuits $\vec{C} = (C^+, C^-)$ in $\omega(M)$, or equivalently $\chi_{\vec{C}}^\top f = e_N$.

Theorem 3.98. *Let M be a matroid which is orientable as a digraphoid, and let $M = M[A], A \in \mathbb{R}^{m \times n}$ be a representation of M resp. a corresponding orientation $\omega(M)$ by a totally unimodular matrix over the reals according to Theorem 3.92.*
Then the elementary row vectors of $\ker(A)$, i.e., the vectors whose support is inclusion-minimal within this vector space, are exactly the scalar multiples of the characteristic vectors of the signed circuits of M, which moreover generate $\ker(A)$.
Furthermore, the row space of A, i.e., $\operatorname{im}(A^\top)$ contains all the characteristic vectors of signed cocircuits in M as its elementary vectors and thus is spanned by them. Consequently, a real-valued assignment $f : E(M) \to \mathbb{R}$ is a flow with respect to $\omega(M)$, iff $Af = \mathbf{0}$.

The latter follows since given a pair of vectors $u, w \in \mathbb{R}^n$ with supports $\operatorname{supp}(u) \subset \operatorname{supp}(w)$, we find a $\lambda \in \mathbb{R}$ so that in $v_\lambda := w - \lambda u$, at least one additional component of w is 0, so that $w = \lambda u + v_\lambda$ with $\operatorname{supp}(\lambda u), \operatorname{supp}(v_\lambda) \subset \operatorname{supp}(w)$, and repeating this construction yields a representation of each vector in the respective vector space as a linear combination of elementary vectors.

Remark 3.99. Given a digraphoid $\omega(M)$, an assignment $f : E(M) \to N$, where $(N, +)$ is an abelian group, is a flow resp. a tension on $\omega(M)$ iff it is a tension resp. a flow on the corresponding dual digraphoid $\omega(M)^*$.

The following is an important fact on flows in matroids only defined on part of the element set. Given an assignment $f : E(M) \to N$ ($(N, +)$ abelian group) on M, we define its *support* to be $\operatorname{supp}(f) := f^{-1}(N \backslash \{e_N\})$.

Observation 3.100. *Let $\omega(M)$ be an orientation of a regular matroid M, $(N, +)$ an abelian group. Then for every element subset $F \subseteq \omega(M)$, an assignment $f : E(M) \to N$ is a flow with $\operatorname{supp}(f) = F$ if and only if $f|_{\overline{F}} = \mathbf{0}|_{\overline{F}}$ (the all-e_N-assignment) and $f|_F$ is a flow ranging in $N \backslash \{e_N\}$ on the corresponding orientation $\omega(M)[F] = \omega(M) - \overline{F}$ of the regular minor $M[F]$.*

Proof. Denote by $\mathcal{S}$ the set of signed cocircuits in $\omega(M)$.
Given a flow f on $\omega(M)$ so that $f|_{\overline{F}} = \mathbf{0}|_{\overline{F}}, f(e) \neq e_N, \forall e \in F$, we have $\sum_{e \in S^+ \cap F} f(e) =$

$\sum_{e \in S^+} f(e) = \sum_{e \in S^-} f(e) = \sum_{e \in S^- \cap F} f(e), \forall S \in \mathcal{S}$ and thus $f|_F$ is a nowhere-zero-flow on $\omega(M)[F]$.
On the other hand, given some flow $g : F \to N$ on $\omega(M)[F], g(e) \neq e_N, \forall e \in F$, the definition

$$f(e) := \begin{cases} 0, & \text{if } e \notin F \\ g(e), & \text{if } e \in F \end{cases}$$

gives rise to an N-flow on $\omega(M)$ with $\operatorname{supp}(f) = F$. $\square$

Corollary 3.101. *Let $(N, +)$ be some abelian group and $\omega(M)$ an orientation of the matroid M. If $f : E(M) \to (N, +)$ is a flow with respect to $\omega(M)$, then for every other orientation $\omega'(M)$ of M, there is a corresponding flow $f_{\omega'(M)} : E(M) \to (N, +)$ on $\omega'(M)$ so that $\operatorname{supp}(f) = f^{-1}(N \backslash \{e_N\}) = f^{-1}_{\omega'(M)}(N \backslash \{e_N\}) = \operatorname{supp}(f_{\omega'(M)})$.*

Proof. According to 3.100, it suffices to prove the statement in the case of $\operatorname{supp}(f) = E(M)$, i.e., when f is nowhere zero. According to a theorem of Tutte (cf. [Tut54], [Tut66]), which is going to be discussed in the case of graphs in 4.3, the number of N-valued flows on a fixed orientation of a matroid such as $\omega(M)$, which range in $N \backslash \{e_N\}$ (i.e., are nowhere zero) is only dependent on $|N|$ and thus equal for different orientations $\omega'(M), \omega(M)$ on M, proving the stated equivalence. $\square$

The following notes that the given definitions of flows on digraphs and digraphoids are indeed consistent.

Remark 3.102. Let $G = (V, E; \delta)$ be a graph and $M(G)$ its corresponding oriented matroid. Then for every corresponding pair of orientations $\mathcal{O}(G)$ resp. $\omega(M(G))$ of G resp. $M(G)$, an assignment $f : E(G) = E(M(G)) \to N$, where $(N, +)$ is an abelian group, is a flow resp. a tension on $\mathcal{O}(G)$ iff it is a flow resp. a tension on $\omega(M)$.

Proof. This is an immediate consequence of the coherence of oriented cycles and minimal cuts in G resp. $\mathcal{O}(G)$ and the corresponding order pairs of circuits and cocircuits on M resp. $\omega(M)$. $\square$

Conclusively, the following is the matroid counterpart to strong connectivity in digraphs:

Definition 3.103. Let M be a regular matroid equipped with an orientation $\omega(M)$. Then M is called *totally cyclic* if there is no directed cocircuit in $\omega(M)$, i.e., $S^+, S^- \neq \emptyset$ for all signed cocircuits $S = (S^+, S^-) \in \mathcal{S}$.

4 Colourings of Graphs, Digraphs, NZ- and NL-flows

This section contains the most essential and central concepts that this master's thesis is concerned with. We start with a short introduction of the usual colouring theory and the chromatic number for general graphs and especially planar graphs and sketch the duality between colourings and Nowhere-Zero-Flows (NZ-flows for short) in this case. In the second part, we generalize these definitions and the duality for planar graphs to digraphs as done by Paul Erdős and Victor Neumann-Lara in [ENL82] resp. [NL82] introducing the dichromatic number of (di-)graphs resp. Hochstättler in [Hoc17] where Neumann-Lara-flows on digraphs (NL-flows for short) are defined. While this section focuses on the notion of colourings and NZ-flows and their most elementary properties, in section 6.9, we also take more elaborate results into account, including some positive existence results for NL-flows under additional edge connectivity assumptions from [Hoc17].
In the last part, we finally demonstrate the generality of the concept of NL-flows in digraphs by presenting a coherent notion for oriented matroids. This subsection again is guided by the results established in [Hoc17].

4.1 Graph Colourings and the Chromatic Number

In general, a colouring or labeling is usually related to a decomposition of a given object, e.g. a graph, into smaller ordered substructures. Obviously, if we want to understand the structural properties of a given graph class, it would be nice to discover such representations using as few sub configurations as possible, which then may help to deal with other open problems.
Problems dealing with graph colourings in the classical sense, especially for planar graphs, have a long history, and were considered already pretty early on in the middle of the 19th century, coming along with and inspired by the famous Four-Colour-Conjecture for plane maps. An often-used picturesque formulation of this problem is as follows: Assume we are given a geographical map with different countries and borders, where each country is connected, i.e., it does not admit exclaves. Our goal is to colour the map with a palette of four different colours such that each area of a country receives a single fixed colour, in such a way, that we can reconstruct the map when dropping all the borders, i.e., if we are only given the coloured areas of the countries. In other words, this means that at each border, the meeting countries admit distinct colours. The simple question the Four-Colour-Problem raises is the following: Is such a colouring always possible, given *any* geographical map with the described additional properties?
Looking at small examples, such colourings always seem to exist, and thus, the problem has been known and studied for a long time, appearing as the famous *Four-Colour-Conjecture* (FCC). Despite its simple formulation, which makes it explainable even to non-experts, the conjecture remained open for many decades, passing generations of mathematicians. It was finally resolved in 1976, when Appel and Haken claimed a proof of the conjecture (cf. [AH76]). It is nowadays widely believed to be true, although some mathematicians rejected the proof, since it makes extensive use of computer calculations (i.e., it verifies a set of about 2000 so-called unavoidable configurations). This has given rise to several

philosophical debates concerning the foundations of mathematics in the past. Although the number of configurations to be checked by computer meanwhile has been reduced to the impressive number of 633 (cf. [RSST96]), a simple proof of the conjecture without using massive computer calculations still is to be found.
Given a planar geographical map as explained above, we can easily embed this into the following graph-theoretic context: Define a simple graph $G = (V, E)$ in such a way that the vertices in V correspond to the different countries of the map and a pair uv of distinct vertices $u \neq v \in V$ is an edge in E if and only if the corresponding countries admit a common border in the map. By placing points representing vertices inside the area of the corresponding countries on the map and connecting them by non-intersecting edges passing through the respective common border whenever they are adjacent, it is easily seen that G admits a plane embedding and thus is a loopless planar graph. A colouring of the countries as required above then corresponds to a colouring of the elements of V using at most four different colours in such a way that the end vertices of each edge in G admit distinct assigned colours. This is illustrated by the following figure.

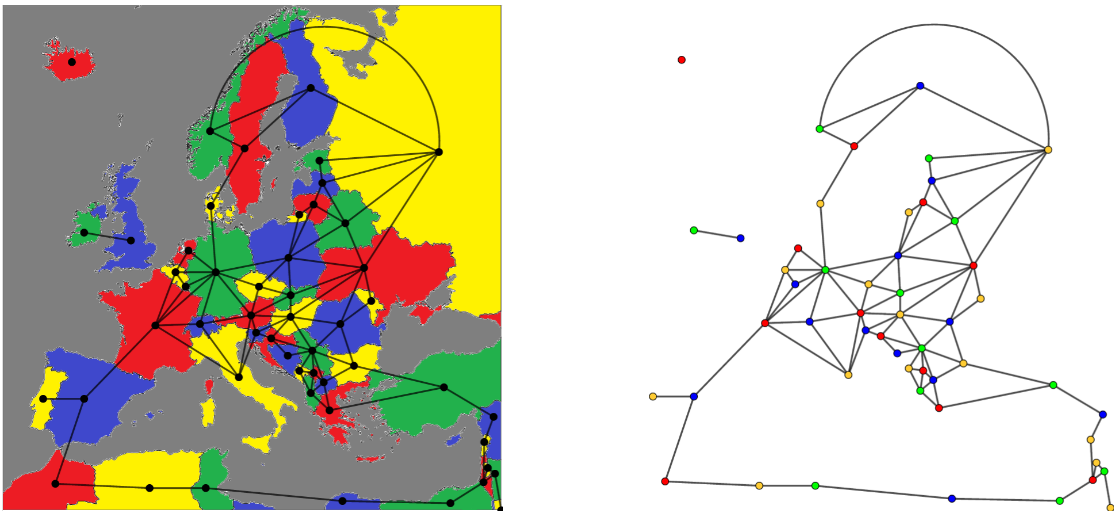


Figure 6: A geographical map of Europe, a legal 4-colouring of it and the corresponding simple 4-coloured planar graph.

This coherence finally gives rise to the following (widely known) definition of graph colourings in the classical sense:

Definition 4.1. Let G be a graph and M some finite set. A legal M-*colouring* of the graph G is defined to be an assignment $c : V(G) \to M$, such that $c(u) \neq c(v)$ for each pair of vertices $u, v \in V(G)$ that are adjacent in G.
If $k \geq 1$ is some natural number, in this master's thesis, we will refer to a legal graph colouring $c : V(G) \to \{0, ..., k-1\}$ as a k-*colouring*.
Obviously, due to isomorphisms, G admits a legal M-colouring if and only if it admits a $|M|$-colouring.

The least natural number $k \in \mathbb{N}$, for which G admits a legal k-colouring (this is defined to be ∞ if there is none) is referred to as the famous *Chromatic Number* of G, denoted by $\chi(G)$.

Remark 4.2. If G is a graph admitting loops, per definition, it does not admit a legal colouring. On the other hand, since graph colourings are defined with respect to the adjacency structure of the vertices, multiple edges between vertices do not change the colouring properties of a graph. Thus, when analysing graph colourings, one may focus on simple graphs.

The Four-Colour-Theorem thus can be expressed as $\chi(G) \leq 4$ for all simple planar graphs G. The simplest graphs, the complete graphs $K_n, n \in \mathbb{N}$, obviously admit chromatic number $\chi(K_n) = n$, since each pair of distinct vertices is adjacent and thus, the assigned colours in a legal graph colouring have to be pairwise distinct. Since the chromatic numbers of subgraphs of a given graph yield lower bounds on its own chromatic number, this already gives the following lower bound for the chromatic number, using the so-called *clique number* of a graph.

Observation 4.3. *Let G be a simple graph and H some subgraph of G. Then $\chi(H) \leq \chi(G)$. Define the* clique number *of G by*

$$\omega(G) := \max\{|V| | V \textit{ is a clique in } G\}.$$

Then $\omega(G) \leq \chi(G)$.

Proof. If H is a subgraph of G, each legal graph colouring of G induces a legal graph colouring of H, proving the inequality. If $V \subseteq V(G)$ is a maximum-size clique in G, then $G[V]$ is a complete subgraph of G, and so $\omega(G) = |V| = \chi(G[V]) \leq \chi(G)$. □

Apart from that, if we are given some simple graph and want to find a legal colouring with preferably few colours in a fast way, one would most likely use the following "greedy" strategy:
Take some ordering $\{v_1, ..., v_n\} = V(G)$ of the vertices of the actual graph G. Start by setting $c(v_1) := 0$. For each $i \in \{2, ..., n\}$, inductively chose its colour as the least possible nonnegative integer which is not yet being used by one of its neighbours, i.e. formally, $c(v_i) := \min(\mathbb{N}_0 \backslash \{c(v_j) | j = 1, ..., i-1 : v_j \in N(v_i)\}) \leq |N(v_i)| \leq \Delta(G)$. This obviously is a legal graph colouring which uses, due to the latter inequality, at most $\Delta(G) + 1$ different colours.

Observation 4.4. *Let G be a simple graph. Then $\chi(G) \leq \Delta(G) + 1$.*

It shall be mentioned (we omit a proof) that in many cases, the upper bound of $\Delta + 1$ can be improved by 1 to Δ. This is specified by the following widely known Theorem due to Brooks:

Theorem 4.5 (cf. [Bro41])**.** *Let G be a simple graph which is not a complete graph and no cycle of odd length. Then $\chi(G) \leq \Delta(G)$.*

There are simple constructions showing that all of the bounds on the chromatic number above can be arbitrarily bad. First of all, bipartite graphs, which are exactly the graphs with chromatic number at most 2, can have arbitrarily high maximum degree and thus, in the latter inequality, the left side may stay bounded while the right hand side tends to infinity. For the lower clique-bound, there are many different constructions, the most prominent probably being the recursive definition of the so-called Mycielski-graphs due to Jan Mycielski:

Definition 4.6. Let $G = (V, E)$ be a simple graph. The *Mycielski-Graph* of G, denoted by $\mu(G)$, is defined as follows: Take some ordering $V = \{v_1, ..., v_n\}$ of the vertices in G and define a "copy set" $W = \{w_1, ..., w_n\}$ containing distinct but corresponding elements. Let furthermore $u \notin V \cup W$ be some additional vertex. $\mu(G)$ now has $V \cup W \cup \{u\}$ as vertex set. The adjacencies in $\mu(G)$ are defined as follows:

- Two vertices $v_1 \neq v_2 \in V$ are adjacent if and only if they are adjacent in G.
- A vertex $w_i \in W$ admits the original neighbours of v_i in G and u as neighbours, i.e., $N_{\mu(G)}(w_i) = N_G(v_i) \cup \{u\}, i = 1, ..., n$.
- $N_{\mu(G)}(u) = W$.

The following now contains the most important structural properties provided by the Mycielski-Construction:

Theorem 4.7. *Let $G = (V, E)$ be a simple graph and $\mu(G)$ its Mycielski-graph. Then $\chi(\mu(G)) = \chi(G) + 1$, and $\mu(G)$ is triangle-free whenever G is triangle-free.*

Proof. Let $V = \{v_1, ..., v_n\}$, $W = \{w_1, ..., w_n\}$, and u be as in definition 4.6. Let $k := \chi(G)$. Then there is a legal k-colouring $c_G : V \to \{0, ..., k-1\}$ of G. Define a $(k+1)$-colouring of $\mu(G)$ by $c_{\mu(G)}(v_i) := c_{\mu(G)}(w_i) := c_G(v_i), i = 1, ..., n$, $c_{\mu(G)}(u) := k$. It follows immediately from the definition that this indeed is a legal graph colouring of $\mu(G)$, proving $\chi(\mu(G)) \leq k + 1 = \chi(G) + 1$. Therefore, for $\chi(\mu(G)) = \chi(G) + 1$, it suffices to prove that $\mu(G)$ does not admit a legal k-colouring. Assume for a proof by contradiction it did, and let $c' : V \cup W \cup \{u\} \to \{0, ..., k-1\}$ be a corresponding colouring. Without loss of generality, we may assume that $c'(u) = k - 1$. Thus, $c'(w_i) \in \{0, ..., k-2\}, i = 1, ..., n$. Define a $(k-1)$-colouring $c : V \to \{0, ..., k-2\}$ by

$$c(v_i) := \begin{cases} c'(v_i), & \text{if } c'(v_i) < k-1 \\ c'(w_i), & \text{if } c'(v_i) = k-1 \end{cases}, i = 1, ..., n.$$

This has to be legal: Let $v_i \neq v_j \in V$ be two arbitrary adjacent vertices. Then either, $c'(v_i), c'(v_j) < k - 1$, in which case we immediately have $c(v_i) = c'(v_i) \neq c'(v_j) = c(v_j)$, or exactly one of the vertices (without loss of generality v_i) is coloured $c'(v_i) = k-1$. But then, since w_i and v_j are adjacent in $\mu(G)$, we again conclude $c(v_i) = c'(w_i) \neq c'(v_j) = c(v_j)$. Therefore, c is indeed a legal $(k-1)$-colouring of G, contradicting $\chi(G) = k > k - 1$. This contradiction shows that $\chi(\mu(G)) = \chi(G) + 1$ as claimed.
For the second part of the assertion, assume that G is triangle-free. We claim that also $\mu(G)$

is triangle-free. Assume for contrary there was a triangle in $\mu(G)$. Since the only adjacencies in $W \cup \{u\}$ are those between u and $w_i, i = 1, ..., n$ and since u does not admit a neighbour in V, such a triangle has to contain at least two vertices in V. Since $G = \mu(G)[V]$ is triangle free, it furthermore has to contain exactly one vertex w_i out of W and two adjacent vertices $v_k \neq v_l \in V$. According to the definition of $\mu(G)$, this means that v_i, v_k, v_l are pairwise adjacent vertices in V, i.e., a triangle, which again gives a contradiction to the initial assumption. This proves the claim. $\square$

Consequently, when starting with some triangle-free graph G_0, the repeated application of the Mycielski-construction according to $G_i := \mu(G_{i-1}), \forall i \in \mathbb{N}$, yields a sequence $(G_i)_{i \geq 0}$ of triangle-free simple graphs (i.e., $\omega(G_i) \leq 2, i = 1, 2, 3, ...$), and $\chi(G_i) = i + \chi(G_0) \to \infty, i \to \infty$.
In order to finalise this little excursion on bounds for the chromatic number, we present an improved upper bound for the chromatic number of a graph in terms of the so-called *Colouring Number* of a graph according to Szekeres and Wilf, which will reappear later on in paragraphs 5.1 and 8.2:

Definition 4.8. Let G be a simple graph. The number

$$d(G) := \max_{H \subseteq G} \delta(H),$$

i.e., the maximal minimum degree of the subgraphs of H (equivalently, we could also take the maximum over all induced subgraphs, as is easily seen), is called the *degeneracy* of G. Furthermore, $\mathrm{col}(G) := d(G) + 1$ is known as the *colouring number of G* according to Szekeres and Wilf (cf. [SW68]). For a given number $k \in \mathbb{N}$, G is called *k-degenerate* if and only if $k \geq d(G)$, i.e., if every subgraph of G contains a vertex of degree at most k.

Corollary 4.9. *If G is a forest, $d(G) \leq 1$. If G is a simple planar graph, then $d(G) \leq 5$.*

Proof. All subgraphs of a forest are forests, and thus, it suffices to show that each forest has a vertex of degree at most 1. But this follows immediately since the non-trivial connected components of forests are trees admitting leafs.
Again, all subgraphs of a planar graph are planar and it suffices to show that every planar graph G admits a vertex of degree at most 5. Assume for contrary that all vertices in G admitted degree at least 6. According to the Handshake-Lemma, this implies $|E(G)| \geq 3|V(G)|$. But according to Theorem 3.30 from the Definitions and Preliminaries, $|E(G)| \leq 3|V(G)| - 6$, contradiction. $\square$

Using the following improved inequality, we derive a first upper bound on the chromatic number of planar graphs, namely 6.

Theorem 4.10. *Let G be a simple graph. Then $\chi(G) \leq d(G) + 1 = \mathrm{col}(G)$.*

Proof. We prove the theorem by strong induction over $n := |V(G)|$. If this is 1, then $\chi(G) = 1$, and the claim holds obviously true. So assume for the inductive step that $n \geq 2$ and that the stated inequality holds true for all simple graphs on at most $n - 1$ vertices. Let

$d(G) := k \in \mathbb{N}_0$. According to the definition of $d(G)$, for all subgraphs H of G, we have $\delta(H) \leq k$, and therefore especially $\delta(G) \leq k$, implying the existence of a vertex $v \in V(G)$ of degree at most k. We consider the simple graph $G - v$ on $n-1$ vertices. Since $G-v$ is a subgraph of G, all the subgraphs H of $G-v$ are also subgraphs of G and thus fulfill $\delta(H) \leq k$, i.e., $d(G-v) \leq k$. According to the inductive assumption, we thus have $\chi(G-v) \leq k+1$, i.e., there is a legal graph colouring $c' : V(G)\backslash\{v\} \to \{0, ..., k\}$ of $G-v$. Since $|N(v)| \leq k$, according to the pigeon-hole principle, at least one of the $k+1$ numbers in $\{0, ..., k\}$ does not appear in $N(v)$, say $i \in \{0, ..., k\}$. By setting $c(u) := c'(u), u \in V(G)\backslash\{v\}, c(v) := i$, we thus end up with a legal $(k+1)$-graph-colouring c of G, and the principle of induction now yields the desired claim. □

The proof of the above result encodes an improved version of the greedy-colouring algorithm, by choosing a special vertex ordering when applying it: Let $V = V(G)$ be the vertex set of a given k-degenerate simple graph G. Define an ordering $V = \{v_1, ..., v_n\}$ inductively as follows:
Choose v_n as a vertex of minimal degree (and thus at most k) in $G_n := G$ and define $G_{n-1} := G_n - v_n$. More generally, for each $i \in \{1, ..., n\}$, given the partial ordering $v_{i+1}, v_{i+2}, ..., v_n$ and the induced subgraphs $G_i, G_{i+1}, ..., G_n = G$, choose v_i as a vertex of minimum degree in G_i (and thus at most k) and define $G_{i-1} := G_i - v_i$ whenever $i > 1$.
Now, it is easily seen that applying the greedy colouring-algorithm described above to G with this special vertex ordering gives rise to a $(k+1)$-colouring of G.
Since the Four-Colour-Theorem will reappear at various points of this master's thesis, we conclude our considerations for colourings of planar graphs with a very elegant argument due to Kempe resp. Heawood (cf. [Hea90]), improving the upper bound on the chromatic number of simple planar graphs to 5:

Theorem 4.11. *Let G be a simple planar graph. Then $\chi(G) \leq 5$.*

Proof. As above, we prove the statement using induction on the number of vertices $n := |V(G)|$ of G: If this is 1, the claim is obvious, so assume $n \geq 2$ and that the claim is true for all simple planar graphs on $n-1$ vertices. As proven above, G admits a vertex v of degree at most 5. According the inductive assumption, $G - v$ admits a 5-colouring $c' : V(G)\backslash\{v\} \to \{0, 1, 2, 3, 4\}$. If there are at most 4 distinct colours appearing in $N(v)$, we can conclude that there is a colour $c_v \in \{0, 1, 2, 3, 4\}$ not appearing in $N(v)$, and $c(u) := c'(u), \forall u \in V(G)\backslash\{v\}, c(v) := c_v$, will give rise to a legal 5-colouring of G as claimed. Else, we may assume that the cyclically ordered neighbours $N(v) = \{v_0, v_1, v_2, v_3, v_4\}$ in a plane embedding of G are labeled $c'(v_i) = i, i = 0, ..., 4$. Denote by $H_{0,2}$ the bipartite induced subgraph of $G - v$ containing all the vertices of colour 0 or 2. If $v_0, v_2 \in V(H_{0,2})$ are contained in different connected components of $H_{0,2}$, we can change c' to a (still legal) 5-colouring c'' of $G - v$ by exchanging the colours 0 and 2 in the connected component of $H_{0,2}$ containing v_0. c'' now has at most 4 distinct colours contained in $N(v)$ in this case, and the claim follows as above. So assume for the next case that v_0 and v_2 are connected by an alternating $0,2$-path P in $H_{0,2}$ resp. G. The cycle $P + v$ in G now separates the vertices v_1 and v_3 and thus, there is no alternating $1,3$-path in $G - v$ connecting them. By renaming the colours, we now can apply the same argument as above and therefore prove the induction hypothesis also in this remaining case. All in all, the principle of induction now yields the claim, G is 5-colourable. $\square$

In general, deciding 3-colourability of a given graph is an NP-complete problem, and for many graphs, finding good estimates for their chromatic number won't be an easy task. Still, the class of *line graphs* as defined in the preliminaries with its corresponding edge-colourings of graphs admits much more precise bounds on chromatic numbers:

Definition 4.12. Let $G = (V, E, \delta)$ be a loopless graph and $L(G)$ its loopless line graph. Then for any finite set M, a legal M-edge-colouring of G is defined as a legal M-vertex-colouring $c : V(L(G)) = E(G) \to M$ of $L(G)$, i.e., so that adjacent edges admit distinct colours. In the case $M = \{0, ..., k-1\}$, an M-edge-colouring c is also referred to as a k-edge-colouring. The minimal $k \in \mathbb{N}_0$ for which G admits a k-edge-colouring, i.e., $\chi(L(G))$, is called the *chromatic index* of G and denoted by $\chi'(G)$.

Since for each vertex v of a loopless graph G, the $\deg_G(v) = |E_G(v)|$ incident edges give rise to a clique in $L(G)$, we have $\chi'(G) = \chi(L(G)) \geq \omega(L(G)) \geq \Delta(G)$. Moreover, we have $\Delta(L(G)) \leq 2\Delta(G) - 2$ and thus, $\chi'(G) \leq 2\Delta(G) - 1$ according to the above estimates. The following result due to Vizing (cf. [Viz64]) shows that in fact, we can do much better and restrict the chromatic index of a simple graph to the two possibilities $\Delta(G), \Delta(G) + 1$:

Theorem 4.13 (Vizing)**.** *Let G be a loopless graph. If G is simple, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. More generally, if $\gamma(G)$ denotes the maximal number of parallel edges between a pair of vertices, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + \gamma(G)$.*

A proof of this theorem was first discovered in [Viz64] and is nowadays part of any standard graph-theory lecture.

Theorem 4.14. *Let $G = (V_1 \dot\cup V_2, E, \delta)$ be a k-regular bipartite graph. Then G admits a k-edge-colouring, or equivalently, its edge set E decomposes to k edge-disjoint perfect matchings.*

Proof. If $k = 0$, the empty mapping defines a 0-colouring of G.
If $k \geq 1$, it follows immediately from the regularity condition that $|V_1| = |V_2|$ and $|N_G(X)| \geq |X|, \forall X \subseteq V_1$. So, according to Hall's Marriage Theorem 3.26, G admits a perfect matching M. Then $G - M$ is a $(k-1)$-regular bipartite graph and by colouring the edges of M with colour $k-1$, every $(k-1)$-edge-colouring of $G - M$ gives rise to a k-colouring of G. The claim now follows using an inductive argument over k. $\square$

4.2 Nowhere-Zero-Flows

This subsection introduces flows in the context of colourings. At first sight, the relation between those topics is not so clear, since flows primarily are used as a way of modelling dynamical processes such as transport systems or electrical networks.
In order to explain this correspondence, it is important to first discuss the notion of coflows. As defined in the preliminaries, a tension on an oriented graph $\vec{G}$ is defined as an assignment

$f : E(\vec{G}) \to \mathbb{R}$ (or analogously ranging in some other abelian group with additive structure) such that for every oriented cycle $C = (C^+, C^-)$ in $\vec{G}$, $\sum_{e \in C^+} f(e) = \sum_{e \in C^-} f(e)$. In contrast to the appearance of flows as a dynamical concept, coflows can be used to describe static *potentials* on the vertex set of the actual graph, just as the tension in an electrical network measures the difference of given potentials. And this again yields a context in which graph colourings are treatable: Consider a legal k-colouring $c : V(G) \to \{0, ..., k-1\}$ of the underlying graph $G := U(\vec{G})$. Then the values of c, treated as real numbers, can be considered potentials on the vertices of $\vec{G}$, giving rise to a well-defined tension f_c on $\vec{G}$ according to $f_c(e) := c(w) - c(u)$, for each directed edge $e = (u, w)$ in $\vec{G}$. The requirement that c is a legal graph colouring now becomes equivalent to $f_c(e) \neq 0, \forall e \in E(\vec{G})$, and moreover f_c only takes on integers within $\{\pm 1, \pm 2, ..., \pm(k-1)\}$. This already explains the term "Nowhere-Zero" in the heading. We are still speaking of tensions or so-called *Nowhere-Zero-coflows* on the orientation $\vec{G}$ of G, so what is the relation of colourings to flows?
For the purpose of explaining this coherence, consider a connected simple planar graph G together with its well-defined planar dual graph G^*. Given some k-colouring c of G, a fixed orientation $\vec{G}$ on G and the corresponding tension f_c as defined above, we can consider $f_c : E(\vec{G}) \to \{\pm 1, ..., \pm(k-1)\}$ also as an assignment on the edges of the directed dual $\vec{G}^*$ of the orientation. f_c, which was a tension on $E(\vec{G})$, i.e., $\sum_{e \in C^+} f_c(e) = \sum_{e \in C^-} f_c(e)$, for all oriented cycles $C = (C^+, C^-)$ in $\vec{G}$, now turns into a flow on $\vec{G}^*$, since the oriented cycles $C = (C^+, C^-)$ now are exactly the oriented minimal cuts in $\vec{G}^*$, i.e., f_c considered on $\vec{G}^*$ according to the above equalities fulfils the Kirchhoff Conservation Law, i.e., is an integer flow, which ranges in $\{\pm 1, ..., \pm(k-1)\}$. Furthermore, given some plane embedding of $\vec{G}^*$, the colouring of G corresponds to a face-colouring of G^*. Given that, we can derive the flow values on the arcs by subtracting e.g. the assigned value of the colour on the left bounding face from the respective colour-value on the right bounding face. This is illustrated in the following figure:

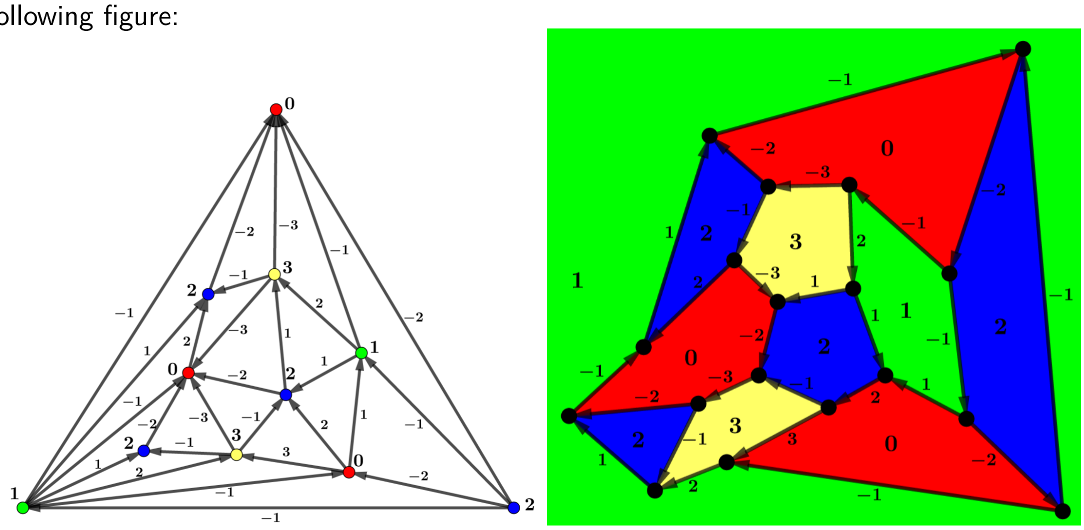


Figure 7: Left: 4-Colouring of an oriented planar triangulation with corresponding NZ-k-coflow as described above. Right: Dualized flow on the 3-regular 3-edge-connected directed planar dual with corresponding face-colouring.

These considerations finally give rise to the following definition:

Definition 4.15. Let G be a graph with some orientation $\vec{G}$ on it. A *Nowhere-Zero-k-coflow* (NZ-coflow for short) on $\vec{G}$ is an integer-valued tension $f : E(\vec{G}) \to \{\pm 1, ..., \pm(k-1)\}$. Analogously, a *Nowhere-Zero-k-flow* on $\vec{G}$ is defined as an integer-flow $f : E(\vec{G}) \to \{\pm 1, ..., \pm(k-1)\}$ on $\vec{G}$.

Above, we sketched how to derive Nowhere-Zero-coflows from graph colourings, using colours out of $\{0, 1, ..., k-1\}$ with $k \in \mathbb{N}$. Obviously, as already provided by the definition of graph colourings, we are not restricted to those colours, instead, we can equivalently use any kind of elements out of some set M of size k. If this set additionally admits some kind of additive structure, i.e., $M = (N, +)$ is an abelian group with neutral element e_N (the inverse of some element $a \in N$ in the following is denoted by $-a$), the same assignment as sketched above works and will, given some graph colouring $c_V(G) \to N$ and an orientation $\vec{G}$ of G, give rise to a tension $f_c : E(\vec{G}) \to (N, +)$, such that $f(e) \neq e_N, \forall e \in N$. Thus, the following definition is justified:

Definition 4.16. Let G be a graph equipped with some fixed orientation $\vec{G}$. Let furthermore $(N, +)$ be a finite abelian group with neutral element e_N. A tension $f : E(\vec{G}) \to N \backslash \{e_N\}$ is defined to be a so-called *Nowhere-Zero-N-coflow* on $\vec{G}$ (NZ-N-coflow for short), while a flow $f : E(\vec{G}) \to N \backslash \{e_N\}$ is called a *Nowhere-Zero-N-flow*.

It is important to notice that in contrast to NL-flows which will be defined later on, the notions of Nowhere-Zero-k and -N-flows are not restricted to the fixed special orientation they are defined on, but are much more objects on the underlying actual graph itself, as explained in more detail by the following

Observation 4.17. *Let G be a graph equipped with some fixed orientations $\mathcal{O}(G), \mathcal{O}'(G)$ and let $(N, +)$ an abelian group with neutral element e_N, $k \in \mathbb{N}$.*
Given a flow $f : E(\mathcal{O}(G)) \to N$ resp. $f : E(\mathcal{O}(G)) \to \{-(k-1), ..., 0, ..., k-1\}$, f is a NZ-N-flow resp. a NZ-k-flow on $\mathcal{O}(G)$ iff the corresponding flow $f_{\mathcal{O}'(G)} : E(\mathcal{O}'(G)) \to N$ resp. $f_{\mathcal{O}'(G)} : E(\mathcal{O}'(G)) \to \{-(k-1), ..., 0, ..., k-1\}$,

$$f_{\mathcal{O}'(G)}(e) := \begin{cases} f(e), & \textit{if } e \textit{ is oriented the same in } \mathcal{O}(G) \textit{ and } \mathcal{O}'(G) \\ -f(e), & \textit{if } e \textit{ is oriented differently in } \mathcal{O}(G) \textit{ and } \mathcal{O}'(G) \end{cases}$$

on $\mathcal{O}'(G)$ is a NZ-N-flow resp. a NZ-k-flow on $\mathcal{O}'(G)$ for any orientation of G.
Thus, we will often simply refer to NZ-flows on some (and thus any) orientation of G as NZ-flows on G.

Analogously to the definition of the chromatic number of a graph in the primal case, we may define the so-called flow index of a given graph in terms of NZ-flows as follows:

Definition 4.18. Let G be a graph. The least natural number $k \geq 1$ for which some (and thus any) orientation of G admits a NZ-k-flow is defined as the *flow index* of G, denoted by $\xi(G)$.

The described relationships between colourings of graphs and NZ-coflows are not only one-sided, moreover, given any NZ-N-coflow f on a graph G (where $(N,+)$ again is a finite abelian group), we can construct a legal N-colouring of G as follows:
Without loss of generality, assume that G is simply connected (if there were different connected components of G, we could apply the following argument for each component and stick the arising colourings together, eventually ending up with a legal N-colouring of G).
Denote by $\vec{G}$ the orientation on G which f is defined on. Starting with some fixed reference vertex v_0 which is going to receive colour $e_N \in N$, for each vertex $v \in V(G)\backslash\{v_0\}$, we define its colour $c(v)$ according to

$$c(v) = \sum_{e \in P^+} f(e) - \sum_{e \in P^-} f(e) \in N,$$

where P is an arbitrarily chosen path in G starting in v_0 and ending in v. The sets P^+ resp. P^- hereby contain the edges oriented forwards resp. backwards in $\vec{G}$ when traversing P starting from v_0.
First of all, this assignment is well-defined. In order to prove this, we show that for each given pair $u \neq w \in V(G)$ of distinct vertices, the value $\sum_{e \in P^+} f(e) - \sum_{e \in P^-} f(e)$ used above does not depend on the choice of the path P: Assume for a proof by contradiction this claim was false. Then there is a pair $u \neq w$ of vertices and of two different (oriented) paths $P_1 = (P_1^+, P_1^-), P_2 = (P_2^+, P_2^-)$ in G starting in u and ending in w, for which

$$\sum_{e \in P_1^+} f(e) - \sum_{e \in P_1^-} f(e) \neq \sum_{e \in P_2^+} f(e) - \sum_{e \in P_2^-} f(e).$$

Given these assumptions, we may assume that u, w, P_1 and P_2 are chosen minimal with respect to $\ell(P_1)$. We claim that now, P_1 and P_2 have to be internally vertex-disjoint: If there was a vertex $x \in (V(P_1) \cap V(P_2))\backslash\{u,w\}$, let $P_1[u,x], P_1[x,w]$, $P_2[u,x], P_2[x,w]$ denote the partial paths ending resp. starting in x. Obviously,

$$P_i^+ = P_i[u,x]^+ \dot\cup P_i[x,w]^+, P_i^- = P_i[u,x]^- \dot\cup P_i[x,w]^-, i = 1,2.$$

Thus, we have for each $i \in \{1,2\}$

$$\sum_{e \in P_i^+} f(e) - \sum_{e \in P_i^-} f(e) = \left(\sum_{e \in P_i[u,x]^+} f(e) - \sum_{e \in P_i[u,x]^-} f(e)\right) + \left(\sum_{e \in P_i[x,w]^+} f(e) - \sum_{e \in P_i[x,w]^-} f(e)\right),$$

and the above inequality implies that either $u, x, P_1[u,x], P_2[u,x]$ or $x, w, P_1[x,w], P_2[x,w]$ is a counterexample to the above assertion with $\ell(P_1[u,x]), \ell(P_1[x,w]) < \ell(P_1)$, contradicting the assumed minimality. Thus, indeed $V(P_1) \cap V(P_2) = \{u,w\}$.
Let C with $E(C) := E(P_1) \cup E(P_2)$ be the unique cycle in G made up by P_1 and P_2. Choosing an appropriate orientation, we have $C^+ = P_1^+ \dot\cup P_2^-, C^- = P_1^- \dot\cup P_2^+$, giving

$$\begin{aligned} e_N &\neq \left(\sum_{e \in P_1^+} f(e) - \sum_{e \in P_1^-} f(e)\right) - \left(\sum_{e \in P_2^+} f(e) - \sum_{e \in P_2^-} f(e)\right) \\ &= \sum_{e \in C^+} f(e) - \sum_{e \in C^-} f(e), \end{aligned}$$

a contradiction, since f is a coflow and thus a tension on $\vec{G}$. All in all, the above assumption is false and hence, $c : V(G) \to N$ indeed is a well-defined vertex-colouring. Furthermore, given any directed edge $e = (x_1, x_2)$ in $\vec{G}$ and some path $P = (P^+, P^-)$ in G starting in v_0 and ending in x_1, then $c(x_2) - c(x_1)$ equals (by using the paths P, $P\Delta\{e\}$ in the definition of $c(x_1), c(x_2)$) $f(e) \neq e_N$. Thus, $c(x_1) \neq c(x_2)$ for each edge e in G and hence, c indeed is a legal N-colouring of G constructed from the given NZ-coflow f:

Proposition 4.19. *Let G be a graph and $(N, +)$ a finite abelian group, $k := |N|$. Then G admits a legal k- resp. N-colouring if and only if G admits a NZ-N-coflow or equivalently a NZ-k-coflow.*

Proof. The equivalence of k-colourings and N-colourings of G and of NZ-N-coflows follows directly from the foregoing argument. In addition, we have already explained how to deduce NZ-k-coflows from a given k-colouring. It only remains to show that each NZ-k-coflow yields a k-colouring of G. But this follows immediately from the fact that any NZ-k-coflow f, taken modulo k and considered as a NZ-$\mathbb{Z}_k$-coflow, gives rise to a legal $\mathbb{Z}_k \cong \{0, ..., k-1\}$-colouring of G. □

Consider now again a planar connected graph G with the planar dual graph G^*. Then the duality of NZ-flows on G and NZ-coflows on G^*, together with the results above yields the following, which is a special case of a much more general theorem, known as *Equivalence Theorem* (Tutte). It is proven in the next subsection when dealing with evaluations of the Tutte polynomial.

Theorem 4.20 (Equivalence Theorem for Planar Graphs)**.** *Let G be a planar graph, $k \in \mathbb{N}$ and $(N, +)$ an abelian group of order k. Then the following are equivalent:*

(i) G admits a NZ-k-flow.

(ii) G admits a NZ-$\mathbb{Z}_k$-flow

(iii) G admits a NZ-N-flow.

Proof. G admits a NZ-k- resp. a NZ-$\mathbb{Z}_k$- resp. a NZ-N-flow if and only if G^* admits the respective NZ-coflows. The equivalence of these dualized statements is the content of proposition 4.19. □

Hence we can derive the following connection between the chromatic numbers and flow indices of planar graphs:

Remark 4.21. Let G be a connected planar graph and G^* a planar dual graph. Then $\chi(G) = \xi(G^*)$.

Proof. $\chi(G)$ is the least $k \geq 1$ for which G admits a legal k-colouring, i.e. equivalently a $\mathbb{Z}_k$-colouring, which according to proposition 4.19 means that G admits a NZ-$\mathbb{Z}_k$-coflow. But since the tensions on some orientation of G (considered as assignments on the edges) are exactly the flows on the respective dual orientation of G^* (see proposition 3.53), this is equivalent to G^* admitting a NZ-$\mathbb{Z}_k$- or equivalently (cf. Theorem 4.20) NZ-k-flow, i.e., this minimal natural number is indeed exactly $\xi(G^*)$. □

We conclude this subsection with a generalization of the above notions of colourings and NZ-flows to the context of (orientations of) regular matroids introduced in the Preliminaries, which will include graphs in the form of their graphic matroids as a special case. First of all, as we have seen in subsection 3.4, tensions and flows can also be defined on oriented regular matroids $\omega(M)$ by using analogues to the Kirchhoff-Laws for flows on graphs by replacing oriented cycles and cuts by signed circuits and cocircuits. Such a generalization of colourings to matroids would not have been possible using the original definition of graph colourings, since vertices in graphs do not admit appropriate generalizations to regular matroids. Again, we can study these flows in the wider context of arbitrary abelian groups as underlying algebraic structures:

Definition 4.22. Let M be a regular matroid and $\omega(M)$ some orientation of it. Let $k \geq 1$ be a natural number and $(N, +)$ some finite abelian group with neutral element e_N. A *Nowhere-Zero-k-coflow* on $\omega(M)$ is a tension on $\omega(M)$ ranging in $\{\pm 1, ..., \pm(k-1)\}$. Analogously, a *Nowhere-Zero-k-flow* is defined as an integer flow on $\omega(M)$ ranging in $\{\pm 1, ..., \pm(k-1)\}$.
In the same way, *(algebraic) Nowhere-Zero-N-coflows* resp. *(algebraic) Nowhere-Zero-N-flows* are the tensions resp. flows on $\omega(M)$ with respect to $(N, +)$ ranging in $N \backslash \{e_N\}$.

As in the case of graphs, we have the following (obvious) relation between NZ-flows on different orientations of the same regular matroid:

Observation 4.23. *Let M be a regular matroid equipped with some fixed orientations $\omega(M), \omega'(M)$ and let $(N, +)$ an abelian group with neutral element e_N, $k \in \mathbb{N}$.*
Given a flow $f : E(\omega(M)) \to N$ resp. $f : E(\omega(M)) \to \{-(k-1), ..., 0, ..., k-1\}$, f is a NZ-N-flow resp. a NZ-k-flow on $\omega(M)$ if and only if the corresponding flow $f_{\omega'(M)} : E(\omega'(M)) \to N$ defined according to Corollary 3.101 (i.e., it admits the same support as f) from the Preliminaries on $\omega'(M)$ is a NZ-N-flow resp. a NZ-k-flow on $\omega'(M)$ for any orientation of G.
Thus, we will often simply refer to NZ-k-flows on some (and thus any) orientation of M as NZ-flows on G.

As in the case of graphs, we may now define the flow index of a regular matroid M as the least $k \in \mathbb{N}$ for which it admits a NZ-flow:

Definition 4.24. Let M be a regular matroid. The quantity

$$\xi(M) := \min\{k \in \mathbb{N} | M \text{ admits a NZ-}k\text{-flow}\}$$

is called the *Flow Index* of M.

Since flows and tensions on an orientation of a graph G are exactly the corresponding flows and tensions of the corresponding orientation of the graphic matroid $M(G)$, the following is an immediate consequence of the definitions of NZ-(co)flows on graphs and matroids:

Remark 4.25. Let G be a graph and $M(G)$ its corresponding (regular) graphic matroid, and $\mathcal{O}(G)$ resp. $\omega(M)$ corresponding orientations. Then for every natural $k \geq 1$ and any finite abelian group $(N, +)$, an assignment $f : E(G) = E(M) \to \mathbb{Z}$ resp. $f : E(G) = E(M) \to N$ is a NZ-k- resp. NZ-N- (co)flow with respect to $\mathcal{O}(G)$, iff it is one with respect to $\omega(M)$. We thus have $\xi(G) = \xi(M(G))$.

Moreover, the duality presented above for the planar case (i.e., $\chi(G) = \xi(G^*)$) now admits a generalization to dual pairs of matroids as follows:

Definition 4.26. Let M be a regular matroid. The flow index of its dual matroid M^* (which is also regular), or equivalently, the minimal $k \in \mathbb{N}$ for which M admits a NZ-k-coflow, is defined as the *chromatic number* of M, denoted by $\chi(G)$.

Using proposition 4.19 and remark 4.25, we immediately get that $\chi(G) = \chi(M(G))$ for every graph G, and thus, the definition above is a fitting generalization of graph colourings to regular matroids.

4.3 Digression: The Tutte Polynomial

4.3.1 Definition

The *Tutte polynomial* is a polynomial in two variables x, y which is assigned to every matroid in a unique way according to the following definition:

Definition 4.27. Let M be a matroid with ground set S and $r : 2^S \to \mathbb{N}_0$ the associated rank function. The *Tutte polynomial* $T_M(x, y) = T(M; x, y)$ of M is defined by:

$$T(M; x, y) := \sum_{A \subseteq S} (x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}.$$

Although this polynomial may be considered over arbitrary rings and fields, in the following, we will focus on the real-valued interpretation. The Tutte-polynomial already contains many important characteristics of a matroid, e.g. we have $T(M; 1, 1) = |\mathcal{B}|, T(M; 1, 2) = |\{A \subseteq S | A \text{ is spanning}\}|, T(M; 2, 1) = |\mathcal{I}|, T(M; 2, 2) = 2^{|S|}$, where $\mathcal{B}, \mathcal{I}$ denote the set systems of bases resp. independent sets of M. The Tutte polynomial additionally encodes much more useful information, especially concerning more special classes of matroids, as will be shortly sketched in paragraph 4.3.3.

4.3.2 Recursion

The representation of the Tutte polynomial by the summation formula presented above is inconvenient for many purposes. This is already due to the simple fact that in general, we might get an exponential ($2^{|S|}$) number of summands, which makes an efficient computation according to this formula almost impossible. In addition, the bases of the summands are $x - 1, y - 1$, which may lead to further difficulties when trying to find a monomial representation of the Tutte polynomial. In many cases it is much easier to define the Tutte polynomial via a recursive deletion-contraction relation, which computes it in terms of the Tutte polynomials of the Matroids arising from M by contraction and/or deletion of an arbitrary but fixed element $e \in E(M)$:

Theorem 4.28. *Let M be a matroid, and let $e \in S := E(M)$. Then the following holds:*

- *If $S = \emptyset$ then $T(M; x, y) = 1$,*
- *If e is a loop of M, then $T(M; x, y) = y \cdot T(M\backslash e, x, y)$,*
- *If e is a coloop of M, we have $T(M; x, y) = x \cdot T(M/e, x, y)$,*
- *If e is neither a loop nor a coloop, the deletion-contraction relation*

$$T(M; x, y) = T(M\backslash e; x, y) + T(M/e; x, y)$$

holds true.

Proof. The case $S = \emptyset$ is trivial. Let at first e be a loop of M. Then $r_M(A) = r_M(A\backslash\{e\}) = r_{M\backslash\{e\}}(A\backslash\{e\}), \forall A \subseteq S$, and therefore

$$\begin{aligned}
T(M; x, y) &= \sum_{A \subseteq S} (x-1)^{r_M(S) - r_M(A)} (y-1)^{|A| - r_M(A)} \\
&= \sum_{A \subseteq S, e \in A} (x-1)^{r_{M\backslash e}(S\backslash\{e\}) - r_{M\backslash e}(A\backslash\{e\})} (y-1)^{|A\backslash\{e\}| + 1 - r_{M\backslash e}(A\backslash\{e\})} \\
&\quad + \sum_{A \subseteq S, e \notin A} (x-1)^{r_{M\backslash e}(S\backslash\{e\}) - r_{M\backslash e}(A)} (y-1)^{|A| - r_{M\backslash e}(A)} \\
&= ((y-1)+1) \sum_{A \subseteq S\backslash\{e\}} (x-1)^{r(M\backslash e) - r_{M\backslash e}(A)} (y-1)^{|A| - r_{M\backslash e}(A)} \\
&= y \cdot T(M\backslash e; x, y).
\end{aligned}$$

The case in which e is a coloop, is dual to the above and thus works analogously.
Let now e be neither a loop nor a coloop. Then we have for all $A \subseteq S$:

$$r_M(A) = \begin{cases} r_{M\backslash e}(A), & \text{if } e \notin A, \\ r_{M/e}(A\backslash\{e\}) + 1, & \text{if } e \in A. \end{cases},$$

and $r(M) = r(M\backslash e)$. This implies:

$$\begin{aligned}
T(M; x, y) &= \sum_{A \subseteq S} (x-1)^{r_M(S) - r_M(A)} (y-1)^{|A| - r_M(A)} \\
&= \sum_{A \subseteq S, e \notin A} (x-1)^{r(M\backslash e) - r_{M\backslash e}(A)} (y-1)^{|A| - r_{M\backslash e}(A)} \\
&+ \sum_{A \subseteq S, e \in A} (x-1)^{r_{M/e}(S\backslash\{e\}) + 1 - r_{M/e}(A\backslash\{e\}) - 1} (y-1)^{|A\backslash\{e\}| - r_{M/e}(A\backslash\{e\})} \\
&= T(M\backslash e; x, y) + T(M/e; x, y),
\end{aligned}$$

and the claim follows. $\square$

On the one hand, this recursion yields the possibility of a recursive computation of a monomial representation, but still this may take exponentially many steps in general and thus is not efficient on the other hand. Nonetheless, in many special and structured cases it allows a fast and sometimes explicit computation of the Tutte polynomial, and is appropriate for many theoretical results (proofs by induction). Unfortunately, most of them have to be omitted here. A simple example is the following result, which will be needed in subsequent paragraphs:

Theorem 4.29. *Let M be a matroid. Then the Tutte polynomial of its dual matroid arises from T_M by interchanging the first and the second variable (x, y), i.e.,*

$$T(M^*; x, y) = T(M; y, x), \forall x, y \in \mathbb{R}.$$

Proof. For each matroid M, define a polynomial P_M by $P_M(x, y) := T(M^*; y, x)$. We have to show that $P_M = T_M$ for all matroids. We do this by verifying that P fulfils all the defining recursive properties of T_M in Theorem 4.28. This follows easily from the same recursive properties of T_{M^*} and if we use the fact that the loops resp. coloops in M are exactly the same as the coloops resp. loops of M^*, as well as the equations $(M/e)^* = M^*\backslash e, (M\backslash e)^* = M^*/e$ for each $e \in E(M) = E(M^*)$:

$$P_M(x, y) = T(M^*; y, x) = yT(M^*/e; y, x) = yT((M\backslash e)^*; y, x) = yP_{M\backslash e}(x, y)$$

if e is a loop in M,

$$P_M(x, y) = T(M^*; y, x) = xT(M^*\backslash e; y, x) = xT((M/e)^*; y, x) = xP_{M/e}(x, y)$$

if e is a coloop of M, and

$$P_M(x, y) = T(M^*; y, x) = T(M^*/e; y, x) + T(M^*\backslash e; y, x) = P_{M\backslash e}(x, y) + P_{M/e}(x, y)$$

if it is neither of both. □

4.3.3 Related Polynomials

The linear recursion described above enables us to highlight the central role of the Tutte polynomial, which appears as the origin of all polynomials in two variables x, y fulfilling a similar linear recursion.

Theorem 4.30. *Let u, v, τ, σ be arbitrary rational functions in $\mathbb{R}(x, y)$, where $\tau \not\equiv 0$. Let $t : \{M$ is a matroid$\} \to \mathbb{R}(x, y)$ denote the function which is recursively defined as follows:*

- *If $E(M) = \emptyset$, then $t(M) = 1$.*
- *If $e \in E(M)$ is a loop of M, then $t(M) = v \cdot t(M\backslash e)$.*
- *If e is a coloop of M, then $t(M) = u \cdot t(M/e)$.*
- *If e is neither a loop nor a coloop, we have $t(M) = \sigma \cdot t(M\backslash e) + \tau \cdot t(M/e)$.*

Then this recursion is consistent (i.e., it does not lead to different values of t when taking different orders of elements while reducing), and $t(M) = \sigma^{|E(M)|-r(M)}\tau^{r(M)}T(M; \frac{u}{\tau}, \frac{v}{\sigma})$. In this sense, it is an evaluation of the Tutte polynomial.

Proof. Its is easy to verify that the right hand expression fulfils the four recursive conditions defining t. Thus, the uniqueness of this definition immediately implies the claimed equality. □

This general property of the Tutte polynomial is the main reason for its appearance in many different areas of mathematics and especially in matroid-related topics such as graph theory. The following few examples illustrate this concept.

4.3.4 The Tutte-Whitney Polynomial for Graphs

We can also assign a two-variable-polynomial to each given graph $G = (V, E)$ according to the recursive definition $T_G(x, y) := x^i y^j$, if E consists of i bridges, j loops and no further edges, and $T_G := T_{G-e} + T_{G/e}$, if $e \in E$ is neither a bridge nor a loop, which we refer to as the *Tutte-Whitney-polynomial* of the graph. It is clear that the evaluation of the Tutte-polynomial of a graphic matroid $M = M[G]$ has to fulfil exactly the same defining conditions, since deletion and contraction of edges resp. elements in graphs resp. the corresponding graphic matroids are equivalent. Hence, the Tutte-Whitney polynomial turns out to be a simple evaluation of the Tutte polynomial at the corresponding graphic matroid.

4.3.5 The Chromatic Polynomial and the Flow Polynomial

Given a graph G, we can assign the so-called *chromatic polynomial* $P_G(x)$ in *one* variable to it. This polynomial, evaluated at a natural number $k \in \mathbb{N}$, counts the number of legal k-colourings of G as defined in the context of the colouring theory of arbitrary graphs. By a simple combinatorial argument, one can show that this property indeed uniquely defines a polynomial. The most interesting thing about the chromatic polynomial is that if we have a fast way of computing it, we can derive the chromatic number of the given graph as the first natural number $k \in \mathbb{N}$ of it not being a root. The relation between the chromatic polynomial of a graph and the Tutte-polynomial of a matroid becomes immediately clear with the following recursive properties at hand:

Theorem 4.31. *Let $G = (V, E)$ be a Graph and P_G the associated chromatic polynomial. Then the following holds:*

- *If $E = \emptyset$, then $P_G(x) = x^{|V|}$.*
- *If $e \in E$ is a loop in G, then $P_G(x) = 0 = 0 \cdot P_{G-e}(x)$.*
- *If $e \in E$ is a bridge of G, then $P_G(x) = (x - 1) \cdot P_{G/e}(x)$.*
- *If $e \in E$ is neither a loop nor a bridge in G, we have $P_G(x) = P_{G-e}(x) - P_{G/e}(x)$.*

Proof. We first verify the statements in the case where $x = k \in \mathbb{N}$ is a natural number.

- If G consists of isolated vertices, then each assignment of colours out of $\{0, ..., k-1\}$ to the $|V|$ vertices of G is legal, and hence, the number of such colourings is given by $P_G(k) = k^{|V|}$.
- If e is a loop in G, then G obviously does not admit a legal colouring, and hence, $P_G(k) = 0$ as claimed.
- If e is a bridge in G, let u, w denote the end vertices of e and V_u, V_w the components of $G - e$ containing u resp. w. Let $V = V_1 \dot{\cup} V_2$ be a set decomposition of V in such a way that V_1 is the extension of V_u by all the remaining vertices and $V_2 = V_w$. Since e is a bridge, it is the only edge in $G[V_1, V_2]$ and hence, legal colourings of G using colours out of $0, ..., k-1$ are the same as the pairs of legal colourings of $G[V_1]$ resp. $G[V_2]$ where u, w receive distinct colours, being stuck together.
The legal colourings of G/e on the other hand are in bijection to the pairs of colourings of $G[V_1]$ and $G[V_2]$ where u and w receive the same colour. But being given such a colouring, we can shift the colours of all vertices in V_2 in $k-1$ possible ways according to $\{0, ..., k-1\} \rightarrow \{0, ..., k-1\} : i \rightarrow (i+l) \bmod k$ in order to end up with $k-1$ distinct colourings of G as described above. On the other hand, we thereby cover all the colouring pairs of $G[V_1], G[V_2]$ exactly once: For each pair of legal k-colourings of $G[V_1], G[V_2]$ where u, w receive distinct colours, there is a unique shift-value $l \in \{1, ..., k-1\}$ such that the reverse colour-transformation $\{0, ..., k-1\} \rightarrow \{0, ..., k-1\} : i \rightarrow (i-l) \bmod k$ applied to all vertices in V_2 maps the colour of w in the original colouring of $G[V_2]$ to the colour of v in $G[V_1]$. Hence, there is a one-to-$(k-1)$ correspondence between the colourings of G/e and G, which gives $P_G(k) = (k-1)P_{G/e}(k)$ as claimed.
- Now let $e \in E$ be neither a loop nor a bridge of G. The colourings of $G-e$ with colours in $\{0, ..., k-1\}$ can be separated into the ones with different resp. identical colours assigned to the end vertices of e. The colourings of the first type are exactly the legal colourings of G, while the ones of the second type are in bijection to the colourings of G/e (the identical colours assigned to the vertices of e correspond to the colour of the contraction-vertex of e in G/e). Hence, we have $P_{G-e}(k) = P_G(k) + P_{G/e}(k)$ and we are done.

Since $P_G, P_{G-e}, P_{G/e}$ are polynomials in one variable, the correctness for infinitely many distinct values already yields the asserted recursions. $\square$

By defining Q_G as the normalization of the chromatic polynomial such that $Q_G(x) = 1$ whenever $E(G) = \emptyset$, i.e. according to $Q_G(x) := \frac{P_G(x)}{x^{|V(G)|}}$, we get that Q_G has the following recursive properties:

- If $E = \emptyset$, then $Q_G(x) = 1$.
- If $e \in E$ is a loop of G, then $Q_G(x) = 0 \cdot Q_{G-e}(x)$.
- If $e \in E$ is a bridge in G, then $Q_G(x) = \frac{x-1}{x} \cdot Q_{G/e}(x)$.

- If $e \in E$ is neither a loop nor a bridge in G, then $Q_G(x) = Q_{G-e}(x) - \frac{1}{x}Q_{G/e}(x)$.

We now consider the mapping t, which assigns a polynomial in two variables to a given matroid according to 4.30 with $u(x,y) = \frac{x-1}{x}, v(x,y) = 0, \sigma(x,y) = 1$ and $\tau(x,y) = -\frac{1}{x}$. We henceforth have $t(M)(x,y) = (-\frac{1}{x})^{r(M)}T(M;1-x,0)$, and obviously, the evaluation $t(M[G])$ in the case of graphic matroids satisfies the same defining properties as Q_G above. This finally implies their identity, i.e.

$$P_G(x) = x^{|V(G)|}Q_G(x) = x^{|V(G)|}t(M[G])(x,y)$$

$$= x^{|V(G)|}(-\frac{1}{x})^{|V(G)|-c(G)}T(M[G];1-x,0) = (-1)^{|V(G)|-c(G)}x^{c(G)}T(M[G];1-x,0).$$

Because of that, we may reduce the computation of the chromatic polynomial to a special evaluation of the Tutte-polynomial of the appropriate graphic matroid resp. the Tutte-Whitney-polynomial.

We now proceed with the *flow polynomial* of a graph G with some arbitrarily fixed orientation D. We motivate this polynomial, which is denoted by C_G resp. C_D (the concrete orientation does not matter for the polynomial, as it does not for Nowhere-Zero-flows), by first considering a planar connected graph H. Let H^* be some planar dual graph of H. From Section 4.2 we know that colourings of H using colours in $\{0,...,k-1\}$ are mapped to NZ-k-coflows of H. Given such a flow, we can take its values modulo k and obtain a NZ-$\mathbb{Z}_k$-coflow. Those flows again are in bijection to the NZ-$\mathbb{Z}_k$-flows of H^*. In this context, the colours of any colouring are uniquely defined by the assigned coflow up to the k cyclic shifts $\{0,...,k-1\} \to \{0,...,k-1\}, i \to (i+l) \mod k, l = 0,...,k-1$ not changing the differences between the colours of end vertices at each edge. Hence, each coflow corresponds to exactly k colourings of G, which means that the number of NZ-$\mathbb{Z}_k$-flows on H^* is given by $\frac{1}{k}P_H(k)$, and consequently also defines a unique extending polynomial in one variable, which will be denoted by $C_{H^*}(x)$. Of course, this gives rise to the question whether there is such a polynomial $C_G(\cdot)$ for each graph G, i.e. which counts the number of $\mathbb{Z}_k$-flows on G when being evaluated at $k \in \mathbb{N}$. The following theorem gives an even more general positive answer to this question: Such a polynomial exists for each underlying abelian group $(N,+)$ with neutral element e_N, and is only dependent on the order $|N| = k$ of this group. This already is a pretty surprising result and is best explained with the following recursive properties of the number of NZ-N-flows on a graph G:

Theorem 4.32. *Let $(N,+)$ with neutral element e_N be an abelian group, and for an arbitrary graph G, denote by $C_G(N), C_{G-e}(N), C_{G/e}(N)$ the number of algebraic NZ-N-flows on specified orientations of $G, G-e, G/e$. Then the following holds:*

- *If $E = \emptyset$, then $C_G(N) = 1$.*
- *If $e \in E$ is a loop of G, then $C_G(N) = (|N|-1) \cdot C_{G-e}(N)$.*
- *If $e \in E$ is a bridge in G, then $C_G(N) = 0 \cdot C_{G/e}(N)$.*

- *If $e \in E$ is neither a loop nor a bridge in G, then $C_G(N) = C_{G/e}(N) - C_{G-e}(N)$.*

Proof. In the following, we assume that G is equipped with an arbitrary orientation D (and $G/e, G-e$ accordingly with $D/e, D-e$), which is fixed as the base of the flows considered in the following. As was noted above, the choice of the orientation has no influence on the numbers $C_G(N), C_{G-e}(N), C_{G/e}(N)$.

- If G does not admit edges, the only NZ-flow on G is the empty mapping on E.

- If e is a loop in G incident to a vertex $v \in V(G)$, it is incoming and outgoing with respect to v at the same time. Hence, the flow values assigned to e cancel out in the Kirchhoff-expression for the excess of the flow at v. Moreover, the NZ-N-flows on G are exactly the NZ-N-flows on $G-e$ with an additional arbitrary assignment of a non-zero-element $n \in N \setminus \{e_N\}$ to e. Hence, the number of such flows is $(|N|-1) \cdot C_{G-e}(N)$ as claimed.

- If $e = (u, w)$ is a bridge of G, there can't be any NZ-N-flow on G: Denote by $V(G) = V_1 \dot\cup V_2$ a decomposition of the vertex set such that $G[V_1, V_2] = \{e\}$ with $u \in V_1, w \in V_2$. Assume there was a NZ-N-flow f on G. Since $G[V_1, V_2] = \{e\}$ is a cut in G and due to the Kirchhoff-law-properties of f, this would mean that also $e_N = f(e)$, immediately contradicting the NZ-condition for f. Hence, the number of NZ-N-flows is zero in this case.

- Assume that $e = (u, w)$ is neither a loop nor a bridge in G. Let f be an arbitrary flow on G/e resp. D/e. Then f has to fulfil Kirchhoff's law of flow conservation at each vertex distinct from u, w, and additionally at the vertex v_e corresponding to the contraction of e, i.e.,

$$0 = \mathrm{exc}_f(v_e) = f^+(v_e) - f^-(v_e) = \sum_{e' \in E^+_{D/e}(v_e)} f(e') - \sum_{e \in E^-_{D/e}(v_e)} f(e')$$

$$= \sum_{e' \in (E^+_D(u) \cup E^+_D(w)) \setminus \{e\}} f(e') - \sum_{e \in (E^-_D(u) \cup E^-_D(w)) \setminus \{e\}} f(e').$$

If g is any assignment $g : E(G) \to N$ so that g agrees with f on all edges $e \in E(G) \setminus \{e\}$, the above can be written as

$$0 = \underbrace{g^+(u) - g^-(u)}_{=:\mathrm{exc}_g(u)} + \underbrace{f^+(v) - f^-(w)}_{=:\mathrm{exc}_g(w)} - g(e) + g(e) = \mathrm{exc}_g(u) + \mathrm{exc}_g(w).$$

The latter property makes it possible to find a unique extension g of f to a flow on G resp. D: g is a flow if and only if $\mathrm{exc}_g(u) = -\mathrm{exc}_g(w) = 0$, and the only additional value fulfilling this condition is

$$g(e) = g(e) + 0 = g(e) + \mathrm{exc}_g(w) = \sum_{e' \in E^+_D(w)} f(e') - \sum_{e' \in E^-_D(w) \setminus \{e\}} f(e').$$

We thereby have established a well-defined mapping F_1 assigning an extending N-flow on D to every NZ-N-flow f on D/e. On the other hand, restricting any N-valued flow on D which is non-zero at every edge $e' \in E(G)\backslash\{e\}$ obviously gives a NZ-N-flow on D/e, and the thereby defined mapping F_2 is the inverse mapping of M_1, wherefore F_1 and F_2 are bijections. Hence, the number $C_{G/e}(N)$ of NZ-N-flows is equal to the number of N-valued flows on D whose only possible "zero" is at e. Those may be subdivided into the NZ-N-flows on D and the N-flows on D being e_N at e and in $N\backslash\{e_N\}$ at each distinct edge, which of course are in one-to-one correspondence with the NZ-N-flows on $D-e$. Hence, $C_{G/e}(N) = C_G(N) + C_{G-e}(N)$ and we are done.

□

Corollary 4.33. *Let, for each graph G, $C_G(x)$ be a one-variable polynomial recursively defined as follows:*

- *If $E = \emptyset$, then $C_G(x) = 1$.*
- *If $e \in E$ is a loop of G, then $C_G(x) = (x-1) \cdot C_{G-e}(x)$.*
- *If $e \in E$ is a bridge in G, then $C_G(x) = 0 \cdot C_{G/e}(x)$.*
- *If $e \in E$ is neither a loop nor a bridge in G, then $C_G(x) = C_{G/e}(x) - C_{G-e}(x)$.*

Then this recursion is consistent (i.e., it does not lead to different values of t when taking different orders of elements while reducing), and

$$C_G(x) = (-1)^{|E(G)|-|V(G)|+c(G)} T(M[G]; 0, 1-x) = (-1)^{|E(G)|-|V(G)|+c(G)} T_G(0, 1-x).$$

If N is an arbitrary abelian group of order k with neutral elment e_N, then the number of NZ-N-flows on any fixed orientation of G is given by $C_G(k)$. Thus, the so-called flow polynomial *$C_G(x)$ of G is an evaluation of the Tutte-polynomial of the corresponding graphic matroid resp. of the Tutte-Whitney polynomial.*

Proof. The first part follows from applying Theorem 4.30 with $v(x,y) := x-1, u(x,y) := 0$, $\sigma := -1, \tau := 1$. The second statement is an immediate consequence of the same defining recursive properties being fulfilled by the numbers $C_G(N)$ and $C_G(k)$ where $k = |N|$. □

All in all, counting the number of legal colourings resp. the number of Nowhere-Zero-N-flows on a fixed orientation of a graph G where N is any abelian group of order N both uniquely give rise to polynomials P_G resp. C_G. These polynomials arise from special evaluations of the Tutte-Whitney-polynomial, which, up to a monomial factor, are symmetric. The duality between legal graph colourings and Nowhere Zero-flows on the dual graph in the planar case can now be rediscovered in the context of matroid duality: While the Tutte polynomials of dual matroids are simply related by an interchange of variables (cf. Theorem 4.29), we analogously evaluate the Tutte-Whitney polynomial of G at the coordinates $(1-x, 0)$ resp. $(0, 1-x)$ when counting colourings resp. NZ-flows. This agrees with the introducing observations for planar graphs made above.

4.3.6 The Jones-Polynomial for Knots

In knot theory, one is interested in invariants of knots and more generally links, which are assigned to plane projections of these and stay identical for equivalent projections, i.e., projections representing the same knot or link. It can be shown that this is equivalent to the fact that those invariants stay the same under application of the so-called *Reidemeister moves*. The invariants may e.g. help to distinguish different knots from one another when only some projection of them is given. One of the strongest known such invariants is the *Jones-polynomial*, which again is a one-variable polynomial assigned to each plane projection. It turns out that the Jones-polynomial emerges from the so-called *Bracket-polynomial by Kaufman* via a transformation of variables, which again can be defined by a couple of recursive relations, which are quite similar to the ones presented in Theorem 4.30. Actually, for appropriately defined auxiliary graphs representing a knot/link-projection, it is possible to affiliate the Kaufman Bracket-polynomial and thereby the Jones-polynomial to an evaluation of the Tutte-polynomial, see [Hub09] for further details.

4.3.7 NP-Hardness

By the representation of the chromatic polynomial as an evaluation of the Tutte-polynomial of the corresponding graphic matroid as explained above, it is easy to see that in general, computing/evaluating the Tutte-polynomial is an NP-hard problem. This is demonstrated by the following reduction:
A graph G is 3-colourable if and only if $P_G(3) \neq 0$, i.e. $T(M[G]; -2, 0) \neq 0$ (see section 4.3.5), and consequently, deciding 3-colourability can be polynomially reduced to computing the Tutte-polynomial for (graphic) matroids. But since determining 3-colourability of a given graph is a well-known NP-complete problem, this immediately implies the NP-hardness of computing a Tutte polynomial of an arbitrarily given matroid, which henceforth is not feasible in polynomial time on a Turing-Machine whenever $P \neq NP$.

4.3.8 Consequences for NZ-Flows

The most important consequence of the Tutte polynomial and especially the flow polynomial of a graph in terms of Nowhere-Zero-flows is the fact that it counts the number of NZ-k-flows on a given graph only depending on the *size* of the underlying abelian group the flows are defined on. Apart from that, it is independent of the algebraic structure of this group. This is why, given a graph G, $P_G(4)$ not only represents the number of NZ-4-flows on G, but at the same time the number of NZ-$\mathbb{Z}_4$- and NZ-$\mathbb{Z}_2 \times \mathbb{Z}_2$-flows. This concept is one of the main ideas used in the proofs of Jaeger's resp. Seymour's 8- resp. 6-flow theorems, see Section 5.3 for further details.
This especially implies the following important result, known as *equivalence theorem*:

Theorem 4.34 (Equivalence Theorem, Tutte 1954, cf. [Tut54])**.** *Let $D = (V, A)$ be a digraph and $(N, +)$ an arbitrary abelian group of order $k \geq 2$. Then the following conditions are equivalent:*

(i) There exists a NZ-$\mathbb{Z}_k$-flow on D.

(ii) There exists a NZ-N-flow on D.

(iii) There exists a NZ-k-flow on D.

Proof. (iii) $\Rightarrow$ (i): By taking all the values of a NZ-k-flow on D modulo k, we will get a NZ-$\mathbb{Z}_k$-flow.
(i) $\Rightarrow$ (ii): According to the above, the number of NZ-$\mathbb{Z}_k$-flows on D is the same as the number of NZ-N-flows on D, namely $C_{U(D)}(k)$. This implies the equivalence of (i) and (ii).
(ii) $\Rightarrow$ (iii): Without loss of generality, assume that $U(D)$ is connected. Let f be a NZ-$\mathbb{Z}_k$ flow. Considering f as an assignment of integers to the edges of D with range contained in $\{0, ..., k-1\}$, we get a "pre-flow" on D in such a way that the *excess* of each vertex v, i.e., $\mathrm{exc}_f(v) = f^+(v) - f^-(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e)$ is a multiple of k, because the same excess evaluated modulo k has to be 0 since f is a $\mathbb{Z}_k$-flow on D. We now show that we can modify f to a NZ-k-flow f' so that $f \equiv f' (\mathrm{mod}\ k)$: Since $f \equiv f (\mathrm{mod}\ k)$, we can find a flow f' on D congruent to f modulo k which is minimal with respect to the value

$$\sum_{v \in V(D)} |\mathrm{exc}_f(v)|.$$

We show that f' has to have excess zero at each vertex and hence is a NZ-k-flow as desired: Assume there was a vertex with non-zero excess. Since $\sum_{v \in V(D)} \mathrm{exc}_f(v) = 0$, this implies the existence of a pair v_1, v_2 of vertices in D with positive resp. negative excess. Let P be an arbitrary path in $U(D)$ connecting v_1, v_2, without loss of generality directed from v_2 to v_1. Adding additional weight $\pm k$ to each edge contained in P gives us a flow f'' congruent to f' and hence f modulo k which leaves the excesses of vertices in $V(D) \setminus \{v_1, v_2\}$ unaffected but decreases the absolute values of the excesses at v_1 and v_2. Consequently, we derive a contradiction to the assumed minimality of f', which means that the assumption was false, f' is indeed a flow and we are done. $\square$

4.4 Digraph Colourings and the Dichromatic Number

In this section, we finally introduce the main problems this master's thesis deals with, especially the 2-Colour-Conjecture due to Victor Neumann-Lara from 1985. Since more elaborate arguments and considerations concerning the dichromatic number of especially planar digraphs are presented in the remaining chapters of the thesis, at this point, we will restrict ourselves to the most basic notions and definitions.
Let $D = (V, A)$ be a digraph. How to define a notion of legal colourings of D in an appropriate way? If we do so, we have to use the orientation information given by the arcs and cannot just restrict to the underlying graph. One possibility (which apparently is the simplest found so far) is to forbid directed cycles fully contained in one colour-class of a vertex-colouring of D.

Definition 4.35. Let D be a digraph and $c : V(D) \to M$ with some finite set M a vertex-colouring of D. We call c a *legal M-digraph-colouring* of D, if the subdigraph $D[A]$, where A denotes the set of monochromatic arcs in A, i.e., they admit the same colours at their end vertices, is acyclic. In other words: There are no monochromatic directed cycles in D with respect to c. If $M = \{0, ..., k-1\}$, then c is called a legal k-digraph-colouring. Obviously, due to isomorphisms, a legal M-colouring corresponds resp. is equivalent to a legal $|M|$-colouring of D. The minimal number $k \in \mathbb{N}$, for which a legal k-digraph-colouring exists (and thus also for all $k' \geq k$), is called the *Dichromatic Number* of the digraph D, denoted by $\vec{\chi}(D)$.

Given a *graph* G, we can furthermore ask for the minimal number $k \in \mathbb{N}$, for which each orientation of G admits a legal digraph-colouring:

Definition 4.36. Let G be a graph. The *Dichromatic Number* of G is defined as the maximum

$$\vec{\chi}(G) := \max_{\mathcal{O}(G)} \vec{\chi}(\mathcal{O}(G))$$

over all orientations of G.

All the defined quantities are considered ∞ in case there is no legal colouring.

Remark 4.37. Let D be an arbitrary digraph. If D admits loops, according to the above definition, D cannot admit legal digraph colourings. Thus, in the analysis of the dichromatic number of digraphs resp. graphs, we may restrict ourselves to loopless digraphs.

The following gives an immediate connection between the dichromatic and the chromatic numbers of loopless graphs, showing that the former is more general and additionally justifying the definition of digraph colourings as defined by Neumann-Lara.

Proposition 4.38 (Relation between the dichromatic and chromatic numbers of graphs)**.** *Let G be a loopless graph and $Z(G)$ its symmetric orientation, i.e., the digraph arising from G by replacing each undirected edge $e \in E(G)$ by a pair of arcs with converse directions connecting the same end vertices. Then a vertex colouring $c : V(G) \to M$ for a finite set M is a legal M-graph-colouring of G if and only if it is a legal M-digraph-colouring of $Z(G)$. Thus, $\chi(G) = \vec{\chi}(Z(G))$.*

Proof. Obviously, every legal M-graph-colouring of G is a legal digraph colouring of $Z(G)$, since the set of monochromatic arcs A in $Z(G)$ is empty. On the other hand, given some legal digraph-colouring of $Z(G)$, this does not admit any monochromatic edges: If there was one, the directed cycle consisting of the two antiparallel edges replacing it in $Z(G)$ would be monochromatic, contradiction. □

Remark 4.39. Let D_1, D_2 be digraphs such that D_1 is a subdigraph of D_2. Then $\vec{\chi}(D_1) \leq \vec{\chi}(D_2)$. For every graph G, $\vec{\chi}(G) \leq \chi(G)$.

Proof. The first inequality follows from the fact that each legal digraph colouring of D_2 induces a legal digraph colouring on D_1. Since any orientation of G is a subdigraph of $Z(G)$, this implies $\vec{\chi}(G) \leq \vec{\chi}(Z(G)) = \chi(G)$. □

Given the general problem of finding dichromatic numbers of graphs, one should usually start off with "simple" graphs in structural or topological terms, e.g., orientations of complete graphs (so-called *Tournaments*) or planarly embedded loopless digraphs. Furthermore, if allowing antiparallel edges between pairs of vertices in the considered digraphs, in any legal digraph colouring, these vertices are forced to admit distinct colours and thus, we may end up with some sort of mixed notion of digraph-colourings and something which might be called "partial" graph-colouring, since only certain edges are required to be non-monochromatic. Thus, the following two questions concerning two of the most prominent classes of simple graphs, are natural:

- What is the value of $\vec{\chi}(K_n)$ for a given natural number $n \in \mathbb{N}$?
- What is the maximal value $\vec{\chi}(D)$ for a simple planar digraph D?

The two question raised above are surprisingly hard to answer. As already mentioned in a famous quote by Paul Erdős,"It is surprisingly difficult to determine $\vec{\chi}(G)$, even for the simplest graphs". First of all, even today, no explicit expressions for the dichromatic numbers of complete graphs, i.e., the maximal number of colours needed for a legal colouring of a tournament on n vertices, are known, and, due to the super-exponential growth of the number of tournaments on n vertices ($2^{\frac{n(n-1)}{2}}$), only the values $\vec{\chi}(K_n), n \leq 11$ are precisely known. Erdős and Neumann-Lara showed that its asymptotic growth is given by $\Theta(\frac{n}{\log n})$, but still, this seems like a very unsatisfactory result.
The following is the (in comparison to the Four-Colour-Theorem less prominent but still widely known) *2-Colour-Conjecture*, which provides an answer to the latter question: While we know that simple planar digraphs always admit legal 3-colourings (cf. section 5.1), this Conjecture claims that we can improve this bound to 2 in this case:

Conjecture 4.40. *Every simple planar digraph D is 2-colourable. Equivalently, for every simple planar graph G, $\vec{\chi}(G) \leq 2$.*

In this master's thesis, we will deal with different attempts to tackle this conjecture and partial positive as well as negative results. Since every planar graph can be triangulated so that it becomes a subgraph of a planar triangulation, when dealing with the Two-Colour-Conjecture, we may restrict ourselves to orientations of 3-connected planar triangulations, which we will often consider in the following sections. The following gives at least some computational evidence for the conjecture:

Remark 4.41 (cf. [KV17])**.** Conjecture 4.40 holds true for all simple planar digraphs on at most 26 vertices.

It is worth mentioning that while for arbitary undirected graphs, deciding whether it is bipartite can be easily done in polynomial time but recognizing 3-colourable graphs is a NP-hard task, here for digraphs, 2 is the critical value: A digraph D is 1-colourable iff it is acyclic, which can be tested in polynomial time in the number of verices. On the other hand, deciding 2-colourability of an arbitrarily given digraph again is NP-hard, as was shown by Bokal et. al. in [BFJ$^+$04]:

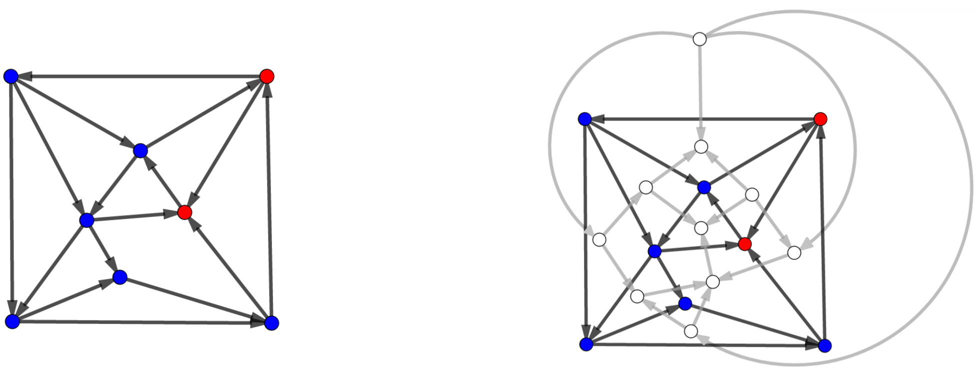
Theorem 4.42 (cf. [BFJ$^+$04], Theorem 3.1)**.** *Deciding whether an arbitrary digraph D admits a 2-digraph-colouring is NP-complete.*

4.5 Neumann-Lara-Flows

In the following we want to introduce dual notions corresponding to colourings of (especially planar) digraphs which are the so-called Neumann-Lara-coflows and Neumann-Lara-flows (NL-coflows and NL-flows for short). The definitions and results presented in this section are taken from [Hoc17] or follow its considerations. Unfortunately, the notion of digraph colourings, except for its correspondence to graph colourings described above, as far as I know, does not admit a nice picturesque description as e.g. in terms of colourings of geographical maps. As in the case of graph colourings and NZ-flows, we sketch this correspondence by considering some digraph D.

Assume for the following that we are given a legal digraph-colouring $c : V(D) \to \{0, ..., k-1\}$ of D. Then, as in the case of graphs, the values of c, treated as real numbers, can be considered potentials on the vertices of $\vec{G}$, giving rise to a well-defined tension f_c on D according to $f_c(e) := c(w)-c(u)$, for each edge $e = (u, w)$ in D. Since the set A of monochromatic edges in D now is exactly the set of edges receiving coflow-value 0, i.e., $\overline{\mathrm{supp}(f_c)}$, the requirement that c is a legal digraph colouring becomes equivalent to $D[A] = D - \mathrm{supp}(f_c)$ being an acyclic digraph. Moreover, f_c only takes on integers within $\{0, \pm 1, \pm 2, ..., \pm(k-1)\}$ (now, in contrast to NZ-flows, 0 is not excluded). This correspondence to the notion of colourings according to Victor Neumann-Lara already explains the term "Neumann-Lara" in the heading.

For the purpose of explaining the coherence of colourings and certain flows, consider a connected simple planar digraph D together with some planar directed dual D^*. Given some k-digraph-colouring c of D, a fixed orientation $\vec{G}$ on G and the corresponding tension f_c as defined above, we can consider $f_c : E(\vec{G}) \to \{0, \pm 1, ..., \pm(k-1)\}$ also as an assignment on the edges of the directed dual D^*. f_c, which was a tension on D, now becomes an integer flow, which ranges in $\{0, \pm 1, ..., \pm(k-1)\}$. The property that $D - \mathrm{supp}(f_c)$ was an acyclic digraph, due to the duality between acyclic and totally cyclic digraphs, now translates to $(D - \mathrm{supp}(f_c))^* = D^*/\mathrm{supp}(f_c)$ being totally cyclic or equivalently, since D, D^* are simply connected, being a strongly connected digraph. This is illustrated by the following figure.



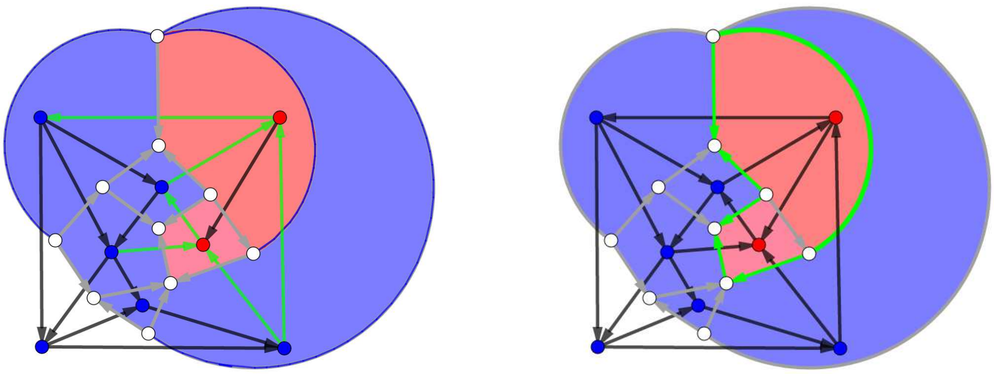


Figure 8: Illustration of the correspondence between digraph colourings and NL-coflows of a digraph. The pictures described the coloured planar digraph resp. its directed dual. Green edges mark the supports of the NL-coflow respective NL-flow.

Finally, the following formally introduces the notions of Neumann-Lara-coflows and -flows on Digraphs motivated above and introduced in [Hoc17]:

Definition 4.43 (Hochstättler, cf. [Hoc17])**.** Let D be a digraph, and $k \in \mathbb{N}$ a natural number. Then a *Neumann-Lara-k-coflow* on D is defined as a tension $f : E(D) \to \{0, \pm 1 ..., \pm(k-1)\}$ such that $D - \text{supp}(f)$ is acyclic.
Analogously, a *Neumann-Lara-k-flow* on D is a flow f ranging in $\{0, \pm 1, ..., \pm(k-1)\}$ such that $D/\text{supp}(f)$ is totally cyclic. In case that D is simply connected, the latter is equivalent to $D/\text{supp}(f)$ being strongly connected.
The minimal number for which D admits a NZ-k-flow is defined as the *NL-flow-index* of D, denoted by $\vec{\xi}(D)$ (which is considered ∞ if there is no legal NL-flow).
If G is some (undirected) graph, then the maximal NL-flow-index of all its orientations is defined as the *NL-flow-index*

$$\vec{\xi}(G) = \max_{\mathcal{O}(G)} \vec{\xi}(\mathcal{O}(G))$$

of G.

When considering the relation between graph and digraph colourings, we could derive the chromatic number of a graph by considering its symmetric orientation $Z(G)$. In the dual setting, we can analogously express NZ-flows of a graph $G = (V, E, \delta)$ in terms of NL-flows on its *directed subdivision* $S(G) = (V', A, \delta')$, defined by $V' := V \cup E$, $A := E^1 \cup E^2$, where $E^1, E^2 \simeq E$ are copy sets of E and $\delta(e_1) = (u, e), \delta(e_2) = (w, e)$ for all edges $e \in E, \delta(e) = \{u, w\}$ with copies $e_i \in E_i, i = 1, 2$:

Proposition 4.44 (cf. [Hoc17])**.** *Let G be a graph and $S(G)$ its directed subdivision. Then for every $k \in \mathbb{N}$, G admits a NZ-k-flow iff $S(G)$ admits a NL-k-flow.*

Proof. Assume first that $f : E(G) \to \{\pm 1, ..., \pm(k-1)\}$ is some NZ-k-flow on an orientation $\mathcal{O}(G)$. Then for each edge $e \in E(G)$ with $\delta_{\mathcal{O}(G)}(e) = (u, w)$, define the values of $g : E(S(G)) \to \{\pm 1, ..., \pm(k-1)\}$ at e by $g((u, e)) := f(e), g((w, e)) := -f(e)$. This obviously defines a NZ-k- and thus a NL-k-flow on $S(G)$.

On the other hand, given any NL-k-flow $g : E(S(G)) \to \{0, \pm 1, ..., \pm(k-1)\}$ on $S(G)$, the flow conservation at each edge-vertex e with $\delta(e) = \{u, w\}$ yields $g((u, e)) = -g((w, e))$, and this has to be non-zero, since e induces a directed 2-cut in $S(G)$ which has to be covered by $\mathrm{supp}(f)$. Consequently, $f(e) := g((u, e))$ for each edge $e \in E(\mathcal{O}(G)), \delta_{\mathcal{O}(G)}(e) = (u, w)$, where $\mathcal{O}(G)$ denotes some arbitrarily chosen orientation, defines a NZ-k-flow on G, proving the reverse implication. □

Again, in the argument above, we do not have to restrict to integer flows. Instead, given some finite abelian group $(N, +)$ with neutral element e_N, any N-colouring of D according to $f(e) := c(w) - c(u), e = (u, w) \in E(D)$, defines a tension on D such that $D - \mathrm{supp}(f)$ with $\mathrm{supp}(f) := f^{-1}(N \backslash \{e_N\})$ is an acyclic digraph. This gives rise to the following extension of the above definitions:

Definition 4.45. Let D be a digraph, and $(N, +)$ a finite abelian group with neutral element e_N. Then an *(Algebraic) Neumann-Lara-N-coflow* on D is defined as a tension $f : E(D) \to N$ such that $D - \mathrm{supp}(f)$ is acyclic.
Analogously, an *(Algebraic) Neumann-Lara-k-flow* on D is a flow f ranging in N such that $D/\mathrm{supp}(f)$ is totally cyclic.
The minimal number for which D admits an NL-k-flow is defined as the *NL-flow-index* of D, denoted by $\vec{\xi}(D)$ (which is considered ∞ if there is no legal NL-flow).

As in the case of graphs, the described relationships between colourings of digraphs and NL-coflows are not only one-sided, moreover, given any NL-N-coflow f on a digraph D (where $(N, +)$ again is a finite abelian group), we can construct a legal N-colouring of D as follows:
Without loss of generality, assume that D is simply connected (if there were different connected components of $U(D)$, we could apply the following argument for each component and stick the arising colourings together, eventually ending up with a legal N-colouring of D). Starting with some fixed reference vertex v_0 which is going to receive colour $e_N \in N$, for each vertex $v \in V(D) \backslash \{v_0\}$, we define its colour $c(v)$ according to

$$c(v) = \sum_{e \in P^+} f(e) - \sum_{e \in P^-} f(e) \in N,$$

where P is an arbitrarily chosen oriented path in $U(D)$ resp. D starting in v_0 and ending in v. As we have seen in section 4.2 when considering NZ-coflows, this assignment is well-defined. Thus, given any edge $e = (x_1, x_2) \in E(D)$ and some path $P = (P^+, P^-)$ in D starting in v_0 and ending in x_1, then $c(x_2) - c(x_1)$ equals $f(e)$(by using the paths P, $P\Delta\{e\}$ in the definition of $c(x_1), c(x_2)$). Thus, the set $A := \overline{\mathrm{supp}(f)}$ of edges receiving coflow-value e_N in D corresponds to the set of monochromatic arcs with respect to c. This means that $D[A] = D - \mathrm{supp}(f)$ is acyclic, implying that c as defined above indeed is a legal digraph colouring of D.

Proposition 4.46. *Let D be a digraph and $(N, +)$ a finite abelian group, $k := |N|$. Then D admits a legal k- resp. N-digraph-colouring if and only if D admits a NL-N-coflow or equivalently a NL-k-coflow.*

Proof. The equivalence of k-colourings and N-colourings of G and of NL-N-coflows follows directly from the foregoing arguments. Additionally, we have already explained how to deduce NL-k-coflows from a given k-colouring. Thus, it only remains to show that each NL-k-coflow yields a k-colouring of G. But this follows immediately from the fact that a NL-k-coflow f, taken modulo k and considered as a NL-$\mathbb{Z}_k$-coflow, gives rise to a legal $\mathbb{Z}_k \cong \{0, ..., k-1\}$-colouring of D. $\square$

Consider now again a connected planar digraph D with a directed dual D^*. Then the duality of NL-flows on D and NL-coflows on D^* together with the above yields the following theorem, which is a directed analogue to the Equivalence Theorem of Tutte for planar digraphs.

Theorem 4.47 (Equivalence Theorem for Planar Digraphs)**.** *Let D be a planar digraph, $k \in \mathbb{N}$ and $(N, +)$ an abelian group of order k. Then the following are equivalent:*

(i) D admits a NL-k-flow.

(ii) D admits a NL-$\mathbb{Z}_k$-flow

(iii) D admits a NL-N-flow.

Proof. D admits a NL-k- resp. a NL-$\mathbb{Z}_k$- resp. a NL-N-flow if and only if D^* admits the respective NL-coflows. The equivalence of these dualized statements is the content of Remark 4.46. $\square$

From the above we can derive the following connection between the chromatic numbers and flow indices of planar graphs:

Remark 4.48. Let D be a simply connected planar digraph and D^* its directed dual. Then $\vec{\chi}(D) = \vec{\xi}(D^*)$.

Proof. $\vec{\chi}(D)$ is defined to be the least $k \geq 1$ for which D admits a legal k-digraph-colouring or equivalently a $\mathbb{Z}_k$-colouring, which according to Proposition 4.46 means that D admits a NL-$\mathbb{Z}_k$-coflow. But since the tensions on D (considered as assignments on the arcs) are exactly the flows on D^* (see Proposition 3.53), this is equivalent to D^* admitting a NL-$\mathbb{Z}_k$- or equivalently (cf. Theorem 4.47) NL-k-flow, i.e., this minimal k is indeed exactly $\vec{\xi}(D^*)$. $\square$

We thereby get the following reformulation of the 2-Colour-Conjecture, which will be a central topic throughout the rest of this thesis.

Theorem 4.49. *The following are equivalent:*

(i) The 2-Colour-Conjecture, i.e., every simple planar digraph is 2-colourable.

(ii) Every 3-edge-connected planar digraph admits a NL-2-flow.

(iii) Every cubic 3-edge-connected planar digraph admits an NL-2-flow.

(iv) Every 3-edge-connected planar digraph D admits an even edge subset E so that D/E is totally cyclic.

(v) Every cubic 3-edge-connected planar digraph admits an even edge subset so that D/E is totally cyclic.

Proof. According to Remark 4.48 above, (i) and (ii) are equivalent, since the dual graphs of the simple planar graphs are exactly the 3-edge-connected planar graphs, cf. the preliminaries. (ii)⇒(iii) is obvious, so assume for the reverse implication that every 3-edge-connected *cubic* digraph admits an NL-2-flow. Then given any simply connected 3-edge-connected digraph D which is not cubic, we may restrict ourselves on proving the existence of a NL-2-flow in the case where D is 2-vertex-connected:
If there was a cut vertex v_{cut} in $U(D)$, we could split D along v_{cut} ending up with two digraphs D_1 and D_2 (both containing v_{cut}). We then proceed by induction on the number of vertices in D and construct an NL-2-flow on D by joining two NL-2-flows f_1 on D_1 resp. f_2 on D_2. The resulting 2-flow f on D indeed has to be a NL-flow, since $D/\text{supp}(f)$ arises from the totally cyclic digraphs $D_1/\text{supp}(f_1), D_2/\text{supp}(f_2)$ by joining them via a common vertex and thus is totally cyclic too.
So assume in the following that $U(D)$ is 2-vertex connected. We define a 3-regular digraph $\hat{D}$ by replacing each vertex $v \in V(D)$ of degree ≥ 4 by a directed cycle C_v of length $\deg_{U(D)}(v)$ and connecting the neighbours of v each to a different vertex of C_v. This digraph is furthermore 3-edge-connected: If S is an arbitrary cut in $U(\hat{D})$, it either contains no edges of any cycle C_v and hence corresponds to a cut in D, wherefore it has size at least 3, or, in the other case, it contains at least one (and thus, since this has to be an even number at least two) edges of a cycle C_v, $\deg_{U(D)}(v) \geq 4$. If $|S|$ was less than 3, i.e., $|S| = 2$ contained only those two edges in C_v, this would imply that v is a cut vertex in $U(D)$, contradicting our additional assumption above. Hence, $\hat{D}$ is a 3-regular 3-edge-connected digraph, which according to (iii) admits a NL-2-flow f. The restriction $f' := f|_{E(D)}$ now gives rise to a NL-2-flow on D, finishing the proof of (iii) ⇒ (ii).
The equivalences of (ii),(iv) and (iii),(v) now can be derived from the fact that the even edge subsets of D are exactly the supports of $\mathbb{Z}_2$-flows on D. □

It is not at all trivial to see why the equivalence theorem above should hold true in general, i.e., even for not necessarily planar digraphs D. The corresponding generalized statement does indeed hold true, and was proven by Barbara Altenbokum and Winfried Hochstättler in a pretty recent research project, cf. [AH18].

Theorem 4.50 (Altenbokum and Hochstättler, cf. [Hoc18] and [AH18])**.** *Let D be a digraph, $k \in \mathbb{N}$ and $(N, +)$ an abelian group of order k. Then the following are equivalent:*

(i) D admits a NL-k-flow.

(ii) D admits a NL-$\mathbb{Z}_k$-flow

(iii) D admits a NL-N-flow.

Idea of a proof. The idea of the proof in [Alt18] is analogous to the one given in Section 4.3 for NZ-flows, by showing that the number of algebraic NL-N-flows on the abelian group N is a polynomial in $|N|$, thereby immediately proving the equivalence of (ii) and (iii). (ii) follows from (i) by taking all values of a given k-flow modulo k, and (ii) $\Rightarrow$ (i) again is done by balancing the excesses of a given $\mathbb{Z}_k$-flow considered as an integer pre-flow to zero, compare the proof of 4.34. $\square$

We conclude this introduction with a generalization of the above notions of digraph colourings and NL-flows to the wider context of oriented regular matroids as introduced in the preliminaries, which will include digraphs in the form of their corresponding oriented graphic matroids as a special case. When stepping from digraphs to oriented matroids, the request of total cyclicity means that there are no non-empty all-$\oplus$ resp. all-$\ominus$ cocircuits. Again, we can study these flows in the wider context of arbitrary abelian groups as underlying algebraic structures:

Definition 4.51. Let $\omega(M)$ be an oriented regular matroid (a digraphoid). Let $k \geq 1$ be a natural number and $(N, +)$ some finite abelian group with neutral element e_N. A *Neumann-Lara-k-coflow* on $\omega(M)$ is defined as a tension on $\omega(M)$ ranging in $\{0, \pm 1, ..., \pm(k-1)\}$, such that $\omega(M) - \mathrm{supp}(f)$ contains no all-$\oplus$- resp. all-$\ominus$-circuits, i.e., there are no directed circuits contained in $\overline{\mathrm{supp}(f)} = f^{-1}(\{0\})$. Analogously, a *Neumann-Lara-k-flow* is defined to be an integer flow on $\omega(M)$ ranging in $\{0, \pm 1, ..., \pm(k-1)\}$, such that there are no directed cocircuits contained in $\overline{\mathrm{supp}(f)} = f^{-1}(\{0\})$, i.e., $\omega(M)/\mathrm{supp}(f)$ is totally cyclic.
In the same way, *(algebraic) Nowhere-Zero-N-coflows* resp. *(algebraic) Nowhere-Zero-N-flows* are the tensions resp. flows on $\omega(M)$ with respect to $(N, +)$ such that $f^{-1}(\{e_N\})$ contains no all-$\oplus$- resp. all-$\ominus$-circuits resp. cocircuits.

As in the case of digraphs, we may now define the NL-flow index of a digraphoid $\omega(M)$ as the least $k \in \mathbb{N}$ for which it admits a NL-flow:

Definition 4.52. Let $\omega(M)$ be a digraphoid. The quantity

$$\vec{\xi}(\omega(M)) := \min\{k \in \mathbb{N} | \omega(M) \text{ admits a NL-}k\text{-flow.}\}$$

is called the *NL-flow index* of $\omega(M)$.
Considering the regular matroid M, its NL-flow index $\vec{\xi}(M)$ is defined as the maximal NL-flow index of all of its orientations.

Since flows and tensions on a digraph D are exactly the flows and tensions of the corresponding orientation of the graphic matroid $M(U(D))$, the following is an immediate consequence of the definitions of NL-(co)flows on digraphs and digraphoids:

Remark 4.53. Let D be a digraph, $G := U(D)$ and $\omega(M(G))$ its corresponding digraphoid. Then for every natural $k \geq 1$ and any finite abelian group $(N, +)$, an assignment $f : E(D) = E(\omega(M(G))) \to \mathbb{Z}$ resp. $f : E(D) = E(\omega(M(G))) \to N$ is a NL-k- resp. NL-N- (co)flow with respect to D, iff it is one with respect to $\omega(M(G))$. We thus have $\vec{\xi}(D) = \vec{\xi}(\omega(M(G)))$.

Moreover, as in the case of NZ-flows, the duality presented above for the planar case (i.e., $\vec{\chi}(D) = \vec{\xi}(D^*)$) now admits a generalization to dual pairs of matroids as follows:

Definition 4.54. Let $\omega(M)$ be a digraphoid. The NL-flow index of its dual digraphoid $\omega(M)^*$ or equivalently, the minimal $k \in \mathbb{N}$ for which $\omega(M)$ admits a NL-k-coflow, is defined as the *dichromatic number* of $\omega(M)$, denoted by $\vec{\chi}(\omega(M))$.
Given a regular matroid such as M, its dichromatic number $\vec{\chi}(M)$ is defined as the maximal dichromatic number of a digraphoid defined on it, or equivalently, $\vec{\chi}(M) := \vec{\xi}(M^*)$.

Using proposition 4.46 and remark 4.53, we immediately get that $\vec{\chi}(D) = \vec{\chi}(\omega(M(G)))$ ($G := U(D)$, ω is the corresponding orientation) for every digraph D, and thus, the above is a fitting generalization of digraph colourings to digraphoids.
It is worth mentioning that the authors of [AH18] also showed a version of the Equivalence Theorem 4.50 presented above for digraphoids, which is restated in the following without giving a proof:

Theorem 4.55 (cf. [Hoc18])**.** *Let $\omega(M)$ be a digraphoid and $k \in \mathbb{N}$, $(N, +)$ an abelian group of order k. Then the following are equivalent:*

(i) $\omega(M)$ admits a NL-k-flow.

(ii) $\omega(M)$ admits a NL-$\mathbb{Z}_k$-flow.

(iii) $\omega(M)$ admits a NL-N-flow.

Although NL-coflows and -flows can be introduced for general oriented matroids as done by Hochstättler in ([Hoc17], Section 5), it does not seem likely that there is a generalization of the equivalence theorem above to the setting of general oriented matroids, since their proof involves specific properties of orientations on regular matroids, e.g. the existence of unimodular representation matrices (cf. Theorem 3.92).

5 Previous Partial Results

5.1 Vertex Arboricity and Coindependent Flows

If we are given some graph and want to find legal digraph colourings of all its orientations using preferably few colours (i.e., determine its dichromatic number), a simple idea could be the following: We do not consider the special respective properties of each orientation, but omit the orientations and simply look for a vertex-colouring of the actual underlying graph, which yields a legal digraph colouring of *all* possible orientations *at the same time*. Since every cycle in the underlying graph can be made directed in some orientations, this is obviously equivalent to forbidding monochromatic cycles in the considered vertex-colouring. This concept was introduced in [CKW68] under the name of *Point Arboricity* and has been studied in several research papers, now mainly appearing as the so-called *Vertex Arboricity* of a graph:

Definition 5.1. Let G be a graph. A colouring $c : V(G) \to \{0, ..., k-1\}$ is called a *legal va-k-colouring* of G, if there are no monochromatic cycles in G with respect to c.
The least natural number $k \in \mathbb{N}$ for which G admits a legal va-k-colouring is called the *Vertex Arborcity* of G, denoted by $\mathrm{va}(G)$.

Observation 5.2. *Let G be a graph and $\mathcal{O}(G)$ some orientation of G. Then any legal va-k-colouring of G is a legal digraph colouring of $\mathcal{O}(G)$. Hence, $\vec{\chi}(G) \leq \mathrm{va}(G)$.*

In the following, we will mainly deal with va-colourings of *simple* graphs: On the one hand we obviously should exclude loops, since they cannot admit legal va-colourings, while on the other hand, multiple edges between a pair of vertices will force us to use distinct colours at those, yielding a mixed version of usual and va- graph colourings, which we exclude from our following considerations.
If we look e.g. at small examples of simple planar graphs, it seems easy to find legal va-2-colourings, as is illustrated by the following example of the icosahedron graph.

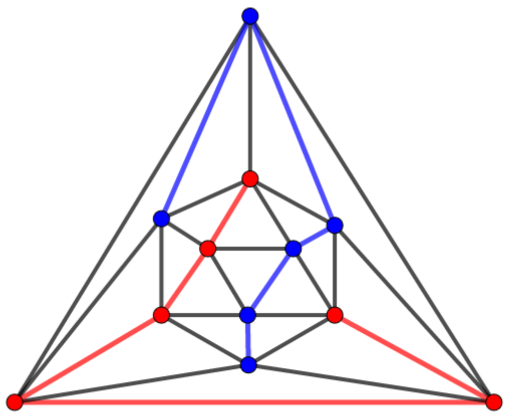


Figure 9: Red and blue edges represent represent the respective monochromatic edges. Since both induced subgraphs are forests, this is a legal va-2-colouring of the icosahedron graph.

On the other hand, as we will see in the following when considering a dual concept of vertex arboricity in the context of Neumann-Lara-Flows, for larger simple planar graphs, three colours might be necessary in some cases. First of all, as was proven in section 4.5, given a planar graph with some orientation $\mathcal{O}(G)$, a legal k-digraph colouring of $\mathcal{O}(G)$ corresponds to a k-flow f on G^*, such that the digraph $\mathcal{O}(G)^*/\mathrm{supp}(f)$ is totally cyclic ($\mathcal{O}(G)^*$ denotes the directed dual of $\mathcal{O}(G)$) and vice versa. Thus, a legal va-k-colouring of G corresponds to a k-flow on some orientation of G^*, such that $\mathcal{O}(G)^*/\mathrm{supp}(f)$ is totally cyclic for *any* orientation of G^* resp. $G^*/\mathrm{supp}(f)$. Since each cut in $G^*/\mathrm{supp}(f)$ can be made directed in some orientations, this is equivalent to requiring that $G^*/\mathrm{supp}(f)$ does not contain any non-empty cuts at all, i.e., it consists of isolated vertices and loops. This again means that each pair of vertices $u \neq v \in V(G^*)$ contained in the connected graph G^* admits a connecting path inside $\mathrm{supp}(f)$, in other words, $\mathrm{supp}(f)$ contains a spanning tree. When dealing with flows, we may obviously restrict ourselves to connected graphs. We thus finally end up with the following notion:

Definition 5.3. Let D be a connected digraph. An integer flow

$$f : E(D) \to \{0, \pm 1, ..., \pm (k-1)\}$$

on D (and thus on any orientation of $G := U(D)$), such that $\mathrm{supp}(f)$ contains a spanning tree, is defined to be a coindependent k-flow on D resp. G. The smallest natural number $k \geq 1$ for which G admits a coindependent k-flow is called the *coindependent flow index* of G and is denoted by $\xi_{\mathrm{coin}}(G)$.
More generally, if $(N, +)$ is an arbitrary abelian group with neutral element e_N, an N-flow $f : E(D) \to (N, +)$ on D (and thus any orientation of G) is called a *coindependent N-flow* of D resp. G, iff $\mathrm{supp}(f) := f^{-1}(N \backslash \{e_N\})$ contains a spanning tree of G.

As in the primal case for the vertex arboricity, coindependent flows give rise to Neumann-Lara-flows on any orientation of a given graph. Again, as in the case of Nowhere-Zero- and Neumann-Lara-flows, it can be shown that the existence of coindependent flows is a pure graph property and does not depend on the underlying algebraic structure, but just on the order of the abelian group the arithmetics are defined on.

Proposition 5.4. *Let G be a connected graph equipped with some fixed orientation $\mathcal{O}(G)$ and $(N, +)$ an abelian group with neutral element e_N. Given a flow $f : E(\mathcal{O}(G)) \to N$, f is a coindependent N-flow on G if and only if the corresponding flow $f_{\mathcal{O}'(G)} : E(\mathcal{O}'(G)) \to N$,*

$$f_{\mathcal{O}'(G)}(e) := \begin{cases} f(e), & \text{if } e \text{ is oriented the same in } \mathcal{O}(G) \text{ and } \mathcal{O}'(G) \\ -f(e), & \text{if } e \text{ is oriented differently in } \mathcal{O}(G) \text{ and } \mathcal{O}'(G) \end{cases}$$

on $\mathcal{O}'(G)$ is a NL-N-flow on $\mathcal{O}'(G)$ for any orientation $\mathcal{O}'(G)$ of G.

Proof. If $\mathrm{supp}(f) = \mathrm{supp}(f_{\mathcal{O}'(G)})$ (for any orientation $\mathcal{O}'(G)$ of G) contains a spanning tree of G, then $\mathcal{O}'(G)/\mathrm{supp}(f_{\mathcal{O}'(G)})$ consists of a single vertex with a couple of directed loops, which is strongly connected and thus, $f_{\mathcal{O}'(G)}$ is a NL-N-flow on $\mathcal{O}'(G)$.

On the other hand, if $f_{\mathcal{O}'(G)}$ is a NL-N-flow for each orientation $\mathcal{O}'$, then any orientation of the graph $G/\operatorname{supp}(f)$ appears as the contraction of $\operatorname{supp}(f) = \operatorname{supp}(f_{\mathcal{O}'(G)})$ in the digraph $\mathcal{O}'(G)$ for some orientations $\mathcal{O}'(G)$ and thus is strongly connected. Since any cut in $G/\operatorname{supp}(f)$ could be made directed in some orientations, this means that the graph $G/\operatorname{supp}(f)$ does not contain non-empty cuts, i.e., every cut in the connected graph G contains at least one edge of $\operatorname{supp}(f)$, implying that $\operatorname{supp}(f)$ contains a spanning tree as claimed. □

Corollary 5.5. *Let $k \geq 1$ be a natural number and $(N, +)$ an abelian group with neutral element e_N of order $|N| = k$. Then for any graph G, the following are equivalent:*

- *There is a coindependent k-flow on G.*
- *There is a coindependent $\mathbb{Z}_k$-flow on G.*
- *There is a coindependent N-flow on G.*

Proof. According to the Equivalence Theorem 4.50 for digraphs, for each orientation $\mathcal{O}'(G)$, the existence of a NL-k-, a NL-$\mathbb{Z}_k$- and of a NL-N-flow are equivalent. The claim now is an immediate consequence of the above proposition 5.4. □

Observation 5.6. *Let G be a connected planar graph and G^* some (connected) planar dual. For each $k \in \mathbb{N}$, G admits a legal va-k-colouring if and only if G^* admits a coindependent k-flow. Thus, $\operatorname{va}(G) = \xi_{\text{coin}}(G^*)$.*

Proof. From the above considerations it becomes clear that each va-k-colouring of G guarantees the existence of a coindependent k-flow on G^*. On the other hand, assume we are given such a coindependent k-flow f on G^*, and let $\mathcal{O}(G), \mathcal{O}(G)^*$ be the dual pair of orientations f is defined on. By taking f modulo k, we get a coindependent $\mathbb{Z}_k$-flow on $\mathcal{O}(G)^*$. Taking the same assignment of integer values for the corresponding edges of $\mathcal{O}(G)$, this gives rise to a tension $g : E(\mathcal{O}(G)) \to \mathbb{Z}_k$ on $\mathcal{O}(G)$. We fix a vertex $v \in V(G)$ and define a k-vertex-colouring of G according to $c : V(G) \to \mathbb{Z}_k \cong \{0, ..., k-1\}, c(w) := \sum_{e \in P^+} g(e) - \sum_{e \in P^-} g(e)$, where $P = (P^+, P^-)$ is an oriented path in G resp. $\mathcal{O}(G)$ starting in v and ending in w, such that P^+ contains the edges which are oriented forwards with respect to $\mathcal{O}(G)$ and $P^- = E(P) \setminus P^+$. Since g is a tension, this colouring is thereby well-defined and additionally satisfies $c(x_2) - c(x_1) = g(e)$ for each arc $e = (x_1, x_2)$ in $\mathcal{O}(G)$. We now claim that this already is a legal va-k-colouring of G: Assume for contrary there was a cycle C in G which is monochromatic with respect to c, i.e., $g(e) = 0$ for each $e \in E(C)$. In G^*, $E(C)$ thus corresponds to the edge set of a non-empty cut which contains no edge of $\operatorname{supp}(f \bmod k) = \operatorname{supp}(f)$, contradicting the fact that $\operatorname{supp}(f)$, as f is a coindependent k-flow on the connected graph G^*, has to contain a spanning tree. This implies the reverse implication and hence, the claim follows. □

After having introduced the most important formal notions and terminology, we now get back to the vertex arboricity of simple planar graphs and show that 3 is the best upper bound we may derive in this case. With the following two statements, we first deduce the existence of legal va-3-colourings for planar graphs, as was already done in [CK69]. The proof of the result is actually very easy and simple, and is (as the proof of 6-colourability of simple planar graphs presented in 4.1) based on an inequality including the degeneracy of the graph:

Theorem 5.7 (cf. e.g. [CK69]). *Let G be a simple graph. Then* $\mathrm{va}(G) \leq \lceil \frac{\mathrm{col}(G)}{2} \rceil$.

Proof. We prove the assertion using strong induction on the number of vertices $n = |V(G)|$ of G. If $n = 1$, the result is trivial, so assume that $n \geq 2$ and the claim holds true for all simple planar graphs on at most $n-1$ vertices. The definition of the colouring number yields a vertex v of degree at most $d(G)$ in G. According to the inductive assumption, the simple planar graph $G - v$ on $n - 1$ vertices, which also has to be $d(G)$-degenerate, admits a va-$\lceil \frac{d(G)+1}{2} \rceil$-colouring $c' : V(G)\backslash\{v\} \to \{0, ..., \lceil \frac{d(G)+1}{2} \rceil - 1\}$. Since $|N(v)| \leq d(G) < d(G) + 1 \leq 2 \cdot \lceil \frac{d(G)+1}{2} \rceil$, there is a colour $c_v \in \{0, 1, ..., \lceil \frac{d(G)+1}{2} \rceil - 1\}$ which appears at most once in $N(v)$. Define $c : V(G) \to \{0, ..., \lceil \frac{d(G)+1}{2} \rceil - 1\}$ by $c(w) := c'(w), w \in V(G)\backslash\{v\}, c(v) := c_v$. This has to be a legal va-$\lceil \frac{d(G)+1}{2} \rceil$-colouring of G: If there was a monochromatic cycle C in G, it would have to pass v, since c' is a legal va-colouring of $G - v$. Therefore, v admits at least two neighbours on C and thus in $N(v)$ coloured c_v, contradicting our choice of c_v.
The principle of induction now yields the claim. □

The 5-degeneracy of planar graphs now immediately yields the following:

Corollary 5.8. *If G is a simple planar graph, then* $\mathrm{va}(G) \leq 3$.

For the purpose of classifying the simple planar graphs with vertex arboricity 2, we introduce the following equivalent notion of coindependent 2-flows in graphs:

Theorem 5.9. *Let G be a graph. Then the coindependent $\mathbb{Z}_2$-flows on G are exactly the assignments $f_E : E(G) \to \mathbb{Z}_2, f_E = \mathbb{1}_E$, on each orientation of G, where E is a connected even spanning subgraph of G.*

Proof. The given assignments are obviously flows (since $n \cdot 1 = 0$ for every even integer $n \in \mathbb{Z}$ in $\mathbb{Z}_2$), and since $\mathrm{supp}(f) = E$ induces a connected graph, it contains the edge set of a spanning tree which then also is a spanning tree of G.
On the other hand, the $\mathbb{Z}_2$-flows on G were defined as $\mathbb{Z}_2 = \{0, 1\}$-flows on some orientation of G, i.e., they are (since $1 = -1$ in $\mathbb{Z}_2$) assignments of the form $\mathbb{1}_E$ where $E \subseteq E(G)$ is an even edge subset. Furthermore, E has to contain a spanning tree in G and therefore must be spanning and connected. □

This immediately yields the following result in the case of planar triangulations:

Corollary 5.10. *Let T be a planar triangulation and T^* its 3-edge-connected cubic planar dual graph. Then T admits a va-2-colouring if and only if T^* is Hamiltonian.*

Proof. According to the above, T admits a legal va-2-colouring if and only if T^* admits a coindependent 2-flow, which again means the existence of a spanning connected even edge subset in T^*. Since this is a cubic graph, the even edge subsets are exactly unions of vertex-disjoint cycles, and hence, T admits a legal va-2-colouring if and only if T^* contains a spanning cycle, i.e., is Hamiltonian. □

This result now immediately gives rise to examples of planar triangulations and thus simple planar graphs which admit no va-2-colouring, i.e., which have vertex-arboricity 3:

Example 5.11. The planar triangulation depicted in the following figure is the dual graph of a non-Hamiltonian cubic planar 3-connected graph with as few as possible edges resp. vertices, i.e., the smallest planar triangulation with vertex arborcity 3 (for more detailed explanations see [RW08]):

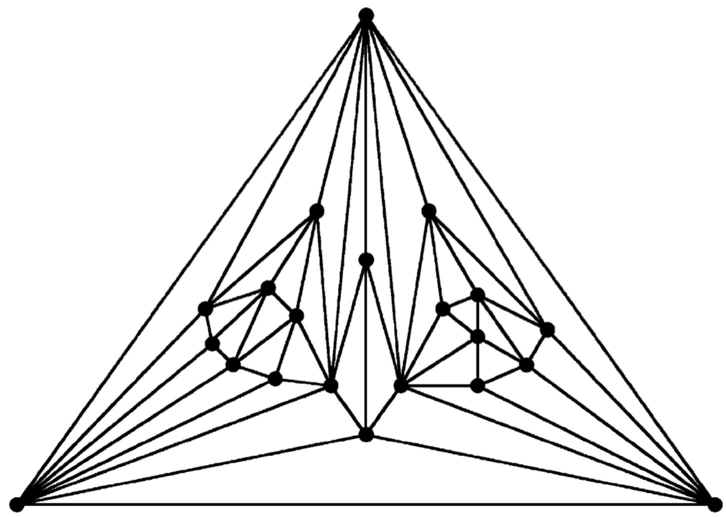


Figure 10: The smallest plane triangulation with vertex arboricity 3. The figure is taken from [RW08].

As in the case of Neumann-Lara and Nowhere-Zero-flows, coindependent flows may be generalized to the setting of regular matroids, which is shortly sketched in the following:

Definition 5.12. Let M be a regular matroid equipped with some orientation $\omega(M)$. Let $k \geq 1$ be a given natural number and $(N, +)$ an abelian group of order k with neutral element e_N.
A *coindependent k-flow* on M resp. $\omega(M)$ is defined as a flow ranging in $\{0, \pm 1, ..., \pm(k-1)\}$ on $\omega(M)$ (and thus on any orientation $\omega'(M)$ of M according to Corollary 3.101) such that the support $\mathrm{supp}(f) = \overline{f^{-1}(\{0\})}$ contains a basis B of M, or equivalently, $\overline{\mathrm{supp}(f)}$ is independent in the dual matroid M^* (which explains the name coindependent).
Analogously, an *(algebraic) coindependent N-flow* on $\omega(M)$ (resp. M resp. any orientation of M) is defined as a flow $f : E(\omega(M)) \to N$ whose support $\mathrm{supp}(f) := f^{-1}(N \backslash \{e_N\})$ contains a basis B of M.
The least natural number $k \in \mathbb{N}$, for which M admits a coindependent k-flow, is defined as the *coindependent flow index of M*, denoted by $\xi_{\mathrm{coin}}(M)$.

Analogous to the graph case, we have the following more or less trivial equivalence results. The proofs are either analogous to the ones for the digraph- resp. graph-versions above or are immediate consequences of well-known relations between graphs, digraphs and their respective regular graphic matroids resp. orientations of them. Most of them will reappear as special cases of results from Subsection 6.2 below, where k-coindependent flows may be rediscovered as so-called coindependent $(k,1)$-flows on regular matroids and graphs.

Proposition 5.13. *Let M be a regular matroid equipped with some fixed orientation $\omega(M)$ and $(N,+)$ an abelian group with neutral element e_N. Given a flow $f: E(\omega(M)) \to N$, f is a coindependent N-flow on M if and only if the corresponding flow $f_{\omega'(M)}$ on $\omega'(M)$ with $\mathrm{supp}(f) = \mathrm{supp}(f_{\omega'(M)})$ defined according to Corollary 3.101 is a NL-N-flow on $\omega'(M)$ for every orientation of M.*

Proof. Completely analogous to the case of digraphs. □

Remark 5.14. Let G be a connected graph equipped with some orientation $\mathcal{O}(G)$, $k \in \mathbb{N}$ and $(N,+)$ some abelian group. Denote by $M := M(G)$ the graphic matroid of G with the orientation $\omega(M)$ corresponding to $\mathcal{O}(G)$. Then an assignment $f: E(G) = E(M) \to \{0, \pm 1, ..., \pm(k-1)\}$ is a coindependent k-flow with respect to $\mathcal{O}(G)$ and G iff it is a coindependent k-flow with respect to $\omega(M)$ and M. At the same time, an assignment $f: E(G) = E(M) \to N$ is a coindependent N-flow with respect to $\mathcal{O}(G)$ and G iff it is one with respect to $\omega(M)$ and M.

Proof. This follows immediately from the equivalence of flows on graphs and flows on their corresponding regular graphic matroids as well as from the correspondence of edge sets of spanning trees in G and the bases in $M = M(G)$. □

Corollary 5.15. *Let $k \geq 1$ be a natural number and $(N,+)$ an abelian group with neutral element e_N of order $|N| = k$. Then for any regular matroid M, the following are equivalent:*

- *There is a coindependent k-flow on M.*
- *There is a coindependent $\mathbb{Z}_k$-flow on M.*
- *There is a coindependent N-flow on M.*

Proof. According to the Equivalence Theorem 4.55 for NL-flows on regular oriented matroids, for each orientation $\omega'(M)$, the existence of a NL-k-, a NL-$\mathbb{Z}_k$- and of a NL-N-flow are equivalent. The claim now is an immediate consequence of the Proposition 5.13 above. □

5.2 Colouring Planar Digraphs of Digirth at least 4

Over a long period of time, besides the techniques explained in the previous chapter and used e.g. in [CK69] and [Hoc17], which more or less try to find structures in the underlying *graphs* of the considered digraphs in order to legally colour *all* the possible orientations of such a graph with the *same* fixed 2-colouring resp. find a 2-flow in the graph which yields a valid NL-2-flow for all the possible orientations (see section 5.1), there hasn't really been much progress on a proof of the 2-colour conjecture. However, the analysis in the previous paragraph 5.1 has shown that using the information given by the special orientation of the actual *di*graph may be necessary for some graphs.

In contrast, newer results, which have been recently published by Mohar et al. in [HM17] (2014) and [LB17] (2017), deal with relaxations of the conjecture in terms of forbidden directed cycles of small length in the digraph, i.e., they consider planar digraphs of so-called digirth at least five resp. four and prove the correctness of the conjecture in these subcases:

Definition 5.16. Let $\vec{G}$ be an arbitrary (multi-)digraph. The *digirth* $\mathrm{digir}(\vec{G})$ of $\vec{G}$ is defined as the minimal length of a directed cycle in $\vec{G}$, which is considered ∞ if $\vec{G}$ is acyclic.

Theorem 5.17 (Harutyunyan and Mohar, 2014, [HM17])**.** *Let $\vec{G}$ be a simple planar digraph of digirth at least five. Then $\vec{G}$ admits a legal 2-colouring.*

Theorem 5.18 (Li and Mohar, 2017, [LB17])**.** *Let $\vec{G}$ be a simple planar digraph with a vertex w_0 such that each directed triangle in $\vec{G}$ contains w_0. Then $\vec{G}$ admits a legal 2-colouring. Especially, each simple planar digraph of digirth at least four is 2-colourable.*

Hence, those two results address simple planar digraphs with excluded directed cycles of length four and three, while the newer result even allows a little strengthening in the sense of allowing directed triangles containing a fixed vertex. Clearly, forbidding directed cycles of length three makes the problem of finding a legal 2-colouring a lot easier, since the shorter the directed cycles in the considered digraph get, the more restrictive the request of a non-monochromatic colouring of these cycles becomes, but still it seems to be the best known result known so far.
In the following, we review the proof of Theorem 5.18. Since this obviously implies the earlier digirth-five result in Theorem 5.17, we concentrate on the former, which also deviates from the elaborate discharging techniques used in the earlier paper by using an elegant and simple argument in the dual graph. The fact that NL-flows as introduced in section 4 appear as the dual concept of digraph colourings makes it therefore much more convenient to present the proof in this dual setting, which we do in the following. The most essential ingredient used in [LB17] is the notion and existence of *Tutte-paths* in planar graphs, discovered by Tutte and later on Thomassen in [Tut46] resp. [Tho83]:

Definition 5.19 (cf. [Tho83])**.**

(i) Let G be a graph, and let H be a subgraph of G. An *H-component* of G is defined as a subgraph B of G, which is either the 2-vertex-graph induced by a chord of H (i.e., an edge $e \in E(G)\backslash E(H)$ with both ends in $V(H)$) or a component of $G - V(H)$ together with all the edges between B and $V(H)$, which are called *attachments*, the vertices in $V(B) \cap V(H)$ are called *vertices of attachment*. B is called *k-attached*, iff there are k vertices of attachment.

(ii) Let additionally C be a cycle in G, and $u, v \in V(G)$. A path P in G starting in u and ending in v is called a *Tutte path* with respect to C, if each P-component is at most 3-attached and moreover at most 2-attached whenever it contains an edge of C.

Theorem 5.20 (Thomassen, [Tho83])**.** *Let G be a 2-connected plane graph with (outer) facial cycle C. Let $u \neq v \in V(G), e \in E(G)$ such that v and e are contained in C. Then there is a Tutte path with respect to C, u, v containing e.*

We prepare the proof with the following lemma, which will reappear later on in section 7 when dealing with minimal counterexamples to a certain conjecture.

Lemma 5.21 (Li and Mohar, [LB17])**.** *Let $\vec{T}$ be an oriented planar triangulation with outer face $a_1a_2a_3$, such that there are no directed triangles. Then for every precolouring of a_1, a_2, a_3 in two colours there is a 2-colouring of $\vec{T}$ extending this precolouring.*

Proof, cf. [LB17]. We first of all prove the lemma in the case that the triangulation $T = U(\vec{T})$ is 4-connected and deduce the general result later on by an inductive argument. So assume in the following that $\vec{T}$ is 4-connected, i.e., it does not have separating triangles. We look at the thereby 3-edge-connected and internally 4-edge-connected directed dual $D := \vec{T}^*$ of $\vec{T}$ and denote by $v_0 \in V(D)$ the vertex representing the outer face of $\vec{T}$. By the duality between 2-colourings and NL-2-coflows resp. NL-2-flows in the dual graph of $\vec{T}$ which we pointed out in section 4.5, the claim is equivalent to the following (since D is connected, totally cyclic and strongly connected are equivalent terms):
Let $g : E(v_0) =: \{e_1, e_2, e_3\} \to \mathbb{Z}_2$ be an arbitrary legal pre-flow-assignment at v_0, i.e., $g(e_1)+g(e_2)+g(e_3) = 0$. Then there is a NL-$\mathbb{Z}_2$-flow f of D with $f(e_i) = g(e_i), i = 1, 2, 3$. Recalling that the supports of $\mathbb{Z}_2$-flows coincide with the even edge subsets of the underlying graph, this again is equivalent to:
For each subset $M \subseteq E(v_0)$ of even size there is an even edge set $E \subseteq E(D)$ in $U(D)$, such that D/E is totally cyclic/strongly connected, and $E \cap E(v_0) = M$.
In the following, our goal will be to construct such an even edge set E.

For this purpose, we differentiate between pre-flow-sets M of size 2 and 0.

In the first case, let $M = \{e_2, e_3\}$ (without loss of generality). Consider an arbitrary plane embedding of the planar digraph D. Since $D = \vec{T}^*$ is 3-regular and 2-connected, there is a unique face with facial cycle C in this embedding containing v_0 and the two incident edges e_2, e_3. Denote by $v_i, i = 1, 2, 3$ the neighbour of v_0 adjacent via e_i. First of all, due to standard results about 2-connected plane graphs (cf. preliminaries), we can find another plane embedding of the graph $U(D)$ such that C gets moved to the outer face, i.e., v_0, v_2, v_3, e_2, e_3 become outer vertices resp. edges. In the following, we will work with this embedding.
According to Theorem 5.20, there is a Tutte path P in D with respect to C, which connects the vertices v_0, v_2 and passes through e_3. Obviously, this path does not contain e_1 or e_2, since v_0 has degree 1 in P. Hence, $P + e_2$ completes P to a cycle in D, whose edge set is denoted by $E \subseteq E(D)$. Obviously, E is an even edge set in $U(D)$. Furthermore, $E \cap E(v_0) = E \cap \{e_1, e_2, e_3\} = \{e_2, e_3\} = M$ according to the above, so it remains to show that D/E is totally cyclic. We first observe that the graph $U(D) - V(E)$ consists of isolated vertices: Assume there was a connected component B of $U(D) - V(E)$ with at least two vertices. Let $S := U(D)[V(B), V(D)\setminus V(B)], |S| \geq 3$ be the corresponding cut in $U(D)$. Each edge $e \in S$ connects a vertex from B to a vertex in E, since else B was no connected component of $U(D) - V(E)$. Each vertex in $V(E)$ has degree $3 - 2 = 1$ in $U(D) - E$, which implies that no two edges of S meet in a common vertex of E and that S is a non-trivial cut. Since D is internally 4-edge-connected, we have $|S| \geq 4$ which means

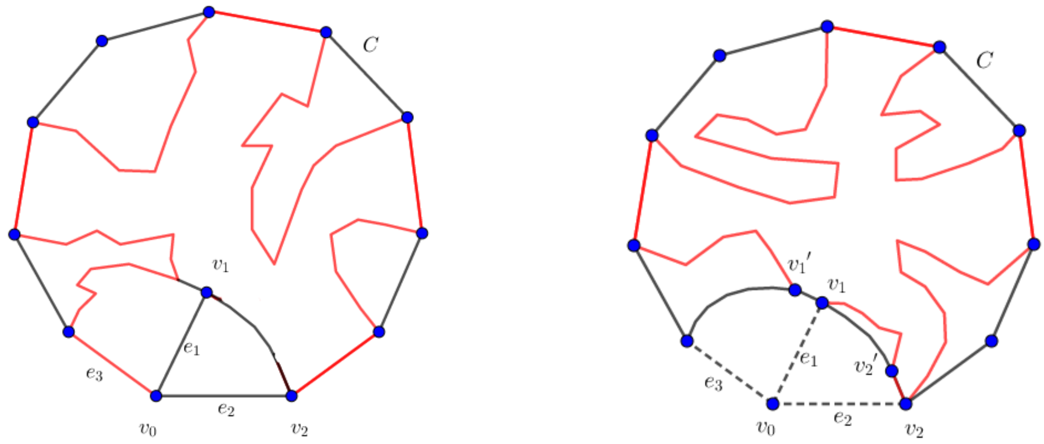


Figure 11: Illustration of the argument in the cases $|M| = 2$ (left) resp. $|M| = 0$ (right). The Tutte Paths used in the proofs are marked red.

that the E-component defined by B has 4 attachments to E. Since $E = P + e_2$, this means that also the P-component induced by B has at least 4 vertices of attachment, but this contradicts the definition of a Tutte path.
Finally this implies that D/E only consists of the vertex w_E corresponding to the contraction of E and the connections between w_E and the vertices in $V(D)\setminus V(E)$ without further adjacencies between each other. Since D does not contain any sources or sinks (since they would correspond to directed triangles in $\vec{T}$), this already implies the strong connectivity of D/E. Thus, we are done in the case of $|M| = 2$.

In the second case, let $M = \emptyset$. We take the same plane embedding of $U(D)$ as described above with the vertices v_0, v_2, v_3 on the outer face, but now consider the plane graph arising from this embedding by deleting v_0. This still is a 2-connected plane embedding, since $U(D)$ as the dual of a 3-connected simple graph was 3-connected. Denote by C' the new arising outer face-cycle, which now additionally contains v_1. Let v_1', v_2' be neighbours of v_1, v_2 on C'. They are distinct from v_1 and v_2, since otherwise v_0, v_1, v_2 would be a triangle in D and hence $\vec{T} = D^*$ not 4-connected, contradiction. According to Theorem 5.20, there is a Tutte path P in $U(D) - v_0$ with respect to C' which connects v_1 and v_1' and passes through the edge v_2v_2'. By adding the edge v_1v_1', this becomes the cycle $E := P + v_1v_1'$, which is an even edge set in $U(D) - v_0$ and hence $U(D)$ not containing v_0. This implies $E \cap E(v_0) = \emptyset = M$. Again, it remains to show that D/E is strongly connected. As above, we first consider non-trivial connected components B of $U(D) - V(E)$. If they do not contain the vertex v_0, they also are connected components of $U(D) - (V(E) \cup \{v_0\})$ and hence can be excluded as in the first case. So now, let B be such a component containing v_0 and hence (since it is non-trivial and $v_1, v_2 \in V(E)$) also v_3. There are two possibilities: Either $V(B) = \{v_0, v_3\}$, or one of the two additional edges incident to v_3, both of which lie on C', is also contained in B. In the latter case, according to definition 5.19 the P-component induced by B is at most 2-attached to $V(P) = V(E)$. But since the vertices of attachment each only contribute one edge to the cut $S := U(D)[V(B), V(D)\backslash V(B)]$ induced by B, this would give $|S| \leq 2$ and hence a contradiction to the 3-edge-connectivity of the planar dual $U(D)$. All in all, this implies that $U(D) - V(E)$ only consists of isolated vertices plus possibly the connected component $\{v_0, v_3\}$ of size 2. Hence, D/E again consists of a vertex w_E representing E and the connections to the components of $U(D) - V(E)$. Since all the isolated vertices are neither sources nor sinks in $U(D)$, this again implies that D/E is strongly connected in the case that v_0v_3 is contained in E and that $D/E - \{v_0, v_3\}$ is strongly connected in case it is not. Then, the only possibility for a directed cut in D/E is the one between the 2-vertex-component $\{v_0, v_3\}$ and $V(E)$. But if this was directed, either v_0 or v_3 would have to be a source or sink, which is not possible. This finishes the proof in the second case, and hence, we are done if $\vec{T}$ is 4-connected.

To show the complete statement, assume there was a minimal counterexample to the statement with respect to the number of vertices. If it was 4-connected, as proven above, the statement would be true, and hence, it has to contain a separating triangle $x_1x_2x_3$. Consider the directed planar triangulations $\vec{T_1}$ and $\vec{T_2}$ arising from deleting all inner resp. outer vertices of $x_1x_2x_3$, which obviously still have digirth at least four. Since they have fewer vertices than the original triangulation, the conjecture holds for them, and hence, for every precolouring of the outer face in two colours, there is a legal 2-colouring of the outer triangulation, inducing a precolouring on $x_1x_2x_3$ for the inner triangulation, which again can be extended to a legal 2-colouring. Together this gives a 2-colouring of the original triangulation. If this admitted a monochromatic directed cycle, it would have to pass $x_1x_2x_3$ two times, which either produces a monochromatic directed cycle in the outer or in the inner triangulation, contradicting the fact that we were given legal 2-colourings of $\vec{T_1}$ and $\vec{T_2}$. All in all, this shows that the claim does not admit counterexamples and hence is true. $\square$

We are now prepared for the proof of the digirth-four-result.

Proof (of Theorem 5.18), cf. [LB17]. We prove the theorem in the case that $\vec{G}$ is an oriented planar triangulation. The general result then simply follows due to the fact that $\vec{G}$ can be triangulated without producing additional directed cycles, by placing additional sources inside regions of length ≥ 4.
We furthermore may restrict ourselves on proving the theorem in the case where every separating triangle contains w_0: In the presence of separating triangles not containing w_0, as above, we can reduce using either the Lemma proved above or the statement of this theorem inductively. So assume that all separating triangles in $\vec{T}$ contain w_0, and consider the directed dual $D := \vec{T}^*$. The vertex w_0 incident to all the directed triangles in $\vec{T}$ corresponds to a facial cycle in $U(D)$, which we call C. Chose three arbitrary consecutive vertices a, b, c on C. According to Theorem 5.20, there is a Tutte path P in $U(D)$ with respect to C which connects a, b and passes through the edge bc on C. Let $E := P + ab$ again be the cycle arising from P by adding the edge ab which cannot be contained yet. E obviously is an even edge set in $U(D)$, and we will now show that D/E is strongly connected which finally implies that D admits a NL-2-flow and hence $\vec{T}$ is 2-colourable, as claimed. We show that analogously to the proof of Lemma 5.21, $U(D) - V(E)$ consists of isolated vertices:

Assume for contrary there was a non-trivial component B of $U(D) - V(E)$ containing at least one edge. According to the definition of a Tutte path, B has a most 3 vertices of attachment to $V(P) = V(E)$. Since E is a cycle in a cubic graph, this implies that the cut $U(D)[B, \overline{B}]$ contains at most three edges, i.e., is a non-trivial 3-cut (due to 3-edge-connectedness). According to the above, the separating triangle in $\vec{T}$ corresponding to this cut must contain w_0 and thus exactly two edges incident to w_0. In other words, two of the three attachment-edges, which are per definition also part of the P-component corresponding to B, are contained in C. This means that contrary to the above, there are only two vertices of attachment. This contradiction proves the partial claim.
Now, for the strong connectivity of D/E it suffices to show that none of those isolated vertices in $U(D) - V(E)$is a source nor a sink. According to the additional requirement, among the vertices in $V(D)\backslash V(E)$ only the vertices on C may be sources or sinks. It thus remains to show that all the vertices of C are contained in E: Assume there was a vertex $v \in V(C)\backslash V(E) = V(C)\backslash V(P)$. Then there is a unique P-component B containing it. As above, according to the definition of a Tutte path, since B contains an edge of C, B can only have two vertices of attachment to P resp. E. But each vertex on E has at most one neighbour in $V(B)\backslash V(E) \neq \emptyset$, and therefore $U(D)[V(B), V(D)\backslash V(B)]$ is a non-trivial cut in $U(D)$ of size at most two, which contradicts the 3-edge-connectivity of D, and hence D/E must be strongly connected. This finally implies the assertion. $\square$

5.3 The 2-Flow Conjecture

In this paragraph we want to discuss possibilities of extending the 2-colour conjecture in the dual context of NL-2-flows as stated in 4.5 to non-planar digraphs. We will first give some evidence for proposing the NL-2-Flow Conjecture as done by Hochstättler in [Hoc17], and later on demonstrate why the conjecture fails in general by giving a counterexample found by Kolja Knauer and Petru Valicov in [KV17]. Subsequently, we discuss possible relaxations of the conjecture and ways of making progress on those.

5.3.1 Four- and Five-Flow Conjecture

First of all, we want to take a look at what happens in the undirected context of NZ-flows of graphs. As is well-known, the Four-Colour Theorem proven in 1976 by Appel and Haken using massive computer calculations states that every loopless planar graph admits a vertex-colouring with at most 4 colours. If we dualize this statement as explained in section 4, we get the following:

Every bridgeless planar graph admits a Nowhere-Zero-4-flow.

If you consider bounds on chromatic/ dichromatic numbers of arbitrary simple graphs, it is easy to see that for example the family of complete graphs admits arbitrarily large (unbounded) chromatic and dichromatic numbers. On the other hand, a dualization of the chromatic or dichromatic number of a graph to an integer flow problem is not possible for non-planar (di-)graphs, since for arbitrary graphs, there is no well-defined way to generalize planar duals. Consequently, the fact that the chromatic numbers of arbitrary graphs are unbounded does not imply the same for the flow index of arbitrary graphs with sufficient edge-connectivity, and surprisingly, as was shown by Jaeger and Seymour in [Jae79] and [Sey81] respectively, a fixed number $k \in \mathbb{N}$ for which every 2-edge-connected (bridgeless) graph G admits a NZ-k-flow indeed exists. While Jaeger was first to show the existence of such a k by proving the existence of 8 or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ -flows respectively, Seymour improved this result by constructing Nowhere-Zero flows in $\mathbb{Z}_2 \times \mathbb{Z}_3$, i.e., showed $k \leq 6$ for 2-edge-connected graphs. What about lower bounds on this number k? Looking at planar graphs, we know that $k \geq 4$. At first sight, one might be tempted to conjecture that the dualized version of the 4-Colour-Theorem as described above extends to arbitrary bridgeless graphs, i.e. $k = 4$ is the minimal upper bound we were looking for, but unfortunately, there is a well-known 2-edge-connected 3-regular but non-planar graph, called *Petersen graph*, which does not admit a NZ-4, but only a NZ-5-flow. It is depicted below. The reason why the Petersen graph

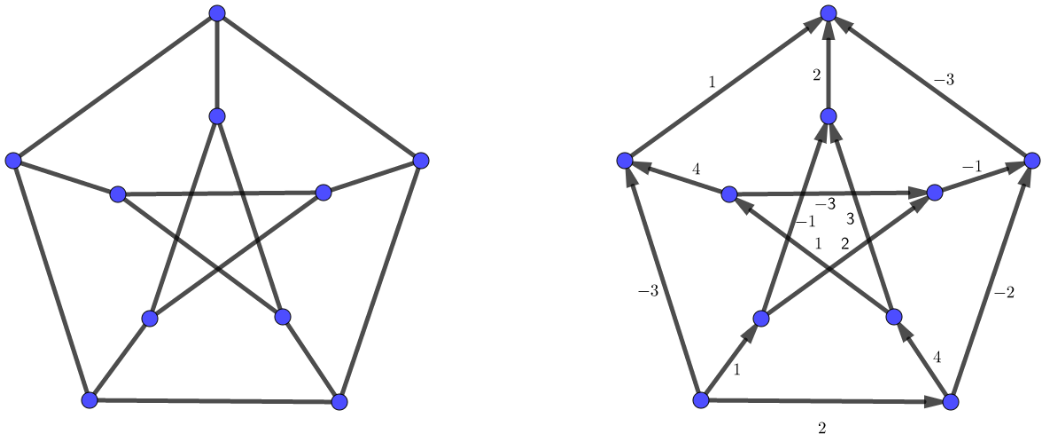


Figure 12: Illustration of the Petersen graph and a NZ-5-flow of it.

does not admit a NZ-4-flow is the following: $\mathbb{Z}_2 \times \mathbb{Z}_2$ is an abelian group of order 4, and hence, a graph G admits a NZ-4-flow if and only if there is a NZ-$\mathbb{Z}_2 \times \mathbb{Z}_2$-flow, which again is equivalent to the existence of a pair f_1, f_2 of 2-flows on G with $E(G) = \mathrm{supp}(f_1) \cup \mathrm{supp}(f_2)$, i.e., the edges of G can be covered by two even edge subsets. Suppose the Petersen graph had a NZ-4-flow. Then each of those edge sets therefore has to cover all the vertices and consists of vertex disjoint cycles. Let E_1, E_2 denote the two even edge sets. Then $E_1 \supseteq \overline{E_2}$. Since E_2 is a 2-factor in a 3-regular graph, this is equivalent to E_1 containing a perfect matching (1-factor), i.e., all the cycles in E_1 are of even length. Since the Petersen graph has 10 vertices, no Hamiltonian cycle but girth 5, this is impossible, and we end up with the desired contradiction.

Thus, the best known lower bound on the number k we are looking for is 5 in the case of general bridgeless graphs. The remaining open question is whether already every bridgeless graph admits a NZ-5-flow (as conjectured by Tutte in [Tut66] and since then known as the still unsolved 5-Flow-Conjecture) or if there is an example of a bridgeless graph only admitting a NZ-6-flow, i.e., Seymours upper bound can not be improved.

Computer calculations have been verifying the 5-Flow-Conjecture for a large number of graphs and it is nowadays widely believed to hold true.
Furthermore, it turned out that the Petersen graph really seems to be the only substantial obstruction to the existence of a NZ-4-flow, i.e., all the counterexamples found so far admit a Petersen Minor. Hence, the following conjecture, known as *4-Flow-Conjecture*, which obviously is a generalization of the 4-Colour-Theorem as formulated above, was again proposed by Tutte in [Tut66]:

Every bridgeless graph G without a Petersen graph as minor admits a Nowhere-Zero-4-flow.

5.3.2 The Conjecture

The considerations for NZ-flows on bridgeless graphs show that dualizing the colouring theory for planar graphs and omitting the planarity condition in this dualized setting turns out to yield constant bounds on a number k for which NZ-k-flows exist. Since a NZ-k-flow can be considered as a NL-k-flow for every orientation of a graph G (cf. section 4), we thereby also know that 6 is an upper bound on the minimal number k for which every bridgeless digraph D admits a NL-k-flow as well. If we think e.g. of the 2-Colour-Conjecture, it seems likely that in the case of NL- instead of NZ-flows, much better and lower bounds on such a minimal k should exist. Actually, in comparison to the still open conjectures on Nowhere-Zero-Flows, in this section we will provide the exact value of such a minimal k in the case of NL-flows on 3-edge-connected digraphs, by first illustrating a positive result derived in [Hoc17] by DeVos and Hochstättler proving the existence of NL-3-flows for each 3-edge-connected digraph D and later on falsifying the 2-Flow-Conjecture by reviewing a recently found example of a 3-edge-connected digraph D without a NL-2-flow by Kolja Knauer and Petru Valicov in [KV17].

We start with some easier results concerning the existence of NL-flows under different edge connectivity assumptions, which are furthermore coindependent flows of the underlying graph. For this purpose, we need the following two statements about the existence of spanning trees with various intersection properties:

Lemma 5.22 (cf. Kundu 1974 [Kun74], Jaeger 1979 [Jae79])**.**

(i) If G is a 4-edge-connected graph, it admits two edge-disjoint spanning trees.

(ii) If G is a 3-edge-connected graph, it admits three spanning trees with empty three-fold edge-intersection.

The following is taken from [Hoc17]:

Theorem 5.23 (Hochstättler, cf. [Hoc17])**.** *Let D be a digraph.*

(i) If D is 2-edge-connected, then D admits a NL-6-flow.

(ii) If D is 3-edge-connected, then D admits a NL-4-flow.

(iii) If D is 4-edge-connected, then D admits a NL-2-flow.

Proof.

(i) If D is 2-edge-connected, then D admits a NZ-6-flow according to Seymours 6-Flow-Theorem (cf. [Sey81]), and hence also a NL-6-flow.

(ii) If D is 3-edge-connected, according to Lemma 5.22, there are three spanning trees B_1, B_2, B_3 of $G := U(D)$ with $B_1 \cap B_2 \cap B_3 = \emptyset$. For each $e \in B_1$ and $i \in \{2, 3\}$, define $\phi_{e,i} = 0$ to be the constant zero-flow on D whenever $e \in B_i$, while in the case of $e \notin B_i$, it is the canonical $\mathbb{Z}_2$-flow on D whose support is the unique chord-cycle $C(e, B_i)$ through e and B_i. We put $\phi_i := \sum_{e \in B_1} \phi_{e,i}$, which again is a $\mathbb{Z}_2$-flow on D whose support contains $B_1 \backslash B_i$. Hence, $\phi := (\phi_2, \phi_3)$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on D, whose support contains $(B_1 \backslash B_2) \cup (B_1 \backslash B_3) = B_1 \backslash (B_2 \cap B_3) = B_1$, and we are done, since this shows the existence of a NL-$\mathbb{Z}_2 \times \mathbb{Z}_2$ and hence a NL-4-flow on D.

(iii) If D is 4-edge-connected, then, according to Lemma 5.22, there are two disjoint spanning trees B_1, B_2 in $G := U(D)$. For each edge $e \in E_1$, let $C(e, B_2)$ be the unique cycle in $B_2 + e$. For each $e \in B_1$, define ϕ_e to be the canonical $\mathbb{Z}_2$-flow on D with support $C(e, B_2)$. We define the $\mathbb{Z}_2$-flow f by $f := \sum_{e \in B_1} \phi_e$. Obviously, $B_1 \subseteq \mathrm{supp}(f)$, and hence, f is a coindependent 2-flow of G, which proves the claim.

□

After having proven those upper bounds on a respective number k for which NL-k-flows exist, we can ask whether the achieved statements are already optimal in each of the three cases, or if they allow improvements. For (i) and (iii), this is actually pretty easy to answer:

- First of all, as was shown in Section 4, Proposition 4.44, for each graph G, the subdivision digraph $S(G)$ has a NL-k-flow if and only if G admits a NZ-k-flow. As was demonstrated above, the 3-edge-connected Petersen graph P_{10} does not admit a NZ-4-flow and therefore, the 2-edge-connected digraph $S(P_{10})$ does only admit a NL-5, but no NL-4-flow. On the other hand, improving the bound of 6 in the theorem above to 5 is equivalent to proving the 5-Flow-Conjecture and hence is not an easy task: If every 2-edge-connected digraph admitted a NL-5-flow, also each digraph $S(G)$, where G is a 2-edge-connected graph, admitted one, which gives a NZ-5-flow for each such graph. Hence, making progress in the 2-edge-connected case for NL-flows is equivalent to proving the 5-flow-conjecture.
- If D is a 4-edge-connected digraph, D admits a NL-1-flow if and only if D is totally cyclic. Since acyclic 4-edge-connected digraphs exist, there is no room for improvement in the 4-edge-connected case.

The question whether there is a NL-3- or even a NL-2-flow on each 3-edge-connected digraph D is much less clear. In the following, we will improve the bound of 4 deduced above to 3 by proving that 3-edge-connected graphs admit coindependent 3-flows as already established in [Hoc17], Theorems 9 and 10. For this purpose, we sketch some of the theory introduced by Seymour in [Sey81] when proving his 6-Flow-Theorem. We start with the definition of the cyclic k-closure of an edge set X in a graph, $k \in \mathbb{N}$:

Definition and Proposition 5.24 (cf. [Sey81])**.** Let G be a graph and $X \subseteq E(G)$ a set of edges in G. We define $\mathcal{Y}(X) \subseteq 2^{E(G)}$ as the set of edge sets $Y \subseteq E(G)$ such that $X \subseteq Y$ and for each cycle C in G, either C is contained in $G[Y]$ or $|E(C)\backslash Y| \geq k+1$, i.e., C has at least $k+1$ edges outside Y.
Obviously $E(G) \in \mathcal{Y}(X) \neq \emptyset$. Let $Y_0 \in \mathcal{Y}(X)$ be of minimal size. Then Y_0 is uniquely determined by this property and is defined to be the cyclic k-closure $\langle X\rangle_k$ of X.
Furthermore, the mapping $\langle\cdot\rangle_k : X \subseteq E(G) \rightarrow \langle X\rangle_k$ defined in this way is a closure operator, i.e., it is extending ($X \subseteq \langle X\rangle_k$), inclusion-monotone and idempotent.

Proof. In order to prove the uniqueness of Y_0, assume for contradiction there was another edge set $Y_0 \neq Y_1 \in \mathcal{Y}(X)$ such that $|Y_0| = |Y_1|$. We show that $X \subseteq Y_0 \cap Y_1 \in \mathcal{Y}(X)$ which then yields the desired contradiction ($|Y_0 \cap Y_1| < |Y_0|$). So let C be an arbitrary cycle in G whose edge set is not contained in $E(Y_0) \cap E(Y_1)$. Then either $E(C) \nsubseteq E(Y_0)$ or $E(C) \nsubseteq E(Y_1)$, which according to the definition of $\mathcal{Y}(X)$ means that $|E(C)\backslash Y_i| \geq k+1$ for some $i \in \{0,1\}$. Hence, we have $|E(C)\backslash(Y_0 \cap Y_1)| \geq |E(C)\backslash Y_i| \geq k+1$ in this case and we are done.
For each pair $X \subseteq Y \subseteq E(G)$, we obviously have $\mathcal{Y}(X) \supseteq \mathcal{Y}(Y)$ and hence $\langle Y\rangle_k \in \mathcal{Y}(X)$. As shown above, this guarantees $\langle Y\rangle_k \cap \langle X\rangle_k \in \mathcal{Y}(X)$, and therefore $|\langle Y\rangle_k \cap \langle X\rangle_k| \geq |\langle X\rangle_k|$ proving the claimed monotonicity $\langle X\rangle_k \subseteq \langle Y\rangle_k$.
For the idempotence it thus suffices to show that $\langle\langle X\rangle_k\rangle_k \subseteq \langle X\rangle_k$. This is, as we know by the closure of $\mathcal{Y}(\cdot)$ under intersections, equivalent to $\langle X\rangle_k \in \mathcal{Y}(\langle X\rangle_k)$. Obviously, $\langle X\rangle_k \subseteq \langle X\rangle_k$. For each cycle C whose edges are not all contained in $\langle X\rangle_k$, since $\langle X\rangle_k \in \mathcal{Y}(X)$, we have that $|C\backslash\langle X\rangle_k| \geq k+1$. Hence, we indeed obtain $\langle X\rangle_k \in \mathcal{Y}(\langle X\rangle_k)$, which finishes the proof. □

The reason why the cyclic k-closure of an edge set in a graph is important for our purposes is provided by the following nice statement:

Theorem 5.25 (cf. [Sey81])**.** *Let G be a graph, $k \geq 2$ and let X be a "spanning" edge set in G, i.e., $\langle X\rangle_{k-1} = E(G)$. Then there is a $\mathbb{Z}_k$-flow ϕ on some (and thus any) orientation of G which is non-zero on $\overline{X}$.*

Proof (cf. [Sey81]). We prove the statement by induction on the size of $|\overline{X}|$. If this is zero, i.e., $X = E(G)$, then the statement is trivial ($\phi = 0$). So assume for the induction step that $|\overline{X}| \geq 1$ and the statement holds true for all $X' \subseteq E(G)$ with $|\overline{X'}| < |\overline{X}|$. In this case, $\langle X\rangle_k = E(G) \neq X$ and so $X \notin \mathcal{Y}(X)$ which means that there is a cycle C in G not contained in $G[X]$ with $|E(C)\backslash E(X)| \leq k-1$. Looking at $X_C := X \cup E(C)$, we thus have $|\overline{X_C}| < |\overline{X}|$ and we can apply the induction hypothesis, which yields a k-flow ϕ' on some orientation D of G which is non-zero on $\overline{X_C} = \overline{X} \cap \overline{E(C)}$. Let ϕ_C be a canonical $\mathbb{Z}_k$-flow in D whose support is $E(C)$, ranging on $\{k-1, 0, 1\}$. We show that there is a $l \in \mathbb{Z}_k$ such that the $\mathbb{Z}_k$-flow $\phi := \phi' + l\phi_C$ is non-zero on $\overline{X}$, in which case we are done. ϕ_C has support $E(C)$ and ϕ' is non-zero on $\overline{X_C} = \overline{X} \cap \overline{E(C)}$ and so ϕ agrees with ϕ' on $\overline{X} \cap \overline{E(C)}$ and is thus non-zero. Consequently, it suffices a find a $l \in \mathbb{Z}_k$ with $-\frac{\phi'(e)}{\phi_C(e)} \neq l$ for all $e \in \overline{X} \cap E(C) = E(C)\backslash X$. But according to the above, we have $|E(C)\backslash X| \leq k-1 < |\mathbb{Z}_k|$, and therefore this is a simple application of the pigeon-hole principle. Using the principle of induction, the claim follows. $\square$

At the end of his paper, Seymour deduces the following strong result which we will use to construct NL-3-flows on 3-edge-connected digraphs:

Theorem 5.26. *Let G be a 3-regular 3-vertex-connected graph. Then there is a partition $E(G) = X_1 \dot{\cup} X_2$ of the edges in G with the following properties:*

$$\langle X_1 \rangle_1 = E(G), \langle X_2 \rangle_2 = E(G).$$

Proof. See [Sey81], (5.1). $\square$

The following simple observation regarding the cyclic 1-closure of an edge set leads us immediately to the context of coindependent flows:

Remark 5.27. Let G be a connected graph and $X \subseteq E(G)$. Then $\langle X \rangle_1 = E(G)$ if and only if X contains a spanning tree.

Proof. $\Rightarrow$: Assume that $\langle X \rangle_1 = E(G)$. If X did not contain a spanning tree, there would be a cut $S = G[Z, \overline{Z}]$ in G not containing any edge of X. Let $Y := \overline{S} \supseteq X$. Then $|Y| < |E(G)|$ and because of $\langle X \rangle_1 = E(G)$, there has to be a cycle C in G with $|E(C) \backslash Y| = |E(C) \cap S| = 1$, but 1 is not even, contradiction.
$\Leftarrow$: Let T be a spanning tree contained in X. Assume that $\langle X \rangle_1 \neq E(G)$, which means that $X \subseteq Y \subseteq E(G) \backslash \{e\}$ for some edge $e \in \overline{X}$ and some $Y \in \mathcal{Y}(X)$. Let $C = C(e, T)$ be the unique cycle in $T + e$. Then $|E(C) \backslash Y| = |E(C) \cap \{e\}| = 1$, contradicting the defining properties of $\mathcal{Y}(X)$. Hence, $\langle X \rangle_1 = E(G)$ as claimed. $\square$

Corollary 5.28 (cf. [Hoc17])**.** *Let G be a 3-regular 3-vertex-connected graph. Then G admits a coindependent $\mathbb{Z}_3$-flow.*

Proof. Let X_1, X_2 be a partition as in Seymour's theorem 5.26. According to the above observation, X_1 contains a spanning tree, and according to Theorem 5.25 with $k = 3$, $X = X_2$ we know that there is a $\mathbb{Z}_3$-flow on G whose support contains $\overline{X_2} = X_1$ and thus a spanning tree. Hence, this is a coindependent $\mathbb{Z}_3$-flow. $\square$

By using standard techniques, we can generalize this result to arbitrary 3-edge-connected graphs:

Theorem 5.29 (cf. [Hoc17])**.** *Let D be a 3-edge-connected digraph. Then D admits a coindependent $\mathbb{Z}_3$-flow and thus a NL-3-flow.*

Proof. The proof is analogous to the one given in subsection 4.5 for Theorem 4.49. First of all, we may confine on proving the theorem in the case where D is 2-vertex-connected: If there was a cut vertex v_{cut} in $U(D)$, we could split D along v_{cut} ending up with two digraphs D_1 and D_2 (both containing v_{cut}). We then proceed by induction on the number of vertices in D and construct a coindependent $\mathbb{Z}_3$-flow on D by joining two coindependent $\mathbb{Z}_3$-flows on D_1 resp. D_2. This is possible, since the spanning trees in $U(D)$ are exactly the

spanning trees of $U(D_1)$ joined with those of D_2.
So assume in the following that $U(D)$ is 2-vertex connected. We define a 3-regular digraph $\hat{D}$ by replacing each vertex $v \in V(D)$ of degree at least 4 by a directed cycle C_v of length $\deg_{U(D)}(v)$ and connecting the neighbours of v each to a different vertex of C_v. This digraph is furthermore 3-edge-connected: If S is an arbitrary cut in $U(\hat{D})$, it either contains no edges of any cycle C_v and hence corresponds to a cut in D, wherefore it has size at least 3, or, in the other case, it contains at least one (and thus, since this has to be an even number, at least two) edges of a cycle C_v, $\deg_{U(D)}(v) \geq 4$. If $|S|$ was less than 3, i.e., $|S| = 2$ contained only those two edges in C_v, this would imply that v is a cut vertex in $U(D)$, contradicting the additional assumption of 2-connectivity above. Hence, $\hat{D}$ is a 3-regular 3-edge-connected and hence also 3-vertex-connected digraph. According to Corollary 5.28, there is a coindependent $\mathbb{Z}_3$-flow on $\hat{D}$. The restriction $f' := f|_{E(D)}$ hence gives rise to a coindependent $\mathbb{Z}_3$-flow on D, and we are done. □

After having improved the upper bound of 4 to 3 for a minimal number k for which NL-k-flows exist on all 3-edge-connected digraphs, the only open question remaining is whether we may even generalize the 2-Colour-Conjecture, which as formulated in Section 4 states that each *planar* 3-edge-connected digraph admits a NL-2-flow, to arbitrary 3-edge-connected digraphs, i.e., omit the planarity condition. If we want to propose such a general conjecture, we should start with considering some small, well-known and non-planar cubic 3-edge-connected graphs, such as the Petersen graph P_{10}. This is also convenient with the observations made in Section 5.3.1 which show that, at least when considering Nowhere-Zero-Flows, the Petersen graph seems to have special properties which make it harder to find integer flows on it fulfilling given range restrictions. The following figure illustrates why "even" the Petersen graph admits a NL-2-flow on all of its orientations: Assume for contradiction there was an orientation D of P_{10} so that no matter which even edge subset in D is contracted, the arising digraph still contains a directed cut, or equivalently, for each even edge set in P_{10}, there is a disjoint directed cut in D. This especially holds for the even edge sets in P_{10} as marked in the figure with red edges and all of its five symmetric rotated versions. Hence, for each of the five even edge subsets, the only cut contained in its complement, with blue marked edges, has to be directed. If we consider the cycle in P_{10} consisting of the five outer vertices in P_{10}, this implies that each non-neighbouring pair of edges on the cycle has different orientations in D (i.e. one of them is oriented clockwise, the other one counterclockwise on the outer 5-cycle). Thus each pair of consecutive edges on the cycle have the same orientation, as both edges have distance two to the same oppositely positioned edge on the cycle, and hence, we get that C has to be directed, contradicting the fact that non-neighbouring edges have to have distinct orientations. Knowing that the 2-Colour-Conjecture implies the existence of NL-2-flows for 3-edge-connected planar digraphs and additionally taking the example of the Petersen graph into account, Hochstättler in [Hoc17] proposed the following conjecture, to which we will refer as the *2-Flow-Conjecture* during the remaining subsection.

Conjecture 5.30 (Two-Flow-Conjecture, cf. [Hoc17]). *Every 3-edge-connected digraph D admits a NL-2-flow.*

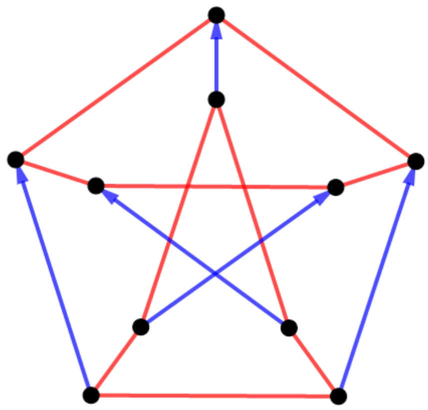


Figure 13: Each orientation of the Petersen graph admits a NL-2-flow.

5.3.3 A Counterexample

Although we have given quite some evidence for the correctness of the conjecture above, due to a counterexample constructed by Kolja Knauer and Petru Valicov in [KV17], it has recently turned out to be false. The arising (3-regular and 3-edge-connected) digraph has 224 vertices and $\frac{3}{2} \cdot 224 = 336$ edges and is constructed by taking many copies of the Petersen graph, glueing them together in a specific way and finally replacing each vertex by a configuration of 7 vertices, which will reappear in section 7. The following review of this construction is oriented at [KV17]. We start off with the following special property of perfect matchings in the Petersen graph. The lemma refers to Figure 14. Note, that in cubic graphs, perfect matchings as the 1-factors are exactly the complements of the spanning even edge subsets, i.e., the 2-factors.

Observation 5.31 (cf. [KV17])**.** *In the partial orientation of the Petersen graph shown above, each perfect matching containing the upper left edge contains a directed cut.*

Proof. This becomes immediately clear by Figure 14. □

As illustrated by Figure 14, we will use this special partial orientation of the Petersen graph (called $P_{(0,1)}$) as replacement configuration for vertices of degree 3 in another cubic graph by splitting the vertex 0 into three copies. We consider the following digraph, where the $P_{(0,1)}$-gadgets are inserted at the three big red vertices. In the gadgets, we choose arbitrary orientations for all the undirected edges not contained in the partial orientation. Their common adjacent 0_1-vertex (cf. Figure 14) is marked:

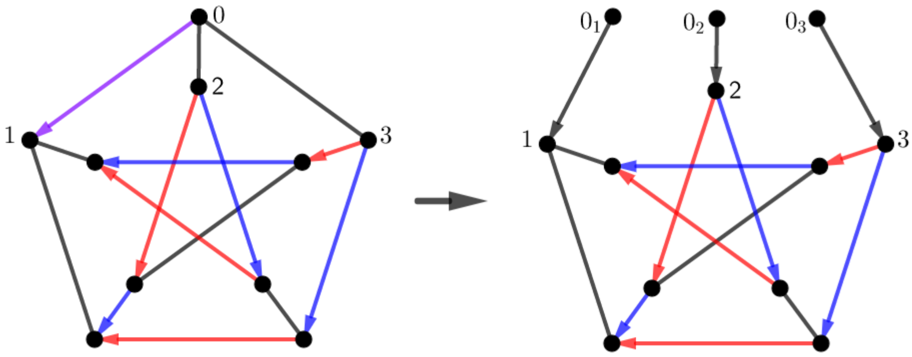


Figure 14: Left: Special orientation properties of Matchings in the Petersen graph P_{10}. The blue resp. red edges mark the only two perfect matchings of the Petersen graph containing the upper left edge. Right: Splitting of the vertex 0 in the gadget $P_{(0,1)}$.

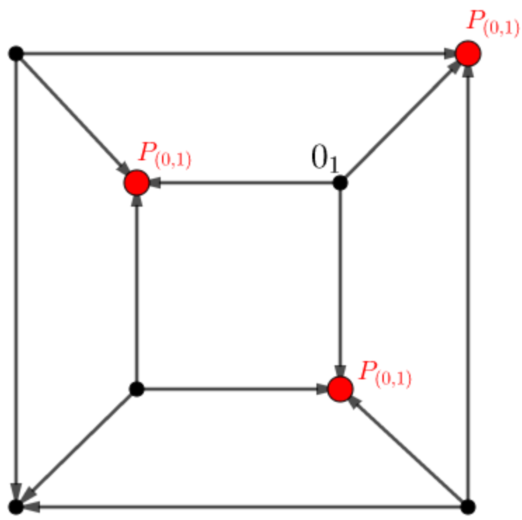


Figure 15: The digraph $D_{(0,1)}$.

The arising oriented graph is denoted by $D_{(0,1)}$. The special properties of the Petersen graph regarding perfect matchings have the following consequence

Lemma 5.32. *In the cubic and 3-edge-connected digraph $D_{(0,1)}$, each perfect matching fully contains a directed cut.*

Proof. The 3-edge-connectivity of $D_{(0,1)}$ is easily verified.
Let now M be an arbitrary perfect matching of $U(D_{(0,1)})$. Then the 0_1-vertex is being matched to exactly one of the gadgets $P_{(0,1)}$ in D. Consider the non-trivial cut in $U(D_{(0,1)})$ of size 3 induced by this gadget. The number of matching edges in the cut has to be odd and thus 1 or 3, since $\overline{M}$ is a 2-factor in $U(D_{(0,1)})$. In the case it is 1, according to observation

5.31, the matching on the considered gadget induced by M corresponds to a perfect matching of the P_{10} containing the edge $(0,1)$, and thus contains a directed cut inside the oriented gadget. Since this cut splits all the edges incident to 0-vertices, it separates some vertices in the gadget distinct from $0_1, 0_2, 0_3$ from the rest of $D_{(0,1)}$ and hence, this gives rise to a directed cut in $D_{(0,1)}$ fully contained in M. If all three edges incident to the gadget are contained in M, those three edges already give rise to a directed cut in $D_{(0,1)}$ contained in M, wherefore we are also done in this case. □

Considering the digraph $D_{(0,1)}$, we have almost reached our goal of constructing a counterexample to the 2-Flow-Conjecture: Each even edge subset E in $D_{(0,1)}$ covering every vertex (i.e., each 2-factor resp. each complement of a perfect matching) has a directed cut in its complement which is a directed cut in $D_{(0,1)}/E$, and thus, there is no NL-2-flow on $D_{(0,1)}$ with E as its support. But still there is one, as you can see by e.g. taking a Hamiltonian cycle in the cube graph and extending it to an even edge set in $D_{(0,1)}$ using fitting 2-factors in the Petersen-gadget, as constructed in figure 13. Thus, for a counterexample, we should force the supports of legal NL-2-flows in some extension of $D_{(0,1)}$ to induce 2-factors in $D_{(0,1)}$. The following local replacement step at each vertex which is neither a source nor a sink shows what such a forcing configuration may look like:

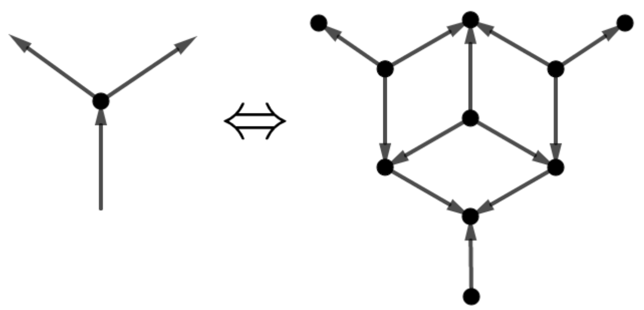


Figure 16: Local replacement of a no-source-no-sink-vertex in $D_{(0,1)}$ by a 7-vertex-configuration inducing a non-trivial cut of size 3.

By applying this replacement step to every vertex in $D_{(0,1)}$ which is neither a source nor a sink, we end up with a cubic 3-edge-connected digraph D. This finally is the desired counterexample:

Theorem 5.33. *The digraph D admits no NL-2-flow and is thus an obstruction to the NL-2-Flow-Conjecture.*

Proof. Assume for contradiction there was an even edge subset $E \subseteq E(D)$ such that D/E is strongly connected. By contracting all the 7-vertex-configurations from the replacement steps above, this gives rise to an even edge subset of $D_{(0,1)}$ whose contraction in $D_{(0,1)}$ is a contraction minor of D/E and thereby strongly connected. According to Lemma 5.32, this even edge subset is not a 2-factor, i.e., it leaves at least one vertex uncovered. This vertex cannot be a source or a sink of $D_{(0,1)}$, since in this case, the three incident edges would give rise to a directed cut in $D_{(0,1)}$ not covered by the edge subset. Therefore, this vertex corresponds to a 7-vertex-configuration in D whose three cut edges are each not covered by E. On the other hand, all the sinks and sources contained in this configuration have to be covered by E. An easy case distinction now shows that this is not possible without using one of the cut edges. □

5.3.4 Other Conjectures

Because of the falsification of the 2-Flow-Conjecture demonstrated above, it remains unclear which 3-edge-connected digraphs admit a NL-2-flow and which do not. We thus want to look for additional requirements and properties under which the existence of NL-2-flows indeed can be guaranteed. A proposal for such a restriction was already done in [KV17]. They claim that the 2-Flow-Conjecture is satisfied for all digraphs with cubic, 3-edge-connected and *bipartite* underlying graph:

Conjecture 5.34 (cf. Knauer and Valicov, [KV17])**.** *If D is a cubic 3-edge-connected digraph with bipartite underlying graph $U(D)$, then D admits a NL-2-flow.*

There is indeed some evidence that NL-2-flows might be easier to find in this case, since cubic bipartite graphs usually provide a "brighter" structure of even subgraphs, and are e.g. additionally 3-edge-colourable, i.e., their edge set can be decomposed into three disjoint perfect matchings M_1, M_2, M_3, each of whose complements are 2-factors of the graph. Moreover, the following well-known conjecture due to Barnette on the Hamiltonicity of cubic 3-connected bipartite planar graphs admits the above conjecture 5.34 in the planar case as a consequence:

Conjecture 5.35 (Barnette, 1969, cf. [Bar69])**.** *Let G be a cubic 3-edge-connected bipartite planar graph. Then G admits a Hamiltonian circuit.*

Observation 5.36. *If Barnette's Conjecture holds true, every cubic 3-connected bipartite planar graph admits a coindependent $\mathbb{Z}_2$-flow. Thus, Conjecture 5.34 is correct for planar graphs.*

Proof. Let $C \subseteq E(G)$ be the edge set of a Hamiltonian circuit in a given cubic 3-connected bipartite graph G. This is a spanning, connected even subgraph. Thus, the claim is a consequence of Theorem 5.9. □

We now want to take another look at this conjecture restricted to the case of *planar* 3-edge-connected cubic digraphs and show that it may be reduced to the case of *balanced* digraphs with the same properties.

Definition 5.37. Given any digraph D, we call it *balanced*, if for every cycle C in $U(D)$, the same (oriented) cycle considered in D has the same number of edges in both directions.

Obviously, each cycle in an underlying graph of a balanced digraph has to be of even length and hence, the underlying graph must be bipartite. Note that for any given planar 3-edge-connected digraph D with a planar directed dual D^*, D is balanced if and only if every cut D^* has the same number of edges in both directions. This means that D^* is a

Eulerian planar digraph, and hence, being balanced is the dual property to that of a Eulerian orientation.
In the following we show that every orientation of a plane triangulation with even underlying graph can be embedded into a Eulerian oriented planar triangulation as an induced subdigraph.
For this purpose, we prove the following lemma.

Lemma 5.38. *Let $e_1, e_2, e_3 \in 2\mathbb{Z}$ be three given even integers, such that $e_1 + e_2 + e_3 = 0$. Then there exists a planar triangulation T with outer face a_1, a_2, a_3 and an orientation $\mathcal{O}$ of its inner edges, such that $\mathrm{exc}_{\mathcal{O}}(v) = 0$ for every inner vertex v and $\mathrm{exc}_{\mathcal{O}}(a_i) = e_i, i = 1, 2, 3$.*

Proof. We prove the lemma by induction on $|e_1| + |e_2| + |e_3| \in \mathbb{N}_0$. If this is zero, i.e., $e_1 = e_2 = e_3 = 0$, we simply take T to be a triangle without orientations. Now assume the statement to be true for all even integers f_1, f_2, f_3 with $|f_1| + |f_2| + |f_3| < k \in \mathbb{N}$, and let e_1, e_2, e_3 be three even integers summing up to zero and $|e_1|+|e_2|+|e_3| = k$, which is even as well. Since $k > 0$, one of the numbers is greater and another one is less than zero, so without loss of generality $e_1 \geq 2, e_2 \leq -2$. Therefore, with $e'_1 := e_1 - 2, e'_2 := e_2 + 2, e'_3 := e_3$, we have $|e'_1| + |e'_2| + |e'_3| = k - 4 < k$ and by the induction hypothesis, there is a planar triangulation T' and an orientation $\mathcal{O}'$ of its inner edges with excess 0 for every inner vertex and e'_i at a_i. Now consider the unique triangular face of T' incident to a_1, a_2 and let w be the third vertex of the face. Now stack the following oriented triangulation with excess 0 at every inner vertex isomorphically into this triangle such that a_1 receives two additional outgoing edges, a_2 receives two additional incoming edges and w receives exactly one additional edge of both types:

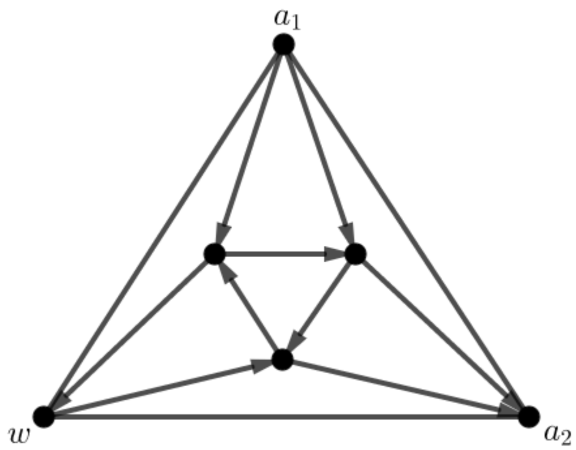


Figure 17: Construction of directed plane triangulations with prescribed excesses.

Obviously, in the resulting Triangulation T with the constructed orientation $\mathcal{O}$ of inner edges we still have $\mathrm{exc}_{\mathcal{O}}(v) = \mathrm{exc}_{\mathcal{O}'}(v) = 0$ for every inner vertex v, and $\mathrm{exc}_{\mathcal{O}}(a_1) = e'_1 + 2 = e_1, \mathrm{exc}_{\mathcal{O}}(a_2) = e'_2 - 2 = e_2, \mathrm{exc}_{\mathcal{O}}(a_3) = e'_3 = e_3$. Therefore the inductive statement is proven, and with the principle of induction the hypothesis follows. $\square$

Theorem 5.39. *Let $\vec{T}$ be an arbitrary orientation of a planar triangulation T, which is even. Then there exists a Eulerian planar directed triangulation $\vec{T}'$ such that $\vec{T}$ is an induced subdigraph of $\vec{T}'$.*

Proof. In the following, we denote by $\mathcal{F}$ the set of bounded triangular faces of T. We show that there exists an assignment $e : \{(f, v)|v \in f \in \mathcal{F}\} \to \mathbb{Z}$ of integer values to the "angles" of T such that:

- For every vertex v we have
$$\mathrm{exc}_{\vec{T}}(v) = \sum_{f \in \mathcal{F}: v \in f} e_{f,v}.$$
- All the assigned numbers are even.
- For every triangular face f in T with vertices v_1, v_2, v_3 we have $e_{f,v_1} + e_{f,v_2} + e_{f,v_3} = 0$.

Notice that since T is even, $\mathrm{exc}_{\vec{T}}(v)$ is even for all $v \in V(T)$. This means that finding integers as described above is equivalent to finding their halfs $a_{f,v} := \frac{e_{f,v}}{2}$ with

- For every vertex v we have
$$\frac{\mathrm{exc}_{\vec{T}}(v)}{2} = \sum_{f \in \mathcal{F}: v \in f} a_{f,v}.$$
- For every triangular face f in T with vertices v_1, v_2, v_3 we have $a_{f,v_1} + a_{f,v_2} + a_{f,v_3} = 0$.

The proof of this works by augmentations along certain paths: We start with the assignment $a_{f,v} = 0$ for every angle (f, v) in T, which obviously fulfills the latter condition. We now decrease the value $\sum_{v \in V(T)} |\frac{\mathrm{exc}_{\vec{T}}(v)}{2} - \sum_{f \in \mathcal{F}: v \in f} a_{f,v}|$ as long as it is greater than zero and thereby end up with an assignment satisfying both conditions.
So assume
$$\sum_{v \in V(T)} \left| \frac{\mathrm{exc}_{\vec{T}}(v)}{2} - \sum_{f \in \mathcal{F}: v \in f} a_{f,v} \right| > 0.$$
Using the second condition, we get that
$$\sum_{v \in V(T)} \left(\frac{\mathrm{exc}_{\vec{T}}(v)}{2} - \sum_{f \in \mathcal{F}: v \in f} a_{f,v} \right) = 0.$$
Hence, there exist two vertices $v_1 \neq v_2$ with
$$\frac{\mathrm{exc}_{\vec{T}}(v_1)}{2} - \sum_{f \in \mathcal{F}: v_1 \in f} a_{f,v_1} > 0 \text{ and } \frac{\mathrm{exc}_{\vec{T}}(v_2)}{2} - \sum_{f \in \mathcal{F}: v_2 \in f} a_{f,v_2} < 0$$
respectively. Since T is connected, there exists a simple path P connecting v_1, v_2 in T. Consider P as ordered from v_1 to v_2. For every edge e on P, choose one side of e belonging

to a bounded triangular face. Consider the angle-pair incident to e on this side, and increase the value of the angle appearing first in the above order by 1 while decreasing the value of the second angle by 1. This operation keeps the second condition above satisfied. For every vertex $v \in V(P)\setminus\{v_1, v_2\}$, the value $\frac{\mathrm{exc}_{\vec{T}}(v)}{2} - \sum_{f \in \mathcal{F}: v \in f} a_{f,v}$ remains unchanged, while $\frac{\mathrm{exc}_{\vec{T}}(v_1)}{2} - \sum_{f \in \mathcal{F}: v_1 \in f} a_{f,v_1} > 0$ decreases and $\frac{\mathrm{exc}_{\vec{T}}(v_2)}{2} - \sum_{f \in \mathcal{F}: v_2 \in f} a_{f,v_2} < 0$ increases by 1. Therefore, $\sum_{v \in V(T)} |\frac{\mathrm{exc}_{\vec{T}}(v)}{2} - \sum_{f \in \mathcal{F}: v \in f} a_{f,v}|$ decreases by 2. Hence, continuing this operation leads us to an integer angle assignment as required.
Given such an assignment $e : \{(f, v) | v \in f \in \mathcal{F}\} \to \mathbb{Z}$, together with Lemma 5.38 we find a triangulation T_f for every face f with oriented inner edges, such that the excess is 0 for every inner vertex and exactly $-e_{f,v_i}$ at the outer vertex a_i^f of T_f which we identify with the corresponding vertex v_i of f. Now we stack T_f with this orientation and identification of vertices into f for all $f \in \mathcal{F}$. The first condition now ensures that the resulting directed planar triangulation $\vec{T}'$ is Eulerian, and $\vec{T}$ is indeed an induced subdigraph. □

Corollary 5.40. *The following conjectures are equivalent:*

- *Every planar cubic 3-edge-connected digraph D with bipartite underlying graph admits a NL-2-flow.*
- *Every planar cubic 3-edge-connected balanced digraph D admits a NL-2-flow.*

Proof. The first statement implies the second, since balanced digraphs do not admit odd cycles. Assume that every cubic balanced 3-edge-connected planar digraph admits a NL-2-flow, i.e., every Eulerian orientation of a planar triangulation is 2-colourable. According to the above, given any directed planar triangulation $\vec{T}$ with even underlying graph, we can find an extending directed triangulation $\vec{T}'$ which contains $\vec{T}$ as an induced subdigraph. Every legal 2-colouring of $\vec{T}'$ now induces a legal 2-colouring of $\vec{T}$, and we are done. □

We now turn to another, way more specialized, conjecture. After having proven Theorem 5.18 in Section 5.2, it remains unclear to what extent generalizations in the dual context of NL-2-flows are possible. The dual NL-flow formulation of the weaker result in Lemma 5.21 is the following.

Let D be a 3-edge-connected planar digraph without directed cuts of size 3, together with a special fixed vertex $v_0 \in V(D)$ and a pre-flow-assignment $g : E(v) =: \{e_1, e_2, e_3\} \to \mathbb{Z}_2$ at v_0, i.e., $g(e_1) + g(e_2) + g(e_3) = 0$. Then there is a NL-$\mathbb{Z}_2$-flow f of D with $f(e_i) = g(e_i), i = 1, 2, 3$.

If we consider NL-flows on digraphs, as in the case of NZ-flows, a natural question is to ask whether we can drop the planarity condition in the above statement. In any generalization of the approach of Mohar and Li in [LB17], we will have to replace the notion of Tutte paths in planar graphs by a wider concept, since general existence results for Tutte paths in non-planar graphs are not known. A detailed analysis of the proofs given in Section 5.2 shows that all the constructed cycles in the dual graphs leave, when being deleted, a set of isolated vertices. In other words, the cycles contain at least one vertex of each edge in the graph. In general, such cycles are called *dominating cycles*:

Definition 5.41. Let G be a graph and C a cycle in G. C is called *dominating*, if $G-V(C)$ consists of isolated vertices.

In the following, we will present ideas of a proof of such a generalization under the assumption of a well-known conjecture on cubic graphs, which is called *Fleischner's Conjecture*. In order to introduce its content, we have to define a prominent class of cubic graphs, which are in particular important when considering the 4-Flow-Conjecture presented above. According to Vizing's theorem on edge-colourability of graphs (cf. Section 3, Theorem 4.13), the maximal number k of colours needed for a k-edge-colouring of a simple graph G is either its maximum degree $\Delta(G)$ or $\Delta(G)+1$, i.e., in the case of a cubic simple graph, 3 or 4. Snarks are essentially the cubic graphs with chromatic index 4:

Definition 5.42. Let G be a cubic 2-edge-connected simple graph. G is called a *snark* iff it has chromatic index 4, i.e., is not 3-edge-colourable.

Fleischner's Conjecture in its original formulation from [Fle84] states that:

Conjecture 5.43 (Fleischner,1984, [Fle84])**.** *Every cubic internally 4-edge-connected snark admits a dominating cycle.*

Since the publication of the conjecture, many equivalent versions, often connected to Hamiltonicity, have been discovered. Some of them are listed in the following equivalence-theorem:

Theorem 5.44 (cf. e.g. [Koc00])**.** *The following statements are equivalent:*

(i) Every internally 4-edge-connected cubic snark admits a dominating cycle. (Fleischner, [Fle84], 1984)

(ii) Every internally 4-edge-connected cubic graph admits a dominating cycle. (Ash and Jackson, [AJ84])

(iii) Every 4-connected line graph, i.e., $L(G)$ where G is a graph, is Hamiltonian. (Thomassen, [Tho86], 1986)

(iv) Every 4-connected claw-free graph is Hamiltonian. (Mathews-Sumner, 1984)

Proof. A proof of the equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ can be found in [Koc00]. A proof of $(i) \Leftrightarrow (iv)$ is contained in [Ryj97]. □

For our purposes, we will furthermore need the following equivalent strengthening of the dominating-cycle conjecture of Ash and Jackson:

Theorem 5.45 (cf. [FT90], Theorem 3)**.** *The following statements are equivalent:*

(i) Every internally 4-edge-connected cubic graph admits a dominating cycle.

(ii) *Given an internally 4-edge-connected cubic graph G and a pair of independent edges $e_1, e_2 \in E(G)$ (i.e. without a common incident vertex), there is a dominating cycle C in G passing through e_1 and e_2.*

The generalization we would like to prove (given the correctness of the equivalent conjectures above) states that every 3-edge-connected digraph D without directed cuts of size 3 admits a NL-2-flow. As in the planar case when first considering 4-connected triangulations, we will first prove this statement in the case of internally 4-edge-connected cubic digraphs in order to use inductive arguments along 3-cuts in the underlying graph later on, thereby trying to generalize it to arbitrary 3-edge-connected cubic digraphs. First of all, by simply taking a dominating cycle in a cyclically 4-edge-connected cubic graph as the support of a $\mathbb{Z}_2$-flow, the following is straightforward:

Proposition 5.46. *Let D be an orientation of a cyclically 4-edge-connected cubic graph without dicuts of size 3. Under the assumption of the Conjecture of Ash and Jackson, D admits a NL-2-flow.*

Proof. As mentioned above, let C be a dominating cycle in $U(D)$ and define a $\mathbb{Z}_2$-flow $f = \mathbb{1}_C$ on D. Then $D/\mathrm{supp}(f) = D/C$ has to be totally cyclic and thus f a NL-$\mathbb{Z}_2$-flow: If there was a minimal directed cut in $D/\mathrm{supp}(f)$, then $|S| \geq 4$ according to the additional assumption. On the other hand, D/C admits a star structure with central vertex v_C corresponding to C and the vertices $v \notin V(C)$ each admitting three parallel incident edges connecting it to v_C. Thus, S as a minimal cut consists of three such parallel edges, contradiction. $\square$

The ability to prescribe a pair of edges which a dominating cycle has to use as given by the above theorem, is, as in the case of Tutte paths, essential for the induction step. Still, there is an important deviation from the proofs in section 5.2: When looking for a Tutte path in the dual graph of the considered triangulation, we were able to prescribe starting and ending vertex and an edge e contained in the path at the same time, such that in fact, we could find a dominating cycle passing through two given and possibly adjacent edges. Thus, with the additional restriction of independence for the two prescription-edges, our task becomes a lot harder. In order to prepare a possible proof, we need the following result on splitting-off-operations in internally 4-edge-connected cubic graphs. Unfortunately, due to lack of time, I haven't yet been able to verify it's correctness.

Conjecture 5.47. *Let G be a cubic internally 4-edge-connected graph, and $u \neq v \in V(G)$ a pair of vertices with distance at least two in G (i.e. they are distinct and not adjacent). Denote by $\{e_1^u, e_2^u, e_3^u\}$, $\{e_1^v, e_2^v, e_3^v\}$ the sets of incident edges of u resp. v. Then there is a bijective assignment $J : \{1,2,3\} \to \{1,2,3\}$, such that the cubic graph G_J arising from G by deleting u and v and adding an edge e_i between the original end vertices of e_i^u and $e_{J(i)}^v$ for $i = 1, 2, 3$, is internally 4-edge-connected.*

Ideas of a proof. The idea of a proof could be to perform well-known splitting-off-operations on graphs preserving given edge-connectivity restrictions, which are a little more complicated here compared with the standard case (internal 4-edge-connectivity instead of k-connectivity for some number $k \in \mathbb{N}$). In order to illustrate this, consider the graph G_{uv} arising from G by identifying the vertices u and v, i.e. formally, adding an edge e_{uv} between the vertices of G and contracting it: $G_{uv} := (G + e_{uv})/e_{uv}$. This graph admits degree 3 at each vertex $v \in V(G_{uv})\backslash\{uv\}$ and degree 6 at $\{uv\}$, and is as a contraction minor of $G + e_{uv}$ at least internally 4-edge-connected. Denote the sets of neighbours of u resp. v in G by Q resp. S. According to well-known sufficient and necessary conditions for the existence of splitting-off-operations between two given disjoint vertex subsets (Q and S) of a graph, as e.g. explained in [BJJ04], which can be easily verified in this case, a complete splitting-off-sequence of three edges between Q and S preserving the *3-edge-connectivity* indeed exists. Unfortunately, I have not yet managed to find an analogous result for internally 4-edge-connected graphs, but maybe there are possibilities for e.g. reducing the problem to corresponding 4-edge-connected graphs. $\square$

The following equivalent formulation of the conjectured statement above (Observation 5.49) might also be easier to handle. It uses the following lemma:

Lemma 5.48. *Let G be a graph and denote by $L(G)$ its line graph. Then the following statements are equivalent:*

- *G is internally 4-edge-connected.*
- *$L(G)$ is 4-vertex-connected.*

Proof. We prove the equivalence of the negated statements: $L(G)$ is not 4-vertex-connected if and only if there is a vertex subset $V \subseteq V(L(G)) = E(G), |V| \leq 3$ such that $L(G) - V = L(G - E_V)$, where $E_V \subseteq E(G), |E_V| = |V| \leq 3$ denotes the set of edges in G corresponding to V, is disconnected. This obviously is equivalent to requiring that there are two disjoint connected components in $G - E_V$ each containing at least one edge. This means the existence of a non-trivial cut in G of size at most 3, proving the claimed equivalence. $\square$

Observation 5.49. *Let G be an internally 4-edge-connected cubic graph, and $u \neq v \in V(G)$ a pair of vertices with distance at least two in G (i.e. they are distinct and not adjacent). Denote by $\{e^u_1, e^u_2, e^u_3\}$, $\{e^v_1, e^v_2, e^v_3\}$ the sets of incident edges of u resp. v. Then the following two statements are equivalent:*

- *The claim of the open problem in 5.47, i.e., there is a bijective assignment $J : \{1, 2, 3\} \to \{1, 2, 3\}$, such that the cubic graph G_J arising from G by deleting u and v and adding an edge e_i between the original end vertices of e^u_i and $e^v_{J(i)}$ for $i = 1, 2, 3$, is internally 4-edge-connected.*
- *There is a bijective assignment $J : \{1, 2, 3\} \to \{1, 2, 3\}$ such that the graph arising from the 4-connected line graph $L(G)$ by identifying the vertices e^u_i and $e^v_{J(i)}$ in $L(G)$ (i.e., adding an edge in between them and contracting it) for $i = 1, 2, 3$ and deleting the edges of the triangle consisting of the three arising vertices, is still 4-connected.*

Proof. This is an immediate consequence of the equivalence between internally 4-edge-connected graphs and 4-vertex-connected line graphs proven above. $\square$

The following sketches an idea how to make use of the splitting-off operation described above.

Conjecture 5.50. *Assume the conjectures contained in Theorem 5.44 and Observation 5.49 are true. Let G be an arbitrary internally 4-edge-connected cubic graph with radius at least 3. Then, for every given pair $u \neq v \in V(G)$ of distinct vertices and a given pair of edges $b_1, b_2 \in E(G)$ incident to u, there is a dominating cycle C in G which passes through b_1, b_2 and v.*

Idea of a Proof. We distinguish between different cases concerning $u \neq v$:

1. For the first case assume that u and v have distance at least three or distance two in G, where in the latter case, the neighbours of u at the ends of b_1, b_2 both are assumed not to be contained in $N(v)$. Then, according to Conjecture 5.47, there is an internally 4-edge-connected cubic graph of the form G_J where the incident edges of u and v are pairwise identified. Denote by e_1, e_2, e_3 the three arising edges, such that e_1, e_2 in G_J correspond to the original incident edges b_1, b_2 of u in G. e_1 and e_2 are independent: If they had a vertex in common, since the end vertices are either neighbours of u or v and since the neighbours of u resp. v are independent sets of vertices (G is triangle-free), this means that $N(u) \cap N(v) \neq \emptyset$, which immediately contradicts our assumption in the case of $\mathrm{dist}_G(u, v) \geq 3$, while in the latter case, this would imply that either b_1 or b_2 is incident to such an intersection vertex in $N(u) \cap N(v)$, which again is a contradiction. Hence, we can apply Theorem 5.45 and deduce the existence of a dominating cycle C_J in G_J passing through e_1 and e_2. Let C be the set of edges in G which are contained in C_J as edges of G_J (if they are not incident to u or v) together with the edges incident to u or v corresponding to e_1 or e_2 in G_J. It is easy to see that C is a cycle and uses at least one vertex of each edge in G. Hence, it is a dominating cycle in G through e_1, e_2 and v and we are done in this case.

2. In the situation where u, v are neighbours or admit distance two, the task seems to become a lot harder. It is possible that in this case, other methods for prescribing a pair of edges/vertices as above have to be discovered.

□

Corollary 5.51. *Assume that statement from 5.50 holds true in general. Let D be an arbitrary 3-regular and 3-edge-connected digraph without directed cuts of size 3, so that $U(D)$ admits radius at least 3. Then for every pair u, v of distinct vertices, such that there is a non-trivial 3-cut in $U(D)$ separating u and v whenever D is not internally 4-edge-connected, and any pair b_1, b_2 of edges incident to u in D, there is a NL-$\mathbb{Z}_2$-flow on D whose support contains b_1 and b_2 and covers v.*

Proof. We prove the assertion using induction on the number of edges in D. It obviously holds true if $|E(D)| = 3$, i.e., if $U(D)$ consists of two vertices with 3 parallel edges in between them.
For the induction step, assume that $|E(D)| > 3$. We distinguish between two cases:

1. If D is internally 4-edge-connected, i.e., there are no non-trivial cuts of size 3 in $U(D)$, then according to Conjecture 5.50, there is a dominating cycle C in $U(D)$ passing through b_1, b_2 and v. Use this even edge subset in $U(D)$ as the support of a $\mathbb{Z}_2$-flow on D. It remains to show that this flow is a NL-$\mathbb{Z}_2$-flow, i.e., $D/E(C)$ is strongly connected. But since C is a dominating cycle, the only edges in $D/E(C)$ connect the vertex v_C corresponding to the contraction of C to vertices in $V(D) \setminus V(C)$. Thus, $D/E(C)$ has a star-structure consisting of a central vertex v_C together with three edges connecting it to each vertex $v \in V(D) \setminus V(C)$. Since those three edges are the incident edges of a vertex in D, they are a cut of size three in D which therefore cannot be directed. Hence, every vertex $v \in V(D/E(C)) \setminus \{v_C\}$ is strongly connected to v_C, implying the srong connectivity of $D/E(C)$, and we are done in this case.

2. If D is not internally 4-edge-connected, then there is a non-trivial cut of size 3 in $U(D)$ separating u and v, which we denote by $S = U(D)[X, \overline{X}]$, $u \in X, v \in \overline{X}$. let D_u, D_v denote the 3-regular 3-edge-connected digraphs arising from contracting $\overline{X}$ resp. X, i.e., $D_u := D/\overline{X}, D_v := D/X$. Each of them contains a vertex $v_{\overline{X}}$ resp. v_X of degree 3 representing the contracted vertices in $\overline{X}$ resp. X so that the incident edges of $v_{\overline{X}}$ and v_X correspond to the cut edges in S. Since each cut in D_u, D_v corresponds to a cut in D, D_u, D_v also do not contain directed cuts of size 3 and each have less vertices and thus fewer edges than D. Therefore, we can apply the induction hypothesis to D_u and D_v, and so there is a NL-$\mathbb{Z}_2$-flow f_u on D_u whose support contains the edges $b_1, b_2 \in U(D_u)$ (since $u \in X$ and b_1, b_2 are incident to u) and covers the vertex $v_{\overline{X}}$. Let b_1', b_2' be the edges of S resp. incident to $v_{\overline{X}}$ contained in $\operatorname{supp}(f_u)$. Then again, the induction hypothesis guarantees the existence of a NL-$\mathbb{Z}_2$-flow f_v on D_v such that $\operatorname{supp}(f_v)$ contains the edges b_1', b_2' and covers v. Let f denote the $\mathbb{Z}_2$-flow on D arising from taking the values of f_u on edges incident to X and of f_v on edges incident to $\overline{X}$ (f_u, f_v agree on S). Since $D_u/\operatorname{supp}(f_u), D_v/\operatorname{supp}(f_v)$ are strongly connected digraphs, and $\operatorname{supp}(f) = \operatorname{supp}(f_u) \cup \operatorname{supp}(f_v)$ and because S contains a pair of edges in converse directions, we know that also $D/\operatorname{supp}(f)$ has to be strongly connected. Consequently, f is a NL-$\mathbb{Z}_2$-flow on D whose support contains b_1, b_2 and covers v, as claimed.

All in all, the principle of induction proves the claim. $\square$

Corollary 5.52. *Assume the equivalent conjectures from Theorem 5.44 as well as the open problems/conjectures in 5.49 and 5.50 are hold true. Then every 3-edge-connected digraph D without directed cuts of size 3 admits a NL-2-flow.*

Proof. In the case that D is 3-regular, the statement immediately follows from Corollary 5.51. If D admits vertices of degree at least 4, we can replace each such vertex w of degree $n \geq 4$ in $U(D)$ by a directed cycle $\vec{C_n}$ in D by connecting the edges which were originally incident to w each with a different vertex of $\vec{C_n}$. The arising 3-regular digraph $\hat{D}$ is still 3-edge-connected and does not contain directed cuts of size 3 (each directed cut in $\hat{D}$ corresponds to the edges set of a directed cut in D, since each replacement cycle $\vec{C_n}$ is directed). Thus, it admits a NL-2-flow $\hat{f}$. We define a NL-2-flow f on D by taking the same flow value for each edge in $E(D)$ as for the corresponding edge in $E(\hat{D})$. Because of $\text{supp}(f) = \text{supp}(\hat{f}) \cap E(D)$, $D/\text{supp}(f)$ is a contraction minor of the strongly connected digraph $\hat{D}/\text{supp}(\hat{f})$ and thus strongly connected itself, which shows that f is a NL-2-flow on D. $\square$

The last two conjectures we propose arise from two simple observations regarding the counterexample constructed in Section 5.3.3: First of all, the Petersen minors of the constructed digraph used as gadgets have a very special structure of perfect matchings as explained in the Observation 5.31, which is not easy to find in other 3-edge-connected cubic graphs. Thus, in analogy to the 4-Flow-Conjecture, we propose:

Conjecture 5.53. *Let D be any 3-edge-connected digraph such that $U(D)$ does not admit a Petersen minor. Then D admits a NL-2-flow.*

This conjecture is obviously stronger than the 2-Colour-Conjecture, since planar graphs do not admit a (non-planar) Petersen minor. Moreover, under the assumption of the 4-Flow-Conjecture, it can be reduced to the following statement which proposes a relationship between NZ-4- and NL-2-flows on a digraph:

Conjecture 5.54. *Let D be a 3-edge-connected digraph. If D admits a NZ-4-flow, then D also admits a NL-2-flow.*

The second very essential tool used in the construction of the counterexample was the ability to expand vertices of degree three to bigger configurations inducing non-trivial cuts of size 3. We thus conjecture the following:

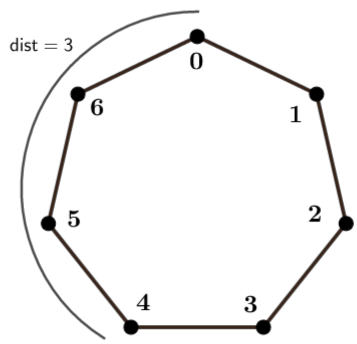
Conjecture 5.55. *Every essentially 4-edge-connected cubic digraph D admits a NL-2-flow.*

6 Circular (Di-)chromatic Number and Fractional NZ- and NL-Flows

In this section, we introduce the notion of circular colourings of graphs and digraphs and the related notions of various flow indices in the dualized and generalized context of regular (oriented) matroids. Later on, we consider a corresponding dual concept in terms of flows on graphs, which yields the notions of fractional Nowhere-Zero-flows, fractional coindependent flows and fractional Neumann-Lara-flows on graphs resp. digraphs.
The main motivation for considering fractional versions of flow indices here is the observation that, as pointed out in the previous section, although (under the assumption of the 2-Colour-Conjecture) 3-edge-connected planar digraphs and probably many more 3-edge-connected digraphs admit NL-2-flows, there still are some which admit NL-flow-index 3. In order to overcome the gap between 2 and 3 for the NL-flow indices and to allow a finer distinction between 3-edge-connected digraphs in terms of the "hardness" of finding a NL-flow, we introduce fractional notions, observing that there might be digraphs "almost" admitting NL-2-flows.

6.1 Circular colourings of graphs and digraphs

Let us initially consider a simple graph G. If we look for a graph colouring of it, we require different colours at both ends of each edge. We may strengthen this condition by treating the colours $\{0, ..., k-1\}$ assigned to the vertices as ordered in a circular manner, i.e., we consider them as positioned along a circle where i neighbours the numbers $(i-1) \bmod k, (i+1) \bmod k$ for each $i \in \{0, ..., k-1\}$. In this context, we may define the *circular distance* of two colours $i, j \in \{0, ..., k-1\}$ by $\mathrm{dist}_k(i, j) := \min\{|(i-j) \bmod k|, |(j-i) \mathrm{mod}\ k|\}$, i.e., it is the length of a shortest path connecting i, j in the circular ordering. This is illustrated by the figure depicted below.



When colouring a graph G, we now may additionally require that the colours of adjacent vertices are not only distinct, but moreover "far apart" in the sense of circular distances. With this intuition at hand, we end up with the following definition:

Definition 6.1. Let G be a simple graph and $k, d \in \mathbb{N}$. A (k, d)*-colouring* of G is an assignment of colours $c : V(G) \to \{0, ..., k-1\}$ to the vertices so that $\mathrm{dist}_k(c(u), c(v)) \geq d$ for each pair $u \neq v$ of adjacent vertices.

If $d = 1$, this is obviously equivalent to the usual notion of graph colourings.
The first appearance of this circular concept in the context of graph colourings is in the 1988-paper [Vin88] of Vince where the so-called *star chromatic number* (also known as *circular chromatic number*) $\chi^*(G)$ of a graph G is introduced. Intuitively, this number is supposed to allow a more accurate classification of graphs with the same chromatic number by capturing e.g. the feeling that some graphs with chromatic number $l + 1$ are "close" to having chromatic number $l \in \mathbb{N}$ while others are not. This is best illustrated by the example of a cycle graph C_n with odd $n \in \mathbb{N}, n \geq 3$. Obviously, $\chi(C_n) = 3$ in this case, but on the other hand, we can find a 2-colouring of it which is "almost" fine in the sense that only one edge is monochromatic. And indeed, as we will see in the following, Vince's circular star chromatic number of the C_n with odd $n \geq 3$ is $\chi^*(C_n) = 2 + \frac{2}{n-1}$ and hence gets closer to 2 as the length of the cycle grows.
But how to define a fractional analogue to the chromatic number appropriately? Given a usual $\chi(G)$- resp. $(\chi(G), 1)$-colouring of a graph G and any natural scaling constant $r \in \mathbb{N}$, we can simply extend our range of possible colours from $\{0, ..., \chi(G) - 1\}$ to $\{0, ..., r\chi(G) - 1\}$ and identify the original colour $i \in \{0, ..., \chi(G) - 1\}$ with the new colour $ri \in \{0, ..., r\chi(G) - 1\}$. Since also all the circular distances between distinct colours thereby get scaled by a factor of r, this always gives rise to a $(r\chi(G), r)$-colouring of G, and hence, we can find (k, d)-colourings of G where d is arbitrarily large and $\frac{k}{d}$ is at most $\chi(G)$. But in addition to that, as d grows, we get more and more freedom how to chose the assigned colours, and it might well be that we can thereby do better than just scaling a usual graph colouring of G. Thus, a natural way to define the star chromatic number of G is the smallest quotient $\frac{k}{d}$ we might reach with a legal (k, d)-colouring of G.

Definition 6.2 (cf. [Vin88]). Let G be a graph. The *star chromatic number* or *circular chromatic number* of G is the real number $\chi^*(G)$ defined by

$$\chi^*(G) := \inf \left\{ \frac{k}{d} \middle| \text{There is a } (k, d)\text{-colouring of } G \right\}.$$

The following theorem captures the most important elementary properties of the star chromatic number as already proven in [Vin88].

Theorem 6.3 (cf. [Vin88]). *Let G be a loopless graph. Then the following holds:*

(i) $\chi^*(G)$ is a positive rational number, and $\chi^*(G) \geq 2$ whenever $E(G) \neq \emptyset$.

(ii) $\lceil \chi^*(G) \rceil = \chi(G)$, i.e., $\chi^*(G) \in (\chi(G) - 1, \chi(G)]$.

(iii) For each rational number $q \in \mathbb{Q}, q = \frac{m}{n} \geq 2$, there is a graph $G_{m,n}$ with $\chi^*(G_{m,n}) = \frac{m}{n} = q$.

(iv) For every $k, d \in \mathbb{N}$, there exists a (k, d)-colouring of G if and only if $\frac{k}{d} \geq \chi^*(G)$.

(v) If $\chi^*(G) = \frac{m}{n}$, then there is a (k, d)-colouring of G with $\frac{k}{d} = \frac{m}{n}$ and $k \leq |V(G)|$.

Proof. Some of the assertions are consequences of the discussion above. The full proofs of all the statements can be found in [Vin88], Section 3. $\square$

We now supplement these general properties with the concrete example of the odd cycle graphs which was already discussed above:

Example 6.4. Let $n = 2l + 1, l \geq 1$ be any odd number and C_n the corresponding cycle graph of length n. Let (k, d) be a pair of natural numbers admitting a legal (k, d)-colouring of C_n with $\frac{k}{d} = \chi^*(G)$. Without loss of generality (due to the scalability explained above) k is even. It furthermore has to fulfil the inequality $\frac{k}{d} \geq \frac{n}{l} = \frac{2n}{n-1} = 2 + \frac{2}{n-1}$: Assume $c : V(G) \to \{0, ..., k-1\}$ is the corresponding assignment. Let $v_1, ..., v_n$ be a cyclic enumeration of the vertices on C_n. Then $\mathrm{dist}_k(c(v_i), c(v_{i+1})) \geq d, i = 1, ..., n, n+1 := 1$, and we get

$$n \cdot d \leq \sum_{i=1}^{n} \mathrm{dist}_k(c(v_i), c(v_{i+1})).$$

Since $\mathrm{dist}_k(c(v_i), c(v_{i+1})) \leq \frac{k}{2}, i = 1, ..., n$, we have

$$k \cdot l - n \cdot d \geq k \cdot l - n \cdot \frac{k}{2} + \sum_{i=1}^{n} \underbrace{\left(\frac{k}{2} - \mathrm{dist}_k(c(v_i), c(v_{i+1}))\right)}_{\geq 0} = \sum_{i=1}^{n} \underbrace{\left(\frac{k}{2} - \mathrm{dist}_k(c(v_i), c(v_{i+1}))\right)}_{\geq 0} - \frac{k}{2}.$$

By assumption, n is odd. For each assigned colour $c(v_i) \in \{0, ..., k-1\}$, where $i \in \{1, ..., n\}$ is odd, let $\overline{c}(v_i)$ be its "opposite" colour, i.e., $\overline{c}(v_i) := (c(v_i) + \frac{k}{2}) \bmod k$ and $\overline{c}(v_i) := c(v_i)$, whenever $i \in \{1, ..., n\}$ is even. Obviously, the circular k-distance as defined in 6.1 fulfils the triangle inequality, i.e., $\mathrm{dist}_k(x, y) \leq \mathrm{dist}_k(x, z) + \mathrm{dist}_k(z, y)$, $\forall x, y, z \in \{0, ..., k-1\}$. We thus have

$$\mathrm{dist}_k(c(v_1), c(v_n)) = \mathrm{dist}_k(\overline{c}(v_1), \overline{c}(v_n)) \leq \sum_{i=1}^{n-1} \mathrm{dist}_k(\overline{c}(v_i), \overline{c}(v_{i+1})).$$

On the other hand, we have $\mathrm{dist}_k(\overline{c}(v_i), \overline{c}(v_{i+1})) = \frac{k}{2} - \mathrm{dist}_k(c(v_i), c(v_{i+1})), i = 1, ..., n$, implying

$$\mathrm{dist}_k(c(v_1), c(v_n)) \leq \sum_{i=1}^{n-1} \left(\frac{k}{2} - \mathrm{dist}_k(c(v_i), c(v_{i+1}))\right).$$

Adding $\frac{k}{2} - \mathrm{dist}(c(v_1), c(v_n))$ at both ends of the inequality now gives that $k \cdot l - n \cdot d \geq 0$, i.e., $\chi^*(G) = \frac{k}{d} \geq \frac{n}{l} = 2 + \frac{2}{n-1}$ as claimed.
On the other hand, putting $c(v_i) := (\underbrace{2^{-1}}_{=l+1} \cdot i) \bmod k, i = 1, ..., n$ gives an (n, l)-colouring of G and thus, $\chi^*(G) = 2 + \frac{2}{n-1}$ for all odd $n \geq 3$.

Looking at the notion of digraph colourings according to Neumann-Lara, it becomes clear that we can introduce a similar definition of (k, d)-colourings and a fractional dichromatic value for each loopless digraph D: A k-digraph colouring of D in the usual sense is an assignment of colours $0, ..., k-1$ to the vertices of D such that there are no monochromatic directed cycles. Given a number $d \in \mathbb{N}$ and a cyclic ordering of the colours as described above, we can replace the term of a "monochromatic edge" in this definition by an edge $e = (u, v) \in E(D)$ whose end vertices have circular distance less than d, i.e., $\mathrm{dist}_k(u, v) < d$. We thereby end up with the following definition of (k, d)-digraph colourings.

Definition 6.5. Let D be a digraph and $k, d \in \mathbb{N}$. A (k,d)-digraph-colouring of D is defined to be an assignment $c : V(D) \to \{0, ..., k-1\}$ of at most k colours to the vertices of G, such that the induced subdigraph $D[A]$ of D, where A denotes the set of directed edges $e = (u, w)$ in D with $\mathrm{dist}_k(c(u), c(w)) < d$, is acyclic.

In complete analogy to the undirected case, given any legal digraph colouring $c : V(D) \to \{0, ..., \vec{\chi}(D) - 1\}$, we can scale those colours with any natural scaling factor $r \in \mathbb{N}$ in order to define a legal $(r\vec{\chi}(D), r)$-colouring, which thus gives us an infinite sequence of (k,d)-pairs with $\frac{k}{d} \leq \vec{\chi}(G)$ for which legal colourings exist. Again, we may ask if we can do better as d grows and thus define star dichromatic numbers of digraphs resp. graphs:

Definition 6.6. Let D be a digraph. The *star dichromatic number* of D is the real value $\vec{\chi}^*(D)$ defined by

$$\vec{\chi}^*(D) = \inf\left\{\frac{k}{d}\middle|\text{There is a } (k,d)\text{-digraph colouring of } D\right\}.$$

If G is a digraph and $\Theta(G)$ the set of its orientations, i.e., the digraphs which have G as underlying graph, we can define the *Star Dichromatic Number* of the *graph* G as the maximum over all the star dichromatic numbers of its orientations:

$$\vec{\chi}^*(G) := \max_{D \in \Theta(G)} \vec{\chi}^*(D).$$

These definitions so far seem very reasonable and natural. Still, they have some surprising and odd properties, as shall be sketched by the following example.

Example 6.7. Let $\vec{C}_n$ for $n \geq 2, n \in \mathbb{N}$ denote the directed cycle of length n. Given any number $d \in \mathbb{N}$, if $\vec{C}_n$ admits a legal (k,d)-digraph colouring c, then there has to be a directed edge (u,v) in $\vec{C}_n$ with circular distance at least d between its end vertices, and hence, $k \geq 2d$ is necessary. On the other hand, by choosing any circular enumeration $1, ..., n$ of $\vec{C}_n$ such that the edges are of the form $(i, i+1), i = 1, ..., n-1, (n, 1)$ and putting $c(1) := d, c(i) := 0, i = 2, ..., n$, we get a legal $(2d, d)$-colouring of $\vec{C}_n$ for each natural number $d \in \mathbb{N}$, and so $\vec{\chi}^*(\vec{C}_n) = \vec{\chi}(C_n) = 2$.

This might seem a little surprising, since, if we compare a digon to a $\vec{C}_{100}$, intuitively, a fractional dichromatic number should be 2 in the first case (since the digon forces distinct colours at the two vertices) and very close to 1 in the second case, since the $\vec{C}_{100}$ is almost directed in the sense that the reversal of any of the 100 edges makes it acyclic. This is not at all reflected in the notion introduced above. Moreover, our way of definition forces a gap of star dichromatic numbers between 1 and 2, so that they can't be used for a finer classification of sparse (in terms of directed cycles) digraphs:

Observation 6.8. *Let D be a digraph. If D is acyclic, then $\vec{\chi}^*(D) = \vec{\chi}(D) = 1$. If D contains a directed cycle, then $\vec{\chi}^*(D) \geq 2$.*

Proof. The first claim is clear, since the all-0-colouring is a legal $(1,1)$-colouring of D in this case. On the other hand, if D contains a directed cycle $\vec{C}_n$ as a subdigraph, any (k,d)-colouring induces a (k,d)-colouring of $\vec{C}_n$ and thus, $\vec{\chi}^*(D) \geq \vec{\chi}^*(\vec{C}_n) = 2$. □

Another definition of fractional dichromatic numbers which overcomes this trouble, called the *circular dichromatic number* of digraphs and graphs, was introduced by Bokal et. al. in [BFJ$^+$04]. Instead of (k, d)-pairs, they use real numbers for their definition:
Given a $p \geq 1$, they consider a plane-circle S_p of perimeter p and define a *strong* p-colouring of D to be an assignment $c : V(D) \to S_p$ of colouring points on S_p to the vertices, in such a way that for every edge $e = (u, w)$ in D, the one-sided distance of $c(u), c(w)$ (i.e., the length of a counter-clockwise arc connecting u to w in S_p) is at least 1. More formally, we can identify S_p with the set $\mathbb{R}/p\mathbb{Z}$ and require that the unique representative of $c(w) - c(u) \in \mathbb{R}/p\mathbb{Z}$ in the interval $[0, p)$ is at least one. Since the term of a strong circular p-colouring turns out to be much less flexible, the authors also defined so-called *weak circular p-colourings* of D, $p \in [1, \infty)$, as mappings $c : V(D) \to S_p$, such that equal colours at both ends of an edge, i.e., $c(u) = c(w)$ where $e = (u, w) \in E(D)$, are allowed, but at the same time, the one-sided distance of $c(u), c(w)$ on S_p is at least 1 whenever they are distinct. Moreover, each so-called colour class, i.e., $c^{-1}(t), t \in S_p$ has to induce an acyclic subdigraph of D. This seems much more intuitive and closer to the definition of legal digraph colourings. The circular chromatic number $\vec{\chi}_c(D)$ according to [BFJ$^+$04] now is defined as the infimum over all real values $p \geq 1$ for which D admits a strong p-colouring, or, which is equivalent (as shown in their paper), as the infimum over all values $p \geq 1$ providing weak circular p-colourings of D.

Definition and Proposition 6.9 (cf. [BFJ$^+$04])**.** Let D be a digraph. The common real value

$$\begin{aligned}\vec{\chi}_c(D) &:= \inf\{p \geq 1 | \exists \text{ legal weak circular } p\text{-colouring of } D\} \\ &= \inf\{p \geq 1 | \exists \text{ legal strong } p\text{-colouring of } D\}\end{aligned}$$

is defined as the *circular dichromatic number* of D. If G is a graph and $\Theta(G)$ the set of its orientations, then we define the maximum

$$\vec{\chi}_c(G) := \max_{D \in \Theta(G)} \vec{\chi}_c(D)$$

to be the *circular dichromatic number* of the graph G.

Proof. A proof of the stated equality can be found in [BFJ$^+$04]. $\square$

After having introduced the alternative notion of Bokal et. al., we may ask some natural questions: What is the relation of the circular dichromatic number to the star dichromatic number resp. the dichromatic number of a digraph? Does it fill the $(1, 2)$-gap for the star dichromatic number gap illustrated in 6.7? What are the circular dichromatic numbers of the $\vec{C}_n, n \geq 2$? And finally, is it possible to find an alternate definition of the circular dichromatic number in terms of some kind of (k, d)-colourings?
The following statements provide answers to most of these problems:

Theorem 6.10 (cf. [BFJ$^+$04])**.** *Let D be a loopless digraph. Then the following holds:*

(i) $\vec{\chi}_c(D) \geq 1$ is a rational number, and the infimum in definition 6.9 is attained, i.e., it can be written as a minimum.

(ii) $\lceil \vec{\chi}_c(D) \rceil = \vec{\chi}(D)$*, i.e.,* $\vec{\chi}_c(D) \in (\vec{\chi}(D)-1, \vec{\chi}(D)]$.

(iii) For each rational number $q \in \mathbb{Q}, q = \frac{m}{n} \geq 1$*, there is a digraph* $D_{m,n}$ *with* $\vec{\chi}_c(D_{m,n}) = \frac{m}{n} = q$.

(iv) If $p \geq 1$ *is any real number, a weak circular* p*-colouring of* D *exists if and only if for every pair* $(k,d) \in \mathbb{N}^2$ *with* $\frac{k}{d} \geq p$*, there exists a legal digraph-colouring* $c_{(k,d)} : V(D) \to \{0,...,k_n - 1\}$ *of* D *such that for each non-monochromatic edge* $e = (u,w)$ *in* D *we have* $(c_{(k,d)}(w) - c_{(k,d)}(u)) \bmod k \geq d$.

Proof. Proofs of the statements (i)-(iii) are contained in [BFJ+04].

(iv) For the first implication, assume that there is indeed a legal weak circular p-colouring of D given as a mapping $c_p : V(D) \to [0,p)$. For each pair $(k,d) \in \mathbb{N}^2$ with $\frac{k}{d} \geq p$, define an assignment $c_{(k,d)} : V(D) \to \{0,...,k-1\}$ according to $c_{(k,d)}(x) = \lfloor k\frac{c_p(x)}{p} \rfloor, \forall x \in V(D)$ ($\frac{c_p(x)}{p} < 1$ implies $\lfloor k\frac{c_p(x)}{p} \rfloor \leq k-1$). First of all, for each edge $e = (u,w)$ in D, either $c_p(u) = c_p(w)$ and thus $c_{(k,d)}(u) = c_{(k,d)}(w)$ or, by definition of a weak circular colouring, $(c_p(w) - c_p(u))$ mod $p \geq 1$, where the latter denotes the unique representative of $c_p(w) - c_p(u) \in \mathbb{R}/p\mathbb{Z}$ in $[0,p)$. But this implies that the unique representative of $k\frac{c_p(w)}{p} - k\frac{c_p(x)}{p}$ as element of $\mathbb{R}/k\mathbb{Z}$ in $[0,k)$ is at least $\frac{k}{p} \geq d$. Rounding down $k\frac{c_p(u)}{p}, k\frac{c_p(w)}{p}$ decreases their respective values by a real number in $[0,1)$, and so,

$$(c_{(k,d)}(w) - c_{(k,d)}(u)) \text{ mod } k > \left((k\frac{c_p(w)}{p} - k\frac{c_p(u)}{p}) \text{ mod } k \right) - 1 \geq d - 1.$$

Since $(c_{(k,d)}(w) - c_{(k,d)}(u))$ mod k is an integer, this already implies

$$(c_{(k,d)}(w) - c_{(k,d)}(u)) \text{ mod } k \geq d$$

for each edge $e = (u,w)$ in D. It therefore remains to show that $c_{(k,d)}$ is a legal digraph colouring of D. Assume by way of contradiction that there was a directed monochromatic cycle C in D with respect to $c_{(k,d)}$. According to the above, for each edge (u,w) in C, $c_p(u) \neq c_p(w)$ would imply $c_{(k,d)}(u) \neq c_{(k,d)}(w)$, wherefore C is also monochromatic with respect to c_p. This finally contradicts the fact that the colour-classes $c_p^{-1}(t)$ induce acyclic subdigraphs of D for all $t \in S_p \cong \mathbb{R}/p\mathbb{Z} \cong [0,p)$. Thus, $c_{(k,d)}$ is a colouring as desired.
For the reverse implication, assume that $p \geq 1$ so that for every pair $(k,d) \in \mathbb{N}^2$ with $\frac{k}{d} \geq p$, there is a legal digraph colouring $c_{(k,d)} : V(D) \to \{0,...,k-1\}$ of D such that $(c_{(k,d)}(w) - c_{(k,d)}(u)) \bmod k \geq d$ for every non-monochromatic edge $e = (u,w)$ in D. Let $((k_n, d_n))_{n \in \mathbb{N}}$ be some sequence in $\mathbb{N}^2$ such that $\frac{k_n}{d_n} \geq p, \forall n \in \mathbb{N}$, and $\frac{k_n}{d_n} \to p, n \to \infty$. According to the assumption, there exists a legal digraph-colouring $c_n = c_{(k_n,d_n)}$ of D ranging in $\{0,...,k_n - 1\}$ with the additional property that $(c_n(w) - c_n(u)) \bmod k_n \geq d_n$ for non-monochromatic edges (u,w). For all $n \in \mathbb{N}$, define $p_n := \frac{k_n}{d_n} \in \mathbb{R}_+$ and consider the colouring $c_{p_n} : V(D) \to [0,p_n), c_{p_n}(x) := p_n\frac{c_n(x)}{k_n} = \frac{c_n(x)}{d_n}$. We can equivalently interpret the colourings $c_{p_n}, n \in \mathbb{N}$ as a vector-sequence in $\mathbb{R}^{V(G)}$. Since $p_n \searrow p, n \to \infty$, $(p_n)_{n \in \mathbb{N}}$ and thus also $(c_{p_n})_{n \in \mathbb{N}}$ are bounded sequences. According to the Heine-Borel-theorem, this

implies the existence of a convergent subsequence of $(c_{p_n})_{n\in\mathbb{N}}$. Without loss of generality, we can thus assume that already $(c_{p_n})_{n\in\mathbb{N}}$ itself converges to a $c \in \mathbb{R}^{V(G)}$. We claim that the thereby defined mapping $c : V(G) \to [0,p]$, taken modulo p, i.e., $c_p : V(G) \to S_p \cong \mathbb{R}/p\mathbb{Z} \cong [0,p)$, $c_p(x) := c(x)$ mod p, is a weak circular p-colouring of D: First of all, for each edge $e = (u,w)$ in D with $c_p(u) \neq c_p(w)$, we have $c(u) \neq c(w)$ and thus, the convergence implies that there is some $n_0(e) \in \mathbb{N}$ such that $c_{p_n}(u) \neq c_{p_n}(w), \forall n \in \mathbb{N}_0, n \geq n_0(e)$ and equivalently $c_n(u) \neq c_n(w), \forall n \geq n_0(e)$. According to the initial assumptions about the colourings c_n, we get that $(c_n(w) - c_n(u))$ mod $k_n \geq d_n, n \geq n_0$. After dividing by d_n, this can be written as $(c_{p_n}(w) - c_{p_n}(u))$ mod $p_n \geq 1, n \geq n_0$. Letting $n \to \infty$ at both sides of the inequality now yields $(c_p(w) - c_p(u))$ mod $p = (c(w) - c(u))$ mod $p \geq 1$ for each non-monochromatic edge $e = (u,w)$. In order to complete the proof, we now only need to check the acyclicity of the colour classes $c_p^{-1}(t), t \in [0,p)$: Assume there was a directed cycle C in D which is monochromatic with respect to c_p. Then for every edge $e \in E(C), e = (u,w)$, we have $\lim_{n\to\infty}((c_{p_n}(w) - c_{p_n}(u))$ mod $p_n) = (c_p(w) - c_p(u))$ mod $p = 0$ implying that given $\varepsilon := 1 > 0$, there is a number $n_1(e) \in \mathbb{N}$ such that $(c_{p_n}(w) - c_{p_n}(u))$ mod $p_n < 1$, for all $n \geq n_1(e)$. As shown above, this already implies that e is monochromatic in c_{p_n} and thus c_n, for $n \geq n_1(e)$. Thus, C is a monochromatic directed cycle in c_n for all $n \geq \max_{e\in E(C)} n_1(e)$, contradicting the choice of c_n as a legal digraph colouring of D. All in all, we know that c_p is indeed a weak circular p-colouring of D, so the stated equivalence follows. ☐

Definition 6.11. Let D be a digraph and $(k,d) \in \mathbb{N}^2$. A legal digraph-colouring $c : V(D) \to \{0,...,k-1\}$ such that for each non-monochromatic edge $e = (u,w)$ in D we have $(c(w) - c(u)) \bmod k \geq d$ is defined as a *circular* (k,d)*-colouring* of D.

Corollary 6.12. *Let D be a digraph. Then*

$$\vec{\chi}_c(D) = \inf\left\{\frac{k}{d}|\exists \text{ a legal circular } (k,d)\text{-colouring of } D.\right\},$$

and for each pair $(k,d) \in \mathbb{N}^2$ with $\frac{k}{d} \geq \vec{\chi}_c(D)$, there is a circular (k,d)-colouring of D.

Proof. The equality is an immediate consequence of Theorem 6.10, (iv). Furthermore, if $\frac{k}{d} \geq p := \vec{\chi}_c(D)$, then according to Theorem 6.10, (i), there is a weak circular p-colouring of D. Theorem 6.10, (iv) now implies the claim. ☐

Theorem 6.13. *Let D be a digraph. Then*

(i) $\vec{\chi}(D) - 1 \leq \vec{\chi}^*(D) \leq \vec{\chi}(D)$.

(ii) Let G be an undirected graph and $Z(G)$ its symmetrical orientation, i.e., the digraph arising from G by replacing each edge $e = uv$ by a bidirectional pair of arcs connecting u and v (This type of digraph was already considered in section 4 when analysing the connections of dichromatic and chromatic numbers). Then the legal (k,d)-(digraph-)colourings of $Z(G)$ are exactly the legal (k,d)-colourings of G, which implies $\vec{\chi}^*(Z(G)) = \chi^*(G)$.

(iii) For each rational number $q = \frac{m}{n} \in \mathbb{Q}, q \geq 2$, there is a digraph $D'_{m,n}$ with star dichromatic number q.

Proof.

(i) The statement holds obviously true in the case that D is acyclic, and so assume in the following that D contains directed cycles.
$\vec{\chi}^*(D) \leq \vec{\chi}(D)$ is an immediate consequence of the fact that each legal k-digraph colouring of D can be considered as a legal $(k, 1)$-colouring of D. On the other hand, we have $\vec{\chi}(D) - 1 \leq \vec{\chi}^*(D)$: Assume for a proof by contradiction that there was a legal (k, d)-colouring of D such that $\frac{k}{d} < \vec{\chi}(D) - 1$. Then $D[A]$ is acyclic, where A is the set of edges in D whose end vertices have circular distance at most $d - 1$ with respect to D. Define $c'(v) := \lfloor \frac{c(v)}{d} \rfloor \in \{0, ..., \lfloor \frac{k-1}{d} \rfloor\}$ for all $v \in V(D)$. We claim that this is a legal $\lfloor \frac{k-1}{d} \rfloor + 1$-colouring of D: Assume for contrary there was a monochromatic directed cycle C in D such that $c'(v) = x, v \in V(C)$. According to the definition of c', this means that $c(v) \in \{dx, dx + 1, ..., dx + d - 1\}, v \in V(C)$. Thus, each pairs of vertices on C has circular k-distance at most $d - 1$ (since D contains a directed cycle, according to 6.8 we have $k \geq 2d$ and thus $k - d \geq d$), and so C is a directed cycle in the acyclic digraph $D[A]$, contradiction. Therefore we have $\vec{\chi}(D) \leq \lfloor \frac{k-1}{d} \rfloor + 1 \leq \vec{\chi}(D) - 2 + 1$, the desired contradiction, and are done.

(ii) The legal (k, d)-colourings of G are defined as assignments $c : V(G) \to \{0, ..., k - 1\}$ for which the circular distance $\mathrm{dist}_k(x, y)$ of two vertices $x \neq y \in V(G)$ is at least D whenever they are adjacent. Every such colouring obviously defines a legal (k, d)-colouring of $Z(G)$, since the set A of arcs with circular distance at most $d-1$ between its end vertices is empty and thus contains no directed cycles. On the other hand, given any legal (k, d)-colouring of $Z(G)$, if $e \in E(G)$ was an edge with circular distance at most $d - 1$ between its end vertices, the digon consisting of the two parallel edges replacing e in $Z(G)$ would be a directed cycle in the acyclic subdigraph $D[A]$ containing arc with circular distance less than d, contradiction, and thus, it again has to be a legal (k, d)-colouring of G, proving the equivalence.

(iii) According to Theorem 6.3, there is a graph $G_{m,n}$ such that $\chi^*(G_{m,n}) = q$. Thus, according to (ii), $Z(G_{m,n})$ is a directed graph with star dichromatic number q.

□

Obviously, after having introduced two different notions of fractional dichromatic numbers of digraphs, it is natural to ask for their relation for different kinds of digraphs. In any case, according to the above, $|\vec{\chi}_c(D) - \vec{\chi}^*(D)| \leq 1$, for all digraphs D. If the 2-Colour-Conjecture holds true, then according to the above, $\vec{\chi}_c(D) \leq 2, \vec{\chi}(D) = 2$ for all simple planar digraphs containing at least one directed cycle. As was shown in [BFJ+04], $\vec{\chi}_c(\vec{C}_n) = 1 + \frac{1}{n-1} < 2 = \vec{\chi}^*(\vec{C}_n), \forall n \in \mathbb{N}, n \geq 3$. Furthermore, it is easy to see that $\vec{\chi}_c(Z(G_{m,n})) = \frac{m}{n} = \vec{\chi}^*(Z(G_{m,n})), \forall m, n \in \mathbb{N}, m \geq 2n$, where $G_{m,n}$ is the graph from Theorem 6.3. So far, I have not managed to find an example of a digraph D with $\vec{\chi}_c(D) > \vec{\chi}^*(D)$. Thus, we pose the following question as an open problem:

Question 6.14. *Does $\vec{\chi}_c(D) \leq \vec{\chi}^*(D)$ hold for all digraphs D?*

6.2 Fractional Nowhere-Zero and Neumann-Lara-Flows

After having introduced the notions of the star chromatic number of graphs and the star dichromatic and circular dichromatic numbers of digraphs, we want to take a look at the dual counterparts of these notions, i.e., explore fractional flow indices of various types.
In the following, we first sketch the theory of *Fractional Nowhere-Zero-Flows* as introduced in [GTZ04]. Since the considerations there are done in the wider context of regular matroids and their orientations, in the following section, we will also stick to that and define NZ-(k,d)-flows, NL-(k,d)-flows and circular NL-(k,d)-flows at first on graphs resp. digraphs in order to extend those definitions later on.

To introduce the topic, let G be an arbitrary planar connected graph equipped with a (k,d)-colouring, $k \geq 2d$, $c : V(G) \to \{0,...,k-1\}$ on it. For any orientation $\vec{G}$ of G, we can define a corresponding NZ-k-coflow on $\vec{G}$, where each edge $e = (u,w)$ in D receives the value $c(w) - c(u) \in \{\pm d, ..., \pm(k-d)\}$ (cf. Section 4). The latter follows from the fact that adjacent vertices in G have to have circular distance at least d in c. On a planar dual graph G^* of G with the corresponding dual orientation $\vec{G^*}$, by taking the same flow values on the edges of $\vec{G^*}$ as for the corresponding edges in $\vec{G}$, we get an integer flow on $\vec{G^*}$ ranging in $\{\pm d, ..., \pm(k-d)\}$. Such coflows resp. flows on one (and thus all) orientations of a graph are what we call (k,d)*-coflows resp. -flows*, thereby obviously generalizing the usual definitions of NZ-k-coflows and -flows on graphs.

Definition 6.15 (cf. [GTZ04])**.**

- Let G be a graph and $\vec{G}$ some orientation on G. Given a pair $(k,d) \in \mathbb{N}^2$ with $k \geq 2d$, a *Nowhere-Zero-*(k,d)*-Coflow* on $\vec{G}$ is defined as a tension on $\vec{G}$ ranging in $\{\pm d, ..., \pm(k-d)\}$. Obviously, by changing the sign of the coflow value at each edge whose orientation is reversed, the existence of a NZ-(k,d)-coflow on some orientation of G implies the existence of corresponding NZ-(k,d)-coflows on every orientation of G, wherefore we often omit the actual orientation of G and speak of NZ-(k,d)-coflows on G itself.

- Let G be a graph and $\vec{G}$ some orientation on G. Given a pair $(k,d) \in \mathbb{N}^2$ with $k \geq 2d$, a *Nowhere-Zero-*(k,d)*-Flow* on $\vec{G}$ is defined as a flow on $\vec{G}$ ranging in $\{\pm d, ..., \pm(k-d)\}$. Obviously, by changing the sign of the flow value at each edge whose orientation is reversed, the existence of a NZ-(k,d)-flow on some orientation of G implies the existence of corresponding NZ-(k,d)-flows on every orientation of G, wherefore we again omit the actual orientation of G and speak of NZ-(k,d)-flows on G itself.

Clearly, in the case $d = 1$, the introduced notions are equivalent to usual NZ-k-coflows and NZ-k-flows. The reason for investigating NZ-(k,d)-flows on graphs obviously is to introduce a fractional NZ-flow index for graphs or more generally regular matroids. As in the case of colourings, we can define such a number as the infimum over all quotients $\frac{k}{d}$ admitting a NZ-(k,d)-flow. Before doing that, we step further and generalize the definition of Nowhere-Zero-(k,d)-Flows in graphs in a natural way to such on regular matroids, thereby again generalizing definitions from paragraph 5.1.

Definition 6.16 (cf. [GTZ04])**.**

- Let M be a regular matroid and $\omega(M)$ some orientation of M, $(k,d) \in \mathbb{N}^2$. A *Nowhere-Zero-*(k,d)-flow on $\omega(M)$ is defined to be an integer flow $f : E(M) \to \{\pm d, ..., \pm(k-d)\}$ according to $\omega(M)$. As in the case of graphs, utilizing Corollary 3.101, the choice of the orientation $\omega(M)$ is irrelevant for the existence of a NZ-(k,d)-flow, and thus, we often refer to such NZ-(k,d)-flows as being defined on M.

- The non-negative real or infinite value defined by

$$\xi^*(M) := \inf\left\{\frac{k}{d}\middle| \exists \text{ a NZ-}(k,d)\text{-flow on } M\right\}$$

is called the *star flow index* of M.

Definition and Proposition 6.17. Let G be an arbitrary graph and $M := M(G)$ its graphic matroid (which is regular). Let $\vec{G}$ be an orientation of G. Then the mappings $f : E(G) = E(M) \to \mathbb{Z}$ defining a NZ-(k,d)-flow on $\vec{G}$ for $(k,d) \in \mathbb{N}^2$ are the same as those defining a NZ-(k,d)-flow on $\omega(M)$, and thus, the star flow index of G defined by $\xi^*(G) := \xi^*(M)$ is the infimum over all quotients $\frac{k}{d}$ where G admits a NZ-(k,d)-flow on some and thus all of its orientations. If G is a connected planar graph with associated dual graph G^*, as explained above, due to the equivalence of (k,d)-colourings and NZ-(k,d)-coflows on graphs, we have $\chi^*(G) = \xi^*(G^*)$.

In order to deduce some properties of the star flow indices of graphs and regular matroids as counterparts of the primal properties presented for the star chromatic number in Theorem 6.3, the authors of [GTZ04] used the so-called Hoffman-Circulation-Lemma as a main tool. This Lemma generally provides equivalent conditions for the existence of rational-valued flows in matroids satisfying some range-restrictions described by upper and lower bounds at each edge:

Lemma 6.18 (Hoffman's Circulation Lemma, cf. [Hof60])**.** *Let M be a regular matroid equipped with an orientation $\omega(M)$. Let $l, u : E(M) \to \mathbb{Q}$ with $0 \leq l \leq u$ be a pair of non-negative rational-valued bounding functions on the elements of M. Then there is a rational flow $f : E(M) \to \mathbb{Q}$ on $\omega(M)$ such that $l(e) \leq f(e) \leq u(e), \forall e \in E(M)$ if and only if for all signed cocircuits $S = (S^+, S^-)$ in $\omega(M)$ the following inequalities hold true:*

$$\sum_{e\in S^+} l(e) \leq \sum_{e\in S^-} u(e), \sum_{e\in S^-} l(e) \leq \sum_{e\in S^+} u(e).$$

In this case, the flow f can additionally be required to be integer-valued if also l and u are.

Proof. The necessity of the inequality-conditions is an immediate consequence of the definition of flows in oriented matroids. The reverse implication is a lot harder, a proof can be found in [Hof60]. $\square$

The Hoffman-Lemma now can obviously be used to find equivalent arithmetic conditions for the existence of NZ-(k,d)-flows in regular matroids.

Theorem 6.19 (cf. [GTZ04]). *Let M be a regular matroid. Then M admits a NZ-(k,d)-flow if and only if there is an orientation $\omega(M)$ such that*

$$\frac{d}{k-d} \leq \frac{|S^+|}{|S^-|} \leq \frac{k-d}{d},$$

for all signed cocircuits $S = (S^+, S^-)$.

Proof. M admits a Nowhere-Zero-(k,d)-flow if and only if it admits one on each of its orientations. Changing negative flow values to positive ones if needed by flipping the orientation of the corresponding elements in each circuit and cocircuit containing them, shows that this again is equivalent to the existence of an orientation $\omega(M)$ admitting a non-negative legal NZ-(k,d)-flow, i.e., an integer flow f on $\omega(M)$ with $f(e) \in \{d, ..., k-d\}, \forall e \in E(M)$. Now applying the Hoffman-Circulation Lemma with the constant lower and upper bounds $l(e) := d, u(e) := k-d, e \in E(M)$ immediately yields the claimed equivalent inequalities. $\square$

Thus we immediately deduce the following central finite maximum-minimum expression of the star flow index $\xi^*(M)$, from which we can easily deduce further properties:

Corollary 6.20. *Let M be a regular matroid. Then*

$$\begin{aligned} \xi^*(M) &= 1 + \min_{\omega(M) \text{ orientation}} \max\left\{ \frac{|S^+|}{|S^-|}, \frac{|S^-|}{|S^+|} \middle| S = (S^+, S^-) \text{ cocircuit} \right\} \\ &= \min_{\omega(M) \text{ orientation}} \max\left\{ \frac{|S|}{|S^-|}, \frac{|S|}{|S^+|} \middle| S = (S^+, S^-) \text{ cocircuit} \right\}. \end{aligned}$$

Proof. According to Theorem 6.19, a NZ-(k,d)-flow on M exists if and only if there is some orientation $\omega(M)$ of M with $\frac{|S^+|}{|S^-|}, \frac{|S^-|}{|S^+|} \leq \frac{k-d}{d} = \frac{k}{d} - 1$ for all signed cocircuits, which immediately implies the stated equality. $\square$

The following theorem captures the most essential properties of the star flow index of regular matroids and corresponds to Theorem 6.3 describing the primal case for graphs:

Theorem 6.21 (cf. [GTZ04]). *Let M be a regular matroid. Then the following holds:*

(i) $\xi^*(M)$ is a positive rational number, and $\xi^*(M) \geq 2$ whenever $E(M) \neq \emptyset$.

(ii) $\lceil \xi^*(M) \rceil = \xi(M)$, i.e., $\xi^*(M) \in (\xi(M) - 1, \xi(M)]$.

(iii) For every $k, d \in \mathbb{N}$, there exists a NZ-(k,d)-flow on M if and only if $\frac{k}{d} \geq \xi^*(M)$.

(iv) There is a (k,d)-colouring of M such that $\frac{k}{d} = \xi^*(M)$ and $k \leq \max_{S \text{ cocircuit}} |S|$.

Proof. (i) This is an immediate consequence of Corollary 6.20, since $|S| = |S^+| + |S^-| \geq 2\min(|S^+|, |S^-|)$ for every signed cocircuit $S = (S^+, S^-)$.

(ii) Setting $d = 1$ in Theorem 6.19, we see that for every $k \in \mathbb{N}$, there is a NZ-k-flow on G if and only if $k \geq \max\left(\frac{|S^-|}{|S^+|} + 1, \frac{|S^-|}{|S^+|} + 1\right) = \max\left(\frac{|S|}{|S^+|}, \frac{|S|}{|S^-|}\right)$, for all signed cocircuits $S = (S^+, S^-)$ in some orientation $\omega(M)$. But this is equivalent to $k \geq \xi^*(M)$, implying $\xi(M) = \min\{k \in \mathbb{N} | \exists \text{NZ-}k\text{-flow on } M\} = \lceil \xi^*(M) \rceil$.

(iii) For every pair $(k,d) \in \mathbb{N}^2$, according to Theorem 6.19, a NZ-(k,d)-flow on M exists if and only if $\frac{k}{d} \geq \max\left(\frac{|S^-|}{|S^+|} + 1, \frac{|S^-|}{|S^+|} + 1\right) = \max\left(\frac{|S|}{|S^+|}, \frac{|S|}{|S^-|}\right)$, for all signed cocircuits $S = (S^+, S^-)$ in some orientation $\omega(M)$, meaning $\frac{k}{d} \geq \xi^*(M)$.

(iv) Apply (iii) to the fraction

$$\frac{k}{d} = \xi^*(M) = \min_{\omega(M) \text{ orientation}} \max\left\{\frac{|S|}{|S^-|}, \frac{|S|}{|S^+|} \middle| S = (S^+, S^-) \text{ cocircuit }\right\}$$
$$= \frac{|S^*|}{\min\{|S^{*-}|, |S^{*+}|\}},$$

whose numerator is of the form $k = |S^*|$ with some designated oriented cocircuit S^* in some orientation of M.

□

As we have seen, considering fractional indices is not only restricted to graphs via (k,d)-colourings in the primal case, but also more generally in the dualized setting of Nowhere-Zero-(k,d)-flows on regular matroids, where, according to the above theorem, still all of the most important properties hold true. We now proceed with similar definitions for the directed case, i.e., we translate the definitions of (k,d)-digraph colourings and circular (k,d)-digraph colourings from the previous paragraph to the context of regular oriented matroids, thereby defining NL-(k,d)-flows and circular NL-(k,d)-flows. Unfortunately, we won't be able to prove analogous results for the arising star NL-flow index and the circular analogon of an oriented regular matroid as was done in Theorems 6.10 and 6.13. The reason for this is explained below.

Given a planar pair of directed dual graphs (D, D^*), the legal (k,d)-colourings of D were defined as the assignments $c : V(D) \to \{0, ..., k-1\}$ for which the set A of arcs whose end vertices admit circular distance at most $d-1$ with respect to k, induces an acyclic subdigraph $D[A]$ of D. If we define the associated tension $f : E(D) \to \{0, \pm 1, \pm 2, ..., \pm(k-1)\}$ on D as in the case of NZ-coflows by $f(e) := c(w) - c(u)$, if $e = (u,w)$, we thus know that the subdigraph $D[A]$ induced by the arc set $A := f^{-1}(\{0, \pm 1, ..., \pm(d-1)\} \cup \{\pm(k-d+1), ..., \pm(k-1)\})$ is acyclic. If we consider f as an integer-assignment on the

edges of D^*, it has to be an integer flow (since f on D was a tension), and since the digraph $D[A] = D - \overline{A}$ was required to be acyclic, the digraph $(D[A])^* = (D - \overline{A})^* = D^*/\overline{A}$, where $\overline{A} = f^{-1}(\{\pm d, ..., \pm(k-d)\})$ is totally cyclic resp. in case that $U(D^*)$ is connected, strongly connected.
On the other hand, if we require the colouring $c : V(D) \to \{0, ..., k-1\}$ in the argument above to be a legal *circular* (k, d)-colouring of D as defined in 6.11, the arising tension $f : E(D) \to \{0, \pm 1, ..., \pm(k-1)\}$ has the property that for each edge $e = (u, w)$, either $f(e) = c(w) - c(u) = 0$ or $f(e)$ mod $k = (c(w) - c(u))$ mod $k \geq d$ and additionally, the set A' of monochromatic arcs, i.e., $A' := f^{-1}(\{0\})$, induces an acyclic subdigraph $D[A']$. Thus, f mod k is a legal NL-$\mathbb{Z}_k$-coflow on D and f mod k ranges in $\{0\} \cup \{d, ..., k-1\} \subseteq \mathbb{Z}_k$. The same assignment f mod k, considered as a mapping on the edges of D^*, therefore gives rise to a legal NL-$\mathbb{Z}_k$-flow on D^* ranging in $\{0\} \cup \{d, ..., k-1\} \subseteq \mathbb{Z}_k$.

The thereby sketched dualities between legal (circular) (k, d)-digraph colourings of a planar digraph and certain integer flows on its directed dual yield the following generalizations of Neumann-Lara-coflows and -flows:

Definition 6.22 (Neumann-Lara-(k, d)-(co-)flows)**.** Let D be a digraph, $(k, d) \in \mathbb{N}^2$.
A *NL-(k, d)-coflow* on D is defined as a tension $f : E(D) \to \{0, \pm 1, ..., \pm(k-1)\}$ with respect to d so that the subdigraph $D[A]$ with $A := f^{-1}(\{0, \pm 1, ..., \pm(d-1)\} \cup \{\pm(k-d+1), ..., \pm(k-1)\})$ is acyclic.
A *NL-(k, d)-Flow* on D is an integer flow $f : E(D) \to \{0, \pm 1, ..., \pm(k-1)\}$ on D, such that $D/\mathrm{supp}_d(f)$ is totally cyclic, where $\mathrm{supp}_d(f) := f^{-1}(\{\pm d, ..., \pm(k-d)\}), \forall d \in \mathbb{N}$ denotes the so-called *d-support* of the flow f.

Definition 6.23 (Circular Neumann-Lara-(k, d)-(co-)flows)**.** Let D be a digraph, $(k, d) \in \mathbb{N}^2$.
A *circular NL-(k, d)-coflow* on D is a legal NL-$\mathbb{Z}_k$-coflow

$$f : E(D) \to \{0\} \cup \{d, ..., k-1\} \subseteq \mathbb{Z}_k$$

on D.
A *circular NL-(k, d)-flow* on D is a legal NL-$\mathbb{Z}_k$-flow

$$f : E(D) \to \{0\} \cup \{d, ..., k-1\} \subseteq \mathbb{Z}_k$$

on D.

Again, the notions of (circular) NL-(k, d)-coflows and -flows on digraphs admit natural extensions to the setting of oriented regular matroids:

Definition 6.24. Let $\omega(M)$ be an orientation of a regular matroid. A *NL-(k, d)-coflow on $\omega(M)$* is a tension $f : E(M) \to \{0, \pm 1, ..., \pm(k-1)\}$ on $\omega(M)$ (i.e., a flow with respect to the dual oriented matroid $\omega(M)^*$), such that the oriented regular matroid $M - \mathrm{supp}_d(f)$ does not contain all-$\oplus$- or all-$\ominus$-circuits (i.e., it is acyclic), where the *d-support* of the flow f is defined by $\mathrm{supp}_d(f) := f^{-1}(\{\pm d, ..., \pm(k-d)\})$ for all $d \in \mathbb{N}$.
Let $\omega(M)$ be an orientation of a regular matroid. A *NL-(k, d)-flow on $\omega(M)$* is an integer flow $f : E(M) \to \{0, \pm 1, ..., \pm(k-1)\}$ on $\omega(M)$, such that the oriented regular matroid $M/\mathrm{supp}_d(f)$ does not contain all-$\oplus$- or all-$\ominus$-cocircuits (i.e., it is totally cyclic).

Definition 6.25. Let $\omega(M)$ be an orientation of a regular matroid.
A *circular NL-(k,d)-coflow* on $\omega(M)$ is a legal NL-$\mathbb{Z}_k$-coflow

$$f : E(\omega(M)) \to \{0\} \cup \{d, ..., k-1\} \subseteq \mathbb{Z}_k$$

on $\omega(M)$. A *circular NL-(k,d)-flow* on $\omega(M)$ is a legal NL-$\mathbb{Z}_k$-flow

$$f : E(\omega(M)) \to \{0\} \cup \{d, ..., k-1\} \subseteq \mathbb{Z}_k$$

on $\omega(M)$.

Now, the following is an immediate consequence of the fact that flows on digraphs correspond to the flows on their corresponding oriented regular matroids, our initial considerations concerning colourings and coflows and of the duality between circuits and cocircuits in matroids.

Remark 6.26.

(i)
- Let $\omega(M)$ be an arbitrary orientation of a regular matroid. Denote by $\omega(M)^*$ the orientation of the (regular) dual matroid M^* corresponding to $\omega(M)$. Then the NL-(k,d)-coflows on $\omega(M)$ are exactly the NL-(k,d)-flows on $\omega(M)^*$ and vice versa.
- For every pair $(k,d) \in \mathbb{N}^2$, a digraph D admits a legal (k,d)-digraph colouring if and only if it admits a NL-(k,d)-coflow. If $\omega(M)$ is the orientation of the graphic matroid $M(U(D))$ corresponding to D, then the NL-(k,d)-coflows resp. -flows on $\omega(M)$ are exactly those of D (if we identify the edges of D with the elements of $\omega(M)$).

(ii)
- Let $\omega(M)$ be an arbitrary orientation of a regular matroid. Denote by $\omega(M)^*$ the orientation of the (regular) dual matroid M^* corresponding to $\omega(M)$. Then the circular NL-(k,d)-coflows on $\omega(M)$ are exactly the circular NL-(k,d)-flows on $\omega(M)^*$ and vice versa.
- For every pair $(k,d) \in \mathbb{N}^2$, a digraph D admits a legal circular (k,d)-digraph colouring if and only if it admits a circular NL-(k,d)-coflow. If $\omega(M)$ is the orientation of the graphic matroid $M(U(D))$ corresponding to D, then the circular NL-(k,d)-coflows resp. -flows on $\omega(M)$ are exactly those of D (if we identify the edges of D with the elements of $\omega(M)$).

Proof. (i) The first item is an immediate consequence of Definition 4.51 from Section 4 and of the duality between cocircuits in M and circuits in M^*.
Considering the second item, we already know according to the considerations above, that every legal (k,d)-digraph colouring gives rise to a NL-(k,d)-coflow, so it remains to prove the converse. Assume f is a legal NL-(k,d)-coflow on D. Without loss of generality, assume that $U(D)$ is connected. By taking f modulo k, we get a $\mathbb{Z}_k$-flow $f' := f \bmod k$ on D, such that $D[A]$ is acyclic, where $A := f^{-1}(\{\pm d, ..., \pm(k-d)\}) = f'^{-1}(\{d, ..., k-d\})$. Since f is a tension on D, also f'

(with respect to $\mathbb{Z}_k$) is one and thus, fixing some reference vertex $v_0 \in V(D)$, the assignment

$c(v) := \sum_{e \in P^+} f'(e) - \sum_{e \in P^-} f'(e) \in \mathbb{Z}_k$, where P is a path in $U(D)$ starting in v_0 and ending in v

and P^+ resp. P^- denote the edges on P in forwards resp. backwards direction, is uniquely defined and provides a vertex-colouring of D with colours $\mathbb{Z}_k = \{0, ..., k-1\}$. We claim that this is a legal (k,d)-colouring of D: For each edge $e = (u,w) \in E(D)$, $\text{dist}_k(c(u), c(w)) = \min((c(w) - c(u)) \bmod k, k - (c(w) - c(u)) \bmod k) \geq d$ if and only if $f(e) \bmod k = (c(w) - c(u)) \bmod k \in \{d, ..., k-d\}$, i.e., A as defined above is exactly the set of edges in D whose end vertices admit circular distance at most $d-1$ in the colouring c. Therefore, c is indeed a legal (k,d)-digraph-colouring, and we are done.

(ii) The first item is again an immediate consequence of Definition 4.51 from Section 4 and of the duality between cocircuits in M and circuits in M^*.
Considering the second item, we know according to the considerations above, that every legal circular (k,d)-digraph colouring gives rise to a circular NL-(k,d)-coflow, so it remains to prove the converse. Assume f is a legal circular NL-(k,d)-coflow on D, i.e., a legal NL-$\mathbb{Z}_k$-coflow without values in $\{1, ..., d-1\}$. According to Proposition 4.46, this NL-flow gives rise to a legal digraph colouring $c : V(D) \to \mathbb{Z}_k$ such that $f(e) = (c(w) - c(u)) \bmod k$, for each edge $e = (u,w) \in E(D)$, i.e., $c(u) = c(w)$ or $(c(w) - c(u)) \bmod k \geq d$. But this already fits the definition of a legal circular (k,d)-digraph colouring, and we are done.

□

We can now finally define the star NL-flow index and the star dichromatic number of a regular oriented matroid $\omega(M)$ as the smallest possible value $\frac{k}{d}$ for which NL-(k,d)-flows resp. -coflows exist. Note that we omit analogous definitions for the circular concept of (k,d)-colourings although such would be possible, since we will restrict our analysis on the concepts of Neumann-Lara-(k,d)-coflows and flows in the rest of this section.

Definition 6.27. Let M be a regular matroid and $\vec{M}$ be some orientation on M. The *star NL-flow index* of the oriented matroid $\vec{M}$ now is defined as the positive real number (resp. considered ∞ if the infimum is empty)

$$\vec{\xi}^*(\vec{M}) := \inf \left\{ \frac{k}{d} \middle| \exists \text{ NL-}(k,d)\text{-flow on } \vec{M} \right\}.$$

The *star dichromatic number* of M is the infimum

$$\vec{\chi}^*(\vec{M}) := \vec{\xi}^*(\vec{M}^*) = \inf \left\{ \frac{k}{d} \middle| \exists \text{ NL-}(k,d)\text{-coflow on } \vec{M} \right\}.$$

The last equality is a consequence of Remark 6.26.
The *star NL-flow index* resp. the *star dichromatic number* of the unoriented matroid M itself are, as in the case of graphs, defined as the maxima

$$\vec{\xi}^*(M) := \max_{\omega(M) \text{ orientation}} \vec{\xi}^*(\omega(M)),$$

$$\vec{\chi}^*(M) := \vec{\xi}^*(M^*),$$

over all orientations of M resp. M^*.

Remark 6.28. Let D be a digraph, and $\vec{M}$ the corresponding orientation of its regular graphic matroid $M = M(U(D))$. Then $\vec{\xi}^*(D) = \vec{\xi}^*(\vec{M})$ and $\vec{\chi}^*(D) = \vec{\chi}^*(\vec{M})$, and especially $\vec{\xi}^*(G) = \vec{\xi}^*(M(G))$, $\vec{\chi}^*(G) = \vec{\chi}^*(M(G))$ for each graph G.

Proof. This follows easily from the definition of the various indices and the observations from Remark 6.26. □

As already mentioned, a nice finite maximum-minimum expression of the defined indices cannot be found as in the case of Nowhere-Zero-Flows via Hoffman's Lemma: When applying it, the integer flows we are looking for have to admit range-restrictions by a pair consisting of lower and upper bounding functions. Obviously, when looking at definition 6.24, we see that since we only restrict the *absolute value* of the flow at certain elements, our range-restrictions may get split up into a negative and positive half: $\{\pm d, ..., \pm(k-d)\} = \{-(k-d), ..., -d\} \cup \{d, ..., k-d\}$ and it is not clear how to handle such restrictions using Hoffman's Lemma or analogues. The main tool we used in 6.19 in order to apply the lemma was the independence of NZ-(k,d)-flows of the concretely used orientation the flows are defined on, i.e., we can flip signs at negative edges without changing the properties of a NZ-(k,d)-flow and thus find an orientation where the left half in the above can be dropped.
Due to those difficulties and lack of space, we have to stop at this point and confine ourselves to the given definitions and the knowledge deduced in Section 6.1 for the case of graphic matroids. Still, it seems likely, that analogous generalizing results for regular (oriented) matroids can be proven.
In the following, we try to overcome the trouble of having a fixed orientation by fractionalizing coindependent flows resp. the concept of vertex arboricity on graphs and regular matroids as described in Section 5.1, thereby deducing some upper bounds for the introduced flow indices without having to use orientation techniques.
To motivate our definitions, let M be an arbitrary regular matroid equipped with an orientation $\omega(M)$. If we look for NL-(k,d)-flows on $\omega(M)$, we might even want to find a single flow f on $\omega(M)$ giving rise to a legal NL-(k,d)-flow on *every* orientation of M, i.e., such that the flow $f_{\omega'(M)}$ arising from f according to Corollary 3.101 is a NL-(k,d)-flow on $\omega'(M)$, for all orientations $\omega'(M)$. This is equivalent to requiring that every orientation of the regular matroid $M/\overline{\mathrm{supp}_d(f)}$ is totally cyclic, i.e., does not contain cocircuits. This means that $\overline{\mathrm{supp}_d(f)} = E(M) \backslash \mathrm{supp}_d(f)$ is the unique basis of $(M/\overline{\mathrm{supp}_d(f)})^* = M^* \backslash \mathrm{supp}_d(f)$, wherefore it is independent in M^* and thus contained in a cobasis $\overline{B} \supseteq \overline{\mathrm{supp}_d(f)}$, where B

denotes some basis of M. All in all, we look for a flow f on some (and thus any) orientation of M, ranging in $\{0, \pm 1, ..., \pm(k-1)\}$ with the property that $\mathrm{supp}_d(f)$ contains a basis:

Definition 6.29. let $\omega(M)$ be an orientation of a regular matroid M, and $(k, d) \in \mathbb{N}^2$. A *coindependent* (k, d)*-flow* on $\omega(M)$ resp. M is defined as an integer flow $f : E(M) \to \{0, \pm 1, ... \pm (k-1)\}$ such that $\mathrm{supp}_d(f) := f^{-1}(\{\pm d, ..., \pm(k-d)\})$ contains a basis of M.

Finally, the following is the fractional analogue to the coindependent index of regular matroids defined in Subsection 5.1:

Definition 6.30. Let M be a regular matroid. The *coindependent star flow index* on M is the positive real number given by

$$\xi^*_{\mathrm{coin}}(M) = \inf \left\{ \frac{k}{d} \middle| \exists \text{ coind. } (k, d)\text{-flow of } M \right\}.$$

Furthermore, the dual matroid M^* is also regular and hence, the quantity $\xi^*_{\mathrm{coin}}(M^*)$ exists and is called the *star vertex arboricity* $\mathrm{va}^*(M)$ of M.

Remark 6.31. As argued above, a coindependent (k, d)-flow on a regular matroid M gives rise to a NL-(k, d)-flow for every orientation of M. We thus have $\vec{\xi}^*(M) \leq \xi^*_{\mathrm{coin}}(M)$. Obviously, the coindependent k-flows on M are exactly the coindependent $(k, 1)$-flows, furthermore implying $\xi^*_{\mathrm{coin}}(M) \leq \xi_{\mathrm{coin}}(M)$.

The definition of the *star vertex arboricity* obviously follows the graphic planar case. As explained in paragraph 5.1, for a planar pair of dual graphs G, G^*, the vertex arboricity $\mathrm{va}(G)$ equals the coindependent flow index $\xi_{\mathrm{coin}}(G^*) = \xi_{\mathrm{coin}}(M(G^*)) = \xi_{\mathrm{coin}}(M(G)^*) \geq \xi^*_{\mathrm{coin}}(M(G)^*) =: \mathrm{va}^*(M(G))$. The latter is a lower bound for the vertex arboricity of the planar graph G and can be interpreted as a fractional version of it. We thus define the *star vertex arboricity* of a (not necessarily planar) graph G by $\mathrm{va}^*(G) := \mathrm{va}^*(M(G))$.

In the following, we present an analogous characterization of the coindependent flow indices as was done for the star flow index in the case of NZ-flows.

Lemma 6.32 (Existence of coindependent (k, d)-flows in regular matroids)**.** *A regular matroid M admits a coindependent (k, d)-flow if and only if there is an orientation $\omega(M)$ and a basis B of M, such that for all cocircuits $S = (S^+, S^-)$ in $\omega(M)$*

$$|S \cap B| \leq \frac{k}{d}|S^-| - \frac{1}{d}|S^- \backslash B|, |S \cap B| \leq \frac{k}{d}|S^+| - \frac{1}{d}|S^+ \backslash B|.$$

Proof. M admits a coindependent (k, d)-flow with respect to a basis B if and only if there is a flow f on an orientation of M with range contained in $\{0, \pm 1, ..., \pm(k-1)\}$ such that additionally $f(a) \in \{\pm d, ..., \pm(k-d)\}$ whenever $a \in B$. By changing the orientation $\omega(M)$ if needed, we can furthermore assume that f is non-negative. In this case, applying the Hoffmann-Lemma to $\omega(M)$ with

$$l(a) := \begin{cases} 0 & \text{if } a \notin B \\ d & \text{if } a \in B \end{cases}, u(a) := \begin{cases} k-1 & \text{if } a \notin B \\ k-d & \text{if } a \in B \end{cases}, a \in E(M),$$

yields that the existence of a coindependent (k,d)-flow is equivalent to the existence of an orientation $\omega(M)$, such that for all signed cocircuits $S=(S^+,S^-)$ in $\omega(M)$ the following holds:

$$\sum_{a\in S^+} l(a) \leq \sum_{a\in S^-} u(a),\ \sum_{a\in S^+} l(a) \leq \sum_{a\in S^-} u(a).$$

This again can be reformulated as

$$d|S^+\cap B| \leq (k-1)|S^-\backslash B| + (k-d)|S^-\cap B| = k|S^-| - d|S^-\cap B| - |S^-\backslash B|$$

and

$$d|S^-\cap B| \leq k|S^+| - d|S^+\cap B| - |S^+\backslash B|.$$

Adding $d|S^-\cap B|$ resp. $d|S^+\cap B|$ to the inequalities and dividing by d now gives the stated equivalent criterion for the existence of coindependent (k,d)-flows. $\square$

Lemma 6.33. *Let M be a regular matroid. Then for each $d_0\in\mathbb{N}$ the following equality holds:*

$$\xi^*_{\mathrm{coin}}(M) = \inf\left\{\frac{k}{d}\middle|\exists \text{ coin. } (k,d)\text{-flow of } M\right\} = \inf\left\{\frac{k}{d}\middle|\exists \text{ coin. } (k,d)\text{-flow of } M, d\geq d_0\right\}.$$

Proof.

$$\inf\left\{\frac{k}{d}\middle|\exists \text{ coind. } (k,d)\text{-flow of } M\right\} \leq \inf\left\{\frac{k}{d}\middle|\exists \text{ coind. } (k,d)\text{-flow of } M, d\geq d_0\right\}$$

is obvious. On the other hand, if (k,d) is any tuple admitting a coindependent flow f, according to the definition, for each $n\in\mathbb{N}$, $n\cdot f$ is a coindependent (nk,nd)-flow of M. If we choose n such that $nd\geq d_0$, since $\frac{k}{d}=\frac{nk}{nd}$, it follows that also

$$\inf\left\{\frac{k}{d}\middle|\exists \text{ coind. } (k,d)\text{-flow of } M\right\} \geq \inf\left\{\frac{k}{d}\middle|\exists \text{ coind. } (k,d)\text{-flow of } M, d\geq d_0\right\},$$

and we are done. $\square$

Theorem 6.34 (Minimum-Maximum-Characterization of the Coindependent Star-Flow-Index)**.** *Let M be a regular matroid with base-set $\mathcal{B}$. The coindependent star flow index $\xi^*_{\mathrm{coin}}(M)$ of M is the minimum*

$$\min_{\omega(M) \text{ orientation}} \min_{B\in\mathcal{B}} \left(\max\left\{\frac{|S\cap B|}{|S^+|},\frac{|S\cap B|}{|S^-|}\middle| S=(S^+,S^-) \text{ cocircuit}\right\}\right).$$

If $(k,d)\in\mathbb{N}^2$ is a pair such that $\frac{k-1}{d}\geq\xi^*_{\mathrm{coin}}(M)$, then M admits a coindependent (k,d)-flow.

Proof. The coindependent star flow index is defined as the infimum over the values $\frac{k}{d}$ for each pair (k,d) admitting a coindependent flow. As in Lemma 6.33, we fix an arbitrary positive integer $d_0 \in \mathbb{N}$ and restrict on those pairs with $d \geq d_0$.
For any orientation $\omega(M)$, denote by $\mathcal{S}_\omega$ the corresponding set of signed cocircuits. Applying Lemma 6.32 now gives that

$$\begin{aligned}
&\xi^*_{\text{coin}}(M) = \inf\left\{\frac{k}{d}\middle|\exists \text{ coind. } (k,d)\text{-flow of } M, d \geq d_0\right\} \\
&= \min_{\omega(M) \text{ orientation}} \inf\left\{\frac{k}{d}\middle| B \in \mathcal{B}, S \in \mathcal{S}_\omega, d \geq d_0, \frac{k}{d} \geq \max\left\{\frac{|S \cap B|}{|S^-|}, \frac{|S \cap B|}{|S^+|}\right\} + o(1)\right\} \\
&= \min_{\omega(M) \text{orientation}} \min_{B \in \mathcal{B}} \left(\max\left\{\frac{|S \cap B|}{|S^+|}, \frac{|S \cap B|}{|S^-|}\middle| S = (S^+, S^-) \text{ cocircuit}\right\}\right) + o(1).
\end{aligned}$$

Letting $d_0 \to \infty$ on both ends of the equality chain now finally proves the claimed equality.
Let now $(k,d) \in \mathbb{N}^2$ and $\frac{k-1}{d} \geq \xi^*_{\text{coin}}(M)$. According to the minimum-maximum expression of $\xi^*_{\text{coin}}(M)$ proven in the first part, we know that there is an orientation $\omega(M)$ of M and a basis $B \in \mathcal{B}$ such that $\frac{|S\cap B|}{|S^+|} \leq \frac{k}{d} - \frac{1}{d}, \frac{|S \cap B|}{|S^-|} \leq \frac{k}{d} - \frac{1}{d}$, for all cocircuits S in M. Since $|S^+ \backslash B| \leq |S^+|, |S^- \backslash B| \leq |S^-|$, we obtain that the conditions in Lemma 6.32 are satisfied and hence, there is a coindependent (k,d)-flow on M. $\square$

Corollary 6.35. *Let M be a regular matroid. Denote by $\mathcal{S}$ the set of cocircuits of M and by $\mathcal{B}$ the set of bases. For each $B \in \mathcal{B}$, let*

$$\beta_M(B) := \max_{S \in \mathcal{S}} \frac{|S \cap B|}{|S|} \in [0,1]$$

and $\beta_M := \min_{B \in \mathcal{B}} \beta_M(B) \in [0,1]$. Then

$$2\beta_M \leq \xi^*_{\text{coin}}(M) \leq \xi^*(M)\beta_M.$$

For each $B \in \mathcal{B}$, $\beta_M(B) < 1$ if and only if there is a basis $B' \in \mathcal{B}$ with $B \cap B' = \emptyset$.

Proof. According to the definition of β_M, we have that for every orientation $\omega(M)$ and basis $B \in \mathcal{B}$

$$\begin{aligned}
&\max\left\{\frac{|S \cap B|}{|S^+|}, \frac{|S \cap B|}{|S^-|}\middle| S = (S^+, S^-) \text{ cocircuit}\right\} \\
&= \max\left\{\underbrace{\max\left\{\frac{|S|}{|S^+|}, \frac{|S|}{|S^-|}\right\}}_{\geq 2} \frac{|S \cap B|}{|S|}\middle| S = (S^+, S^-) \text{ cocircuit}\right\} \\
&\geq 2\beta_M(B) \geq 2\beta_M.
\end{aligned}$$

This proves the first inequality. On the other hand, we have for every fixed orientation $\omega(M)$ of M:

$$\min_{B \in \mathcal{B}} \max\left\{\frac{|S \cap B|}{|S^+|}, \frac{|S \cap B|}{|S^-|}\middle| S \in \mathcal{S}\right\} = \min_{B \in \mathcal{B}} \max\left\{\max\left\{\frac{|S|}{|S^+|}, \frac{|S|}{|S^-|}\right\} \frac{|S \cap B|}{|S|}\middle| S \in \mathcal{S}\right\}$$

$$\leq \min_{B\in\mathcal{B}} \left(\max\left\{ \frac{|S|}{|S^+|}, \frac{|S|}{|S^-|} \middle| S \in \mathcal{S} \right\} \max\left\{ \frac{|S\cap B|}{|S|} \middle| S \in \mathcal{S} \right\} \right)$$
$$= \beta_M(B) \max\left\{ \frac{|S|}{|S^+|}, \frac{|S|}{|S^-|} \middle| S \in \mathcal{S} \right\}.$$

Taking the minimum over all the orientations at both ends of the inequality using Corollary 6.20 now gives the second claimed inequality.
Given a basis B, $\beta_M(B) < 1$ is equivalent to B not fully containing any cocircuit $S \in \mathcal{S}$. In the dual matroid M^*, this is equivalent to requiring that B does not contain circuits and henceforth that B is independent with respect to M^*. This again is the same as requiring B to be contained in a cobasis $\overline{B'}, B' \in \mathcal{B}$, which means the existence of a $B' \in \mathcal{B}$ with $B \cap B' = \emptyset$ as stated, and we are done. □

Corollary 6.36. *Let M be a regular matroid. Then we have $\xi_{\text{coin}}(M) - 2 < \xi^*_{\text{coin}}(M) \leq \xi_{\text{coin}}(M)$, i.e., $\lceil \xi^*_{\text{coin}}(M) \rceil \in \{\xi_{\text{coin}}(M) - 1, \xi_{\text{coin}}(M)\}$.*

Proof. If we put $d = 1$ in Lemma 6.32, we get that $\xi_{\text{coin}}(M)$ is the smallest integer k such that there is an orientation $\omega(M)$ of M and a basis B fulfilling

$$|S \cap B| \leq (k-1)|S^-| + |S^- \cap B|, |S \cap B| \leq (k-1)|S^+| + |S^+ \cap B|,$$

i.e.

$$|S^+ \cap B| \leq (k-1)|S^-|, |S^- \cap B| \leq (k-1)|S^+|$$

for all signed cocircuits S of $\omega(M)$. Thus we have

$$\xi_{\text{coin}}(M) - 1 = \left\lceil \underbrace{\min_{\omega(M) \text{ orient.}} \min_{B\in\mathcal{B}} \left(\max\left\{ \frac{|S^- \cap B|}{|S^+|}, \frac{|S^+ \cap B|}{|S^-|} \middle| S = (S^+, S^-) \text{ cocirc.} \right\} \right)}_{<\xi^*_{\text{coin}}(M)} \right\rceil.$$

Since $\xi^*_{\text{coin}}(M) \leq \xi_{\text{coin}}(M)$ follows directly from the definition, we get that $\lceil \xi^*_{\text{coin}}(M) \rceil \in \{\xi_{\text{coin}}(M) - 1, \xi_{\text{coin}}(M)\}$, which is the claim. □

This result unfortunately seems very unsatisfactory, since, compared to the results for digraphs in the previous paragraph, we would not expect a "gap" between $\xi^*_{\text{coin}}(M)$ and $\xi_{\text{coin}}(M)$ exceeding 1. Thus, we pose the following as an open problem:

Question 6.37. *Does there exist a gap between $\xi^*_{\text{coin}}(M)$ and $\xi_{\text{coin}}(M)$ exceeding one for some regular matroid M? Equivalently, does or does not $\lceil \xi^*_{\text{coin}}(M) \rceil = \xi_{\text{coin}}(M)$ hold for all regular matroids?*

While the above was mostly about general results for the different fractional flow indices of regular (oriented) matroids, we now want to discuss the case when M is a graphic matroid. According to the above, NZ-(k,d)-flows, coindependent (k,d)-flows and NL-(k,d)-flows are equivalent for graphs, digraphs and their corresponding regular resp. oriented matroids. The following already provides a full classification of the different possible fractional coindependent flow indices for cubic 3-edge-connected graphs, proving that only the values $2, 2.5$ and 3 are possible.

Theorem 6.38. *Let G be a 3-edge-connected cubic graph. Then the following holds:*

(i) $2 \leq \xi^*_{\text{coin}}(G) \leq \xi_{\text{coin}}(G) \leq 3$.

(ii) If each Hamiltonian Path in G fully contains a cut of size 3, then $\xi^*_{\text{coin}}(G) = \xi_{\text{coin}}(G) = 3$.

(iii) If G admits a Hamiltonian Path not fully containing a cut of size 3 in G but no Hamiltonian cycle, then $\xi^*_{\text{coin}}(G) = 2.5$, $\xi_{\text{coin}}(M) = 3$.

(iv) If G admits a Hamiltonian cycle, then $\xi^*_{\text{coin}}(G) = \xi_{\text{coin}}(G) = 2$.

Proof. (i) For the left side, consider the inequality $2\beta_M \leq \xi^*_{\text{coin}}(M)$ from Corollary 6.35, where $M = M(G)$ denotes the graphic matroid of G. According to Corollary 6.35, $\beta_M = 1$ whenever there are no two disjoint bases in $M(G)$, i.e., no two edge disjoint spanning trees of G. Let $n := |V(G)|$. Since G is 3-regular, $\frac{3}{2}n$ is the number of edges of G, while two disjoint spanning trees would lead to at least $2(n-1) = 2n-2$ edges, which is only possible if $n = 4$, i.e., $\beta_M = 1$ if G is not a K_4 and we are done in this case. In the case that $G = K_4$, it is easy to verify that $\xi^*_{\text{coin}}(G) = 2$ and hence, the left hand inequality also holds in this case. The remaining inequalities are immediate consequences of the facts that 3-edge-connected graphs admit coindependent 3-flows (see Theorem 5.28) and that a coindependent k-flow can always be considered as a coindependent $(k, 1)$-flow for all $k \in \mathbb{N}$.

(ii) According to the definition of the fractional coindependent index, $\xi^*_{\text{coin}}(G) < 3$ is equivalent to the existence of a pair $(k, d) \in \mathbb{N}^2$ with $\frac{k}{d} < 3$, for which a coindependent (k, d)-flow exists. This again means that there is a spanning tree T in G and a nonnegative flow f on some orientation D of G with $f(a) \in \{0, ..., k-1\}$, for $a \in E(D)$ and $f(a) \in \{d, ..., k-d\}$, for $a \in E(T)$.

Assume for contradiction that there was such a spanning tree T and a corresponding flow f. We show that in this case, T does not fully contain any cut of size 3 in G and hence additionally is a Hamiltonian Path in G (since vertices of degree 3 in T would induce a fully contained cut of size 3):
So assume it did contain a cut S in G of size 3. Let a_1, a_2, a_3 be the arcs contained in S. Since $f \geq 0$ has to fulfil the Kirchhoff-law of flow conservation at S, we can assume (without loss of generality resp. due to symmetry) that regarding a_1; a_2 and a_3 are both oppositely oriented arcs with respect to S, and that $\{d, ..., k-d\} \ni f(a_1) = f(a_2) + f(a_3) \in \{2d, ..., 2k-2d\}$ and hence $2d \leq k - d \Leftrightarrow \frac{k}{d} \geq 3$ contradicting the initial assumption on the pair (k, d). Hence, T is indeed a path not containing a cut of size 3 and we are done.

(iii) Let P be a Hamiltonian path in G with end vertices $u \neq v$, which does not fully contain a cut in G of size 3, and assume that G is not Hamiltonian. We first show that $\xi^*_{\text{coin}}(G) \leq 2.5$ and prove $\xi^*_{\text{coin}}(G) \geq 2.5$ further below. Assume that D is an orientation of G so that P is directed from u to v in D. Our first goal is to show the following partial claim, which is later on used to construct a coindependent $(5,2)$-flow on G which then yields the desired result.

Claim. There are two paths P_1, P_2 in G starting in v and ending in u with the following properties:

- $E(P) \cap E(P_1) \cap E(P_2) = \emptyset$.
- If $e \in E(P) \cap E(P_i)$ for an $i \in \{1,2\}$, then P_i traverses e (as oriented in D) in forward direction.

Proof (of the claim). In the following, we consider the digraph $\tilde{D}$ arising from D by contracting $\overline{E(P)} = E(D) \backslash E(P)$, i.e., $\tilde{D} := D/\overline{E(P)}$. Denote by $\tilde{u}, \tilde{v} \in V(\tilde{D})$ the vertices $u, v \in V(D)$ are contracted to. If $\tilde{u} = \tilde{v}$, then there is a chord between u and v in P which together with P forms a Hamiltonian cycle in G. But this already contradicts our additional assumption. So assume in the following that $\tilde{u} \neq \tilde{v}$. As introduced in the preliminaries, $\lambda'(\tilde{D}, \tilde{v}, \tilde{u})$ is the maximal number of directed edge disjoint paths in $\tilde{D}$ starting in $\tilde{v}$ and ending in $\tilde{u}$. We claim that $\lambda'(\tilde{D}, \tilde{v}, \tilde{u}) \geq 2$. According to Menger's Theorem for digraphs (see 3.44), we have $\lambda'(\tilde{D}, \tilde{v}, \tilde{u}) = \kappa'(\tilde{D}, \tilde{v}, \tilde{u})$ and hence, it suffices to show that each minimal cut $S = U(\tilde{D})[X, \overline{X}]$ with $\tilde{u} \in X, \tilde{v} \in \overline{X}$ has at least 2 edges in backward direction (i.e. starting in $\overline{X}$ and ending in X) in $\tilde{D}$. Note that the cuts in $U(\tilde{D}) = G/\overline{E(P)}$ are exactly the cuts of G contained in $E(P)$. Since S separates $\tilde{u}, \tilde{v}$ in $\tilde{D}$, it separates u and v considered as cut in G and is fully contained in $E(P)$. Since P is a directed path in D, it follows that the numbers of edges in S oriented forwards resp. backwards in D or equivalently $\tilde{D}$ differ by at most 1. Since P does not contain cuts of size 3, we furthermore know that $|S| \geq 4$. Since S is completely contained in $E(P)$ and since P is a path starting and ending on different sides of S, $|S|$ has to be odd, and therefore, we even have $|S| \geq 5$. According to the above, this means that S has at least 2 edges in forwards and backwards direction, and thus, $\lambda'(\tilde{D}, \tilde{v}, \tilde{u}) \geq 2$ as claimed.
Thus we know that there are two edge-disjoint directed paths $\tilde{P}_1, \tilde{P}_2$ in $\tilde{D}$ starting in $\tilde{v}$ and ending in $\tilde{u}$. Since the vertices of $\tilde{D}$ correspond to the connected components of $G - E(P)$, each path $\tilde{P}_i$ can be extended to a path P_i in G from v to u so that $E(P_i) \cap E(P) = E(\tilde{P}_i), i = 1,2$ and the edges of D in $E(P) \cap E(P_i)$ are traversed in forward direction by P_i (since $\tilde{P}_i$ is directed in $\tilde{D}$). We thus have

$$E(P) \cap E(P_1) \cap E(P_2) = E(\tilde{P}_1) \cap E(\tilde{P}_2) = \emptyset.$$

Finally, this proves the Claim. $\square$

First of all, we define a pre-flow f' (which is not yet a flow) on D according to $f' : E(D) \to \{0, ..., 4\}$, $f' := 2 \cdot \mathbb{1}_{E(P)}$, i.e., f' is 2 on P and zero elsewhere. Obviously, $\mathrm{exc}_{f'}(w) = 0$ for all vertices $w \in V(G) \backslash \{u, v\}$ and $\mathrm{exc}_{f'}(u) = 2, \mathrm{exc}_{f'}(v) = -2$. Hence, the only thing to do in order to complete f' to a flow is balancing it at u and v.

For this purpose, define $f := f' + g_1 + g_2$, where g_i denotes a ± 1 augmenting partial flow on P_i, i.e.,

$$g_i(e) := \begin{cases} 1 & \text{if } e \in E(P_i) \text{ and } e \text{ is directed forwards in } P_i, \\ -1 & \text{if } e \in E(P_i) \text{ and } e \text{ is directed backwards in } P_i, \\ 0 & \text{else.} \end{cases}$$

First of all, this gives rise to a flow on D resp. G: Again, we have $\mathrm{exc}_{g_i}(w) = 0$ for all $w \in V(G)\backslash\{u, v\}$. On the other hand, since the P_i were considered as starting in v and ending in u, we have $\mathrm{exc}_{g_i}(u) = -1, \mathrm{exc}_{g_i}(v) = 1$. All in all, we get that f has excess 0 at every vertex and hence is an integer flow on D. Since $|f'| \leq 2, |g_i| \leq 1, i = 1, 2$, we have $f(e) \in \{0, \pm 1, ..., \pm 4\}, \forall e \in E(G)$. In order to show that f is a coindependent $(5, 2)$-flow on D, it suffices to show that P is contained in $\mathrm{supp}_2(f)$, i.e., $f(e) \in \{2, 3\} \subseteq \{\pm 2, \pm 3\}, \forall e \in E(P)$. As assumed, $g_i(e) > 0$ for all $e \in E(P_i) \cap E(P)$. Thus it remains to show that $f(e) \geq 4$ is not possible when $e \in E(P)$. But according to the definition of the g_i, this would imply $e \in E(P) \cap E(P_1) \cap E(P_2) = \emptyset$, which is not possible, and we are done, f is indeed a coindependent $(5, 2)$-flow.

We now prove that $\xi^*_{\mathrm{coin}}(G) \geq 2.5$. For a proof by contradiction, assume that $\xi^*_{\mathrm{coin}}(G) < 2.5$. According to Theorem 6.34, for each fixed $d \in \mathbb{N}, d \geq 3$ large enough such that $\frac{5d-2}{2d} = 2.5 - \frac{1}{d} \geq \xi^*_{\mathrm{coin}}(G)$, there is a $(5d - 1, 2d)$-flow f on some orientation D of G. According to the definition, there is a spanning tree T of G contained in $\mathrm{supp}(f)$ and, as was explained in (ii), $T = P$ has to be a Hamiltonian path since $\frac{5d-1}{2d} \leq 2.5 < 3$. Let u be one of the two end vertices of P, and denote by e_1, e_2 and e_P the two edges in G incident to u that are not contained in P resp. the unique edge of P incident to u. We have $|f(a)| \in \{2d, ..., 3d - 1\}$, for all $a \in E(P)$. Denote by $v_i \in V(P)$ the second vertex contained in e_i, $i = 1, 2$. Obviously, v_i for $i = 1, 2$ is an inner vertex of P, since the fact that v_i was the other end vertex of P would imply that $P + e_i$ is a Hamiltonian cycle in G, contradicting our assumption. The flow values $f(e_i)$ are, according to the flow conservation, either sums or differences of numbers in $\{\pm 2d, ..., \pm(3d - 1)\}$ and thus, $|f(e_i)|$ can range in $\{0, ..., d - 1\} \cup \{4d, ..., 5d - 1\}$. With the same argument at the vertex u, we get that $|f(e_P)|$ is either a difference or a sum of numbers out of $\{0, ..., d - 1\} \cup \{4d, ..., 5d - 1\}$. Since $|f(e_P)| \leq 3d - 1$ and $4d - (d - 1) = 3d + 1$, we deduce that none of those two numbers is contained in the part $\{4d, ..., 5d - 1\}$ of this set. But this again implies $|f(e_P)| \leq 2(d - 1) < 2d$, which is a contradiction to $|f(e_P)| \in \{2d, ..., 3d - 1\}$. This finally proves our claim. Indeed, we have $\xi^*_{\mathrm{coin}}(G) = 2.5$.

(iv) This follows immediately from (i) and the fact that each Hamiltonian cycle C gives rise to a canonical coindependent 2-flow on G with support C.

□

In conclusion, we provide examples fulfilling the conditions for the three cases above, showing that they are non-trivial. First of all, non-traceable cubic 3-edge-connected graphs

do exist and are thus examples for (ii) in the above theorem, cf. e.g. [Zam80].
Internally resp. cyclically 4-edge-connected cubic graphs do not admit non-trivial cuts of size 3, and since paths cannot contain vertices of degree 3, each traceable internally 4-edge-connected cubic graph which is not Hamiltonian is an example for (iii), i.e., admits coindependent star-flow-index 2.5. Thus, also e.g. the Petersen graph admits coindependent star-flow-index 2.5. Planar cubic internally 4-edge-connected graphs which are traceable but not Hamiltonian are also known, such an example is e.g. given in [ABHM00].
Finally, each cubic 3-edge-connected Hamiltonian graph is an example for (iv), of which there are many. It is e.g. known that cubic 3-connected planar graphs up to order 36 admit Hamiltonian cycles.

7 A Conjecture for Directed Planar Triangulations

After having introduced various ways of getting along with the falsification of the 2-Flow-Conjecture, we now take a straight-forward approach of tackling the 2-Colour-Conjecture into account.
We are concerned with the following statement, which is a slight strengthening of the 2-colour-conjecture for planar digraphs: We conjecture that every orientation of a planar triangulation admits a 2-colouring, where the colours of the three outer vertices can be arbitrarily prescribed in a non-monochromatic way. In order to motivate this approach, we first restate the "digirth four" -result by Mohar and Li from section 5.2.

Theorem 7.1. *[LB17] Let $\vec{T}$ be a directed planar triangulation with outer face $a_1a_2a_3$, such that there are no directed cycles of length 3. Then for every pre-colouring of a_1, a_2, a_3 in two colours there is a 2-colouring of $\vec{T}$ extending this pre-colouring.*

While this theorem is about directed planar triangulations without directed triangles, we now consider somehow the converse, i.e., every triangular face is directed. The following theorem for the dual case shows more generally that k-regular digraphs with special orientations admit "extending" NL-2-flows.

Theorem 7.2. *Let $k \geq 2$ and D be a k-regular connected digraph, such that every vertex $v \in V(D)$ is either a source or a sink. Let v_0 be a fixed vertex of D and e_i, e_j two edges incident with v_0. Then there exists a NL-2-flow f of D such that $f(e_i) = 1$ and $f(e_j) = -1$.*

Proof. By assumption D must be bipartite. Since D is k-regular it is k-edge colourable. Let I, J denote the colour classes containing e_i resp. e_j. Then $I \cup J$ is a 2-factor in the graph underlying D. Contracting this 2-factor yields a connected digraph. We will show that it is Eulerian and hence $D/(I \cup J)$ is strongly connected, proving the assertion. To do so consider a vertex w in $D/(I \cup J)$ which arises from a cycle C in $I \cup J$. Clearly, $d^+(w) = d^-(w) = 1/2|C|(k-2)$ and the claim follows. □

Corollary 7.3. *Let $\vec{T}$ be a directed planar triangulation such that every face is directed. Let a_1, a_2, a_3 be the vertices of the outer face. Then there is 2-colouring c of $\vec{T}$ where $c(a_1) = 0$, $c(a_2) = c(a_3) = 1$ and there is no monochromatic triangular face with respect to c in $T := U(\vec{T})$.*

Proof. Consider the planar dual $\vec{T}^*$ of $\vec{T}$. We may assume that (a_ia_{i+1}) are the directions of the edges (indices modulo 3). Setting $f((a_3a_1)) = 1$ and $f((a_1a_2)) = -1$ $\vec{T}^*$ satisfies the preconditions of Theorem 7.2. Hence there exists a NL-2-coflow in $\vec{T}$ which induces an extension of the pre-colouring. Moreover, since the NL-2-flow on $\vec{T}^*$ as constructed in Theorem 7.2 has the union of two disjoint perfect matchings as its support which hence covers all the vertices of $\vec{T}^*$, we may assume that all the triangular faces in T are not monochromatic in the corresponding NL-2-coflow on $\vec{T}$. □

Another prominent class of plane triangulations consists of the so-called *stacked triangulations*, i.e., the plane triangulations which can be built up from a simple triangle by successively stacking additional vertices in existing triangular faces and connecting them to all three incident vertices. We show an analogous positive result for this class of simple digraphs:

Theorem 7.4. *Let $\vec{T}$ be an arbitrary orientation of a stacked triangulation with outer triangle $a_1a_2a_3$. If $c : \{a_1, a_2, a_3\} \to \{0, 1\}$ is any pre-colouring of the outer face which is not monochromatic if $a_1a_2a_3$ is directed, there is a legal 2-colouring of $\vec{T}$ extending c.*

Proof. We prove the assertion using strong induction on the number of vertices of $\vec{T}$. If $T := U(\vec{T})$ is a simple triangle, the claim is obvious. So assume that $|V(\vec{T})| = n > 3$ and that the claim holds true for all stacked directed triangulations with at most $n-1$ vertices. Let v be an inner vertex in $\vec{T}$ of degree 3. Then also $\vec{T} - v$ is a stacked directed triangulation with $n - 1$ vertices, which thus admits a legal 2-colouring extending c. The triangle in $\vec{T} - v$ consisting of the original neighbours of v is coloured with 0 and 1 and so, one colour $c' \in \{0, 1\}$ appears at most once on this triangle. By colouring all vertices in $V(\vec{T})\setminus\{v\}$ as in $\vec{T} - v$ and v with c', we thus end up with a 2-vertex-colouring of $\vec{T}$ extending the pre-colouring of $a_1a_2a_3$ which has to be legal: Monochromatic cycles in $\vec{T}$ cannot pass through v, since this would require at least two neighbours of v being coloured c', which is not the case. Thus, they are monochromatic cycles in $\vec{T} - v$ and therefore not directed. Applying the principle of induction, this proves the claim. $\square$

At first sight, one might be tempted to generalize the corollary above and the result of Mohar and Li by the following slight strengthening of the 2-Colour-Conjecture:

Claim. For each directed planar triangulation $\vec{T}$, we can find a legal 2-colouring of it extending an arbitrary given pre-colouring on the outer three vertices a_1, a_2, a_3 which is legal in the sense that it is not monochromatic whenever the outer triangle is directed.

Although this conjecture seems pretty reasonable, it turns out to be false already when considering planar triangulations on 6 vertices. In order to demonstrate this, consider the following orientation of the octahedron graph:

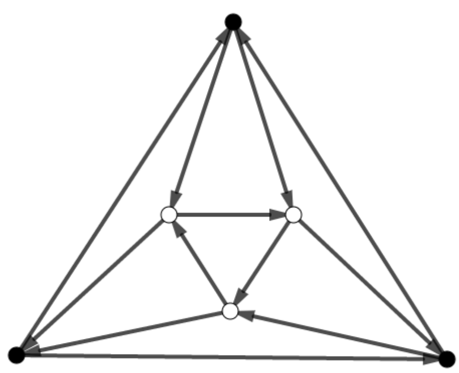


Figure 18: Counterexample to a first attempt of generalizing the 2-Colour-Conjecture.

If we colour each of the three outer vertices with black, this is a legal pre-colouring according to the above conjecture, since the outer triangle is not directed. On the other hand, in any legal extension of this pre-colouring, all the three inner vertices have to be coloured white in order to prevent the three directed triangular faces neighbouring the outer edges from being monochromatic. But this means that the directed triangular face formed by the three inner vertices is coloured all white, which proves that there can't be any extending legal 2-colouring.
On the other hand, when considering any non-monochromatic pre-colouring of the outer face of this directed triangulation, it turns out that extending legal 2-colourings indeed can be found, as illustrated by the next figure.

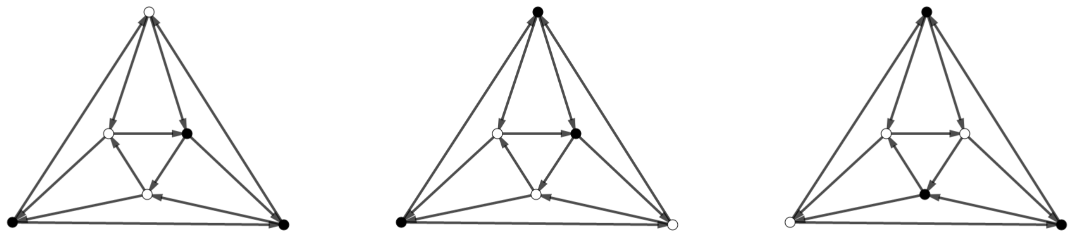


Figure 19: Extending 2-colourings on the octahedron graph given a non-monochromatic pre-colouring. Since the given colourings are even more legal in terms of vertex arboricity, they yield legal colourings for any orientation of the octahedron graph.

The proof of the first reduction operation (R1) used in subsequent considerations, which is presented further below, requires the following observation concerning extending 2-colourings resp. NL-2-flows of small directed planar triangulations resp. 3-edge-connected cubic planar digraphs, showing that the example from Figure 18 is the smallest counterexample to the conjecture stated above:

Observation 7.5. *Let $\vec{T}$ be a directed planar triangulation with outer face $a_1a_2a_3$ with at most 6 vertices, and denote by D the 3-edge-connected cubic planar directed dual with at most 8 vertices, where v_0 denotes the special vertex representing the outer face. If $\vec{T}$ does not correspond to an orientation of the octahedron graph as presented above, i.e., where $a_1a_2a_3$ is not directed but all three facial triangles including an outer edge as well as the triangle consisting of the three inner vertices are directed, then $\vec{T}$ admits a legal 2-colouring extending any given legal precolouring of the outer vertices. Equivalently, given any legal pre-flow assignment $g : \{e_1, e_2, e_3\} \to \mathbb{Z}_2$ on the edges incident to v_0, i.e., $g \not\equiv 0$ whenever v_0 is a source or a sink, there is a NL-$\mathbb{Z}_2$-flow f on D such that $f(e_i) = g(e_i), i = 1, 2, 3$.*

Proof. We prove the assertion only in the primal setting. The dual formulation then immediately follows from the duality of digraph-colourings and Neumann-Lara-flows.
We start with the case where $\vec{T}$ is an oriented octahedron graph resp. D an orientation of the cube graph.
If the given pre-colouring of $a_1a_2a_3$ is not monochromatic, the assertion was already considered and proven in Figure 19. Hence, it suffices to consider the case where the given

pre-colouring is black at each outer vertex (and thus, since the pre-colouring has to be legal in the above sense, the outer triangle is not directed). According to the additional assumption, either the unique face in the inside of $\vec{T}$ or one of the three inner triangles containing an outer edge is not directed. Now, in the first case, colouring all inner vertices with white gives rise to a legal extending 2-colouring, while in the second case, colouring the unique inner vertex contained in the considered non-directed inner face with black and the two remaining inner vertices with white avoids monochromatic directed cycles.
As is easy to verify, all plane triangulations on at most 6 vertices other than the octahedron graph are stacked triangulations, and thus, the claim follows from Theorem 7.4 in the remaining cases. □

Considering these examples, we eventually end up with the following statement, which is the main conjecture we are going to deal with in this section.

Conjecture 7.6.

(i) Let $\vec{T}$ be a directed planar triangulation and a_1, a_2, a_3 the outer face of $\vec{T}$. Then for every non-monochromatic (hereafter also called legal) precolouring of a_1, a_2 and a_3 with two colours there is a 2-colouring of $\vec{T}$ extending it.

(ii) Let D be a 3-regular 3-edge-connected planar digraph with special vertex $v_0 \in V(D)$. Let e_1, e_2, e_3 be the three incident edges of v_0. Let $g : \{e_1, e_2, e_3\} \to \mathbb{Z}_2$ be a legal pre-flow-assignment at v_0, i.e., $g(e_1) + g(e_2) + g(e_3) = 0$ and $g \not\equiv 0$. Then there is a NL-$\mathbb{Z}_2$-flow f on D with $f(e_i) = g(e_i), i = 1, 2, 3$.

(i) and (ii) are equivalent statements if D is the directed dual of $\vec{T}$ and v_0 is the vertex corresponding to the outer face. Thus, in the following we will treat the statements as a common primal-dual conjecture.

So far, we did not manage to find any counterexample, i.e., including a non-monochromatic pre-colouring of the outer face. This is not too surprising, since the Conjecture 7.6 already turns out to be equivalent to the 2-Colour-Conjecture itself, as we are going to show in the following theorem which additionally provides some equivalent but stronger formulations of the 2-colour-conjecture:

Theorem 7.7. *The following statements are equivalent:*

(i) The 2-Colour-Conjecture, i.e., every simple planar digraph is 2-colourable.

(ii) Each oriented planar triangulation admits a 2-colouring without monochromatic bounded facial triangles.

(iii) Conjecture 7.6, i.e., every oriented planar triangulation admits an extending 2-colouring given a non-monochromatic pre-colouring of the outer face.

(iv) Each oriented planar triangulation admits a 2-colouring without monochromatic triangles where the colours of the vertices at any fixed special facial triangle can be prescribed in a non-monochromatic way.

Proof.
$(i) \Rightarrow (ii)$: Assume the 2-Colour-Conjecture is true, and let $\vec{T}$ be an arbitrary directed planar triangulation. We modify $\vec{T}$ to a bigger directed triangulation $\vec{T}'$ by stacking a copy of the special orientation of the octahedron graph as depicted in Figure 18 into each non-directed triangular face of $\vec{T}$ such that the orientations of the three edges of the respective face agree with the orientation of the outer face-edges of the copy (for this purpose, one might have to consider the mirror image of the triangulation in the figure, which does not change the colouring properties). Now, according to the 2-Colour-Conjecture, $\vec{T}'$ admits a legal 2-colouring which induces a legal 2-colouring on $\vec{T}$. In this colouring, none of the facial triangles can be monochromatic: Each such triangle is either directed and hence dichromatic by definition, or it is non-directed and therefore equipped with a copy of the directed oriented octahedron graph as described. But in this case, as argued above, any monochromatic colouring of it would force a monochromatic directed cycle in $\vec{T}'$, which is impossible, and we are done.

$(ii) \Rightarrow (iii)$: Let $\vec{T}$ be an arbitrary directed planar triangulation. We want to show that we can find a legal 2-colouring of $\vec{T}$ extending a specified pre-colouring $c : \{a_1, a_2, a_3\} \to \{0, 1\}$ on the outer vertices a_1, a_2, a_3 (enumerated in clockwise order around the outer triangle) of $\vec{T}$ which is not monochromatic. We distinguish two cases:

1. Assume that the outer face-triangle $a_1a_2a_3$ is not directed. Without loss of generality and due to symmetry, we can assume that the outer edges of $\vec{T}$ are oriented according to $a_1 \to a_2, a_2 \to a_3, a_1 \to a_3$ (cf. figure to the top right). We now consider the orientation and planar embedding of the tetrahedron graph shown in Figure 20 at the top left. Up to rotation, the three bounded facial triangles admit the same orientation as the outer triangle of $\vec{T}$, and hence, we can stick three copies of $\vec{T}$ into these triangles in a unique way so that the orientations of the identified edges agree. During this process, which is illustrated in the rest of Figure 20, the vertices a_1 of the three copies are associated with the same unique inner vertex w of the planarly embedded tetrahedron graph. According to (ii), the arising simple planar digraph (V, A) admits a legal 2-colouring $c : V \to \{0, 1\}$. By flipping colours if needed, we may assume that $c(w) = 0$. Furthermore, since the three outer vertices form a directed triangle, there is exactly one colour change from 0 to 1 and exactly one from 1 to 0 when following the directions of the arcs along this cycle. The corresponding two copies of $\vec{T}$ thereby admit legal 2-colourings c_1, c_2 such that $c_1(a_1) = c(w) = 0$, $c_1(a_2) = 0, c_1(a_3) = 1$ resp. $c_2(a_1) = c(w) = 0, c_2(a_2) = 1, c_2(a_3) = 0$. Thus, the claim holds true whenever the edge a_2a_3 is a dichromatic edge in the given pre-colouring, or equivalently, whenever the unique vertex with distinct prescribed colour is a_2 or a_3, i.e., the sink or the balanced vertex of the outer triangle.
The same has to be true for the planar triangulation $\overleftarrow{T}$ where all edge orientations get reversed. In this case, a_1 is the sink while a_3 is the source. Again, $\overleftarrow{T}$ admits a legal extending 2-colouring whenever the unique outer vertex with distinct prescribed colour is a_1 or a_2. But since the legal 2-colourings of $\vec{T}$ and $\overleftarrow{T}$ are equivalent, these statements together imply that $\vec{T}$ admits a legal extending 2-colouring for every legal pre-colouring, which finally proves Conjecture 7.6 for $\vec{T}$.

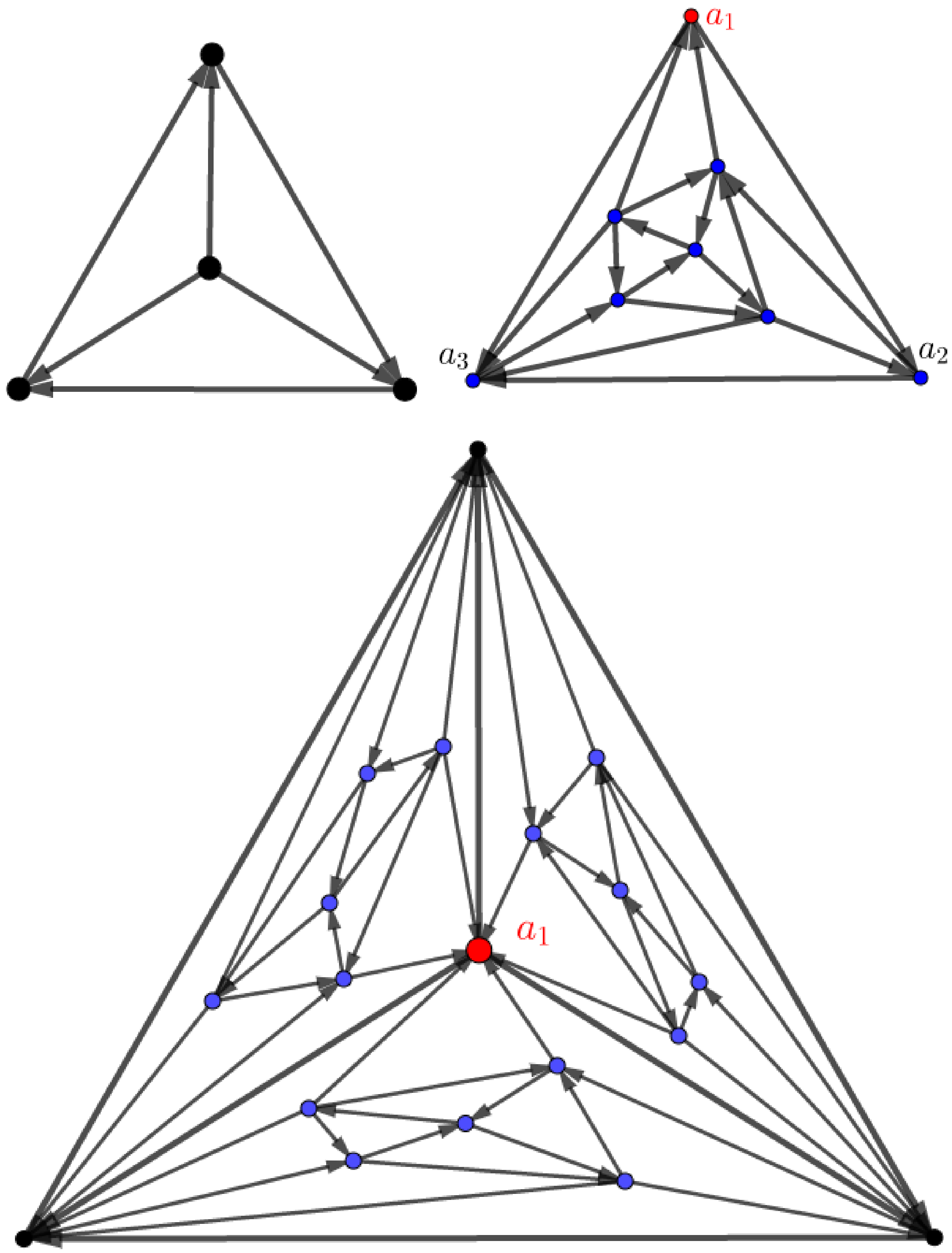


Figure 20: Stacking three copies of a directed planar triangulation into an oriented tetrahedron graph.

2. In this case, the outer triangle $a_1a_2a_3$ of $\vec{T}$ is directed. Let any legal pre-colouring of the outer vertices be given, such that $a_i, i \in \{1,2,3\}$ is the unique vertex with distinct assigned colour. Denote by $\vec{T_i}$ the planar triangulation arising from $\vec{T}$ by reversing the orientation of a_ia_{i+1}. According to case 1, $\vec{T_i}$ admits a legal 2-colouring c extending the pre-colouring of a_1, a_2, a_3. We take the same 2-colouring c for $\vec{T}$ and claim that it has to be legal: Assume there was a monochromatic directed cycle in $\vec{T}$. According to the given pre-colouring, out of the outer edges, such a cycle can only possibly use $a_{i-1}a_{i+1}$ and hence would also be a monochromatic directed cycle in $\vec{T_i}$, contradiction, wherefore we are also done in the second case.

$(iii) \Rightarrow (iv)$: We prove the assertion in the case that the given special facial triangle is the outer face. Since for each given face, we can find an embedding in which it gets moved to the outside, this can be done without loss of generality.
Consider an arbitrary directed planar triangulation $\vec{T}$. We first use the additional assumption of 4-connectivity of the underlying planar triangulation T, i.e., we consider the case where no separating triangles exist. As in $(i) \Rightarrow (ii)$, we can stack oriented octahedron graphs into the non-directed facial triangles of $\vec{T}$ such that in any legal 2-colouring of the arising directed triangulation $\vec{T}'$, all the original facial triangles (and thus all triangles in T) are forced to be non-monochromatic. According to (iii), we can furthermore require such a 2-colouring to extend any given legal pre-colouring on the outer vertices, which already proves the assertion in this special case.
For the general case, we proceed by induction on the number of vertices in $\vec{T}$. The implication is obviously true when $|V(\vec{T})| = 3$, so assume $|V(\vec{T})| > 3$. Either, $T := U(\vec{T})$ is 4-connected, in which case we are done, or it contains a separating triangle $x_1x_2x_3$. As in the proof of the digirth-four-result of Mohar and Li in section 5.2, we can split $\vec{T}$ into directed triangulations $\vec{T}_{\text{in}}, \vec{T}_{\text{out}}$ containing all the vertices on or inside $x_1x_2x_3$ resp. on or in the outside of $x_1x_2x_3$. Given any legal pre-colouring at the outer face-vertices of $\vec{T}$, according to the induction hypothesis, we can find a legal 2-colouring of $\vec{T}_{\text{out}}$ extending this pre-colouring such that all the triangles in $\vec{T}_{\text{out}}$ are not monochromatic, which is especially true for $x_1x_2x_3$. Hence, the colours at x_1, x_2, x_3 can be used as a legal pre-colouring for $\vec{T}_{\text{in}}$ which again can be extended to a legal 2-colouring to the inside not admitting monochromatic triangles. Since the triangles in T lie either in T_{in} or T_{out}, the arising 2-colouring is legal, extends the given pre-colouring and admits no monochromatic triangles. Thus, the claim follows according to the principle of induction.

$(iv) \Rightarrow (i)$: Obvious. $\square$

In the following, we want to analyse and explore properties of minimal counterexamples to this conjecture, in order to make progress on a proof of it and to deduce partial positive results. The technique we use in order to derive these results are reduction operations: Starting with a directed planar triangulation $\vec{T}$, given specified local patterns in $\vec{T}$, we construct one or more reduced triangulations $\vec{T_1}, ..., \vec{T_k}$ with less vertices and edges than $\vec{T}$, such that the following holds: If $\vec{T_1}, ..., \vec{T_k}$ admit extending legal 2-colourings as proposed in the conjecture, also $\vec{T}$ does. The reverse of the statement implies the following: If $\vec{T}$ is a counterexample to Conjecture 7.6 with respect to the number of vertices/edges, then at least one $\vec{T_i}$ is a smaller counterexample, and therefore, $\vec{T}$ is no minimal counterexample. Hence, exploring properties of directed planar triangulations where the various reduction operations are not applicable yields structural results about minimal counterexamples to the conjecture. These may be eventually used to prove the correctness of the conjecture (or to find a counterexample). The following two theorems sum up our knowledge derived by this technique. The proof of it and a detailed analysis of the reductions follow subsequently.

Theorem 7.8. *Let $\vec{T}$ be a minimal counterexample to Conjecture 7.6 with respect to the number of edges (or equivalently vertices). Denote by $a_1a_2a_3$ its outer (designated) face. Then the following holds:*

(i) If $x_1x_2x_3$ is a separating triangle in $T := U(\vec{T})$, then the subdigraph of $\vec{T}$ induced by $x_1x_2x_3$ and its inner vertices is an octahedron graph with an orientation as depicted in Figure 18, i.e. $x_1x_2x_3$ is not directed, but the three facial triangles containing an edge of $x_1x_2x_3$ and the triangular face consisting of the three inner vertices are directed.

(ii) The bounded directed triangular faces cover all inner vertices.

(iii) $\vec{T}/\{a_1, a_2, a_3\}$ (the contraction of the outer face) is strongly connected.

(iv) Every inner vertex $v \in V(\vec{T}) \setminus \{a_1, a_2, a_3\}$ has at least two incoming and two outgoing edges. More generally, every directed cut not containing an edge from the outer face has at least two edges in both directions (i.e., $\vec{T}/\{a_1, a_2, a_3\}$ is moreover 2-arc-connected).

(v) The bounded non-directed triangular faces of $\vec{T}$ cover all inner vertices of degree 4. Together with the separating 4-gons in $\vec{T}$ which have three edges with the same and one with reversed orientation, they cover all inner vertices.

It will be handy, to also have some of the properties formulated in the dual setting ready:

Theorem 7.9. *Let D be the directed planar dual of $\vec{T}$ and v_0 the vertex corresponding to the outer face. If $\vec{T}$ is a minimal counterexample to Conjecture 7.6 then the following holds:*

(i) $U(D)$ is planar, cubic and 3-edge-connected. The only cuts in $U(D)$ of size 3 are either the edges neighbouring a single vertex (trivial cut) or appear as the non-directed cut induced by a cube-like configuration X of 7 vertices in D of which the three vertices incident to cut edges and the unique vertex not adjacent to one of those vertices are sources or sinks, as illustrated in Figure 21.

(ii) Let Q and S denote the sources resp. sinks of D and

$$W = V(D) \setminus (Q \cup S \cup \{v_0\}).$$

Then $U(D)[W]$ is a forest. In particular $D - v_0$ is acyclic.

(iii) If C is an oriented cycle in $D - v_0$, it has at least two edges in both directions.

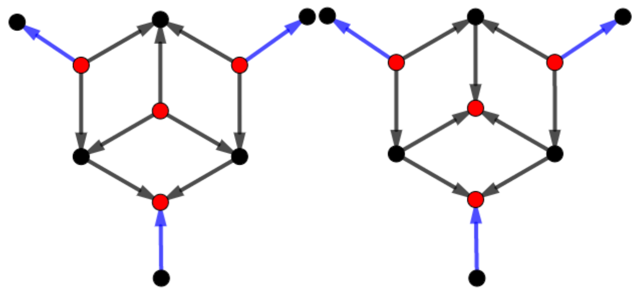


Figure 21: Up to rotational symmetry and symmetry by reversing the orientations of all edges, these are the only non-trivial vertex configurations inducing cuts of size 3 in $U(D)$. The vertices which are required to be sources or sinks are marked red, the cut edges are marked blue.

The following lemmas introduce the main four reduction operations, labeled R1,R2,R3,R4.

Lemma 7.10 (R1). *Let $\vec{T}$ be a directed planar triangulation with outer face $a_1a_2a_3$ and D its directed dual with the vertex v_0 representing the outer face. Let $S = U(D)[X, \overline{X}]$ be a non-trivial 3-edge-cut such that $v_0 \in X$. Denote by $\tilde{D_1}, \tilde{D_2}$ the digraphs defined by $\tilde{D_1} := D/\overline{X}$ and $\tilde{D_2} := D/X$ and let $v_{\overline{X}}, v_X$ be the vertices of degree 3 in $\tilde{D_1}$ resp. $\tilde{D_2}$ corresponding to $\overline{X}$ resp. X. Assume that $D[\overline{X}]$ does not correspond to one of the 7-vertex-configurations depicted in Figure 21 and define two digraphs D_1, D_2 according to the following case distinction:*

(i) If $|\overline{X}| \leq 7$, $D_i := \tilde{D_i}, i = 1, 2$.

(ii) If $|\overline{X}| \geq 8$, let D_1 be the digraph that is $\tilde{D_1}$ in the case that S is directed and else arises from $\tilde{D_1}$ by replacing $v_{\overline{X}}$ by a configuration of seven vertices as depicted in figure 21, such that the cut edges of this configuration are identified with the edges in S and $D_2 := \tilde{D_2}$.

If Conjecture 7.6 holds true for (D_1, v_0) and (D_2, v_X) resp. their duals, it is also true for (D, v_0) resp. $\vec{T}$, and $|E(D_1)|, |E(D_2)| < |E(D)|$.

Proof. D_1 and D_2 are are obviously 3-edge-connected (so is D) and 3-regular. This is because of $|S| = 3$ and because $X, \overline{X}$ are contracted to single vertices: $U(D)[X], U(D)[\overline{X}]$ have to be 2-edge-connected, since the converse would give a contradiction to the 3-edge-connectivity of D. $X, \overline{X}$ have at least two elements and therefore $D_2 = \tilde{D_2}$ has less vertices or equivalently (3-regularity) less edges than D in any case. If $|\overline{X}| \leq 7$, according to the definition, we have $|E(D_1)| = |E(\tilde{D_1})| < |E(D)|$, while in case that $|\overline{X}| \geq 8$, $|V(D_1)| = |V(\tilde{D_1})| + 6 < |V(D/\overline{X})| - 1 + \underbrace{|V(D[\overline{X}])|}_{\geq 8} = |V(D)|$, and therefore $|E(D_1)| < |E(D)|$ also in this case. As provided, the conjecture holds for (D_1, v_0) and (D_2, v_X). Let now $g \not\equiv 0$ be an arbitrary legal pre-flow-assignment on the three incident edges of v_0. We distinguish between the two possible cases (i) and (ii) determining the definition of D_1 and D_2:

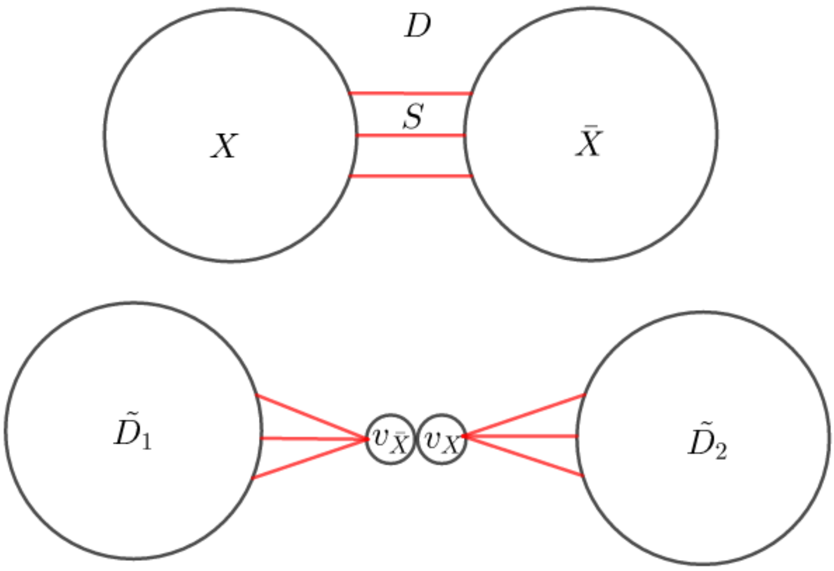


Figure 22: Splitting D along S.

(i) Assume for the first case that $|\overline{X}| \leq 7$. Then, according to the conjecture, there is a NL-$\mathbb{Z}_2$-flow f_1 on $D_1 = \tilde{D}_1$ which extends g. Let g_1 be the restriction of f_1 onto the three edges originally contained in S, and let again $v_X, v_{\overline{X}}$ denote the vertices in D_2 and D_1 representing X and $\overline{X}$ respectively. Consider g_1 as a pre-flow-assignment for v_X and $D_2 = \tilde{D}_2$. This does not necessarily have to be legal, in the sense that $g_1 \equiv 0$ is possible when S is not directed, but, since f_1 is a legal NL-$\mathbb{Z}_2$-flow on $\tilde{D}_1 = D/\overline{X}$ and since S appears as a 3-cut in $\tilde{D}_1$ (the three edges neighbouring $v_{\overline{X}}$), $g_1 \not\equiv 0$ whenever S is a dicut.
Since $|\overline{X}| \leq 7$, $\tilde{D}_2 = D_2$ is a cubic 3-edge-connected planar digraph on at most 8 vertices. Thus, applying the claim of Observation 7.5 to D_1 with special vertex v_X and the pre-flow assignment g_1, we deduce the existence of an NL-$\mathbb{Z}_2$-flow f_2 on D_2 extending g_1. Since S contains the only common edges of $\tilde{D}_1 = D_1$ and $\tilde{D}_2 = D_2$, sticking the two flows $\tilde{f}_1 := f_1, f_2$ together along $\tilde{f}_1|_S = g_1 = f_2|_S$ yields a $\mathbb{Z}_2$-flow f on D with $f|_{D_1} = f_1, f|_{D_2} = f_2$.

(ii) In the second case, we have $|\overline{X}| \geq 8$. Then, according to the conjecture, there is a NL-2-flow f_1 on D_1 which extends g. Let g_1 be the restriction of f_1 onto the three edges originally contained in S, and let again $v_X, v_{\overline{X}}$ denote the vertices in $\tilde{D}_2$ and $\tilde{D}_1$ representing X and $\overline{X}$ respectively. Then g_1 is a legal pre-flow-assignment for v_X and D_2, because either S is directed and so $g_1 \not\equiv 0$ since $D_1/\mathrm{supp}(f_1)$ is totally cyclic, or else, S is not directed implying that S corresponds to the cut edges of a seven-vertex-configuration in $\tilde{D}_1$ as described above. Then the even edge subset $\mathrm{supp}(f_1)$ in $U(D_1)$ has to cover all the four vertices in the configuration required to be sources or sinks and thus contains at least one of the cut edges, i.e. $g_1 \not\equiv 0$ also in this case. Using the claim of the conjecture for D_2, we deduce the existence of an NL-2-flow f_2 on

D_2 extending g_1. By contracting the edges contained in the seven-vertex configuration and keeping the flow values on the remaining edges in case that S is not directed, f_1 induces an NL-2-flow $\tilde{f}_1$ on $\tilde{D}_1$ agreeing on the edges corresponding to S. Since S contains the only common edges of D_1 and D_2, sticking the two flows $\tilde{f}_1, f_2$ together along $\tilde{f}_1|_S = f_1|_S = g_1 = f_2|_S$ again yields a $\mathbb{Z}_2$-flow f on D with $f|_{\tilde{D}_1} = \tilde{f}_1, f|_{D_2} = f_2$.

We now show that (in both cases) the defined flow f is a legal NL-$\mathbb{Z}_2$-flow which then extends the pre-flow-assignment g (since it extends $\tilde{f}_1$ and $\tilde{f}_1$ as f_1 extends g). We do this by proving that $D/\mathrm{supp}(f)$ is strongly connected: If S is not directed, this immediately follows from the fact that $D/\mathrm{supp}(f)$ is a contraction minor of the strongly connected digraph arising from the strongly connected digraphs $\tilde{D}_1/\mathrm{supp}(f_1)$ and $\tilde{D}_2/\mathrm{supp}(f_2)$ by identifying the edges incident to v_X resp. $v_{\overline{X}}$ (i.e., building up the cut S again), since they are connected in both directions via S. If S is directed, at least two edges in S belong to $\mathrm{supp}(f), \mathrm{supp}(\tilde{f}_1), \mathrm{supp}(f_2)$ at the same time, which means that the vertices those two edges are contracted to are contained in $\tilde{D}_1/\mathrm{supp}(\tilde{f}_1)$ and $\tilde{D}_2/\mathrm{supp}(f_2)$ and are identified in $D/\mathrm{supp}(f)$, which means that also in this case $D/\mathrm{supp}(f)$ is strongly connected. $\square$

Corollary 7.11. *Let $\vec{T}, (D, v_0)$ be a pair of a directed planar triangulation and the dual directed graph with special outer-face vertex.*

(i) If this pair is a minimal counterexample to Conjecture 7.6, then (R1) does not apply to $\vec{T}$/its directed dual D with outer face-vertex v_0 respectively.

(ii) If (R1) does not apply to the pair $\vec{T}, (D, v_0)$, then each cut in $U(D)$ of size 3 either corresponds to the three edges neighbouring a vertex or separates v_0 from a configuration of 7 vertices as described in Figure 21.

Proof. The claims of both (i) and (ii) are immediate consequences of Lemma 7.10. $\square$

Since the type of digraphs as described above will reappear several times in the following considerations, it is convenient to introduce the following terminology:

Definition 7.12. Let $\vec{T}, (D, v_0)$ be a pair of a directed planar triangulation and the dual directed graph with special outer-face vertex. We say that $\vec{T}$ resp. (D, v_0) admits the property *(O)*, if (R1) is not applicable, i.e., if the plane triangulation induced by the inner vertices of any separating triangle $x_1x_2x_3$ in $T := U(\vec{T})$ is an octahedron graph with an orientation as depicted in Figure 18 or equivalently, each non-trivial cut in $U(D)$ of size 3 separates v_0 from a configuration of 7 vertices as described in Figure 21.

For the next reduction operation, we introduce the following terminology:

Definition 7.13. Let $\vec{G} = (V, A)$ be a directed graph, let S be a minimal cut in $G = U(\vec{G})$ and C a cycle in $\vec{G}$. We call S respectively C *almost directed*, if it is directed or can be made directed by the reversal of exactly one arc contained in it.

Before we introduce the next reduction operation, we need the following preparatory observation/definition concerning the process of triangulating certain reduced directed planar triangulations.

Definition and Proposition 7.14. Let $\vec{T}$ be a directed planar triangulation with outer face $a_1a_2a_3$ admitting property (O). Let $S = T[X, \overline{X}]$ be an almost directed cut so that $a_1, a_2, a_3 \in \overline{X}$. Then $\vec{T} - X$ is a 2-connected planar graph with outer face $a_1a_2a_3$ which admits at most one face of length greater than 3, namely the one containing the interior" X of S. Analogously, $\vec{T} - \overline{X}$ is a 2-connected planar graph which is triangulated except for the outer face arising from the splitting along S, which may have length greater than 3.
If $|X| \geq 2$, then we define the triangulation $\vec{T}_{\text{out}}$ with respect to $\vec{T}$ and S by sticking an additional vertex into the unique non-triangulated face of $\vec{T} - X$ and connecting it to all vertices of the face, so that it becomes a source.
If instead $|X| = 1, X = \{v\}$, then the vertex v admits outdegree or indegree at most 1 (we may assume without loss of generality that we are in the first case). Let e_{rev} be an edge in $E_{\vec{T}}(v)$ so that $E^+(v) \subseteq \{e_{\text{rev}}\}$, and denote by e_1, e_2 the edges entering v next to e_{rev} in the cyclic ordering around v. Then we define the planar digraph $\vec{T}_{\text{out}}$ as $(\vec{T} - \{e_1, e_2\})/e_{\text{rev}}$. This is a directed planar triangulation with outer face $a_1a_2a_3$.
In any case, we define the directed planar triangulation $\vec{T}_{\text{in}}$ by sticking a new vertex into the outer face of $\vec{T}$ and connecting it to all outer-face vertices by outgoing edges, so that it becomes a source in $\vec{T}_{\text{in}}$.
The directed planar triangulations $\vec{T}_{\text{in}}, \vec{T}_{\text{out}}$ as defined above furthermore fulfil
$|V(\vec{T}_{\text{in}})|, |V(\vec{T}_{\text{out}})| < |V(T)|$.

Proof. It is clear from the definitions of $\vec{T}_{\text{in}}, \vec{T}_{\text{out}}$ that these are planar digraphs in any case and that $\vec{T}_{\text{in}}$ as well as $\vec{T}_{\text{out}}$ in the case of $|X| \geq 2$ admit planar triangulations as underlying graphs. The only critical part of the proof deals with the case $|X| = 1$. Although it is clear that in an appropriate planar embedding of $(\vec{T} - \{e_1, e_2\})/e_{\text{rev}}$ all the faces have to be triangles, it remains to show that this is indeed simple resp. 3-connected, i.e., that this process does not give rise to multiple edges between the contraction vertex of e and a vertex $w \in V(\vec{T}) \backslash v$. Since $(\vec{T} - \{e_1, e_2\})/e_{\text{rev}}$ is planar and does not admit the edges e_1, e_2 with end vertices $w_1, w_2 \in N^-(v)$ any more, such a vertex w would have to be contained in $N^-(v) \backslash \{v, w_1, w_2\}$, and admit each on adjacency to v and its neighbour v_{rev} via e_{rev}. Since $w \neq w_1, w_2$, the vertices w, v, v_{rev} thus would have to form a separating triangle in $\vec{T}$, which is not of the form as required by property (O): The facial triangle in the inside of it containing wv_1 cannot be directed, since wv_1 as well as its two neighbours in $E_T(v)$ are entering v in $\vec{T}$. This is a contradiction which finally proves that $\vec{T}_{\text{out}}$ indeed is a well-defined planar triangulation with outer face $a_1a_2a_3$. $|V(\vec{T}_{\text{in}})|, |V(\vec{T}_{\text{out}})| < |V(T)|$ again is an immediate consequence of the definitions when considering the case distinction $|X| = 1 \leftrightarrow |X| \geq 2$ (note that $|\overline{X}| \geq |\{a_1, a_2, a_3\}| = 3$ in any case). $\square$

Lemma 7.15 (R2)**.** *Let $\vec{T}$ be a directed planar triangulation with outer face $a_1a_2a_3$ admitting property (O). Let $S = T[X, \overline{X}], a_1, a_2, a_3 \in \overline{X}$ be an almost directed cut in $\vec{T}$ not containing an outer edge.*

Let now $\vec{T}_{\text{out}}$ and $\vec{T}_{\text{in}}$ be the directed planar triangulations defined according to 7.14 with respect to $\vec{T}$ and S.
If $c_{\text{out}} : V(\vec{T}_{\text{out}}) \to \{0,1\}$ and $c_{\text{in}} : V(\vec{T}_{\text{in}}) \to \{0,1\}$ are legal 2-colourings, then there is a legal 2-colouring $c : V(\vec{T}) \to \{0,1\}$ of $\vec{T}$ such that $c|_{\{a_1,a_2,a_3\}} = c_{\text{out}}|_{\{a_1,a_2,a_3\}}$.
Therefore, if conjecture 7.6 holds true for $\vec{T}_{\text{out}}$ and $\vec{T}_{\text{in}}$, it is also true for $\vec{T}$.

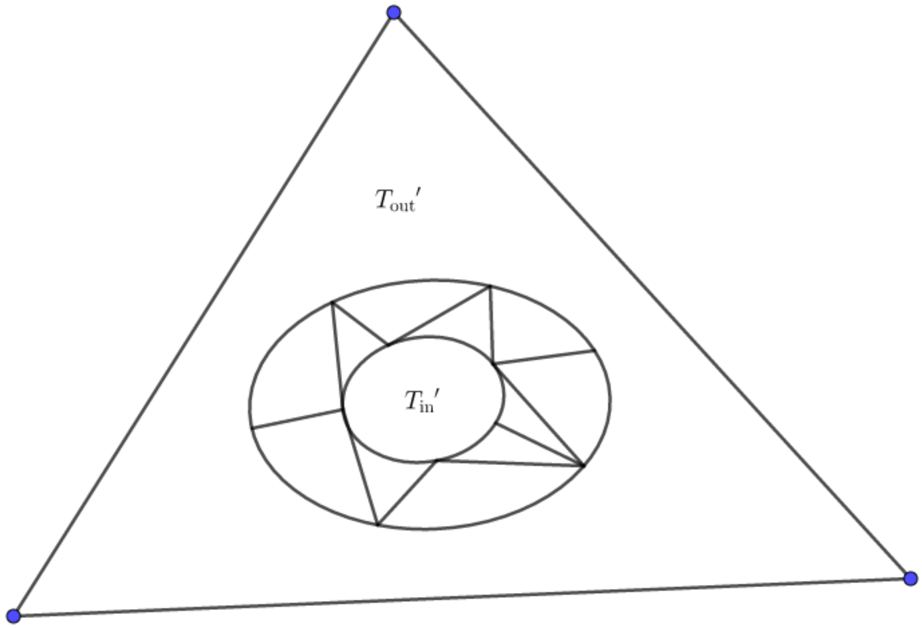


Figure 23: Illustration of the "split-along-cut" operation

Proof. Let an arbitrary pre-colouring of the three outer vertices of $\vec{T}$ be given. This can be understood as a precolouring for $\vec{T}_{\text{out}}$, and therefore there is a 2-colouring of $\vec{T}_{\text{out}}$ extending this precolouring. This induces a 2-colouring on the subdigraph $\vec{T}'_{\text{out}} := \vec{T} - X$ extending the precolouring. We now choose an arbitrary 2-colouring of $\vec{T}_{\text{in}}$ which induces a 2-colouring on $\vec{T}'_{\text{in}} := \vec{T} - \overline{X}$. We stick these two colourings together in order to get a 2-colouring of $\vec{T}$. If there is a unique edge in S in opposite direction, we can assume that the vertices of this edge are coloured differently, since we can switch the colours from black to white and vice versa in $\vec{T}'_{\text{in}}$ if needed. This colouring then extends the precolouring and indeed does not produce monochromatic directed cycles, since every directed cycle either stays inside one of the components, which immediately gives a contradiction, or it has to pass at least one edge in both directions of S, and therefore the unique edge oriented opposed. This again is not possible because this edge is not monochromatic. $\square$

Corollary 7.16.

(i) Let $\vec{T}$ be a directed planar triangulation which is a minimal counterexample to Conjecture 7.6. Then (R1), (R2) do not apply to $\vec{T}$/its directed dual D with outer face-vertex v_0 respectively.

(ii) Let $\vec{T}, (D, v_0)$ be a dual pair of a directed planar triangulation $\vec{T}$ and its directed dual D with special vertex v_0 representing the outer face, which admit property (O) and for which the reduction operation (R2) does not apply. Then every minimal cut in $T := U(\vec{T})$ not containing an outer edge and every cycle C in $U(D) - v_0$ have at least two edges in both directions.

Proof.

(i) Follows immediately from Lemma 7.15.

(ii) The statement obviously holds true for $\vec{T}$, since there are no almost directed cuts in $\vec{T}$ if (R2) is not applicable. By dualization, the statement for (D, v_0) follows.

□

Lemma 7.17 (R3). *Let $\vec{T}, (D, v_0)$ be a dual pair of a planar directed triangulation and its directed dual with property (O), where v_0 denotes the vertex in D corresponding to the outer face. Let e be an inner edge of $\vec{T}$ or an edge in D not incident to v_0 respectively. Let v_1, v_2 be the vertices of e in D, with three incident edges e_i, e_i', e, $i = 1, 2$, such that $e_i \to v_i \to e_i'$ is a directed path for $i = 1, 2$.*
Then we can delete e from D and contract the two edges e_1', e_2', replacing the two directed paths $e_i \to v_i \to e_i'$ by directed edges e_i and connecting the end vertices of these paths with the same orientation. The resulting digraph, which we call $D_e := (D - e)/\{e_1', e_2'\}$, has two vertices and three edges less than D but is still 3-regular and 3-edge-connected. We denote by $\vec{T}_e$ the directed dual of D_e with the same outer face as $\vec{T}$ (corresponding to v_0).
If $f' : E(D_e) \to \mathbb{Z}_2$ is a NL-$\mathbb{Z}_2$-flow on D_e, then there is a NL-$\mathbb{Z}_2$-flow f on D such that f and f_e agree on the edges incident to v_0.
Thus, if conjecture 7.6 holds true for $\vec{T}_e$ and (D_e, v_0) respectively, it is also true for $\vec{T}$ and (D, v_0) respectively.

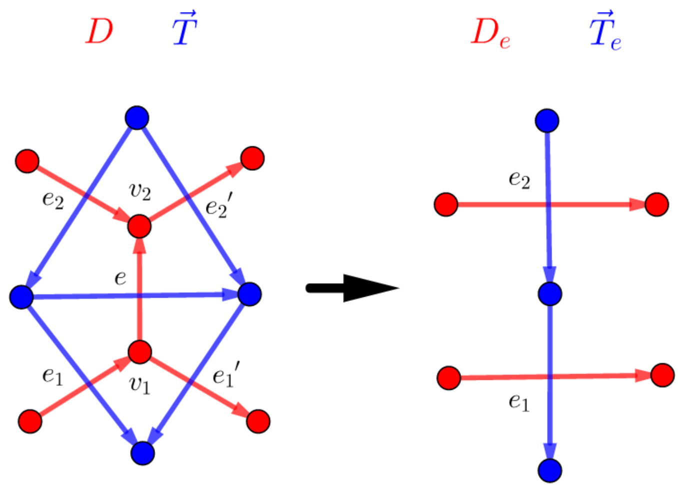


Figure 24: Illustration of the reduction operation for the primal and dual cases.

Proof. The 3-regularity of D_e is trivial. In the following we prove that it is additionally 3-edge-connected. Let S again be an arbitrary cut in $U(D_e)$ with cut set $X \ni v_0$, i.e., a cut in $U(D) - e$ not containing e'_1 and e'_2. If v_1 and v_2 are both in X or both in $\overline{X}$, S itself is a cut in D and therefore $|S| \geq 3$. If the converse is true, $S \cup \{e\}$ is a cut in D. S is especially not the set of the three edges incident to a vertex (i.e., it is non-trivial), since for $v_i, i = 1, 2$, $e \in S$ and $e'_i \notin S$ are incident edges. Since D was required to admit property (O), this means that either $|S \cup \{e\}| = |S| + 1 \geq 4$ and hence $|S| \geq 3$, in which case we are done, or $S \cup \{e\}$ is a non-trivial 3-cut separating v_0 from a cube-like configuration of seven vertices, of which all three vertices incident to a cut edge are sources or sinks. Since either v_1 or v_2 is such a vertex (note that $e = (v_1, v_2) \in S \cup \{e\}$), this would imply that either v_1 or v_2 is a source or a sink, which is obviously not the case. Thus, D_e is indeed 3-edge-connected.
If now D_e admits a NL-2-flow f' on D_e, we define the flow f on D by $f(a) := f'(a)$ for $a \notin \{e, e'_1, e'_2\}$ and $f(e) := 0, f(e'_i) := f'(e_i), i = 1, 2$. Obviously this definition fulfils Kirchhoff's conservation law, and since e is not incident to v_0, f and f' agree on the three edges incident to v_0. It remains to show that f can be completed to a NL-2-flow, which is equivalent to showing that $D/\mathrm{supp}(f)$ is strongly connected. By the assumption we already know that $D_e/\mathrm{supp}(f')$ is strongly connected, and by the definition of f_1 we see that $D/\mathrm{supp}(f)$ consists of $D_e/\mathrm{supp}(f')$ with the additional edge e, where the edges e_1, e_2 may be replaced by directed paths of length two with the same orientation. Therefore also $D/\mathrm{supp}(f)$ has to be strongly connected, which proves the claim. □

Definition 7.18. In the following, we will refer to edges $e \in E(D)$ as described in the lemma above as *deletable edges*, in the dual case ($e \in E(\vec{T})$) we call them *contractible*. In this situation, we equivalently call the graph induced by the two facial triangles incident to e a *reducible diamond*.

Corollary 7.19.

(i) *Let $\vec{T}$ be a directed planar triangulation which is a minimal counterexample to Conjecture 7.6. Then (R3) does not apply to $\vec{T}$/its directed dual D with outer face-vertex v_0 respectively.*

(ii) *Let $\vec{T}$ be a directed planar triangulation with outer face $a_1a_2a_3$ and D its directed dual, where v_0 denotes the vertex in D corresponding to the outer face. If neither (R1) nor (R3) applies to this pair, then there are no reducible diamonds in $\vec{T}$ and no deletable edges in (D, v_0).*

Proof.

(i) Follows immediately from Lemma 7.17.

(ii) This follows from the fact that per definition, the pair admits property (O) if and only if (R1) is not applicable and from Lemma 7.17.

□

The next reduction operation deals with inner vertices and more generally cuts in the triangulation with separated incoming/outgoing edges:

Definition 7.20. Let $\vec{T}$ be a directed planar triangulation with outer face $a_1a_2a_3$, $T := U(\vec{T})$. Let $S = T[X, \overline{X}], S^+ := \vec{T}(X, \overline{X})$, $S^- := \vec{T}(\overline{X}, X)$ be a minimal cut in T not containing one of the three outer edges. In the dual graph $\vec{T}^*$, S corresponds to the edge set of a cycle. Hence, the edges contained in S admit a cyclic ordering. Denote by $E_N \subseteq S^-$ the set of edges ending in X neighbouring at least one edge in S^+ in this ordering. We call S a *reducible cut*, if the following holds:

- The edges $e \in S^+$ of S starting in X are separated from each other, i.e., no two of them are consecutive in the described cyclic order of S.
- For each pair of edges $e_1, e_2 \in S^+$ with distance 2 in the cyclic order, the two facial triangles formed by e_1, e_2 with the edge $e_3 \in E_N$ in between them, are directed.
- Each connected component of $T[S^+]$ is an induced subgraph of T.
- There are no two antiparallel edges in $E(\vec{T}) \setminus E_N$ between connected components of $T[S^+]$.

If v is an inner vertex, we call v *reducible*, if the cut $S = T[X, \overline{X}]$ with $X = \{v\}$ or $X = V(\vec{T}) \setminus \{v\}$ is reducible as defined above.

Lemma 7.21 (R4)**.** *Let $\vec{T}$ be a directed planar triangulation with outer face $a_1a_2a_3$. Let $S = T[X, \overline{X}], a_1, a_2, a_3 \in \overline{X}$, be a reducible cut in $\vec{T}$. Then the planar digraph $(\vec{T} - E_N)/S^+$ is loopless and does not have antiparallel edges. Hence, every planar digraph $\vec{T}_S^+$ arising from $(\vec{T} - E_N)/S^+$ by identifying parallel edges and triangulating faces of length at least 4 admits an embedding as a directed planar triangulation with outer face $a_1a_2a_3$, and the legal 2-colourings of $\vec{T}_S^+$ are also legal colourings for $(\vec{T} - E_N)/S^+$. Let now $c_S : \vec{T}_S^+ \to \{0, 1\}$ be such a legal 2-colouring. Then the 2-colouring $c : \vec{T} \to \{0, 1\}$ of $\vec{T}$, defined by $c(w) := c_S(v_w), \forall w \in \overline{X}$, $c(w) := 1 - c_S(v_w), \forall w \in X$, where v_w corresponds to the contraction vertex of $w \in V(\vec{T})$ in $\vec{T}_S^+$, is also legal, and $c|_{\{a_1,a_2,a_3\}} = c_S|_{\{a_1,a_2,a_3\}}$. The same holds true if we replace all the out-orientations with in-orientations and vice versa.*

Proof. First of all, we have to show that $(\vec{T} - E_N)/S^+$ is a directed planar graph without loops and antiparallel edges. The first part follows from the fact that planar graphs are closed under taking minors. The non-existence of loops and antiparallel edges is an immediate consequence of the definition of a reducible cut.
It remains to show that c is a legal digraph colouring of $\vec{T}$: Assume, there was a monochromatic directed cycle C in $(\vec{T}, c)$. Every edge $e = w_1 \to w_2 \in S^+$ is dichromatic in c_S, since it has one end vertex $w_1 \in X$ and $w_2 \in \overline{X}$, both belonging to the same connected component of $T[S^+]$. Hence, $v_{w_1} = v_{w_2}$ and therefore $c(w_1) = 1 - c_S(v_{w_1}) \neq c_S(v_{w_2}) = c(w_2)$. Consequently, C is either contained in $\vec{T}[X]$ or $\vec{T}[\overline{X}]$ and henceforth also gives rise to a directed cycle in $(\vec{T} - E_N)/S^+$ and thus in $\vec{T}_S^+$, which is monochromatic with respect to c_S according to the relation defining c, contradiction. $\square$

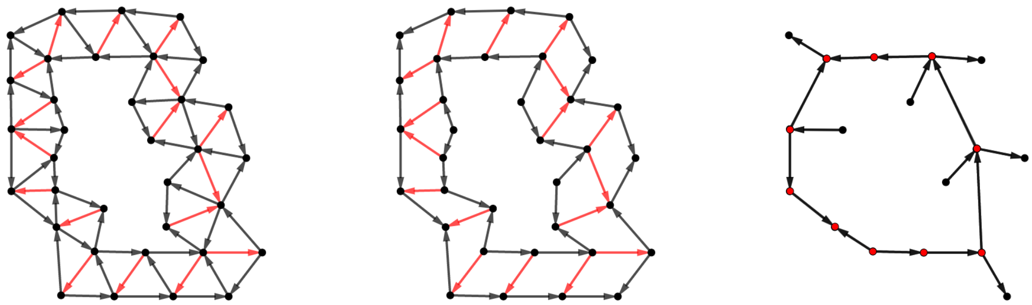


Figure 25: Illustration of the reduction $\vec{T} \Rightarrow \vec{T}_S^+$ for a reducible cut S. Red edges correspond to elements of S^+, the edges deleted in the second figure are the elements of E_N.

The vertex-version of the theorem is the following:

Corollary 7.22. *Let $\vec{T}$ be a directed planar triangulation with outer face $a_1a_2a_3$. Let $v \in V(\vec{T})\backslash\{a_1, a_2, a_3\}$ be a reducible vertex. Denote by $E_N \subseteq E^-(v)$ the set of incoming edges neighbouring at least one outgoing edge. Then the planar digraph $\vec{T}_v^+ := (\vec{T} - E_N)/E^+(v)$ is simple and henceforth a directed triangulation with outer face $a_1a_2a_3$. Let now $c_v : \vec{T}_v^+ \to \{0, 1\}$ be an arbitrary legal 2-colouring of $\vec{T}_v^+$. Then the 2-colouring $c : \vec{T} \to \{0, 1\}$ of $\vec{T}$, defined by*

$$c(w) := \begin{cases} c_v(w), & \text{if } w \in V(\vec{T})\backslash(\{v\} \cup N^+(v)) \\ c_v(v_{E^+(v)}), & \text{if } w \in N^+(v) \\ 1 - c_v(v_{E^+(v)}) & \text{if } w = v \end{cases},$$

where $v_{E^+(v)}$ corresponds to the contraction of $E^+(v)$ in $\vec{T}_v^+$, is also legal, and $c|_{\{a_1,a_2,a_3\}} = c_v|_{\{a_1,a_2,a_3\}}$. The same holds true if we replace all the out-orientations with in-orientations and vice versa.

Proof. Apply Theorem 7.21 to the cut $S := T[\{v\}, V(T)\backslash\{v\}]$ in $T := U(\vec{T})$. $\square$

Corollary 7.23.

(i) Let $\vec{T}$ be a directed planar triangulation which is a minimal counterexample to Conjecture 7.6. Then (R4) does not apply to $\vec{T}$/its directed dual D with outer face-vertex v_0 respectively.

(ii) Let $\vec{T}$ be a directed planar triangulation with outer face $a_1a_2a_3$ such that (R4) is not applicable. Then every minimal cut in $\vec{T}$ not containing an outer edge is non-reducible.

Proof.

(i) Follows immediately from Lemma 7.21.

(ii) Trivial.

$\square$

Proof (of Theorem 7.8 and 7.9). In the following, we assume $\vec{T}, (D, v_0)$ to be a dual pair of a directed planar triangulation with outer face $a_1a_2a_3$, where v_0 is the special vertex of D corresponding to this outer face, and prove the claimed properties for the primal case in 7.8 for $\vec{T}$ and for the dual case in 7.9 for (D, v_0) respectively.

Furthermore, keep in mind that the graph D has to be simple, i.e., there are no loops or multiple edges except if $U(D)$ consists of two vertices connected by three parallel edges: If there was a loop at a vertex v, since D is 3-regular, v would have only one connecting edge to other vertices, which is a 1-cut. If there are three multiple edges, since $U(D)$ is 3-regular and connected, it has to be the graph excluded above. In the remaining case, i.e., there are two edges between the same two vertices v_1, v_2, the third edges incident to v_i, which we call $e_i, i = 1, 2,$ together form a 2-cut in $U(D)$ which again contradicts the 3-edge-connectivity.

Claim 1. $\vec{T}$ and (D, v_0) have property (O).

Follows from Corollary 7.11.
Furthermore, since $\vec{T}$ was proven to admit property (O), the minimality of $\vec{T}, D$ as counterexamples together with Corollary 7.19 directly yields the following.

Claim 2. $\vec{T}$ does not contain reducible diamonds, and equivalently, D does not contain deletable edges.

Claim 3. $U(D)[\overline{Q \cup S}\backslash\{v_0\}]$ is a forest.

For a proof by contradiction assume there was a (not necessarily directed) cycle C in D which does not use any source or sink and does not use v_0. For the first case, assume that C is directed. Corollary 7.16 together with the minimality of D as a counterexample gives us the following and therefore a contradiction in this case.

Claim 4 (i). $U(D)[\overline{Q \cup S}\backslash\{v_0\}]$ does not contain directed cycles. More generally, every cycle C in $U(D) - v_0$ has at least two edges in both directions.

We now consider the case that C is a non-directed cycle and derive a contradiction.

Claim 4 (ii). There is no non-directed cycle C in $U(D)[\overline{Q \cup S}\backslash\{v_0\}]$.

Since C is not directed, it contains a maximal directed subpath $P = v, f_1, v_2, f_2, ..., v_l, f_l, w$ directed from a vertex $v \in V(C)$ to a vertex $v \neq w \in V(C)$, such that the respective second incident edge of v, w on C is directed out or in respectively (see figure below). Since v, w are no sinks or sources, the respective unique third incident edge, which is not contained in C, is incoming for v and outgoing for w.

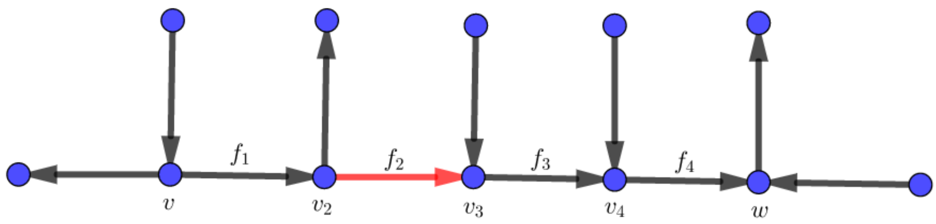


Figure 26: Illustration of the situation on P.

Because f_1, f_l are not deletable (Claim 3), this means that the third incident edges of v_2 and v_l have to be outgoing and incoming respectively. Hence, moving along P, there have to be two consecutive vertices v_i, v_{i+1} on P such that the respective unique third incident edge of the vertices is outgoing or incoming respectively. But this means that (since P is a directed path) f_i is deletable, which is a contradiction.

Claim 4 (iii). The bounded directed triangular faces cover all inner vertices.

Assume for contradiction there was an inner vertex $v \in V(\vec{T}) \setminus \{a_1, a_2, a_3\}$ which is only incident to non-directed triangular faces. These faces correspond to the vertex set of a cycle in $U(D)[\overline{Q \cup S}\setminus\{v_0\}]$, which contradicts Claim 3.

Claim 5. For every inner vertex v of $\vec{T}$ there are at least two in- and at least two outgoing edges. More generally, every cut in $\vec{T}/\{a_1, a_2, a_3\}$ has at least two edges in both directions. Hence, $\vec{T}/\{a_1, a_2, a_3\}$ is strongly connected.

This immediately follows from Claim 4(i) by dualization. The concrete reduction used in lemma 7.15 is illustrated in the following figure for the vertex-case.

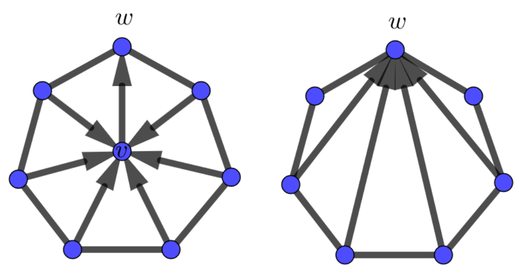


Figure 27: Reducing the planar digraph in case of outdegree 1.

Claim 6. The bounded non-directed triangular faces of $\vec{T}$ cover all inner vertices of degree 4. Together with the separating 4-gons in $\vec{T}$ which have three edges with the same and one with opposite orientation, they cover all inner vertices.

For the second part of the statement, it suffices to prove that every inner vertex $v \in V(\vec{T})\setminus\{a_1, a_2, a_3\}$ not contained in a separating 4-gon as described and only incident to directed triangles would be reducible, which according to Corollaries 7.22 and 7.23 cannot exist in $\vec{T}$. We check the definition in 7.20. The first two items are immediate consequences of the fact that all the incident triangles of v are directed. Each edge $e \in E(\vec{T})\backslash E^+(v)$ connecting two different vertices in $N^+(v)$ together with v would form a separating triangle in $\vec{T}$, which according to property (O) would have exactly three vertices in its inside of which exactly two are adjacent to v. But this contradicts the fact that all the triangular faces incident to v are directed and thus, the number of edges between any two outgoing edges has to be odd. Hence, this contradiction shows that $T[E^+(v)]$ indeed is an induced subgraph of T.

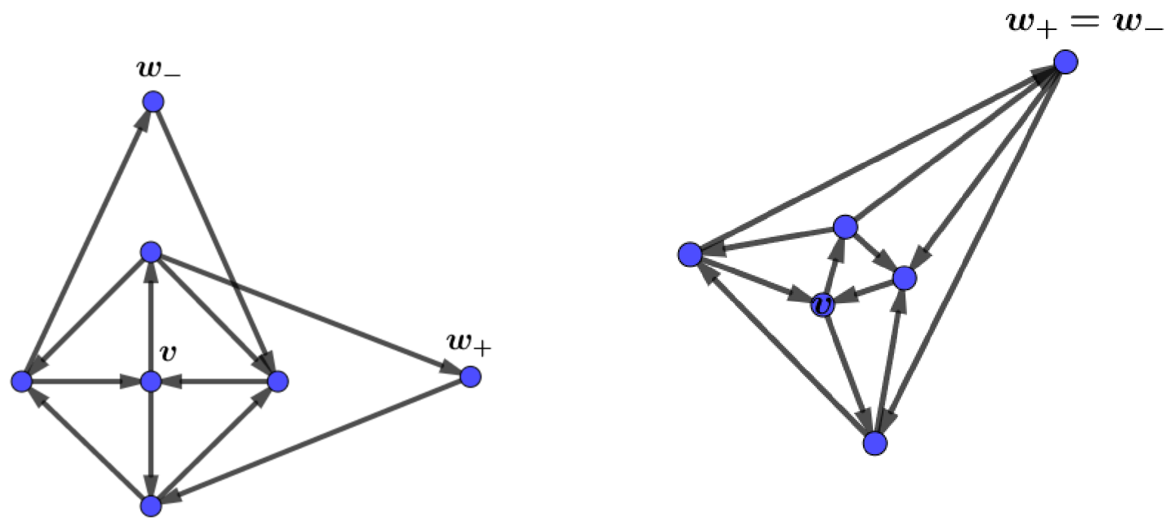


Figure 28: Non-existence of uncovered vertices of degree 4.

Furthermore, $T[E^+(v)]$ is connected and so, there are no antiparallel edges between different connected components. Additionally, if there was a vertex $w \in V(\vec{T})\backslash(\{v\} \cup N^+(v))$ incident to two different vertices in $N^+(v)$ via edges e_1, e_2 not contained in $E^-(v)$ with different respective orientations, those three vertices together with v would form a separating 4-gon containing 3 edges with the same and one with opposite orientation, contradiction. This is because w is not adjacent to v, since in this case $w \in N^-(v)$ and hence the two incident edges of w on the minimal cycle surrounding v have the same respective orientation. This means that at least one of e_1, e_2 is not contained in this minimal cycle, and again, this edge together with its connections to v contradicts the fact that $T[E^+(v)]$ is an induced subgraph as proven above. All in all, also the fourth item of the definition is verified, and we are done with the second claimed statement.

For the first part, assume there was an inner vertex v of degree 4 not covered by a non-directed facial triangle. According to the latter statement, this means that the pairs of vertices in $N^+(v)$ and $N^-(v)$ each have to be adjacent to a common vertex $w_+, w_- \in V(\vec{T})\backslash(\{v\} \cup N(v))$, and thereby build separating 4-gons together with v and $w_{+/-}$. Due to planarity and since v has degree 4, $w_+ = w_-$, since otherwise, there would be edge

crossings according to the Jordan curve theorem: One of the two vertices in $N^+(v)$ lies inside the separating 4-gon containing w^-, the other one lies in the outside. Hence, the vertex $w^+ = w^-$ is adjacent to all neighbours of v, and two of those adjacencies have to form a separating triangle in $\vec{T}$, which is not of the type required by property (O): Assume for contradiction it was of the type (O). Then, since all the faces incident to v were assumed to be directed, and because all the three faces bounding the separating triangle from the inside have to be directed according to (O), this forces the separating triangle itself to be directed, contradicting the definition of the special orientation of the octahedron graph according to (O). This situation is illustrated in the figure above.
The claims (1) to (6) together now prove all the statements of the Theorems 7.8 and 7.9. $\square$

There is a nice positive statement which we can deduce from the properties of minimal counterexamples to the above conjecture. In order to prove this, we make the following general observation, which will also be important later on: Consider a minimal counterexample $\vec{T}$ to the primal conjecture, or more generally, a directed planar triangulation $\vec{T}$ of type (O) without contractible edges (see Theorem 7.8). We therefore know that for every edge $e \in E(T)$ one of the two incident triangular faces has to be transitive (i.e., the two other edges contained in it together built a directed path of length two which is parallel to e) or directed, or e is an outer edge, since else e was contractible. Now assume f to be an incident transitive triangle. In the following, we always have a directed trail $\vec{P}$ parallel to e on the same side as f which we initialize by e and update while we do the construction. As a first step, we update $\vec{P}$ to be the transitive path of e contained in f. We consider the edges e_0^f, e_1^f additionally contained in f. If we look at the orientations of e, e_{i+1}^f with respect to $e_i^f, i = 0, 1$ (algebra modulo 2), we see that they are the same, which means (since e_i^f is not contractible) that the triangle f_i on the other side of e_i^f (if e_i^f is an inner edge) is transitive or contained in a directed bounded facial triangle. If the latter is the case, we stop our construction locally. If f_i is transitive, we update $\vec{P}$ by replacing e_i^f by the transitive path of length two contained in f_i. With the same argumentation as above, additional triangles bordering f_i have to be transitive or directed. We continue this construction until it stops everywhere locally. Due to planarity, we thereby end up with a directed trail $\vec{P}(e, f)$ parallel to e and lying on the same side as f, such that all the facial triangles bounding $\vec{P}$ from the side opposite to e are either directed or the outer face. Furthermore, the length of the trail increases by one with every step, i.e., whenever we replace an edge by a transitive path of length two:

Proposition 7.24. *Let $\vec{T}$ be a directed planar triangulation with property (O) such that there are no contractible edges resp. reducible diamonds. Let $e = u \to v$ be an edge of $\vec{T}$ and f an incident transitive facial bounded triangle in $\vec{T}$ with transitive edges e_0^f and e_1^f and third vertex v_f. Then there is a sequence $\vec{P}_1, ..., \vec{P}_l$ of directed trails in $\vec{T}$ each of them starting in u and ending in v with $l(\vec{P}_i) = i, i = 1, ..., l$, $\vec{P}_1 = (u, e, v), \vec{P}_2 = (u, e_0^f, v_f, e_1^f, v)$ such that $\vec{P}_i$ is obtained from $\vec{P}_{i-1}$ by replacing an edge by a transitive facial triangle for $i \geq 2$ and such that the sets of faces R_i enclosed by e and $P_i, i = 1, ..., l$ (i.e., the set of facial bounded triangles connected to e and an edge $e' \in P_i$ by a path of facial triangles in the dual graph not using any other edges out of P_i) are connected and inclusion-monotone.*

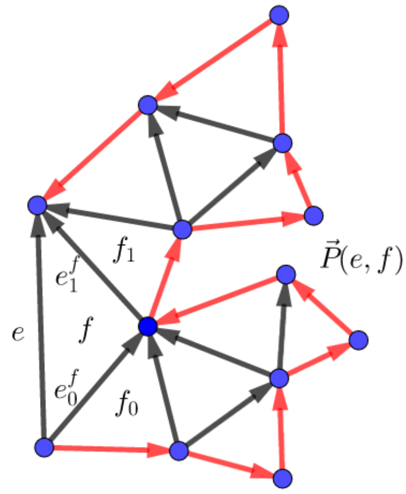


Figure 29: The parallel-path translation.

Moreover, every triangle bordering an inner edge on the boundary of R_l, which is not e, from the outside, is directed.

Definition 7.25. We denote $\vec{P}_l$ in the above context by $\vec{P}(e, f)$, and call it the *parallel path translation* (though it might be a directed trail in general) of the edge e on the side of f.

In the following partial result, which proves Conjecture 7.6 in the case of directed planar triangulations where all directed cycles admit the same orientation, applying the reduction (R1) as introduced above for the general case is not convenient, since we might have to modify the actual planar directed triangulation by plugging in a copy of a directed octahedron graph as shown in figure 18, which contains directed cycles with different planar orientations (clockwise and counter-clockwise).
Instead, we will use an even simpler reduction method by splitting along given separating triangles as was done by Mohar et. al. when proving their digirth-four result (cf. section 5.2). Note, that in the general case of Conjecture 7.6, we have to use the more elaborate reduction (R1) in order to deal with the forbidden monochromatic pre-colourings, which we will omit in the following considerations for oriented plane triangulations with equal orientations of directed cycles. This is formally captured by the following statement:

Lemma 7.26 (R1'). *Let $\vec{T}$ be a directed plane triangulation with outer face $a_1a_2a_3$ and $x_1x_2x_3$ a separating triangle in $\vec{T}$. Denote by $\vec{T}_{\text{out}}$ resp. $\vec{T}_{\text{in}}$ the directed plane triangulations arising from $\vec{T}$ by deleting all vertices in the interior resp. exterior of $x_1x_2x_3$. If $c_{\text{out}} : V(\vec{T}_{\text{out}}) \to \{0,1\}, c_{\text{in}} : V(\vec{T}_{\text{in}}) \to \{0,1\}$ are legal digraph-colourings of $\vec{T}_{\text{out}}$ resp. $\vec{T}_{\text{in}}$ agreeing on $x_1x_2x_3$, i.e., $c_{\text{out}}(x_i) = c_{\text{in}}(x_i), i = 1,2,3$. Then the colouring $c : V(\vec{T}) \to \{0,1\}$ which extends c_{out} and c_{in} on $V(\vec{T}_{\text{out}})$ resp. $V(\vec{T}_{\text{in}})$, is a legal 2-colouring of $\vec{T}$.*

Proof. Assume for a proof by contradiction there was a monochromatic directed cycle C in $\vec{T}$. Since $c_{\text{in}}, c_{\text{out}}$ are legal digraph colourings, C has to cross $x_1x_2x_3$ and thus contains one of the edges $x_i x_{i+1}$ as chord. This chord splits C into two directed paths fully contained in $\vec{T}_{\text{out}}$ resp. $\vec{T}_{\text{in}}$, of whom at least one together with $x_i x_{i+1}$ forms a monochromatic directed cycle, contrary to the fact that $c_{\text{in}}, c_{\text{out}}$ are legal digraph colourings. □

Remark 7.27. Let $\vec{T}$ be a directed planar triangulation for which (R1') is not applicable. Then $T := U(\vec{T})$ is 4-connected and thus especially admits property (O).

Proof. The reduction operation is not applicable if and only if there are no separating triangles in T, i.e., iff T is 4-connected. □

Theorem 7.28. *Let $\vec{T}$ be a directed plane triangulation with at least four vertices such that all the directed cycles in $\vec{T}$ have the same orientation (without loss of generality clockwise). Then at least one of the three reduction operations (R1'), (R2),(R3) is applicable, such that the resulting reduced triangulations still only admit clockwise directed cycles.*

Proof. Assume by way of contradiction none of the three reduction operations is applicable to $\vec{T}$ without violating the clockwise-orientation-condition. Our goal is to lead this to a contradiction.
The idea of the proof is to first verify that (R1'), (R2) and (R3), each keep the additional condition above (all directed cycles are oriented the same) satisfied. From that it becomes immediately clear that all the properties for non-reducible plane triangulation stated in Remark 7.27 and Corollaries 7.16 and 7.19 also hold for $\vec{T}$.

Claim 1. All the reduction operations keep the orientation of directed cycles.

This is almost trivial and easy checkable for (R1') and the $D \to D_e$ operation (R3). It remains to prove that the same is true for the "split along an almost directed cut-operation" (R2). According to definition 7.14, $\vec{T}_{\text{in}}$ arises from a subdigraph of $\vec{T}$ by adding a source, which cannot be contained in any directed cycle. Thus, all directed cycles of $\vec{T}_{\text{in}}$ have to be clockwise too. The same is true for $\vec{T}_{\text{out}}$ in the case of $|X| \geq 2$ (X denotes the cut set of the almost directed cut). If $X = \{v\} \subseteq V(\vec{T}) \backslash \{a_1, a_2, a_3\}$, then any directed cycle in $\vec{T}_{\text{out}} = (\vec{T} - \{e_1, e_2\})/e$ passing the contraction vertex of e (notation as in the definition) corresponds to a directed cycle in $\vec{T}$ (by decontracting e), since otherwise v admitted an outgoing edge other than e_{rev}. Thus, each directed cycle in $\vec{T}_{\text{in}}$ also has to be oriented clockwise in this remaining case, proving claim 1.

Applying 7.27, 7.16 and 7.19 now yields the following (note that the 4-connectivity of T especially implies that $\vec{T}$ admits property (O)):

Claim 2. $\vec{T}$ is 4-connected, does not contain contractible edges resp. reducible diamonds and each cut in $\vec{T}$ not containing one of the outer edges has at least two edges in both directions.

We now proceed with a couple of basic properties of triangular (bounded) faces in $\vec{T}$.

Claim 3. For every inner edge e, the triangle on the right side of it is directed or transitive. The facial triangle on the left side of e is neither of both.

Given an arbitrary inner edge e of $\vec{T}$ (i.e., it is not contained in the outer face), the bounded triangular face to the left of e can not be transitive (i.e., the two remaining edges built a directed path of length two parallel to e), because in this case we could apply the parallel-translation operation as described to e and this face and would end up with a directed path $\vec{P}$ parallel and to the left of e, such that for every edge f on $\vec{P}$ either f is contained in the outer face or the triangular face incident to it on the left is directed (and therefore counterclockwise). But since the latter case is not possible, this means that $\vec{P}$ is fully contained in the outer face. This furthermore implies (since e is an inner edge) that $\vec{P}$ consists only of the single edge e (since every parallel translation step increases the length of the parallel path), which has to lie in the outer face, contradiction.
Therefore the face incident to e on the left is (since it also cannot be directed) non-directed, such that the two additional edges have the same orientation with respect to e. Since e is not a contractible edge in the directed dual D of $\vec{T}$, the triangular face incident to e on the right is either directed or transitive.

We now use these facts in order to derive a contradiction. For this purpose, let again D be the directed dual of $\vec{T}$ where $v_0 \in V(D)$ corresponds to the outer face, and let $Q \cup S = Q$ be the set of sources and sinks (there are no sinks, since they would give rise to a counterclockwise oriented cycle in $\vec{T}$). As deduced in the proof of Theorem 7.8, since the cubic 3-edge-connected planar digraph D admits property (O) and is not reducible by (R1),(R2) or (R3), $U(D)[Q \cup S \backslash \{v_0\}]$ is a forest.
Another (trivial) observation is that the vertices in $Q \cup S = Q$ are isolated from each other, which means that the (clockwise) directed bounded triangular faces in $\vec{T}$ never share a common edge, since of two incident triangular directed faces, one has to be clockwise while the other must have counter-clockwise orientation, which is not possible in $\vec{T}$.

In the rest of the proof, we will consider a simple planar graph F defined as follows: The vertices of F consist of the bipartition classes corresponding to the bounded directed facial triangles in $\vec{T}$ on the one and the vertices of $\vec{T}$ incident to at least two such triangles on the other hand. The adjacencies in F now are defined as the incidences in $\vec{T}$, i.e., a vertex is connected to all the bounded facial triangles containing it.

Obviously, by placing the triangle-vertices inside the corresponding triangle, we get a planar embedding of F.
The (undirected) subgraph of T, obtained by taking all the edges on the boundaries of directed bounded triangular faces, does not admit a cycle of length at least 4, since the corresponding cycle of directed triangles in $\vec{T}$ would always give rise to two directed cycles in $\vec{T}$ with opposite orientation, which is not possible. Hence, F is a forest. We now consider an arbitrary leaf of F, i.e., at least two of the three vertices on the corresponding directed facial triangle are not incident to any other directed facial triangles. In such a situation, we

can generally show that at least one of the vertices of this pair has to be an outer vertex a_i:

Claim 4. The simple graph F is a forest, such that every leaf of it corresponds to a directed facial triangle in $\vec{T}$ containing at least one outer vertex.

Lemma 7.29. *Let $\vec{T}$ be a 4-connected directed planar triangulation admitting property (O) without contractible inner edges and without almost directed cuts.*
Let $e = v_1 \to v_2$ be an inner edge in $\vec{T}$. Let f_1, f_2 be the two bordering facial triangles, and assume that f_1 is directed while f_2 is not. Furthermore, denote by e_1, e_2 the edges contained in f_2 and incident to v_1, v_2.

(i) If v_1, v_2 are inner vertices, then f_1 is not the only directed facial triangle incident to v_1 or v_2.

(ii) If exactly one vertex $v_i, i = 1, 2$ is an outer vertex, and if f_1 is the only directed facial triangle incident to v_1 or v_2, then e_i and e have a different orientation with respect to v_i. Moreover, all the edges incident to v_i in the closed interval between e_i and the outer edge o on the same side of e as f_2 are oriented the same.

Proof.

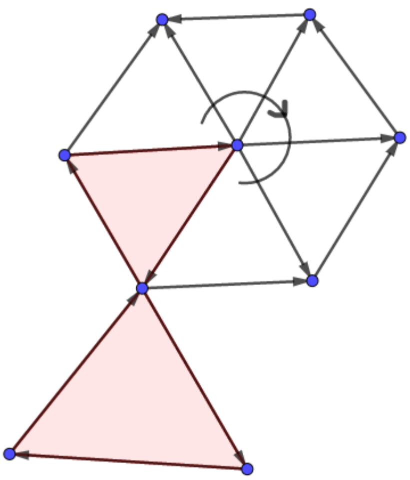


Figure 30: Finding vertices of out- or indegree one.

(i) Assume by way of contradiction f_1 was the only directed facial triangle incident to v_1 or v_2. Without loss of generality, assume f_1 lies to the right of e while f_2 is on the left.
We look at the orientations of the two edges e_1 and e_2 with respect to e. If they are both incoming, we consider the edge e_1 incident to e via v_1. According to what was

shown above, the next edge incident to v_1 in counterclockwise cyclic order is the first of the two transitive edges from the triangular face to the right of e_1 and therefore is also incoming. This line of argument continues inductively until we reach the directed triangle f_1 containing e again. Therefore v_1 has exactly one outgoing edge, which is a contradiction to the cut-condition above. If both edges are outgoing, the same line of argument works symmetrically for v_2, which gives us that v_2 has exactly one incoming edge, which again is not possible. In the remaining case where f_2 is transitive, we can even chose the side for the same line of argument, since e, e_i have the same respective orientation to $e_{3-i}, i = 1, 2$, which again yields a contradiction to the initial assumption, showing that it has to be false. This verifies the claim.

(ii) In this case, again, if the edges e, e_i were both incoming or both outgoing with respect to v_i, the other facial triangle (except f_2) bordering the edge e_{3-i} would have to be transitive, since otherwise, e_{3-i} was a reducible inner edge. Continuing this reasoning inductively, since v_{3-i} is an inner vertex, we see that v_{3-i} has in- or outdegree 1, contradiction. Hence, e, e_i have different orientation, which means that e, e_{3-i} have the same respective orientation to v_{3-i}, since else f_2 would be directed. But now again, we know that the other bounding triangle of e_i has to be transitive, and continuing this process inductively as moving around v_i proves the second statement.

□

In the case of a leaf as described above, assume both vertices not incident to another directed triangle were inner vertices. Let $e = v_1 \to v_2$ be the edge connecting the vertex-pair on the directed triangle, which then is incident to e from the right. Applying the lemma above, part (i) to e, v_1, v_2 then would give a contradiction to $\vec{T}$ being a minimal counterexample.

Claim 5. F is connected, i.e., it is a tree.

Assume to the contrary that the directed-triangle graph was separated into at least two components. This means that there is a set of connected undirected triangles corresponding to a cycle in $U(D)$ separating it into two components each of them containing at least one source.
If this cycle did not contain the outer face-vertex v_0, it would also be a cycle in $U(D)[\overline{Q} \backslash \{v_0\}]$ which is a forest according to Theorem 7.8, contradiction. Therefore it has to contain v_0 and hence corresponds to a path of adjacent undirected triangles in $\vec{T}$ starting and ending with two different outer edges, such that the subdigraphs “above” and “below” it contain at least one directed triangle. Assume without loss of generality that this path starts at a_1a_3 and ends at a_1a_2. We consider the subgraph of F which lies above (i.e. on the side containing a_1) the described path-cut. We may assume this subgraph to be minimal under the made assumptions and therefore connected, since we could otherwise decrease the size of this subgraph by choosing a path “above” the actual one. Every leaf of this tree-component of F has to be incident to a_1 (as was shown above). Since leafs are vertices of degree 1, this means that at most two leafs exist, and since they have to be adjacent in F, this component

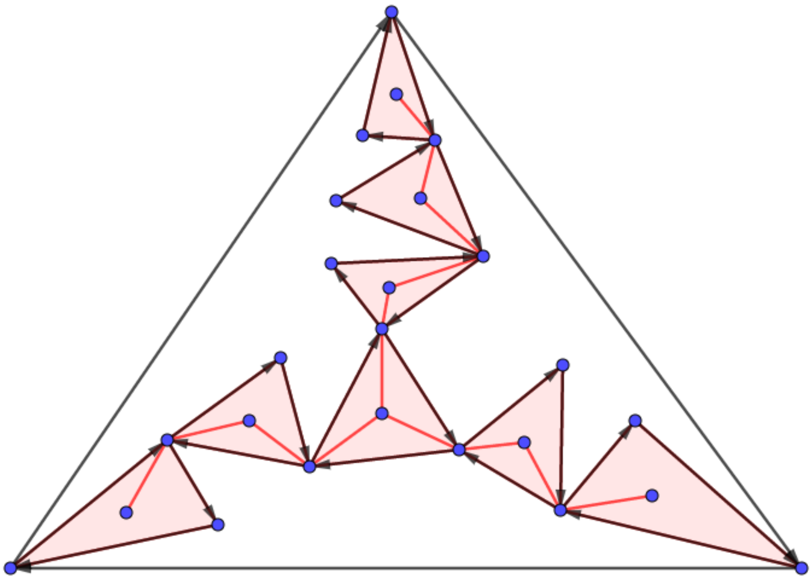


Figure 31: The 3-star-case.

is either a K_1 or a P_3. In both cases, we can choose one of the directed facial triangles and consider the edge e on them which is not incident to a_1. Now, since there are no further directed triangles incident to v_1 or v_2, we get a contradiction according to Lemma 7.29,(i).

The results above show that F is a tree such that every leaf contains at least one outer vertex a_i. Additionally, for each outer vertex a_i, there can be at most one incident directed bounded triangular face corresponding to a leaf of F: If there were two different leafs incident to the same outer vertex, since leafs are of degree 1, this would imply that these two faces are the only two directed triangular faces, which leads us to a contradiction in the same way that was shown above. This means that:

Claim 6. F admits at most 3 leafs, each of them, considered as faces in $\vec{T}$, being incident to a different outer vertex.

This result and the tree structure of F immediately imply the following: Every connected component of $U(D)[\overline{Q}\backslash\{v_0\}]$ has to contain at least one facial triangle containing an outer edge.
We now first consider the case that F has 3 leafs, i.e., exactly one leaf at every outer vertex. Then F has exactly one vertex of degree 3 which corresponds to a directed facial triangle, and is topologically equivalent to a 3-star. The three facial triangles incident to the outer edges $a_ia_{i+1}, i = 1, 2, 3$ now cannot be directed, since in this case there would be at most two leafs. Furthermore, each of those three triangles is contained in exactly one connected component of $U(D)[\overline{Q}\backslash\{v_0\}]$, which already make up all components of $U(D)[\overline{Q}\backslash\{v_0\}]$ (as was shown above). Each of them corresponds to a region in $\vec{T}$ bounded by a cycle containing the corresponding outer edge and such that all the other edges are bounded by a clockwise directed triangle from the outside. This means that all these edges on the cycle are oriented counterclockwise, and therefore all the three outer edges have to be oriented clockwise in order to avoid counterclockwise boundary cycles. Now consider one of the leafs, say the one corresponding to the directed facial triangle containing a_1. Let e be the edge on this triangle

incident to a_1 but not incident to any other directed triangle. Applying Lemma 7.29 part (ii) to e, we conclude that the outer edge incident to a_1 which lies on the same side as e, has to be oriented counterclockwise. This is a contradiction to what was shown above.
In the second case, F has exactly two leafs and therefore is a path, starting and ending with directed triangles incident to different outer vertices, without restriction we assume that the first leaf is attached to a_3, while the second is attached to a_2. Let us first assume that

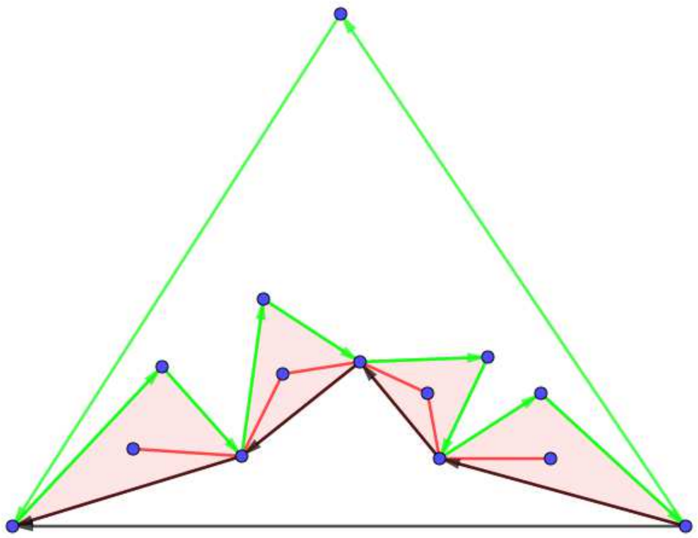


Figure 32: The path-case.

there is no directed triangular face containing a_1. Then there are two components of non-directed triangular faces, one above and the other below the directed-triangle-path between a_1 and a_2.
We can assume that the component containing a_2 and a_3 is non-trivial, because in this case, F would correspond to the single directed facial triangle containing the edge a_2a_3, i.e. $n \leq 4$ since the directed triangles cover all inner vertices (cf. Theorem 7.8,(ii)), but in this case, $\vec{T}$ has to be a simple oriented triangle (a K_4 cannot be a minimal counterexample, since it allows a deletion of the inner vertex) contradicting our initial assumption of $|V(\vec{T})| \geq 4$.
If we consider the two leafs of the path incident to a_2, a_3, with the exact same line of argument as we had above in the case of three leafs, using (ii) of Lemma 7.29, we get that for each of a_2, a_3 one of the both incident outer edges is directed counterclockwise. But since a_2a_3 has to be oriented clockwise (in order not to produce a counterclockwise directed cycle around the component of $U(D)[\overline{Q \cup S}\backslash\{v_0\}]$ containing it), this means that both a_1a_2 and a_2a_3 have to be oriented counterclockwise. But then the boundary of the component containing a_1 is directed counterclockwise, which again gives a contradiction.
In the case that there is a (and therefore exactly one) directed triangular face containing a_1, we can use a similar line of argument as in the 3-star-case in order to derive a contradiction. Finally, this implies the claim. $\square$

Corollary 7.30. *Let $\vec{T}$ be a directed plane triangulation with at least four vertices such that all the directed cycles in $\vec{T}$ have the same orientation (without loss of generality clockwise). Then conjecture 7.6 holds for $\vec{T}$, and even stronger, for any given pre-colouring $c_{\text{pre}} : \{a_1, a_2, a_3\} \to \{0, 1\}$ which is not monochromatic in case that $a_1a_2a_3$ is directed, there is a legal 2-colouring of $\vec{T}$ extending c_{pre}.*

Proof. Assume there was a counterexample to the reinforced statement above, i.e.

If $\vec{T}$ is a directed plane triangulation where all the directed cycles admit clockwise orientation, then there is a legal 2-colouring of $\vec{T}$ extending a pre-colouring of the outer vertices, as long as it is not monochromatic if the outer triangle is directed.

Choose such a counterexample to be minimal with respect to the number of vertices/edges. Then none of the three reduction operations can be applicable: This follows immediately from Lemma 7.15 and Lemma 7.17 for (R2) and (R3). We now prove that also (R1') isn't applicable: Assume for contrary that (R1') was applicable at some separating triangle $x_1x_2x_3$. Since $\vec{T}$ is a *minimal* counterexample to the above statement, it has to hold true for the smaller triangulations $\vec{T}_{\text{out}}$ and $\vec{T}_{\text{in}}$ arising from the reduction. But then, given any pre-colouring of the outer face of $\vec{T}$ which is not monochromatic if the outer triangle is directed, we can find an extending legal 2-colouring of $\vec{T}_{\text{out}}$ which again induces a legal pre-colouring on the outer face of $\vec{T}_{\text{in}}$, namely $x_1x_2x_3$. Extending this pre-colouring to a legal 2-colouring of $\vec{T}_{\text{in}}$ and sticking the two colourings together along $x_1x_2x_3$, Lemma 7.26 now guarantees that also $\vec{T}$ admits a legal extending 2-colouring for any pre-colouring given as above, contradiction.
Now, according to Theorem 7.28, $\vec{T}$ is an orientation of a triangle for which the conjecture obviously does hold true, contradiction. □

The nice thing about the result above is that it is not just a pure existence result. In fact, given an arbitrary directed plane triangulation $\vec{T}$ we can apply the reduction operations as long as we want to. Our proof points out that we have to end up with an orientation of a simple triangle. We then choose the given pre-colouring as a legal extending 2-colouring for this triangle and then reconstruct the colourings while reversing the reductions, in order to end up with a legal 2-colouring of the original directed plane triangulation extending the precolouring. The following algorithm describes this (recursive) procedure more formally.

Algorithm 7.31 (FindTwoColouring). **FindTwoColouring**
Input: $\vec{T}$, a directed plane triangulation where all directed cycles are oriented clockwise, together with a legal precolouring $c_{\text{pre}} : \{a_1, a_2, a_3\} \to \{0, 1\}$.
if $V(\vec{T}) = \{a_1, a_2, a_3\}$ **then**
- $c := c_{\text{pre}}$.
- STOP.

end if
if $\exists$ separating triangle $x_1x_2x_3$ in $\vec{T}$ **then**
- Define $\vec{T}_1$ by deletion of all vertices inside $x_1x_2x_3$.

- Define $\vec{T}_2$ by deletion of all vertices outside $x_1x_2x_3$.
- Define c_1 as the colouring given by application of FindTwoColouring to $\vec{T}_1, c_{\text{pre}}$.
- Define $c'_{\text{pre}} : \{x_1, x_2, x_3\} \to \{0, 1\}$ as restriction of c_1 to x_1, x_2, x_3.
- Define c_2 as the colouring given by application of FindTwoColouring to $\vec{T}_2, c'_{\text{pre}}$.
- Define c by $c(v) = c_1(v)$ for vertices outside the triangle and $c(v) = c_2(v)$ for vertices of $\vec{T}_2$.
- STOP.

end if

if $\exists$ contractible edge $e = uv \in E(\vec{T})\backslash\{a_1a_2, a_2a_3, a_1a_3\}$ with incident edge pairs e_1, e'_1 and e_2, e'_2 **then**

- $\vec{T}_e := (\vec{T} - \{e'_1, e'_2\})/e$
- Define c' as the colouring of $\vec{T}_e$ given by applying FindTwoColouring to $\vec{T}_e$.
- $c(x) := c'(x), x \in V(\vec{T})\backslash\{u, v\}$
- $c(u) := c(v) := c'(e)$.
- STOP.

end if

if $\exists$ almost directed cut $S = T[X, \overline{X}]$, $a_1, a_2, a_3 \in \overline{X}$ with the edge $(v_1v_2, v_1 \in X, v_2 \in \overline{X})$ in converse direction if applicable **then**

- $\vec{T}'_{\text{in}} := \vec{T} - \overline{X}$
- $\vec{T}'_{\text{out}} := \vec{T} - X$
- Define $\vec{T}_{\text{in}}$ according to 7.14.
- Define $\vec{T}_{\text{out}}$ according to 7.14.
- Define c_{out} by application of FindTwoColouring to $(\vec{T}_{\text{out}}, c_{\text{pre}})$.
- Define c_{in} by application of FindTwoColouring to $(\vec{T}_{\text{in}}, c_{\text{pre,in}})$, where $c_{\text{pre,in}}$ is an arbitrary legal precolouring of the outer three vertices of $\vec{T}_{\text{in}}$.
- $c(x) := c_{\text{out}}(x), x \in X$.
- $c(x) := (c_{\text{in}}(x) + c_{\text{out}}(v_1) + c_{\text{in}}(v_2) + 1) \mod 2, x \in \overline{X}$.
- STOP.

end if

Output: A legal 2-colouring c of $\vec{T}$ extending c_{pre}.

Theorem 7.32. *The algorithm described above works correctly and runs in polynomial time in the number $n := |V(\vec{T})|$ of vertices.*

Proof. The number of possible reductions made in the algorithm is at most $r_1 + r_2 + r_3$, where $r_1 = \mathcal{O}(n^2)$ is the number of separating triangles in $\vec{T}$, $r_2 \leq 3n$ is the number of edges in $\vec{T}$ and $r_3 = n$ denotes the number of vertices in $\vec{T}$ (since every cut-split divides the triangulation into two non-trivial parts). Hence, the number of reduction steps in a run of the algorithm is at most $\mathcal{O}(n^2)$. Since each reduction step has a constantly bounded number of operations, the run-time of the algorithm is $\mathcal{O}(n^2) \cdot f(n)$, where $f(n)$ denotes the worst-case runtime needed for a one-time search for separating triangles, almost directed cuts or reducible edges in a directed triangulation with at most n vertices, which is polynomial. This proves the claim. $\square$

8 Subtrees, Bipartite Subgraphs and Feedback Arc Sets

In this final section, we want to deal with the question of 2-colourability of planar digraphs/existence of NL-2-flows in 3-edge-connected digraphs from a very special but interesting perspective: We show that we can construct 2-colourings if we find special subtrees/bipartite subgraphs of planar directed triangulations or spanning trees with special properties in the dual case. We illustrate this for some special subclasses of directed planar triangulations and digraphs in order to use it afterwards to make further progress on the analysis of minimal counterexamples as started in the previous section.

8.1 Dijoins and Feedback Arc Sets

We first give the following definition, which introduces a central concept related to colourings of digraphs:

Definition 8.1. Let $\vec{G}$ and D be digraphs. A *feedback arc set* in $\vec{G}$ is an arc set $A \subseteq E(\vec{G})$ containing at least one directed edge of every directed cycle in $\vec{G}$.
A *dijoin* of D is an edge set $B \subseteq E(D)$ covering each directed cut in D with at least one edge.

Obviously, feedback arc sets and dijoins are dual concepts when considering planar digraphs and their directed duals. A natural question is to ask for a minimum-sized dijoin in an arbitrary digraph D. Surprisingly, this admits a simple Min-Max-characterization due to Lucchesi and Younger (see [LY78]) which we review in the following. The proof we give is a pretty short one found by Lovasz in [Lov76]. As Lovasz, we prepare it with an interesting lemma about families of disjoint dicuts in a digraph avoiding three-fold intersections:

Lemma 8.2 (cf. Lovasz, [Lov76]). *Let D be a digraph, denote by $\vec{\mathcal{S}}$ the set of non-empty directed cuts in D and let F be a collection of non-trivial directed cuts (not necessarily disjoint) in D, which cover each edge of D at most twice. Let*

$$k_{\max} := \max\{k : S_1, ..., S_k \in \vec{\mathcal{S}} \text{ pairwise disjoint}\}.$$

Then $|F| \leq 2k_{\max}$.

Proof. Without loss of generality, assume that $U(D)$ is connected.
Fix the digraph D and assume by way of contradiction there was a collection F of directed cuts containing each edge at most twice with $|F| \geq 2k+1$ ($k = k_{\max}$). In this case, we can choose such a collection F maximal with respect to the value

$$\sum_{S \in F} |X(S)|^2,$$

where for each non-trivial dicut S in D $X(S)$ denotes the unique subset of $V(D)$ such that $D(X(S), \overline{X(S)}) = S$ (the uniqueness is an immediate consequence of the assumed connectivity of D). In the following, we will first show that such a maximal collection must have a so-called *laminar* structure, i.e., for each pair of dicuts $S_1, S_2 \in F$, we have

$X(S_1) \cap X(S_2) = \emptyset$, $X(S_1) \subseteq X(S_2)$, $X(S_2) \subseteq X(S_1)$ or $X(S_1) \cup X(S_2) = V(D)$. So assume to the contrary that there was a pair $S_1, S_2 \in F$ not fulfilling these conditions, which we call *crossing*. We now define two new cuts in D by

$$S_1' := D[X(S_1) \cup X(S_2), \overline{X(S_1) \cup X(S_2)}], S_2' := D[X(S_1) \cap X(S_2), \overline{X(S_1) \cap X(S_2)}],$$

which are obviously still directed, and admit (according to the fact that D is connected and since S_1, S_2 are crossing) non-trivial cut sets. Furthermore, the number of times an arbitrary edge $e \in E(D)$ is covered by the pair S_1, S_2 equals the one for S_1', S_2', which follows immediately from an easy case distinction for the end vertices of the edge e. Hence, the collection $F' := (F \setminus \{S_1, S_2\}) \cup \{S_1', S_2'\}$ fulfills $|F'| = |F| \geq 2k+1$ and still covers each edge in D at most twice. Since $|F|$ was chosen maximal with respect to the sum above, it follows that

$$\begin{aligned}
0 &\geq \sum_{S \in F'} |X(S)|^2 - \sum_{S \in F} |X(S)|^2 = |X(S_1')|^2 + |X(S_2)'|^2 - |X(S_1)|^2 - |X(S_2)|^2 \\
&= \underbrace{|X(S_1) \cup X(S_2)|}_{=:a}{}^2 + \underbrace{|X(S_1) \cap X(S_2)|}_{=:b}{}^2 - |X(S_1)|^2 - |X(S_2)|^2 \\
&= \frac{1}{2}\left((a+b)^2 + (a-b)^2 - \underbrace{(|X(S_1)| + |X(S_2)|)}_{=a+b}{}^2 - (|X(S_1)| - |X(S_2)|)^2\right) \\
&= \frac{1}{2}\left((a-b)^2 - (|X(S_1)| - |X(S_2)|)^2\right) > 0,
\end{aligned}$$

contradiction. The latter follows from

$$\begin{aligned}
|a-b| &= ||X(S_1)| - |X(S_2)|| + 2\left(\underbrace{\min(|X(S_1)|, |X(S_2)|) - |X(S_1) \cap X(S_2)|}_{>0}\right) \\
&> ||X(S_1)| - |X(S_2)||,
\end{aligned}$$

which is a consequence of the crossing condition. Thus, we have the desired contradiction, and it follows that F indeed has to be laminar.

In the following, we work with the simple graph H_F with vertex set $V_F = \{v_1, ..., v_m\}$ where $F = \{S_1, ..., S_m\}$, and two vertices $v_i \neq v_j \in V_F$ are adjacent whenever the corresponding dicuts intersect, i.e. $S_i \cap S_j \neq \emptyset$. Our goal is to show that this is a bipartite graph. Having proved this, we are done: H_F has at least $2k+1$ vertices and hence one of the bipartition classes has at least $k+1$ vertices. But these vertices form an independent vertex set in H_F and hence correspond to a collection of more than k disjoint dicuts in D, which finally yields the desired contradiction.

Let therefore $C = v_1, v_2, ..., v_n, v_{n+1} = v_1$ (without loss of generality, if needed we change the enumeration of V_F) be an arbitrary cycle of length n in H_F. This corresponds to the sequence $S_1, ..., S_n$ of consecutively intersecting cuts, i.e., $S_i \cap S_{i+1} \neq \emptyset, i = 1, ..., n$, which implies $X(S_i) \cap X(S_{i+1}) \neq \emptyset, \overline{X(S_i)} \cap \overline{X(S_{i+1})} \neq \emptyset$. According to the laminarity of F,

we get that $X(S_i) \subseteq X(S_{i+1})$ or $X(S_{i+1}) \subseteq X(S_i), i = 1, ..., n$. Note that the dicuts S_i have to be pairwise distinct, since $S_i = S_j, i \neq j$ would imply that v_i and v_j are vertices of degree 1 in H_F, since otherwise there was a three-fold intersection of dicuts in an edge of $S_i = S_j$. But this again contradicts the fact that v_i, v_j are contained in C. Hence, the inclusions above for the $X(S_i)$ are always proper.
Our goal is to prove that n is even. For this purpose, we want to show that the inclusions appear alternately, i.e., $X(S_1) \subset X(S_2) \supset X(S_3) \subset ...$ or the other way round. Assume for contrary there were three consecutive cuts, without loss of generality S_1, S_2, S_3, such that $X(S_1) \subset X(S_2) \subset X(S_3)$. In the following, for each $S_i, i = 1, ..., n$ we will refer to S_i as being to the left of S_2, if $X(S_i) \subseteq X(S_2)$ or $\overline{X(S_i)} \subseteq X(S_2)$ resp. as being to the right of S_2, if $X(S_i) \subseteq \overline{X(S_2)}$ or $\overline{X(S_i)} \subseteq \overline{X(S_2)}$. The laminarity of F implies that each S_i is either to the left or to the right of S_2. Because of $X(S_1) \subset X(S_2) \subset X(S_3)$, $S_1 = S_{m+1}$ is to the left of S_2, while S_3 is to the right. Hence, there is an index $i' \in \{3, ..., m\}$ such that $S_{i'}$ is to the right of S_2 and $S_{i'+1}$ is to the left. Let $e \in S_{i'} \cap S_{i'+1} \neq \emptyset$ be arbitrarily chosen. Since $S_{i'}$ is to the right of S_2, at least one of the end vertices of e is contained in $\overline{X(S_2)}$, and since $S_{i'+1}$ is to the left of S_2, at least one of the end vertices is contained in $X(S_2)$, i.e., $e \in S_2$ and hence $S_{i'} \cap S_{i'+1} \cap S_2 \neq \emptyset$, contradiction. □

Theorem 8.3 (Lucchesi and Younger, [LY78]). *Let D be a (multi-)digraph and $\vec{\mathcal{S}}$ the set of non-trivial directed cuts in D. Then*

$$\min\{|B| : B \textit{ is a dijoin}\} = \max\{k : S_1, ..., S_k \in \vec{\mathcal{S}} \textit{ pairwise disjoint}\}.$$

Proof (cf. Lovasz [Lov76]). First of all, it is obvious that a dijoin has to contain a distinct edge of each $S_i, i = 1, ..., k$ in a maximal collection dicuts, i.e., the claim above holds with "$\geq$". In the following, we thus restrict on proving the reverse inequality, i.e., show the existence of a dijoin of size $\max\{k : S_1, ..., S_k \in \vec{\mathcal{S}}$ pairwise disjoint$\}$.
We prove the assertion by induction on the number $|E(D)|$ of edges. If $E(D) = \emptyset$, the statement is true. Let now $n := |E(D)| > 0$ and assume the assertion to hold true for all multi-digraphs D with up to $n - 1$ edges.
Let $k_{\max} := \max\{k : S_1, ..., S_k \in \vec{\mathcal{S}}$ pairwise disjoint$\}$. If there is an edge $e \in E(D)$ such that the digraph D/e has no $k_{\max}$ disjoint directed cuts, then, since $|E(D/e)| \leq n-1$, D/e admits a dijoin B_e of size $k_{\max} - 1$. But since the directed cuts of D are exactly those of D not containing e, it follows that $B_e \cup \{e\}$ is a dijoin of D of size $k_{\max}$, which immediately proves the induction hypothesis (the minimality of $B_e \cup \{e\}$ follows from the fact that every dijoin has to contain at least one edge of each of $k_{\max}$ disjoint dicuts in D).

This allows us to assume that D/e contains $k_{\max}$ disjoint directed cuts for each edge $e \in E(D)$.
Consider the digraph $\dot{D}$ arising from D by subdividing all the edges, i.e., replacing them by directed paths of length two with the same orientation. Then $\dot{D}$ admits $2k_{\max} \geq k_{\max} + 1$ disjoint dicuts. Take $e_1, ..., e_l$ as a numbering of the edges of D, and denote by $\dot{D}_i, i = 0, ..., l$ the digraphs arising from D by subdivision of the edges $e_1, ..., e_i$. Since $\dot{D}_0 = D$ has at most $k_{\max}$ disjoint directed cuts and $\dot{D}_l = \dot{D}$ admits more than $k_{\max}$ disjoint directed cuts, there

is an index $i_0 \in \{0, ..., l-1\}$ such that the graph $\dot{D}_{i_0}$ admits at most $k_{\max}$ disjoint directed cuts while $\dot{D}_{i_0+1}$, with the edge e_{i_0+1} being additionally subdivided, admits a collection of at least $k_{\max}+1$ disjoint dicuts. Let $S_1, ..., S_{k_{\max}+1}$ be such a collection in $\dot{D}_{i_0+1}$. Then two of the dicuts, say S_1, S_2, have to contain the two parts of the subdivided edge e_{i_0+1}: If there was at most one such dicut, this would be a collection of $k_{\max}+1$ disjoint dicuts not containing one part p of the subdivided edge in $\dot{D}_{i_0+1}$ and thus also in $\dot{D}_{i_0+1}/p \cong \dot{D}_{i_0}$, which is a contradiction to the above.
If we now identify both edge parts of e_{i_0+1} with e_{i_0+1}, $S_1, ..., S_{k_{\max}+1}$ can be considered as a collection of $k_{\max}+1$ dicuts in $\dot{D}_{i_0}$ which have only one intersection, namely the common edge e_{i_0+1} of S_1 and S_2.
There are $k_{\max}$ disjoint dicuts in D/e_{i_0+1} or equivalently in D not containing e_{i_0+1}. Since $\dot{D}_{i_0}$ is a subdivision of D, this also yields a collection $S_{k_{\max}+2}, ..., S_{2k_{\max}+1}$ of $k_{\max}$ disjoint dicuts in $\dot{D}_{i_0}$ not containing e_{i_0+1}.
All together, $F := \{S_1, ..., S_{k_{\max}+1}, S_{k_{\max}+2}, ..., S_{2k_{\max}+1}\}$ is a collection of $2k_{\max}+1$ dicuts in $\dot{D}_{i_0}$, such that each edge $f \in E(\dot{D}_{i_0})$ is contained in at most two of them: If $f \neq e_{i_0+1}$, then f is contained in at most one of the dicuts $S_1, ..., S_{k_{\max}}, S_{k_{\max}+1}$ (since their only intersection is $S_1 \cap S_2 = \{e_{i_0+1}\}$), and obviously in at most one of the disjoint dicuts $S_{k_{\max}+2}, ..., S_{2k_{\max}+1}$. If on the other hand $f = e_{i_0+1}$, then f is only contained in S_1 and S_2 out of $S_1, ..., S_{k_{\max}}$ and in none of the remaining dicuts, which proves this statement. Application of Lemma 8.2 to the digraph $\dot{D}_{i_0}$, which has at most $k_{\max}$ disjoint dicuts, now gives $|F| \leq 2k_{\max}$, contradiction. This proves the induction step and henceforth the claim. $\square$

We can deduce some simple results about NL-2-flows and 2-colourability from this theorem.

Proposition 8.4. *Let D be a (multi-)digraph. Then the following holds:*

(i) Let there be no (not necessarily directed) odd cut S together with a set $S_1, ..., S_l$ of mutually disjoint directed cuts in D, such that $S \subseteq \bigcup_{i=1}^{l} S_i, S \cap S_i \neq \emptyset, i = 1, ..., l$ and $l = |S|$. Then D admits a NL-2-flow.

(ii) If every directed cut in D has even size, then D admits a NL-2-flow.

(iii) If D admits a NZ-4-flow and every directed cut in D has odd size, then D admits a NL-2-flow.

Proof. (i) Let B be a minimal-sized dijoin of D and let $S_1, ..., S_k \in \mathcal{S}$ be a maximal-sized collection of disjoint directed cuts in D. According to Lucchesi-Younger, $k = |B|$. Moreover, every directed cut S_i has to be covered by at least one edge $e_i \in B$. The mapping $f : \{S_1, ..., S_k\} \to B, S_i \to e_i$ is injective, since the S_j are mutually disjoint, and hence, it is also bijective. We want to show that there is a $\mathbb{Z}_2$-flow f on D with $\mathrm{supp}(f) \supseteq B$. This obviously gives us a NL-2-flow, since the digraph $D/\mathrm{supp}(f)$ has to be totally cyclic: If there was a non-trivial directed cut in $D/\mathrm{supp}(f)$, this would also be a directed cut in D not containing any edge of B, contradiction.
Such a flow exists if and only if every (not necessarily directed) cut S in D with $S \subseteq B$

has even size. So assume such a cut S was odd-sized. Let $\{S_1, ..., S_l\} = f^{-1}(S)$. Then obviously $S \subseteq \bigcup_{i=1}^{k} S_i$ and $l = |S|$, which is a contradiction to the assumption in the theorem.

(ii) Let every directed cut in D be of even size. Assume there were cuts $S, S_1, ..., S_l$ as forbidden in (i). We decompose S into the edges $S = S^+ \cup S^-$ in forward- and backward-direction. As above, the mapping $f : \{S_1, ..., S_l\} \to S, S_i \to e_i : \{e_i\} = S \cap S_i$ defines a bijection. We consider the edge subset $E(D) \supseteq E' := S^+ \cup (\bigcup f^{-1}(S^-) \setminus S^-)$. Putting $g(e) := \begin{cases} 1, \text{ if } e \in E', \\ 0, \text{ else} \end{cases}$, i.e., $g = \mathbb{1}_{E'}$ now defines a tension on D, since $g = \left(\sum_{X \in f^{-1}(S^-)} \mathbb{1}_X\right) + (\mathbb{1}_{S^+} - \mathbb{1}_{S^-})$ is a sum of tensions. Henceforth, E' admits a decomposition into edge-disjoint directed cuts, which all have even size, and therefore $|E'| \bmod 2 = 0$. But on the other hand, we have

$$\begin{aligned} |E'| \mod 2 &= (|S^+| + \sum_{e \in S^-} |f^{-1}(e) \backslash \{e\}|) \mod 2 \\ &= (|S^+| + \sum_{e \in S^-} (|f^{-1}(e)| - 1)) \mod 2 \\ &= (|S^+| + \sum_{e \in S^-} (0 - 1)) \mod 2 \\ &= |S| \mod 2 = 1, \end{aligned}$$

contradiction.

(iii) Let f be a NZ-4-flow on D. By taking all the values of f mod 2, we get a $\mathbb{Z}_2$-flow f' on D. Assume $D/\text{supp}(f')$ contained a directed (and therefore odd) cut S, i.e., it was not totally cyclic. Since f is a flow, we have

$$0 = \sum_{e \in S} f(e) \mod 4 = \sum_{e \in S} 2 \mod 4 = 2|S| \mod 4,$$

i.e., $|S| \mod 2 = 0$, contradiction.

□

The following (very simple but useful) observation is the starting point of our considerations in the following subsections:

Theorem 8.5. *Let $\vec{G}$ be a simple digraph. Then $\vec{G}$ is 2-colourable if and only if there is a bipartite subgraph H of $G := U(\vec{G})$ such that H, considered as arc set in $\vec{G}$, is a feedback arc set.*
Therefore $\vec{G}$ is especially 2-colourable if there is a spanning tree of G which is a feedback arc set.
The dual counterpart of the latter statement is the following: Every 3-edge-connected digraph D admitting a spanning tree in D which does not fully contain any directed cut in D already admits a NL-2-flow.

Proof. For the stated equivalence for planar digraphs, consider first a legal 2-colouring of $\vec{G}$. Take H to be the bipartite subgraph of G which arises from deleting all monochromatic edges. Then obviously every directed cycle has to use an edge of H, which proves the first implication.
On the other hand, if there is such a subgraph H, we can take a 2-colouring of H (in the undirected sense) and extend it to a 2-colouring of $\vec{G}$. This has to be legal indeed, because every directed cycle has to use an edge of H which has two vertices with different colours.
For the dual counterpart, assume that there is a spanning tree T of D such that there is no directed cut fully contained in T. For every chord $e \in E(D)\backslash E(T)$ of T let C_e be the unique (not necessarily directed) cycle in $T \cup \{e\}$. Take f_e to be the canonical $\mathbb{Z}_2$-flow on D which has C_e as its support. We now set $f := \sum_{e \in E(D)\backslash E(T)} f_e$ which still is a $\mathbb{Z}_2$-flow on D. The support $\mathrm{supp}(f)$ contains the edge set $E(D)\backslash E(T)$, since for every chord e there is exactly one flow evaluating to 1 in the sum above. This means that $D/\mathrm{supp}(f)$ is totally cyclic, because every directed cut in $D/\mathrm{supp}(f)$ would be a directed cut fully contained in $\overline{\mathrm{supp}(f)} \subseteq E(T)$, which is a contradiction to our assumption in the theorem. Therefore f is a NL-$\mathbb{Z}_2$-flow on D, which according to chapter 4.5 gives us a NL-2-flow on D. This proves the claim. □

We now want to look at some applications of the previous theorem. We introduce a special subclass of directed plane triangulations called *3-orientations*, which are roughly speaking orientations with outdegree three at every inner vertex. They correspond to the combinatorial concept of *Schnyder woods*, which is widely studied in the dimension theory of posets arising from planar graphs and appears in various problems of graph drawing. See [Fel04] for a survey on this topic.

Definition 8.6. Let T be a plane triangulation with outer face a_1, a_2, a_3 in clockwise order and let $\mathcal{O}$ be an orientation of the inner edges of T such that the following holds:

- For every outer vertex a_i, there are no outgoing edges.
- The out-degree of every inner vertex equals 3.

Then we call $\mathcal{O}$ a 3-orientation on T.

3-orientations with an additional edge colouring in three colours $1, 2, 3$ according to the following rules are called Schnyder woods (usually denoted by S):

- For every outer vertex a_i all the incident edges are coloured i.
- For every inner vertex $v \in V(T)\backslash\{a_1, a_2, a_3\}$ the three outgoing edges are coloured $1, 2, 3$ in clockwise order. The incoming edges between the outgoing edges of colours $i-1$ and $i+1$ are coloured i, for $i = 1, 2, 3$.

It can be shown that every 3-orientation of a plane triangulation T admits a unique Schnyder wood, which means that there is a bijection between Schnyder woods and 3-orientations (see [Bre00]). For preparation purposes, we need a couple of basic facts on Schnyder woods which we mention in the following theorem:

Theorem 8.7 (cf. [Fel04], [Ruc11], [Bre00]). *Let S be a Schnyder wood on a plane triangulation T with outer face $a_1a_2a_3$ in clockwise order. Let T_i for every $i \in \{1,2,3\}$ denote the subdigraph of S induced by the edges of colour i. Then T_i is a rooted tree in a_i which covers all vertices of T except for a_{i-1} and a_{i+1}. Furthermore, every directed cycle C in the orientation induced by S contains at least one edge of each colour.*

Proof. See e.g. [Fel04]. □

We will now use the previous statement about Schnyder woods to prove that our conjecture 7.6 from section 7 also holds true for 3-orientations:

Theorem 8.8. *Let $\vec{T}$ be a directed planar triangulation with outer face $a_1a_2a_3$ in clockwise order such that $\mathcal{O} := \vec{T} - \{a_1a_2, a_2a_3, a_1a_3\}$ is a 3-orientation. Then for every legal precolouring of the outer face, there is a 2-colouring of $\vec{T}$ extending this precolouring. This means that especially conjecture 7.6 holds for $\vec{T}$.*

Proof. We first show that $\vec{T} - \{a_2, a_3\}$ admits a 2-colouring: Let S be the unique Schnyder wood corresponding to $\mathcal{O}$. Then the tree T_1 in S is a spanning tree in $\vec{T} - \{a_2, a_3\}$ which is a feedback arc set (according to Theorem 8.7). The claim therefore follows from Theorem 8.5. For an arbitrary legal precolouring of the outer triangle we now simply extend it by the colouring of $\vec{T} - \{a_2, a_3\}$ (where we flip colours such that the colourings agree in a_1 if needed) to a colouring of the whole directed triangulation. This now still has to be a 2-colouring of $\vec{T}$, since the outer triangle is separated from the inner vertices by a directed cut. This proves the claim. □

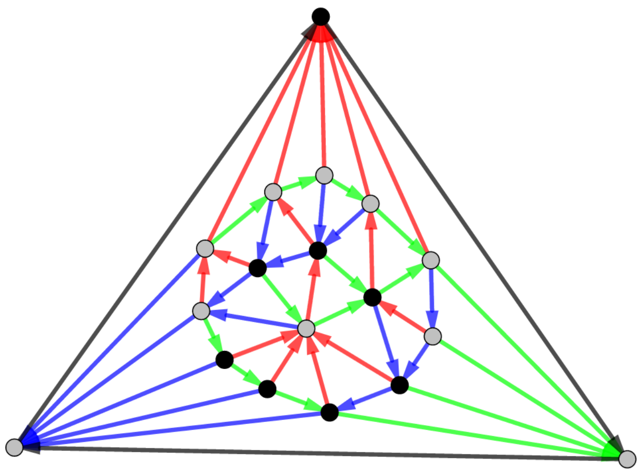


Figure 33: Two-Colouring of a 3-orientation using the red spanning tree of a corresponding Schnyder wood.

8.2 Forcing Sets and Essential Cycles

In the following, we take a look at more general kinds of (not necessarily planar) digraphs and try to derive sufficient conditions for 2-colourability utilizing Theorem 8.5. We use the observations made there as a motivation for finding and eliminating redundancies appearing during the colouring task. This leads us to the notions of *forcing sets* of directed cycles resp. dicuts in digraphs and of *essential cycles* and *essential cuts* in the case of planar digraphs. For this purpose, we start with the following definition which is used in the so-called Rado-Hall-condition from matroid theory:

Definition 8.9 (Transversal). Let $(S, \mathcal{A})$, $\mathcal{A} = \{A_1, ..., A_r\} \subseteq 2^S$ be a set system with ground set S and let $T \subseteq S$ be some subset.
T is called a *partial transversal* of $(S, \mathcal{A})$, if there is an injection (also called a matching) $\phi : T \to \{1, ..., r\}$ such that $x \in A_{\phi(x)}, \forall x \in T$.
T is called a *transversal*, if ϕ is additionally bijective.

Transversals are a widely used concept, particularly in the theory of so-called *transversal matroids*.

Theorem 8.10 (Rado, [Rad71]). *Let $(S, \mathcal{A})$, $\mathcal{A} = \{A_1, ..., A_r\} \subseteq 2^S$ be a set system with the ground set S and M a matroid with $E(M) = S$ and associated rank function $r : 2^S \to \mathbb{N}_0$. There is an independent (with respect to M) transversal $T \subseteq S$ of $(S, \mathcal{A})$, iff*

$$\forall K \subseteq \{1, ..., r\} : r\left(\bigcup_{i \in K} A_i\right) \geq |K|.$$

Proof. The necessity of the conditions can be seen as follows: If $T \subseteq S$ is an independent transversal with associated matching $\phi : T \to \{1, ..., r\}$, then for every $K \subseteq \{1, ..., r\}$, $\phi^{-1}(K)$ is a subset of T of size $|K|$, and is thus independent. Moreover, we have

$$\phi^{-1}(K) = \bigcup_{i \in K} \underbrace{\phi^{-1}(\{i\})}_{\subseteq A_i} \subseteq \bigcup_{i \in K} A_i$$

proving $r\left(\bigcup_{i \in K} A_i\right) \geq |K|$ as claimed.
For the reverse implication, cf. e.g. [Rad71]. □

This characterization, applied to Theorem 8.5, yields the following result which may be used to construct 2-colourings of certain (planar) digraphs. The idea of the proof is to find an independent transversal with respect to the graphic matroid of the underlying graph which covers each directed cycle out of a given set of such at least once.

Theorem 8.11. *Let $\vec{G}$ be a digraph, and let $\mathcal{C}$ be a fixed set of directed cycles in $\vec{G}$. Assume that for every collection $C_1, ..., C_k \in \mathcal{C}, k \geq 1$ of mutually distinct cycles, the following holds:*

$$|V(C_1) \cup ... \cup V(C_k)| \geq k + 1.$$

Then there is a forest in $G := U(\vec{G})$ covering each cycle in $\mathcal{C}$ with at least one edge.

Proof. Let $G := U(\vec{G})$, $S := E(G)$ and denote by $M := M(G)$ the graphic matroid with ground set S. Then the associated rank function r_M of M is, as is well known, given by: $r_M(X) = |V(G[X])| - c(G[X]), \forall X \subseteq S$. Let now $K = \{C_1, ..., C_k\} \subseteq \mathcal{C}$ be an arbitrary subset. For the first case, assume that the subgraph $C_1 \cup C_2 \cup ... \cup C_k$ of $G := U(\vec{G})$ is connected. According to the inequality above, we obtain

$$r_M\left(\bigcup_{C \in K} C\right) = |V(C_1) \cup ... \cup V(C_k)| - 1 \geq k.$$

In the general case, where $C_1 \cup ... \cup C_k$ consists of l connected components with each $k_1, ..., k_l$ cycles, we now conclude from the above that

$$r_M\left(\bigcup_{C \in K} C\right) = |V(C_1) \cup ... \cup V(C_k)| - l \geq (k_1 + 1) + ... + (k_l + 1) - l = k,$$

and thus, the Rado-Hall-conditions from Theorem 8.10 are satisfied. Consequently, there is a transversal T of $(S, \mathcal{C})$, which is an independent set with respect to M and thus the edge set of a forest $H \subseteq G$. In particular, every $C \in \mathcal{C}$ contains an element of T, since T is a transversal. This proves the claim. $\square$

What is the concrete use of the above theorem? First of all, since forests in the graph G are bipartite subgraphs, they (as in the case of 3-orientations) admit a 2-colouring in the graph sense, which, arbitrarily extended to the whole vertex set $V(G)$, does not admit any monochromatic directed cycles inside $\mathcal{C}$: each cycle in this collection is covered by at least one edge of the bipartite subgraph, which is therefore not monochromatic.
A very simple idea in order to guarantee 2-colourings would be to put $\mathcal{C}$ as the full set containing all the directed cycles in $\vec{G}$, in which case any digraph $\vec{G}$ fulfilling the inequalities of Theorem 8.11 is 2-colourable. But those conditions can only possibly be fulfilled by digraphs $\vec{G}$ containing at most $|V(\vec{G})| - 1$ directed cycles (if we take $\{C_1, ..., C_k\}$ to be a list of all directed cycles). Unfortunately, in general, even planar digraphs may admit exponentially many different directed cycles, as is illustrated by the following example:

Example 8.12. Let $n \in \mathbb{N}$ be an arbitrary natural number. Define the simple planar digraph $\vec{G}_n$ as follows: We start with a planarly embedded directed cycle $\vec{C}_n$, but replace each edge on $\vec{C}_n$ by a pair of parallel directed paths of length two:

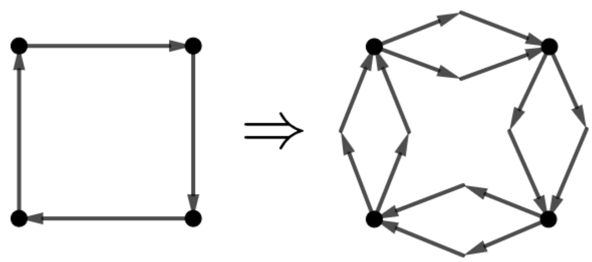


Figure 34: Construction of the digraph $\vec{G}_4$.

This digraph is planar and simple, has $3n$ vertices but exactly 2^n directed cycles: The directed cycles in $\vec{G}_n$ are subdivisions of the $\vec{C}_n$, where at each edge, we may independently choose which of the two directed replacement-paths of length two to use. Thus, the number of directed cycles in $\vec{G}_n$ is $(2^{1/3})^{|V(\vec{G}_n)|}, \forall n \in \mathbb{N}$ and henceforth exponential.

This example shows that the planar digraphs satisfying the above inequalities in the case that $\mathcal{C}$ contains all the directed cycles will only be satisfied by small and sparse digraphs (with respect to the directed cycles). If we want to get more general results, we should try to avoid too much redundancy, i.e., look for proper subsets $\mathcal{C}$ of directed cycles in digraphs $\vec{G}$ such that every 2-colouring of the vertices of $\vec{G}$ which does not contain any monochromatic directed cycles in $\mathcal{C}$ already is a legal 2-colouring. We call such subsets of directed cycles *forcing*:

Definition 8.13. Let $\vec{G}$ be a digraph and $k \in \mathbb{N}$. Let $\mathcal{C}$ be a set of directed cycles in $\vec{G}$. We call $\mathcal{C}$ *k-forcing*, if every k-vertex-colouring $c : V(\vec{G}) \to \{0, ..., k-1\}$ containing a monochromatic directed cycle also contains a monochromatic directed cycle in $\mathcal{C}$.

Obviously, given a digraph, we would like to find preferably small (minimum) forcing sets of directed cycles in order to make the task of finding a legal digraph colouring easier. The following result, using a probabilistic argument, is another example which demonstrates why the concept of forcing sets helps to improve many results which might seem weak at first sight:

Theorem 8.14. *Let D be a digraph without loops. Let $\vec{\mathcal{C}}$ be a k-forcing set of directed cycles in D, $k \in \mathbb{N}$, and denote by $L(C)$ the length of a cycle $C \in \vec{\mathcal{C}}$. Let*

$$q_D := \min\{q \in \mathbb{N} | \sum_{C \in \vec{\mathcal{C}}} q^{-L(C)+1} < 1\}.$$

Then $q_D \leq k \Rightarrow \vec{\chi}(D) \leq q_D \leq k$.

Proof. Assume that $q_D \leq k$. We consider a uniform distribution p on the discrete measure space (Ω, p) with $\Omega := \{1, ..., q_D\}^{V(D)}$. For each cycle $C \in \vec{\mathcal{C}}$, let $A_C \subseteq \Omega$ be the event of a monochromatic colouring of C. Obviously, $p(A_C) = q_D^{-L(C)+1}$. We therefore have

$$p(\bigcup_{C \in \vec{\mathcal{C}}} A_C) \leq \sum_{C \in \vec{\mathcal{C}}} q_D^{-L(C)+1} < 1.$$

Hence, $\bigcup_{C \in \vec{\mathcal{C}}} A_C \neq \Omega$ and we can find a legal q_D-colouring of D. $\square$

Following up the definition of forcing sets of cycles, we can replace the notion of colourings on graphs by the one of Neumann-Lara-flows and consider dicuts instead of directed cycles in this dual setting. Given a digraph D, recall that a NL-k-flow on D is an integer flow on D ranging in $\{0, \pm 1, ..., \pm(k-1)\}$, such that $D/\mathrm{supp}(f)$ is totally cyclic, i.e., $\mathrm{supp}(f)$ covers each directed cut with at least on edge. Analogously, instead of requiring the support of a NL-k-flow to be a dijoin, we can look for one which only covers a given set of certain dicuts. If this is equivalent, we call the appropriate dicut-set a k-forcing set of dicuts:

Definition 8.15. Let D be a digraph and $\mathcal{S}$ a set of dicuts in D. For $k \in \mathbb{N}$, we call $\mathcal{S}$ k-*forcing*, if for every integer flow on D ranging in $\{0, \pm 1, ..., \pm(k-1)\}$ so that $\overline{\text{supp}(f)}$ contains a dicut, it also contains a dicut out of $\mathcal{S}$.

Obviously, when dealing with the 2-Colour-Conjecture and 2-Flows in various kinds of 3-edge-connected digraphs, we are mainly interested in 2-forcing sets of directed cycles resp. dicuts.
The following definitions introduce a very special kind of non-trivial forcings sets in planar digraphs in the primal and arbitrary digraphs in the dual setting:

Definition 8.16.

- Let $\vec{G}$ be a simple planarly embedded digraph and let C be a directed cycle in $\vec{G}$. We call C an *essential cycle*, if the subdigraph of $\vec{G}$ induced by all the vertices and edges of $\vec{G}$ on or in the inside of C admits a 2-colouring such that C is the only monochromatic directed cycle.
  Furthermore, given a directed cycle C in $\vec{G}$, we denote the subdigraph described above by $\vec{G}_{\text{int}}[C]$.
- Let D be a digraph and $v_0 \in V(D)$ a fixed special vertex. A non-empty directed cut $S = U(D)[X, \overline{X}]$ in D, where $v_0 \in X$, is called an *essential cut* in D with respect to v_0, if the digraph $D[\overline{X}]$ admits a NL-2-flow.

The following theorem justifies the term "essential" in the definitions above.

Theorem 8.17.

- *Let $\vec{G}$ be a simple planarly embedded digraph and let $c : V(\vec{G}) \to \{0, 1\}$ be an arbitrary two-colouring of the vertices. Then the following are equivalent:*

  (i) c is a legal digraph colouring.

  (ii) There are no monochromatic essential cycles in $\vec{G}$.

  In other words, $\mathcal{C}_{\text{ess}} := \{C | C$ is an essential cycle $\}$ is 2-forcing.

- *Let D be a digraph with special fixed vertex $v_0 \in V(D)$ and let $f : E(D) \to \mathbb{Z}_2$ be an arbitrary $\mathbb{Z}_2$-flow on D. Then the following are equivalent:*

  (i) f is a NL-2-flow.

  (ii) $\text{supp}(f)$ covers each essential cut with respect to v_0 in D with at least one edge.

  In other words, $\mathcal{S}_{\text{ess}} := \{S | S$ is an essential cut in D with respect to $v_0\}$ is 2-forcing.

Proof.
- (i) $\Rightarrow$ (ii) is trivial. For the converse, assume we are given a two-colouring c without monochromatic essential cycles. If there was a monochromatic directed cycle C, we could choose such a cycle minimal with respect to the number of edges inside of or on it. But this means that the two-colouring of the subdigraph of $\vec{G}$ induced by c does only contain C as a monochromatic directed cycle, and hence, C has to be essential, contradiction.

- (i) $\Rightarrow$ (ii) is trivial. For the converse, assume we are given a $\mathbb{Z}_2$-flow f such that $\operatorname{supp}(f)$ covers all essential cuts in D. If there was an uncovered dicut $S = U(D)[Y, \overline{Y}]$, we could choose such a cut minimal with respect to $|\overline{Y}|$. But since all the essential cuts are covered by $\operatorname{supp}(f)$, this means that S is not essential and thus, the restriction of f to $D[\overline{Y}]$ (which is still a $\mathbb{Z}_2$-flow, since f is 0 on S) is no NL-$\mathbb{Z}_2$-flow. Consequently, there has to be an uncovered directed cut $S' = U(D)[Z, \overline{Y} \setminus Z], \emptyset \neq Z \subset \overline{Y}$ in $D[\overline{Y}]$, and hence, either $U(D)[Y \cup Z, \overline{Y \cup Z}]$ or $U(D)[\overline{Z}, Z]$ induces a directed cut in D, which is contained in $S \cup S'$ and thus not covered by $\operatorname{supp}(f)$. But since $|\overline{Y} \setminus Z|, |Z| < |\overline{Y}|$ this is a contradiction to the minimality condition above, as desired. Therefore f is indeed a NL-$\mathbb{Z}_2$-flow, and we are done.

$\square$

Remark 8.18. Let $\vec{G}$ be a simple planarly embedded digraph.

(i) Essential cycles do not have chords.

(ii) Every facial directed cycle of $\vec{G}$ and more generally every directed cycle in $\vec{G}$ without further directed cycles in $\vec{G}_{\text{int}}[C]$ is essential.

(iii) If C is an interior-inclusion-minimal essential cycle in $\vec{G}$, i.e., there are no other essential cycles contained in $\vec{G}_{\text{int}}[C]$, then C is of the type described above, i.e., C is the only directed cycle in $\vec{G}_{\text{int}}[C]$.

Proof. (i) A directed cycle C with a chord will always give rise to at least two monochromatic directed cycles in $\vec{G}_{\text{int}}[C]$ if the vertices on $\vec{C}$ are all coloured the same.

(ii) Given such a cycle C, define a colouring c on the vertices of $\vec{G}_{\text{int}}$ to be 0 everywhere. This will only give rise to one monochromatic cycle, namely C.

(iii) If C is an interior-inclusion-minimal essential cycle in $\vec{G}$, take a colouring of the vertices of $\vec{G}_{\text{int}}[C]$ to be 0 at every vertex. For each edge $e \in E(C)$, consider this as a two-colouring of the digraph $\vec{G}_{\text{int}}[C] - e$. If this colouring was not legal, according to 8.17, there is an essential cycle in $\vec{G}_{\text{int}}[C] - e$ which then is also essential with respect to $\vec{G}$ but is different from C, and hence, this would contradict the interior-inclusion minimality of C. Therefore, $\vec{G}_{\text{int}}[C] - e$ is acyclic for each edge $e \in E(C)$. But this means that C is the only directed cycle in $\vec{G}_{\text{int}}[C]$, and the claim follows.

$\square$

Theorem 8.19. *Let $\vec{G}$ be a simple planarly embedded digraph, which is an edge-minimal (and under the assumption of edge minimality vertex-minimal) counterexample to the 2-colour-conjecture. Then $\vec{G}$ is connected, and each edge $e \in E(\vec{G})$ is contained in an essential cycle. Furthermore, the interior-inclusion-minimal essential cycles are exactly the directed facial cycles of $\vec{G}$.*

Proof. The connectivity is obvious. If there was an edge $e \in E(\vec{G})$ not covered by any essential cycle in $\vec{G}$, we could look at the planarly embedded digraph $\vec{G} - e$, which has less edges than the minimal counterexample $\vec{G}$ and therefore admits a legal two-colouring.

Take the same two-colouring of the vertices of $\vec{G}$. This can't be legal, i.e., there is a monochromatic essential cycle in $\vec{G}$. But this cycle does not use e, i.e., it is a monochromatic directed cycle in $\vec{G} - e$, contradiction.
If C is an interior-inclusion-minimal cycle, according to the remark above, $\vec{G}_{\text{int}}[C]$ does only contain $\vec{C}$ as directed cycle. Hence, C does not admit inner vertices: If this was the case, we could delete these vertices and would end up with a planarly embedded digraph $\vec{G}'$, such that any extension of a legal two-colouring of $\vec{G}'$ to a two-colouring of $\vec{G}$ is legal too (since no directed cycle in $\vec{G}$ uses a vertex interior to C), which again contradicts the assumed minimality of $\vec{G}$. $\square$

In the following, we want to sketch how to use the colouring property of essential cycles in planar digraphs in order to find further reductions for the 2-colouring task:

Definition 8.20. Let $\vec{G}$ be a simple planarly embedded digraph, and let C be an essential cycle in $\vec{G}$ with at least one inner vertex. We call C *reducible*, if there is a two-colouring $c : V(G_{\text{int}}[C]) \to \{0,1\}$ with $c(v) = 0, v \in V(C)$ such that C is the only monochromatic directed cycle in $\vec{G}_{\text{int}}[C]$ and additionally, there is no non-trivial directed trail in $\vec{G}_{\text{int}}[C]$ starting and ending with (possibly identical) vertices on C, such that all the remaining vertices of the tour are coloured 1 in c.

The following observation justifies the term reducible:

Theorem 8.21. *Let $\vec{G}$ be a simple planarly embedded digraph and let C be a reducible cycle in $\vec{G}$ with an appropriate colouring c_{int} of $\vec{G}_{\text{int}}[C]$ as in the above definition. Let $\vec{G}'$ be the simple planarly embedded digraph arising from $\vec{G}$ by deleting the inner vertices of C.*

- *If $c' : V(\vec{G}') \to \{0,1\}$ is some (not necessarily legal) 2-vertex-colouring of $\vec{G}'$, then the 2-vertex-colouring c of $\vec{G}$ defined by*

$$c(v) := \begin{cases} c'(v), & \textit{if } v \in V(\vec{G}') \\ c_{\text{int}}(v), & \textit{if } v \in V(\vec{G}) \backslash V(\vec{G}') \end{cases}, \forall v \in V(\vec{G}),$$

  admits the same monochromatic directed cycles as c'. Thus, $\vec{G}$ is 2-colourable iff $\vec{G}'$ is 2-colourable.

- *The essential cycles in $\vec{G}'$ are exactly those of $\vec{G}$ only using edges of $\vec{G}'$.*
- *The reducible cycles of $\vec{G}'$ are those of $\vec{G}$ contained in $\vec{G}'$.*

Proof. First of all, notice that since C is reducible and thus essential, according to the above remarks, C does not admit chords. Therefore, in $\vec{G}'$, there are no edges interior to C, i.e., C is a facial cycle.
Let now $c' : V(\vec{G}') \to \{0,1\}$ be a 2-vertex-colouring of $\vec{G}'$ (not necessarily legal). Let c be the colouring of $\vec{G}$ defined above. Assume for contrary there was a monochromatic directed cycle with respect to c which is not yet contained in $\vec{G}'$. This cycle would have to use vertices

inside of and on C, since c_{int} does not admit monochromatic directed cycles in $\vec{G}_{\text{int}}[C]$ other than C. But this especially means that there is a monochromatic directed non-trivial trail in $\vec{G}_{\text{int}}[C]$ containing two vertices on C as its end vertices. If this was coloured 0, in the colouring c_{int} of $\vec{G}_{\text{int}}[C]$ this would produce a monochromatic directed cycle different from C, which contradicts the definition. Hence, this path has to be coloured 1. But again, this is a contradiction to the additional requirement in the definition of a reducible cycle in $\vec{G}$. This proves the first claim.

For the latter two claims, it is clear that for every essential cycle C' in $\vec{G}$ with $E(C') \subseteq E(\vec{G}')$, $\vec{G}'_{\text{int}}[C'] = \vec{G}_{\text{int}}[C'] - (V(\vec{G}_{\text{int}}[C]) \setminus V(C))$, and thus a 2-vertex-colouring of $\vec{G}_{\text{int}}[C']$ so that C' is the unique monochromatic directed cycle, induces a 2-vertex-colouring of $\vec{G}'_{\text{int}}[C']$ where C' is the unique directed monochromatic cycle again, showing that C' indeed is an essential cycle with respect to G'. Moreover, if C' was reducible with respect to $\vec{G}$, it also is with respect to $\vec{G}'$, since any non-trivial directed trail in $\vec{G}'_{\text{int}}[C']$ starting and ending in a 0-vertex on C and admitting colour 1 at each other vertex would give rise to one in $\vec{G}_{\text{int}}[C']$, contradicting the reducibility of C' in $\vec{G}$.
On the other hand, given any essential cycle C' of $\vec{G}'$, this is a directed cycle in $\vec{G}$. We have to prove that $\vec{G}_{\text{int}}[C']$ has a 2-colouring so that C' is the only contained monochromatic cycle, while we already now that $\vec{G}'_{\text{int}}[C'] = \vec{G}_{\text{int}}[C'] - (V(\vec{G}_{\text{int}}[C]) \setminus V(C))$ does admit such a colouring.
There are two possibilities: Either, the interior of C' is disjoint from that of C, in which case $\vec{G}'_{\text{int}}[C'] = \vec{G}_{\text{int}}[C']$ admits a 2-colouring as required, and it is also clear that the reducibility of C' in $\vec{G}$ resp. $\vec{G}'$ are equivalent.
Else (due to planarity and since C is a facial cycle of $\vec{G}'$), C is fully contained in $\vec{G}_{\text{int}}[C']$. Now, applying the first claim of the theorem proven above to the planarly embedded graphs $\vec{G}_{\text{int}}[C'], \vec{G}'_{\text{int}}[C']$ instead of $\vec{G}$ and $\vec{G}'$ yields a 2-vertex-colouring of $\vec{G}_{\text{int}}[C]$ which admits the same monochromatic directed cycles as the colouring of $\vec{G}'_{\text{int}}[C']$, i.e., only C', and thus, C' again is essential with respect to $\vec{G}$. Assume now for the reverse implication of the third assertion, that additionally, C is reducible with respect to $\vec{G}'$. Then in the above colouring of $\vec{G}'_{\text{int}}[C']$, we may assume that there are no non-trivial directed trails in $\vec{G}'_{\text{int}}[C'] = \vec{G}_{\text{int}}[C'] - (V(\vec{G}_{\text{int}}[C]) \setminus V(C))$ connecting two 0-vertices on C by 1-vertices. We claim that there is also none within $\vec{G}_{\text{int}}[C']$. Assume by way of contradiction there was one, which then has to use at least one vertex interior to C. Then there is a subtrail in $G_{\text{int}}[C]$, starting and ending with vertices on C (which are thus coloured 0 in c_{int}) so that all the traversed vertices in between are interior to C and coloured 1. This now finally contradicts the initial assumption that c_{int} was chosen as a valid colouring with respect to the reducibility of C. Thus, we are also done in this case, which finally proves the latter two statements. $\square$

Corollary 8.22. *Let $\vec{G}$ be a simple planarly embedded digraph, which is an edge-minimal (and under the assumption of edge minimality vertex-minimal) counterexample to the 2-colour-conjecture. Then every essential cycle of $\vec{G}$ is non-reducible.*

The following is (independently of forcing sets and essential cycles) a useful statement in terms of 2-flows, guaranteeing the existence of spanning even edge subsets in 3-edge-connected graphs without small components:

Theorem 8.23 (Jackson and Yoshimoto, cf. [JY09])**.** *Every 3-edge-connected graph on at least* 5 *vertices admits a spanning even edge subset* E *so that every component of* $G[E]$ *admits at least* 5 *vertices.*

Corollary 8.24. *Let* G *be a simple planarly embedded graph. Then* G *admits a 2-vertex colouring* $c : V(G) \to \{0, 1\}$ *without monochromatic facial cycles.*

Proof. Since G is simple, (one of its) planar dual graphs G^* is 3-edge-connected. We thus need to show that there is a $\mathbb{Z}_2$-flow on G^* whose support covers each vertex at least once, which is immediate from the above theorem if using a spanning even edge set in G^* as the support of a $\mathbb{Z}_2$-flow. □

Corollary 8.25. *Let* $\vec{G}$ *be a simple planarly embedded digraph, so that all essential cycles in* $\vec{G}$ *with at least one inner vertex are reducible. Then* $\vec{G}$ *admits a 2-colouring.*

Proof. We prove the theorem by induction on the number of vertices of $\vec{G}$ and consider the situation where all essential cycles of $\vec{G}$ are facial as the base case: In that case, according to theorem 8.17, $\vec{G}$ is 2-colourable if we can find a 2-vertex-colouring without monochromatic facial cycles, which is provided by Corollary 8.24.
Assume now for the inductive step that the conjecture holds for all simple planar digraphs with up to $n - 1$ vertices, $n \geq 1$ and let $\vec{G}$ be a simple planarly embedded digraph on n vertices, so that all essential cycles with at least one inner vertex are reducible. We may assume that there is at least one such cycle C since else, we are in the base case. Then consider the planarly embedded digraph $\vec{G}'$ defined according to theorem 8.21 with respect to C, which admits at most $n-1$ vertices. According to 8.21, the essential cycles in $\vec{G}'$ with at least one inner vertex are exactly those of $\vec{G}$ contained in $\vec{G}'$. Thus, they are all reducible with respect to $\vec{G}$ and hence also with respect to $\vec{G}'$. We therefore may apply the inductive assumption to $\vec{G}'$, proving that it and thus also $\vec{G}$ is indeed 2-colourable as claimed, proving the induction hypothesis. Finally, the principle of induction yields the claim. □

The following illustrates another application of the notion of *forcing sets* of cycles.
As explained in paragraphs 4.1 and 5.1, the *degeneracy* of a graph appears in upper bounds on the chromatic number and vertex arboricity of the respective graph, by applying an improved version of the greedy colouring algorithm. In the following, we will use it in order to derive upper bounds on the dichromatic number of a given digraph.

Theorem 8.26. *Let* D *be a digraph with* $\vec{\chi}(D) > k$ *and* $\mathcal{C}$ *a* k*-forcing set of directed cycles in* D*. Let* $G[D, \mathcal{C}]$ *be the simple graph containing the directed cycles in* $\mathcal{C}$ *as vertices so that two cycles* $C_1 \neq C_2 \in \mathcal{C}$ *are adjacent if and only if* $|V(C_1) \cap V(C_2)| = 1$*. Then*

$$d(G[D, \mathcal{C}]) \geq \mathrm{digir}(D)(k - 1),$$

where $\mathrm{digir}(D)$ *denotes the digirth of* D *as defined in section 5.2.*

Proof. Since $\mathcal{C}$ is k-forcing, there is an inclusion-minimal subset $\mathcal{C}' \subseteq \mathcal{C}$, which is also k-forcing. This gives rise to a subgraph H of $G[D, \mathcal{C}]$ containing exactly the cycles in $\mathcal{C}'$ as vertices. We show that $\delta(H) \geq \mathrm{digir}(D)(k-1)$, proving the assertion: Assume there was a directed cycle $C \in \mathcal{C}'$ of degree less than $\mathrm{digir}(D)(k-1)$. Since $|V(C)| \geq \mathrm{digir}(D)$, according to the pigeon-hole principle, there is a vertex $v \in V(C)$, such that there are less than $k-1$ directed cycles $C' \in \mathcal{C}$ with $V(C) \cap V(C') = \{v\}$.
On the other hand, since $\mathcal{C}' \subseteq \mathcal{C}$ was chosen inclusion-minimal, the set $\mathcal{C}' \backslash \{C\}$ is not k-forcing, i.e., there is a vertex-colouring $c : V(D) \to \{0, ..., k-1\}$ of D containing a monochromatic directed cycle, but none of the cycles in $\mathcal{C}' \backslash \{C\}$ is monochromatic at the same time, i.e., C is the only monochromatic directed cycle in $\mathcal{C}'$. Assume without loss of generality that $c(x) = 0, x \in V(C)$. For each $i \in \{1, ..., k-1\}$, consider the vertex-colouring $c_i : V(D) \to \{0, ..., k-1\}$ arising from c by changing the colour at v to i. In c_i, C is not monochromatic and thus, at least one cycle $C_i \in \mathcal{C}' \backslash \{C\}$ has to be monochromatic (since $\vec{\chi}(D) > k$), and since only the colour of v has been changed, it needs to have v as the only common intersection with C, i.e., $V(C) \backslash V(C_i) = \{v\}, i = 1, ..., k-1$. Since the vertices in $V(C_i) \backslash \{v\}$ are coloured i in c, the cycles C_i have to be pairwise distinct. This finally contradicts our initial assumption about C resp. v, which proves the claim. □

Corollary 8.27. *Let D be a digraph, and denote by $\vec{\mathcal{C}}$ the set of all directed cycles in D. Let $G_D := G[D, \vec{\mathcal{C}}]$ be the simple graph as defined in Theorem 8.26. Then*

$$\vec{\chi}(D) \leq 2 + \left\lfloor \frac{d(G_D)}{\mathrm{digir}(D)} \right\rfloor .$$

Proof. This is an immediate consequence of the inequality proven in Theorem 8.26, setting $k := \vec{\chi}(D) - 1$ and since $\vec{\mathcal{C}}$ is obviously l-forcing for all $l \in \mathbb{N}$. □

Concerning 2-colourings, the estimate above immediately yields the following

Corollary 8.28. *Let D be a simple digraph and assume that the simple graph G_D as defined in Corollary 8.27 is $(\mathrm{digir}(D) - 1)$-degenerate. Then D admits a legal 2-colouring.*

8.3 Applications to Minimal Counterexamples

We now want to relate the concept of finding bipartite subgraphs, which are feedback arc sets as introduced above, to our considerations and results for minimal counterexamples which we made in section 7. We thereby know that those minimal counterexamples cannot be too "far away" from the case where every triangular face is directed, since in the dual graph the subgraph induced by the non-directed triangular faces is a forest. Obviously, our method for proving the correctness of the conjecture for the extremal case in Theorem 7.3 can't simply be applied in the same way. Although we can colour the subdigraph which contains exactly the edges on directed triangular faces (and thereby already colour all the vertices, see Theorem 7.8), in general we cannot be sure that this does not produce monochromatic directed cycles if we add the edges between undirected triangular faces. The following theorem is an attempt to make this transition from the subdigraph only containing the directed triangles to the whole planar triangulation possible by not only requiring a 2-colouring of this subdigraph but a strengthening of the equivalent formulation of two-colourings in terms of bipartite subgraphs, cf. 8.5.

Theorem 8.29. *Let $\vec{T}$ with outer triangle $a_1a_2a_3$ be a directed planar triangulation which is not reducible by (R1)-(R3). Let L denote the set of inner edges in $\vec{T}$ where one bounding face is directed and the other is neither directed nor transitive. Let K denote the set of edges not bounding any directed triangle. Then any feedback arc set $H \subseteq E(\vec{T}) \setminus (L \cup K)$ for $\vec{T} - K$ is already a feedback arc set for $\vec{T}$.*

Proof. Assume there was a directed cycle in $\vec{T}$ disjoint from H. Choose a non-trivial closed directed edge sequence C using as few edges from K as possible. If there was an $e \in C \cap K \neq \emptyset$, since T has no contractible edge, due to Theorem 7.8 and Proposition 7.24 we can replace e by the parallel path translation $\vec{P}(e, f) \subseteq L \subseteq \overline{H}$, where f is a transitive triangle containing e. This gives rise to a non-trivial directed edge sequence disjoint from H containing less edges out of K than C, contradicting the assumed minimality. □

This theorem raises the following question: Given a connected region of directed triangles (the underlying graph is a k-triangulation with even degree at every inner vertex), and a fixed subset A of the edges on the boundary of the region, when does there exist a bipartite subgraph H of this region which uses only inner edges and edges in A, such that the edges in H form a feedback arc set for the whole digraph?
If $A = \emptyset$, this question can only be answered positively if the boundary cycle of the region is not directed. The following theorem, which we prepare with an elementary result about graph colourings, states that the converse under some additional assumptions indeed holds to be true.

Proposition 8.30. *Let $G = (V, E)$ be a cubic graph which has a vertex $v_0 \in V$ such that $\hat{G} := G - v_0$ is bipartite. Then G itself is bipartite.*

Proof. Let V_1, V_2 denote the colour classes of $\hat{G}$. Then

$$\sum_{v \in V_1} d_{\hat{G}}(v) = \sum_{v \in V_2} d_{\hat{G}}(v).$$

Let $k \in \{0, 1, 2, 3\}$ denote the number of neighbours of v_0 in V_1. Then

$$\begin{aligned}\sum_{v \in V_1} d_G(v) &= k + \sum_{v \in V_1} d_{\hat{G}}(v) \\ \sum_{v \in V_2} d_G(v) &= (3 - k) + \sum_{v \in V_2} d_{\hat{G}}(v)\end{aligned}$$

Subtracting the second equation from the first, taking the 3-regularity of G into account, yields

$$3 \mid \left(\sum_{v \in V_1} d_G(v) - \sum_{v \in V_2} d_G(v) \right) = 2k - 3.$$

Hence k is either 0 or 3 and the claim follows. □

Proposition 8.31. *Let $G = (V, E)$ be a graph which has a vertex $v_0 \in V$ such that $G - v_0$ is bipartite. Assume furthermore that $\deg(v_0) \geq 3$ and $\deg(v) = 3$ for all $v \in V \setminus \{v_0\}$. Then G has a NZ-3-flow.*

Proof. Without loss of generality, assume G is connected. We proceed by induction on $k \geq 3$. If $k = 3$, then G is bipartite with colour classes V_1 and V_2 by Proposition 8.30. Direct all edges from V_1 to V_2 and choose a perfect matching M of G. Sending a flow of -2 along M and of 1 along all the other edges yields a NZ-3-flow founding the induction.

Thus assume $k \geq 4$. Then v_0 must have two edges, say e_1, e_2, which are adjacent to the same colour class, say V_1. We split these two edges off, by first removing e_1, e_2 and adding a new vertex w to V_2 which we make adjacent to the endpoints of e_1 and e_2 in V_1 and to v_0. The resulting graph $\tilde{G}$ satisfies the assumption of our proposition and furthermore $d_{\tilde{G}}(v_0) = k - 1$ as well as $G = \tilde{G}/v_0 w$. Hence the claim follows from the inductive assumption. $\square$

Corollary 8.32. *Let R be a k-triangulated plane graph with even degree at every inner vertex. Then R is 3-colourable in the graph sense.*

Proof. Let v_0 be the vertex corresponding to the outer face, which has degree $\deg(v_0) \geq 3$. Consider a dual graph $G := R^*$ of R. Since every facial cycle in $G - v_0$ corresponds to the edges incident to an inner vertex of R, such a cycle always has even length and hence, $G - v_0$ is bipartite. Therefore G admits a NZ-3-flow, i.e., $R = G^*$ is 3-colourable. $\square$

Theorem 8.33. *Let R be a k-triangulated planar graph with an orientation of the edges such that every facial triangle is directed and such that the outer cycle on the boundary of R is not directed. Furthermore, let there be no directed cycle C using only outer vertices without consecutive inner edges such that the maximal directed subpaths of C on the boundary of R have length $2 \mod 3$. Then there is a bipartite subgraph H of R using only inner edges, such that the edges of H form a feedback arc set of the orientation.*

Proof. First of all, we notice that the triangulation R has even degree for every inner vertex and is therefore (according to Proposition 8.32) 3-colourable. Assume we are given such a colouring with colours $1, 2, 3$. We now consider the bipartite subgraph H' of R which contains all edges of R incident to a 1-coloured vertex, and define H by $E(H) := E(H') - E(C)$. H now is a bipartite subgraph of R only containing inner edges. We show that $E(H)$ has to be a feedback arc set of the orientation of R, i.e., the orientation on the graph $R - H$ is acyclic: First of all, every inner vertex of R has to be a source or a sink in $R - H$, since for every such vertex v, the colours of the neighbouring vertices appear alternately, i.e., the set of outgoing or the set of incoming edges of v is completely contained in H and deleted in $R - H$. Directed cycles in the orientation of $R - H$ therefore only use vertices from the outer face. Assume there was a directed cycle C not covered by H which uses only outer vertices. Since we have a 3-colouring of R and since the edges incident to vertices in R are alternately incoming and outgoing in the orientation, without loss of generality, C has to use the colours $1, 2, 3, 1, 2, 3, ..., 1, 2, 3, 1$ in clockwise direction and is oriented clockwise. Hence, inner edges of R on C cannot be consecutive, since otherwise at least one of them would be contained in H. Moreover, the tails of inner edges on C are coloured 2, while the heads are coloured 3. This implies that the maximal directed subpaths of C on the boundary between a pair of inner edges have length $2 \mod 3$, which again is a contradiction. This proves the claim. $\square$

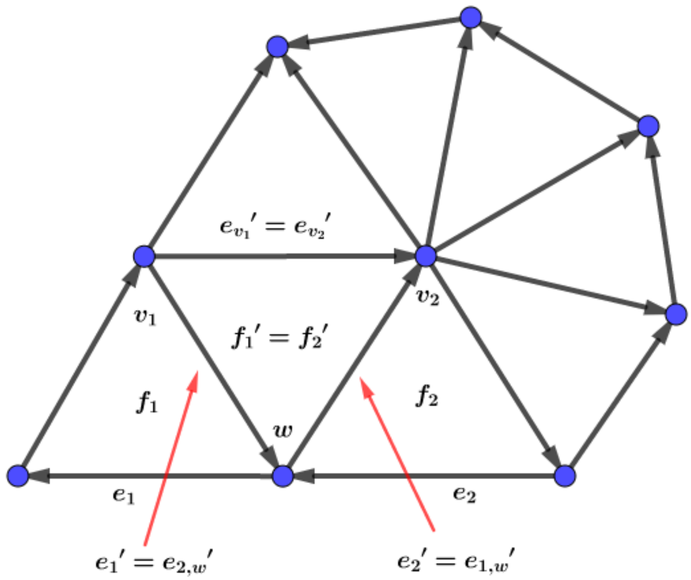
If we look at the argument used to prove the Theorem 8.33, after considering a 3-colouring of the region R, the construction of the bipartite subgraph H of R which is supposed to be a feedback arc set was pretty much straight forward and not really elegant, since the additional condition in the theorem is very technical and special, and, in any case, won't be easily verifiable if it is needed for theoretical purposes. In the following, we want to look at this construction process more carefully, aiming at more general results. We will need the following terminology:

Definition 8.34. Let R be a k-triangulated planar graph. An edge e of R is called a *chord*, if it connects two outer vertices of R but is no outer edge (on the boundary cycle of R). We call e a *proper chord*, if additionally e is not contained in a facial triangle of R, which consists of e and two outer edges.
The facial triangles containing the non-proper chords and two outer edges will be called *peak triangles*.

The rest of this section presents an outline of further possibilities of constructing bipartite feedback arcs sets as required by Theorem 8.29. Many of the results have "experimental" character and lack a general context. Still, they illustrate tools of which some might be helpful in future developments.
The following structural property concerning peak triangles in connected regions of directed triangular faces within a directed planar triangulation which is a minimal counterexample to conjecture 7.6, motivates the subsequent results on the existence of bipartite feedback arc sets in triangulations admitting similar properties.

Definition and Proposition 8.35. Let $\vec{T}$ be a directed planar triangulation so that none of the three reduction operations (R1)-(R3) presented in section 7 does apply. Let L, K denote the edge subsets as in Theorem 8.29. Assume that R is a subdigraph of $\vec{T}$ consisting of a maximal connected region of directed triangular faces in $\vec{T}$ which admits no holes and cut vertices, in other words, R is a k-triangulated graph where k is the length of the boundary cycle in $\vec{T}$, and all the directed facial triangles incident to a vertex on the boundary of R are contained in R.
Given a peak triangle of R, we call it *reducible*, if not both of the outer edges contained in it are members of L.
Let R' be the triangulated graph arising from R by deleting the two outer edges contained in some reducible peak triangle. Then any bipartite feedback arc set of R' contained in $\vec{T}\backslash(L \cup K)$ gives rise to such a set in R.
Moreover, any pair of consecutive non-reducible peak triangles in R (i.e., peak triangles of R') admits equal (ccw/cw) orientation, and there is exactly one bounded triangular face in between them.

Proof. Assume we are given a pair of consecutive peak triangles in R which are both non-reducible, i.e., all of the four boundary edges on the two triangles are contained in L. We show that they admit the structure described above. Denote by f_1, f_2 the incident triangles, where f_1 comes first in clockwise order along the boundary cycle of R, and let e_1, e_2 denote the corresponding non-proper chords of R as well as v_1, v_2 the vertices of f_1, f_2 not incident to e_1 resp. e_2 and $w \in \delta_T(e_1) \cap \delta_T(e_2)$ the unique common vertex of the peak triangles. Let $e_i', i = 1, 2$ be the unique edge in f_i incident to w other than e_i. Since (R2) is (by assumption) not applicable, v_1 as well as v_2 have both at least two incoming and two outgoing edges in $\vec{T}$. For f_1, f_2, denote by f_1', f_2' the adjacent triangular faces admitting e_1', e_2' as a common boundary. Denote by $e_{v_i}', e_{i,w}'$ the two remaining edges of f_i' so that e_{v_i}' is incident to v_i and $e_{i,w}'$ is incident to w for $i = 1, 2$. Then, since $e_1', e_2' \in L$, f_i' is neither directed nor transitive with respect to e_i', $i = 1, 2$. Now, for each $i \in \{1, 2\}$, either $e_{i,w}', e_i'$ or e_{v_i}', e_i' admit the same orientation with respect to the remaining edge within f_i'. Since there are no reducible diamonds in $\vec{T}$, this, as already applied in Lemma 7.29, gives rise to a chain of adjacent transitive triangles around v_i in the first resp. w in the latter case, which stops when reaching the next directed triangle in the corresponding cyclic order around v_i resp. w. In the first case, this already would imply that v_i admits out- or indegree 1 in $\vec{T}$, i.e. gives rise to an almost directed cut in the first case, contradicting the above. Hence only the latter cases are possible, for $i = 1, 2$. But this implies that all the (non-directed) facial triangles incident to w in between f_1, f_1' and f_2', f_2 have to be transitive with respect to each other (whenever they share an edge).



It is easy to see that a facial triangle in $\vec{T}$ can not be transitive with respect to two adjacent triangular faces at the same time, and so, there are less than 3 faces between f_1 and f_2 incident to w, i.e., only $\{f_1', f_2'\}$. $f_1' \neq f_2'$ is not possible, since according to the above, in this case, with respect to $e_{1,w}' = e_{w,2}'$, both pairs e_{v_1}', e_1', e_{v_2}', e_2' would admit the same orientation, so that $e_{1,w}' = e_{2,w}'$ would be a reducible edge, contradiction. Thus, $f_1' = f_2'$.

But then this triangle admits a source resp. a sink at v_1 resp. v_2 and is consequently balanced at w. Finally, we conclude that e_1', e_2' admit distinct orientation with respect to w and henceforth, f_1, f_2 admit the same (cw/ccw) orientation, yielding the claimed properties for the pair of consecutive peak triangles. The figure above illustrates the described situation where $f_1' = f_2'$.
The last claim follows directly from the fact that the vertices of degree 2 in R inside the reducible peak triangles, joined to H via one incident edge not contained in L, give rise to an edge subset H' of R contained in $\vec{T} \backslash (L \cup K)$, which has to be bipartite and a feedback arc set, since the added vertices admit degree 1 in H' as well as $R - H'$, thus not allowing it to be contained in any cycle of odd lengh in H' resp. any directed cycle in $R - H'$. □

The following lemma now finally shows how the above considerations, in a slightly strengthened version (here, consecutive peak triangles with the same orientation are assumed to not exist at all, while above, we could only show that they must appear within a very special configuration), can be used to find bipartite feedback arc sets even in the presence of non-proper chords:

Lemma 8.36. *Let R be a k-triangulated planar graph without proper chords. Let $\vec{R}$ be an orientation of the edges such that every facial triangle is directed, and such that the boundary cycle C of $\vec{R}$ is not directed. Assume there are no two consecutive peak triangles in C with the same orientation (both (counter)clockwise).*

(i) There exists a bipartite subgraph H of R only containing inner edges such that the edges of H form a feedback arc set of $\vec{R}$.

(ii) Let $a_1 = u_1w_1, a_2 = u_2w_2$ be two arbitrarily chosen arcs on the boundary of R, and assume that for each boundary edge e of R incident to a peak triangle but not contained in it, the orientations of the two edges on the peak triangle agree with the orientation of e on C (i.e., all the three edges are in clockwise/counterclockwise direction).
Then we can chose a bipartite feedback arc set H consisting of inner edges in such a way that

$$\operatorname{dist}_H(u_i, w_i) \mod 2 = 1 \text{ or } \operatorname{dist}_H(u_i, w_i) = \infty, i = 1, 2.$$

Proof. (i) As in the proof of 8.33, we consider a 3-colouring of R with $0, 1, 2$. We want to construct a bipartite subgraph H of R such that every (always non-proper) chord of R is contained in H and such that all inner vertices of R are sources or sinks in $\vec{R} - H$. If we are given such a subraph, it also has to define a feedback arc set of $\vec{R}$, since any directed cycle in $\vec{R} - H$ would need to only use outer vertices, i.e., outer edges and chords. But since all chords are covered by H, the only cycle fulfilling these conditions could be the boundary cycle C, which is not directed by the assumption, and we are done.
Now for the construction. First of all, we consider the bipartite subgraph H' of R containing all the inner $0, 1$-edges of the 3-colouring and all edges spanned between a vertex of colour 0 and an inner vertex coloured 2. In $\vec{R} - H'$, already all the inner

vertices are sources or sinks, and hence, this remains true for all subgraphs containing H'. We therefore define H as the extension where all (non-proper) chords of R are joining H', if they are not contained yet. In order to finish the proof, it now suffices to show that this process keeps H a bipartite subgraph. So assume there was an odd cycle in H. Since H' is bipartite, this cycle has to use at least one non-proper chord not contained in H'. Let e be an arbitrary such chord. It connects two outer vertices and by the definition of H', this means that e has exactly one end vertex, denoted by v_e, which is coloured 2. Since v_e (as an outer vertex coloured 2) was not covered by H', such a vertex always is incident to two consecutive edges on the cycle, i.e., for each chord e contained in the cycle, there is a consecutive edge e' with $v_e = v'_e$. Moreover, since e, e' are non-proper chords, there are two peak triangles consisting of two boundary edges and e, e' respectively. According to the assumption in the theorem, these two triangles and especially e, e' cannot have the same orientation, i.e., either both e, e' are outgoing or both are incoming edges with respect to v_e. All the neighbouring vertices of v_e are coloured 0 and 1 and henceforth all the edges between neighbours of v_e are contained in H' and together build a path between the other end vertices v and v' of e and e' respectively. We replace the edges e, e' in the actual odd cycle/closed edge sequence of odd length by this path. Since the edges incident to v_e between e and e' are alternately outgoing and incoming, and since e and e' have the same respective orientation, this means that the length of this path was even and furthermore, the parity of the edge sequence does not change by applying this replacement step. If we do this for all chords e contained in the original cycle, we end up with a closed edge sequence in the bipartite graph H' which is of odd length, contradiction.

(ii) Given one of the two edges a_i, we can add a peak triangle containing a_i from the outside to R, which gives a region R_i containing a_i as a non-proper chord. Obviously, we can extend the orientation $\vec{R}$ to an orientation $\vec{R}_i$ of R_i such that each facial triangle is directed, by simply orienting the two additional edges in such a way that the additional peak triangle is made directed. It is important to notice that the additional requirement in (ii) implies that $R_i, \vec{R}_i$ fulfil all the requirements of (i) except for the fact that now the boundary cycles may be directed.
First of all, not both of the boundary cycles of $\vec{R}_1, \vec{R}_2$ can be directed at the same time, so we may assume that R_1 fulfils the conditions required for (i).
We now use the construction of the subgraphs H and H' from the first part, where we may choose the 3-colouring of R in such a way that u_1, w_1 receive colours 0 and 1 respectively. The actual 3-colouring of R can be canonically extended to a 3-colouring of R_1 by labeling the additional vertex with 2. Let H_1 be the bipartite subgraph of inner edges of R_1 which contains all inner $0,1$-edges as defined in (i). By definition, H_1 contains exactly the edges of H plus a_1. Since this was proven to be a bipartite subgraph, we get that either $\mathrm{dist}_H(u_1, w_1) = \infty$ or $\mathrm{dist}_H(u_1, w_1) \mod 2 = 1$, since if this number was even, a path connecting u_1, w_1 together with a_1 would form a cycle in H_1 of odd length.
It remains to be shown that $\mathrm{dist}_H(u_2, w_2) \mod 2 = 1$ or $\mathrm{dist}_H(u_2, w_2) = \infty$. The latter is obviously true if a_2 is contained in a peak triangle (then u_2,w_2 are disconnected

in H), so in the following, assume it is not.

Either, a_2 is a $0, 1$-edge or it contains a vertex labeled 2.

In the first case, denote by v_2 the third vertex contained in the triangular face incident to a_2. We may assume that there is a path P in H starting in u_2 and ending in w_2. Assume this path had even length. By doing the exact same replacement procedure for the non-proper chords contained in P as described in (i) when considering cycles in H, we get that there is a path P' in H' connecting the end vertices u_2, w_2 of a_2 of even length. But this is impossible: The path P' only contains edges incident to vertices of colour 0, which means that the colours of the vertices on P' are alternately 0 and 1 or 2, contradiction.

In the second case, by switching colours 0 and 1 if needed, we may assume that a_2 is a $0, 2$-edge. Again, assume there was a path P in H between the end vertices of even length. As above, this gives rise to a path P' between the vertices of even length which lies in H', which again is impossible. This proves the claim also in this case and hence, we are done.

□

The second statement is an attempt to prove more general existence results for bipartite feedback arc sets even in the case of proper chords.

Conjecture 8.37. *Let R be a k-triangulated planar graph. Let $\vec{R}$ be an orientation of the edges such that every facial triangle is directed, and such that the boundary cycle C of $\vec{R}$ is not directed. Assume there are no two consecutive peak triangles in C, and for every outer edge incident to a peak triangle but not contained in it, assume that it admits the same (cw/ccw) orientation as the two outer edges. Then there is a bipartite feedback arc set in R only consisting of inner edges.*

Idea of a Proof. We proceed by induction by the number of proper chords contained in R and claim that:

For every directed region $\vec{R}$ as given above and any pair $e_1 = u_1w_1 \neq e_2 = u_2w_2$ of outer edges contained in it, so that they are separated by at least one non-proper chord if there exists one, there is a bipartite feedback arc set in $\vec{R}$ only consisting of inner edges, so that $\operatorname{dist}_H(u_i, w_i) \mod 2 = 1$ or $\operatorname{dist}_H(u_i, w_i) = \infty, i = 1, 2$.

If this is zero, then we may apply Theorem 8.36,(i) in order to derive the claim. So assume for the inductive step that R contained $p \geq 1, p \in \mathbb{N}$ proper chords. Consider some proper chord e separating the given pair e_1, e_2 of edges chosen minimal with respect to the set of triangular faces enclosed by e and e_2, and denote by R_1, R_2 the triangulated graphs consisting of directed triangular faces and an unbounded face, which arise from R by splitting along e, i.e., both of them contain e as an outer edge. If R_1, R_2 both also meet the additional requirements for R in the claimed statement (this is the detail lacking in order to complete the proof), then we could apply the induction hypothesis to them equipped with the pairs e_1, e resp. e, e_2. The requirement that there is a proper chord separating the pairs of edges if there is one at all (in R_1 resp. R_2) is a consequence of the minimality assumption for e. The inductive assumption now yields bipartite feedback arc sets $H_i, i = 1, 2$ in R_i not containing outer edges (i.e, especially not e) so that $\mathrm{dist}_{H_i}(\mathrm{tail}(e_i), \mathrm{head}(e_i)) = \mathrm{dist}_{H_i}(\mathrm{tail}(e), \mathrm{head}(e)) \in \{2k+1|k \geq 0\} \cup \{\infty\}, i = 1, 2$. This implies that the arc set $H := H_1 \cup H_2$ in R does not contain inner edges, is bipartite and does not contain e, moreover, we have $\mathrm{dist}_H(\mathrm{tail}(e_i), \mathrm{head}(e_i)) = \mathrm{dist}_{H_i}(\mathrm{tail}(e_i), \mathrm{head}(e_i)) \in \{2k+1|k \geq 0\} \cup \{\infty\}$ as required. It now remains to show that H is a feedback arc set. Assume to the contrary there was a directed cycle in $\vec{R}$ not containing any edge of H_1 or H_2. Since H_1, H_2 are feedback arc sets in $\vec{R}_1, \vec{R}_2$, this cycle must cross $\delta_R(e)$ using each of the both end vertices of e exactly once. Thus, e is a chord of this cycle which gives rise to a directed cycle not covered by H_i in R_i for some $i \in \{1, 2\}$, which again yields a contradiction (note that $e \notin H_i, i = 1, 2$). $\square$

Conclusively, the above considerations (although they lack generality at some points and especially are not applicable in the case of directed outer cycles or holes within the connected components of directed triangles in a minimal counterexample $\vec{T}$) show that indeed, bipartite feedback arc sets as required by theorem 8.29 exist in many cases. Thus, I believe that more elaborate considerations connected to the approach presented in this section have quite some potential to lead to substantial (partial) results for 2-colourings of directed planar triangulations.

9 Conclusive Remarks and Future Work

Recapitulating this master's thesis, we have seen that although the 2-Colour-Conjecture, as the 4-Colour-Theorem, admits a very simple description, making substantial progress on it is harder than it might seem at first sight. We have discussed various ways and techniques of deriving partial or alternative results, also in the wider context of Neumann-Lara flows on 3-edge-connected digraphs. Furthermore, a lot of open problems and conjectures concerning the existence of NL-2-flows were pointed out, and I believe that especially a proof of the dual version of the digirth-four-result from Mohar and Li under the assumption of Fleischner's Conjecture as presented in subsection 5.3.4 is within reach.

The introduction and analysis of different fractional notions of the dichromatic number and the NL-flow index have entailed numerous open and basic questions concerning e.g. the relationship between star- and circular indices, which remains open, as well as the problem of closing the gap between $\xi^*_{\text{coin}}(M), \xi_{\text{coin}}(M)$ for regular matroids, as mentioned in 6.37. It seems likely that this problem might be resolved in near future by either constructing examples of regular (graphic?, cographic?) matroids with $\xi^*_{\text{coin}}(M) \leq \xi_{\text{coin}}(M) - 1$ or giving a proof of the converse by improving on the bounds used in the proof of Corollary 6.36.

Finally, an integral part of this thesis has been the investigation of minimal counterexamples to an equivalent reformulation of the 2-Colour-Conjecture. We have derived many restrictive structural results, which show that minimal counterexamples, if they exist, have to admit very special properties. On the one hand, as we illustrated by proving partial positive results e.g. for directed planar triangulations admitting equal orientations of directed cycles or other planar orientations such as 3-orientations with a rich structure of directed cycles, these properties may even lead to positive results in some cases. This is not least because Theorem 8.29 provides a reduction of the original colouring task to a subgraph admitting a certain orientation structure, whose properties seem to make the task of finding bipartite feedback arc sets a lot easier and tangible. In addition, the insights provided by the introduction of forcing sets, (reducible) essential cycles and essential cuts in 8.2 could be connected to the study of minimal counterexamples, which may lead to further results.

Additionally, even if the considerations mentioned above do not lead to a proof of the 2-Colour-Conjecture in its whole generality, as sketched by the algorithm 7.31, the described reduction operations can be performed in polynomial time and thus could, if implemented, speed up the verification of the 2-Colour-Conjecture for small planar digraphs a lot: By applying the reductions as long as possible, we end up with non-reducible configurations of planar digraphs, and looking for 2-colourings of them resp. NL-2-flows of their duals by standard methods (Hamiltonicity, search on even subgraphs) will already be sufficient for verifying the Conjecture for many examples, possibly for digraphs of order considerably larger than 26 (cf. 5.34).

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