Diplomarbeit

Matrix Colorings of P_4-sparse Graphs

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Introduction

A lot of problems that are difficult to solve on graphs in general were proved to be efficiently solvable on specific classes of graphs. For example, graph isomorphism and graph coloring are NP-complete in general, but can be solved in polynomial time on the class of P_4-free graphs.

A promising paradigm to solve difficult problems on a class of graphs involves a unique tree representation for each graph in the class. Every leaf in the tree represents one vertex in the graph and the inner nodes of the tree represent operations with which the graph for this node and eventually the graph for the whole tree can be constructed from the graphs of its subtrees. If the class of a graph as well as its tree representation can be determined in polynomial time, this tree representation can be used to create polynomial-time algorithms for problems that are hard on general graphs.

P_4-free graphs are also known as cographs, hence the name cotree for their tree representation. The class of cographs is a subclass of the class of P_4-sparse graphs that allows only a "low density" of induced P_4 subgraphs. P_4-sparse graphs also have a unique tree representation, which indicates that the problems efficiently solvable on cographs may also be efficiently solvable on P_4-sparse graphs.

Some of the problems that are NP-complete for all graphs and efficiently solvable for cographs involve a generalized coloring known as matrix partition. In the standard coloring problem, an algorithm has to decide whether a graph is k-colorable, that is whether its vertices can be partitioned into k independent sets. The matrix partition problem allows additional conditions, such as whether the vertices of a graph can be partitioned into k independent sets and l cliques and there can even be restrictions on whether the vertices of a specific part may or must or must not be adjacent to the vertices of another part.

An even further generalization of the matrix partition problem is the matrix partition problem with lists. In addition to the conditions of the matrix partition, a vertex may only be placed in parts allowed by its list. The tree representation of a graph can be used to find such a matrix partition or determine that no partition exists for the graph.

A graph G is an M-obstruction if there is no partition of the vertices of G such that the coloring conditions defined by the the partition matrix M are satisfied. A minimal M-obstruction is an M-obstruction that contains no M-obstructions other than itself as an induced subgraph. The possible size of minimal M-obstructions has an upper bound depending on the class of G and M. Specific values for these upper bounds can be proved with the help of a tree representation.

This thesis is based upon an article by Feder, Hell, and Hochstättler [FHH06] featuring a linear-time algorithm to decide the matrix partition problem with lists on cographs and that proves upper bounds for the size of minimal matrix obstructions cographs. The theorems and algorithms can be generalized to also include P_4-sparse graphs. This generalisation is the main part of this thesis.

Chapters 1 and 2 define the terms needed in the rest of this thesis. A polynomial-time algorithm for the matrix partition problem with lists for P_4-sparse graphs is presented in Chapter 3, next to a proof that the problem becomes NP-complete if the matrix defining the coloring conditions is part of the problem, even for complete graphs. The rest of this thesis focuses on minimal matrix obstructions. It is proved that the size of a P_4-sparse minimal matrix obstruction with lists grows at most factorially. If no lists are involved, an exponential upper bound for the size of minimal matrix obstructions that are P_4-sparse is shown in Chapter 4. By restricting to constant matrices, a polynomial upper bound can be proved in Chapter 5.

Part I.

Preliminaries

1. Types of graphs

Although some graph problems are generally hard to solve, there are classes of graphs allowing these problems to be solved efficiently. This chapter defines some of these classes and shows how they relate to each other.

Before that, some basic conventions are defined: $\mathbb{N}$ shall contain all positive whole numbers including 0, as opposed to $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. For $m \in \mathbb{N}$, let the sets $\mathbb{N}_m \subset \mathbb{N}$ and $\mathbb{N}_m^* \subset \mathbb{N}^*$ be defined as $\mathbb{N}_m := \{x \in \mathbb{N} \mid x \leq m\}$ and $\mathbb{N}_m^* := \{x \in \mathbb{N}^* \mid x \leq m\}$.

When X is a set, its power set is written as $\mathcal{P}(X) := \{Y \mid Y \subseteq X\}$.

The symbol $\wedge$ stands for the logical AND, the symbol $\vee$ is a logical OR.

The magnitude of a function's slope can be described with the big O notation: Let $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$ be functions, then $f \in O(g) \Leftrightarrow \limsup_{n \leftarrow \infty} \left| \frac{f(n)}{g(n)} \right| < \infty$. This may less formally be written as $f(n) \in O(g(n))$, where the parameter name is usually n or m.

1.1. Simple Graphs

All graphs in this thesis are simple and the term graph is used as a short form for simple graph.

Definition 1. *A **simple graph** G is a tuple (V, E) with $V \neq \emptyset$ (the vertices) and E (the edges) being finite sets. The set of edges E is a subset of $\{\{v, w\} \mid v, w \in V, v \neq w\}$, which is the set of two-element subsets of V.*

*The **size** $|G|$ **of a graph** $G = (V, E)$ is the number of its vertices, $|G| := |V|$.*

*Two vertices $v_1, v_2 \in V$ are **adjacent** if $\{v_1, v_2\} \in E$ and **non-adjacent** if $\{v_1, v_2\} \notin E$.*

$v_1 \barwedge v_2 := \{v_1, v_2\}$ is a short form for an edge.

The set of all vertices adjacent to $v_1 \in V$ is $N(v_1) := \{v_2 \in V \mid v_1 \barwedge v_2 \in E\}$, the set of all vertices non-adjacent to v_1 is $\overline{N}(v_1) := \{v_2 \in V \mid v_1 \barwedge v_2 \notin E\}$.

*$v_1 \in V$ **distinguishes** between $v_2 \in V$ and $v_3 \in V$ if v_1 is adjacent to exactly one of the vertices v_2 and v_3: $v_1 \barwedge v_2 \in E \Leftrightarrow v_1 \barwedge v_3 \notin E$.*

Remark 1. *Simple graphs have no parallel edges: For every two adjacent vertices $v_1, v_2 \in V$, there is exactly one edge $e \in E$ with $e = v_1 \barwedge v2$.*

Remark 2. *Simple graphs have no loops: If $v_1, v_2 \in V$ are adjacent vertices, then $v_1 \neq v_2$.*

Definition 2. *The **complement graph** $\overline{G} = (V, \overline{E})$ of a graph $G = (V, E)$ is the graph with the set of edges $\overline{E} := \{v_1 \bar{\times} v_2 \mid v_1, v_2 \in V \wedge v_1 \bar{\times} v_2 \notin E\}$. That means that two vertices $v_1, v_2 \in V, v_1 \neq v_2$ are adjacent in $\overline{G}$ if and only if they are non-adjacent in G.*

Definition 3. *A graph $G = (V, E)$ containing every possible edge is called a **complete** graph, i.e. $E = \{v_1 \bar{\times} v_2 \mid v_1, v_2 \in V \wedge v_1 \neq v_2\}$. The complete graph with n vertices is K_n. The complement $\overline{K_n} = (V, \varnothing)$ of a complete graph is an **empty** graph.*

Definition 4. *A graph $G' = (V', E')$ is called an **induced subgraph** of $G = (V, E)$ if $V' \subseteq V$ and any two vertices in G' are adjacent if and only if they are adjacent in G. This may also be written as $G' \subset G$ or $G' = G \cap V'$. Similarly, $G \setminus V'$ is the induced subgraph $G \cap (V \setminus V')$ of G.*

Definition 5. *The set of vertices of a complete induced subgraph is called a **clique**, therefore any set of pairwise adjacent vertices is a clique.*

*Similarly, a set of pairwise non-adjacent vertices is called an **independent set**.*

Definition 6. *A graph $G = (V, E)$ is k-**colorable** if there is a partition $A_1, \ldots, A_k$ of its vertex set $V = \bigcup_{i=1}^{k} A_i$, consisting of $k \in \mathbb{N}$ pairwise disjoint parts, such that every part A_i ($1 \leq i \leq k$) is an independent set.*

*A graph $G = (V, E)$ is **perfect** if, for every induced subgraph $G' = (V', E) \subset G$, there is a $k \in \mathbb{N}$ such that the largest clique of G' has size k and G' is k-colorable.*

1.2. Cographs

Cographs are a class of simple graphs that allow a lot of hard graph algorithms to be solved efficiently. In order to define cographs, the graph P_4, the chordless path of length 3, will be defined first:

Definition 7. *The **chordless path** of length 3, P_4, is a simple graph (V, E) with four vertices $\{v_1, v_2, v_3, v_4\} = V$ and exactly three edges $\{v_1 \bar{\times} v_2, v_2 \bar{\times} v_3, v_3 \bar{\times} v_4\} = E$.*

Remark 3. *The complement graph of P_4 is P_4.*

In the 1970s, various researchers discovered cographs independently, although they used different definitions and names. Jung used the term D^*-graph [Jun78], Seinsche worked on them as graphs that have no induced subgraph isomorphic to P_4 [Sei74], and Sumner called them HD (or Hereditary Dacey) graphs [Sum74].

Definition 8. *A **cograph** is a graph (V, E), which does not have a P_4 as an induced subgraph. This means, that for any four vertices $v_1, v_2, v_3, v_4 \in V$ with $v_1 \bar{\times} v_2, v_2 \bar{\times} v_3, v_3 \bar{\times} v_4 \in E$, there must be another edge $e \in E \cap \{v_1 \bar{\times} v_3, v_1 \bar{\times} v_4, v_2 \bar{\times} v_4\}$.*

There are a couple of other characterizations of cographs. The most important characterization is the cotree representation of a cograph. [CLB81]

Definition 9. *A* ***cotree*** $T(G) = (V_{T(G)}, E_{T(G)})$ *is a tree representing a graph* $G = (V, E)$*.* $T(G)$*'s inner nodes are labeled* $\cup$ *or* $+$ *and its leaves are exactly the vertices of* G*. Two vertices of* G *are connected if and only if their least common ancestor in* $T(G)$ *is labeled* $+$*.*

A normalized cotree will have its inner nodes labeled $+$ and $\cup$ in alteration, which means that a $\cup$ node's parent will always be labeled $+$ and the other way around. All graphs represented by a cotree can also be represented by a normalized cotree since a node and its child having the same label can be contracted to one node. This will obviously not change the label of the least common ancestor for any two leaves. Two normalized cotrees representing the same graph will be isomorphic. Normalized cotrees are therefore unique representations of cotree-representable graphs.

A binary cotree is a cotree in binary form, that means every node has exactly two child nodes or it is a leaf. This cotree representation is also unique if a proper algorithm for splitting nodes with more than two child nodes is used for its construction. Figures 1.1 and 1.2 show a cograph and its binary cotree representation.

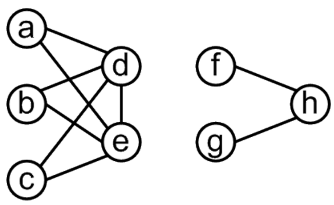

Figure 1.1.: An example of a cograph

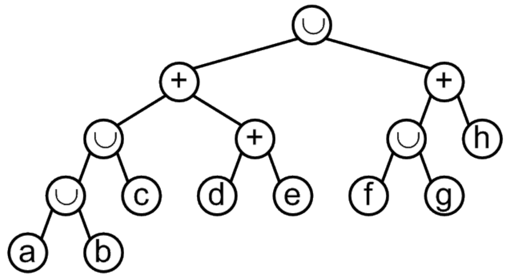

Figure 1.2.: Binary cotree corresponding to the cograph

Corneil et al. have proved that a graph is a cograph if and only if it has a cotree representation [CLB81]. A graph can be built from its cotree in polynomial time by assigning each node $t \in V_{T(G)}$ a graph G_t. For each leaf $v \in V$, the graph G_v is the single vertex graph $G_v = (\{v\}, \emptyset)$ containing only the leaf vertex itself.

For an inner node $t \in V_{T(G)} \setminus V$ labeled $\cup$, the graph G_t is the disjoint union $G_{u_1} \cup \ldots \cup G_{u_n}$ of the graphs of its n child nodes $u_1, \ldots, u_n$ in $T(G)$.

For an inner node $t \in V_{T(G)} \setminus V$ labeled $+$, the graph G_t is the join

$$G_{u_1} + \ldots + G_{u_n} := (V_{u_1} \cup \ldots \cup V_{u_n}, E_{u_1} \cup \ldots E_{u_n} \cup \left\{ x \barwedge y \mid 1 \leq i, j \leq n \wedge i \neq j \wedge x \in V_{u_i} \wedge y \in V_{u_j} \right\}$$

of the graphs of t's child nodes $u_1, \ldots, u_n$ in $T(G)$.

The graph G_r assigned to the root $r \in V_{T(G)}$ of $T(G)$ is the cograph that $T(G)$ represents. Corneil et al. have proven that the unique cotree of a cograph can be computed in linear time [CPS85], Bretscher et al. have developed an easier cograph recognition algorithm that runs in linear time [BCHP08].

1.3. P_4-sparse Graphs

Another class of simple graphs are the P_4-sparse graphs. P_4-sparse graphs constitute a superclass of cographs. Hoàng first defined this type of graph and showed besides other things that P_4-sparse graphs are perfect [Hoà85].

Definition 10. *A simple graph $G = (V, E)$ is called a P_4**-sparse graph** if every subgraph of G induced by any five vertices in V contains at most one P_4 as an induced subgraph.*

Extending the available operations in the cograph's cotree representation by additional node operations yields a unique tree representation of P_4-sparse graphs. Jamison and Olariu have shown how to acquire this tree representation in polynomial time [JO92a]. An important definition for this tree representation is the graph class called a spider:

Definition 11. *A simple graph $G = (V, E)$ is called a **spider** if its vertex set splits into an even number $2\nu \geq 4$ of vertices $c_1, \ldots, c_\nu, s_1, \ldots, s_\nu$ and a vertex set R, such that*

- *$c_1, \ldots, c_\nu$ comprise a clique. Every two vertices c_i, c_j $(1 \leq i, j \leq \nu)$ are adjacent.*
- *$s_1, \ldots, s_\nu$ comprise an independent set. Every two vertices s_i, s_j $(1 \leq i, j \leq \nu)$ are non-adjacent to each other.*
- *Each vertex c_i $(1 \leq i \leq \nu)$ is adjacent to all vertices in R.*
- *No vertex s_i $(1 \leq i \leq \nu)$ is adjacent to a vertex in R.*
- *Either*
  - *the vertices c_i and s_j are adjacent if and only if $i = j$ (**slim spider**), or*
  - *the vertices c_i and s_j are adjacent if and only if $i \neq j$ (**fat spider**).*

*The vertices $c_1, \ldots, c_\nu$ are the **body** of the spider, the vertices $s_1, \ldots, s_\nu$ are the spider's **legs**, and the set R is the spider's **head**. If R is an empty set, the graph is called a **headless spider**.*

An example of a spider graph is shown in Figure 1.3. The class of spider graphs is the basis for two new graph operations.

Definition 12. *Let $s_1, \ldots, s_\nu, c_1, \ldots, c_\nu$ be vertices with $\nu \geq 2$. Let $R = (V_R, E_R)$ be a graph. Let $E_{CR} := \{c_i \times r | 1 \leq i \leq \nu, r \in V_R\}$.*

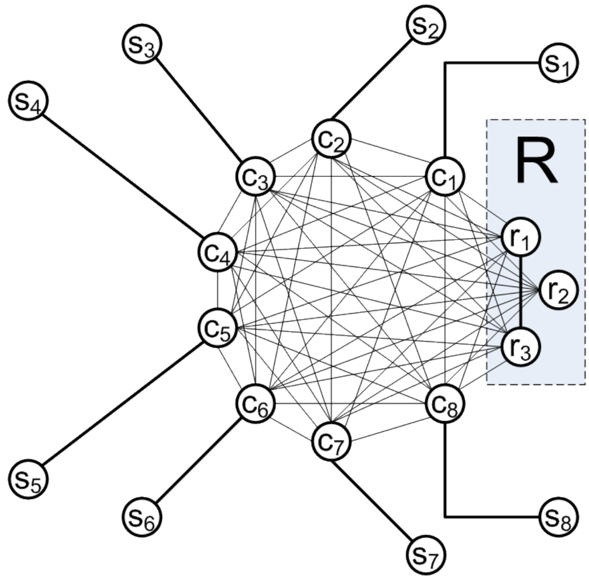

Figure 1.3.: Example of a slim spider graph with $\nu = 8$ and $R = \{r_1, r_2, r_3\}$

The functions $\circledast$ and $\star$ are defined as

$$
\begin{aligned}
\circledast : c_1, \ldots, c_\nu, s_1, \ldots, s_\nu, R &\mapsto (\{c_1, \ldots, c_\nu, s_1, \ldots, s_\nu\} \cup V_R, E_{slim} \cup E_R \cup E_{CR}) \\
\circledast : c_1, \ldots, c_\nu, s_1, \ldots, s_\nu &\mapsto (\{c_1, \ldots, c_\nu, s_1, \ldots, s_\nu\}, E_{slim}) \\
\star : c_1, \ldots, c_\nu, s_1, \ldots, s_\nu, R &\mapsto \left(\{c_1, \ldots, c_\nu, s_1, \ldots, s_\nu\} \cup V_R, E_{fat} \cup E_R \cup E_{CR}\right) \\
\star : c_1, \ldots, c_\nu, s_1, \ldots, s_\nu &\mapsto \left(\{c_1, \ldots, c_\nu, s_1, \ldots, s_\nu\}, E_{fat}\right)
\end{aligned}
$$

where $E_{slim} := \{c_i \times s_i | 1 \le i \le \nu\}$ and $E_{fat} := \left\{c_i \times s_j \middle| 1 \le i, j \le \nu \wedge i \neq j\right\}$. Obviously, $\circledast$ maps to slim spider graphs and $\star$ maps to fat spider graphs.

The P_4-sparse graph's tree representation extends cotrees by the two labels $\circledast$ and $\star$ for inner nodes. When constructing the graph $G = (V, E)$ represented by the tree $T(G) = (V_{T(G)}, E_{T(G)})$, every node $t \in V_{T(G)}$ is assigned a graph G_t. Again, the tree's leaves are the graph G's vertices. An inner node $t \in V_{T(G)}$ now has four possible operations to create its graph out of its children, depending on its label:

- A node labeled $\cup$ is assigned the graph created by the disjoint union of the graphs of its child nodes, which is the same as for cotrees.
- A node labeled $+$ is assigned the join of the graphs of its child nodes, again identical to cotrees
- The graph $G_t := \circledast(c_1, \ldots, c_\nu, s_1, \ldots, s_\nu, G_r)$ of a node labeled $\circledast$ is the operation $\circledast$ applied to its leaf child nodes $c_1, \ldots, c_\nu, s_1, \ldots, s_\nu$ and the graph of its child r. If a node labeled $\circledast$ has an even number of leaf child nodes $c_1, \ldots, c_\nu, s_1, \ldots, s_\nu$ and no further children, then $G_t := \circledast(c_1, \ldots, c_\nu, s_1, \ldots, s_\nu)$ is a slim headless spider graph.

- Similarly, the graph $G_t \coloneqq \star(c_1, \ldots, c_v, s_1, \ldots, s_v, G_r)$ of a node labeled $\star$ is the operation $\star$ applied to its child leaves $c_1, \ldots, c_v, s_1, \ldots, s_v$ and the graph of the its child node r. If a node labeled $\star$ has an even number of leaf child nodes, $c_1, \ldots, c_v, s_1, \ldots, s_v$, then $G_t \coloneqq \star(c_1, \ldots, c_v, s_1, \ldots, s_v)$ is a fat headless spider graph.

This definition of a tree representation for P_4-sparse graph differs a little from the original tree representation defined by Jamison and Olariu, but their proof that a graph is P_4-sparse if and only if it can be represented by such a tree applies here as well [JO92a]. They have also shown in a later publication that the tree representation of a P_4-sparse graph can be computed in linear time [JO92b]. Figure 1.4 shows an example of a P_4-sparse graph next to its tree representation in Figure 1.5.

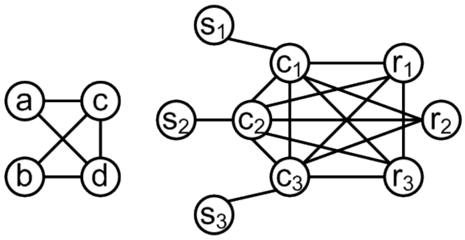

Figure 1.4.: An example of a P_4-sparse graph

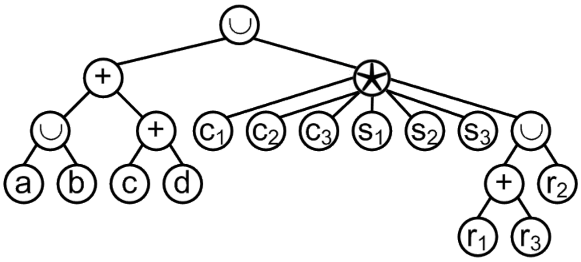

Figure 1.5.: Tree representation corresponding to the P_4-sparse graph

Definition 13. *A **split graph** $G = (C \cup S, E)$ consists of a clique C and an independent set S. G is called **avoiding** if for all $s \in S$ and all $c \in C$, the number of vertices in C non-adjacent to s is $|\overline{N}(s) \cap C| \geq 2$ and the number of vertices in S adjacent to c is $|N(c) \cap S| \geq 2$. G is called **minimal avoiding** if G has the following properties:*

1. *Every vertex $c \in C$ is non-adjacent to a vertex $s \in S$ such that $|\overline{N}(s) \cap C| = 2$.*
2. *Every vertex $s \in S$ is adjacent to a vertex $c \in C$ such that $|N(c) \cap S| = 2$.*
3. *G is avoiding.*

Remark 4. *Let the split graph $G = (C \cup S, E)$ be avoiding. Then the vertices of G split into a clique C and an independent set S with $C \neq \emptyset$, $S \neq \emptyset$, and $C \cap S = \emptyset$.*

Proposition 1. *Let $G = (C \cup S, E)$ be an avoiding split graph. Then G contains a minimal avoiding graph as an induced subgraph.*

Proof. This proof will show that every avoiding split graph that does not already satisfy the properties of a minimal avoiding split graph contains a vertex $x \in C \cup S$ such that $G \setminus \{x\}$ is an avoiding split graph. This way, vertices can be removed from the graph until we get a minimal avoiding induced subgraph of G. As G contains only a finite number of vertices, such an induced subgraph can always be found.

Thus, assume that for the avoiding split graph G does not satisfy at least one of the first two properties of minimal avoiding graphs. By complementing G if necessary, we may assume that property 1 is not satisfied, hence there is a vertex $c \in C$ for which all vertices $s \in \overline{N}(s)$ have $|\overline{N}(s) \cap C| \geq 3$. Any induced subgraph of a split graph obviously is a split graph and $G \setminus \{c\}$ also is avoiding as the condition for avoiding graphs is affected only for vertices in S and of course only those that have been non-adjacent to c before its removal; since these vertices $s \in \overline{N}(c)$ had the property $|\overline{N}(s) \cap C| \geq 3$ in G, they still satisfy the avoiding condition $|\overline{N}(s) \cap (C \setminus \{c\})| \geq 2$ in $G \setminus \{c\}$. □

Proposition 2. *An avoiding split graph $G = (V, E)$ is not P_4-sparse.*

Proof. Assume that the proposition is false and let $G = (V, E)$ be a counterexample with a minimum number of vertices. By Proposition 1, G has a minimal avoiding subgraph and, as G has a minimum number of vertices, G itself must be minimal avoiding.

Since G is a split graph, there is a vertex $s_1 \in S$. G is minimal avoiding, so s_1 is adjacent to a vertex $c_1 \in C$ with $N(c_1) \cap S = \{s_1, s_2\}$. Let $s_3 \in S$ be a vertex non-adjacent to c_1 with $\overline{N}(s_3) \cap C = \{c_1, c_2\}$. Let $c_3 \in C$ be a vertex adjacent to s_3 such that c_3 is adjacent to exactly two vertices in S. The graph $G \cap \{s_1, s_2, s_3, c_1, c_2, c_3\}$ that was constructed so far is depicted in Figure 1.6 with a legend to be found in Figure 1.11.

If c_3 is adjacent neither to s_1 nor s_2, then $G \cap \{s_1, s_2, s_3, c_1, c_3\}$ is a graph of size 5 with two induced P_4s, contradicting that G is P_4-sparse. This contradiction is shown in Figure 1.7. Thus, we may assume that c_3 is adjacent to s_2 and non-adjacent to s_1.

If c_2 is adjacent to s_1, then $G \cap \{s_1, s_3, c_1, c_2, c_3\}$ would be a graph of size 5 with two induced P_4s, which is shown in Figure 1.8.

As G is avoiding, c_2 is adjacent to a vertex $s_4 \in S$, $s_4 \neq s_2$. Because $s_4 \notin \{s_1, s_2\} = N(c_1) \cap S$, we see $c_1 \times s_4 \notin E$. Another consequence is $c_2 \times s_2 \in E$ as otherwise $G \cap \{s_1, s_2, s_4, c_1, c_2\}$ would be a graph of size 5 with two induced P_4s. Figure 1.9 shows the contradiction if $c_2 \times s_2 \notin E$.

G is avoiding, so there are two vertices $c_4, c_5 \in C$, $c_4 \neq c_5$ that are non-adjacent to s_2. Since $\{c_4, c_5\} \cap \{c_1, c_2\} = \emptyset$ and $\overline{N}(s_3) \cap C = \{c_1, c_2\}$, we conclude that s_3 is adjacent to c_4 and c_5. $G \cap \{s_2, s_3, c_1, c_4, c_5\}$ yields the final contradiction, as shown in Figure 1.10. □

Theorem 1. *Every P_4-sparse graph with vertices contains a maximum clique C and a maximum independent set S that meet in one vertex, $|C \cap S| = 1$.*

Proof. Let $G = (V, E)$ be a P_4-sparse graph with $C \subset V$ being a maximum clique and $S \subset V$ being a maximum independent set. We may assume $C \cap S = \emptyset$ as a clique and an independent set may intersect only in at most one vertex and in that case we would already have $|C \cap S| = 1$.

G is not avoiding, so there is a vertex $s \in S$ with $|\overline{N}(s) \cap C| < 2$ or a vertex $c \in C$ with $|N(c) \cap S| < 2$. By complementing G if necessary, we may assume $s \in S$ with $|\overline{N}(s) \cap C| \leq 1$. Actually we have $|\overline{N}(s) \cap C| = 1$ since $|\overline{N}(s) \cap C| = 0$ implies that $C \cup \{s\}$ would be a larger clique than C. Thus,

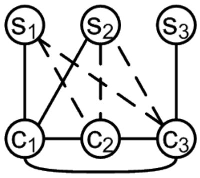

Figure 1.6.: Six vertices of G

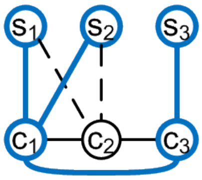

Figure 1.7.: The subgraph is not P_4-sparse

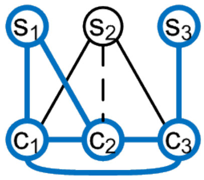

Figure 1.8.: Again, G is not P_4-sparse

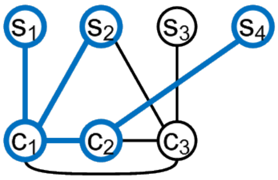

Figure 1.9.: Next subgraph that is not P_4-sparse

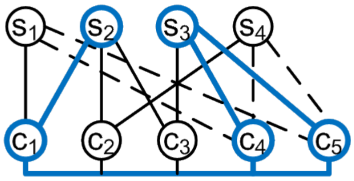

Figure 1.10.: The final contradiction, G is not P_4-sparse

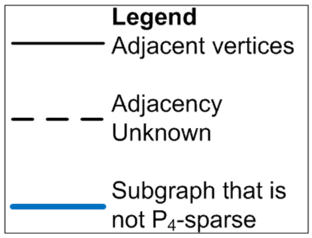

Figure 1.11.: Legend for the other figures

there is exactly one vertex $c_s \in C$ that is non-adjacent to s. As a result, $C \setminus \{c_s\} \cup \{s\}$ induces a maximum clique in G that meets the maximum independent set S in exactly one vertex, s. □

2. Generalized Colorings

The original n-coloring problem asked whether the vertices of a graph can be partitioned into n independent sets. Although there are different generalizations for this problem, this thesis focuses on matrix partitions, which does not only add parts of cliques, but can also place restrictions on the edges between vertices of different parts. Therefore, several well-known graph partition problems can be represented as a matrix partition, which was first defined by Feder et al. [FHKM99].

Definition 14. *Let $M \in \{0, 1, *\}^{m \times m}$ be a symmetric matrix with m rows and m columns with entries either $0, 1,$ or $*$. Then M is called a **partition matrix**.*

*The graph $G = (V, E)$ **admits** M if its vertices can be partitioned into m disjoint sets $A_1, \dots, A_m$ such that for every $i, j \in \mathbb{N}_m^*$, $i \neq j$,*

- *if $M_{i,i} = 0$, then A_i is an independent set,*
- *if $M_{i,i} = 1$, then A_i is a clique,*
- *if $M_{i,i} = *$, then there is no restriction on the edges between vertices of A_i,*
- *if $M_{i,j} = 0$, then every vertex in A_i is non-adjacent to every vertex in A_j,*
- *if $M_{i,j} = 1$, then every vertex in A_i is adjacent to every vertex in A_j, and*
- *if $M_{i,j} = *$, then there is no restriction on the edges between A_i and A_j.*

*The partition $A_1, \dots, A_m$ is called an M-partition. The numbers $1, \dots, m$ uniquely identify a set of vertices in the partition $A_1, \dots, A_m$. Therefore these number may also be called **parts** in order to highlight their meaning in this context. If a graph G does not admit the matrix M, then G **obstructs** M. G is a minimal M-obstruction if G obstructs M and all induced subgraphs of G except G itself admit M.*

*A **partition submatrix** M' of a matrix M is a submatrix of M such that the diagonal entries in M' are diagonal entries in M.*

Definition 15. *Let $\mathcal{M} \subset \bigcup_{\mu=1}^{m} \{0, 1, *\}^{\mu \times \mu}$ be a set of symmetric matrices and let G be a graph. G **admits** $\mathcal{M}$ if there is a matrix $M \in \mathcal{M}$ that G admits. If G obstructs all matrices $M \in \mathcal{M}$ then G **obstructs** $\mathcal{M}$. G is a **minimal** $\mathcal{M}$**-obstruction** if G obstructs $\mathcal{M}$ and all induced subgraphs of G except G itself admit $\mathcal{M}$.*

The decision whether a graph admits a matrix M or not is the M-partition problem. The general matrix partition problem may also be called M-partition problem, even if M is not explicitly defined.

Some problems require the sets A_i ($i \in \mathbb{N}_m^*$) to be non-empty, which will not be the focus of this thesis, although some results may also apply to these problems.

Definition 16. *Let $M \in \{0,1,*\}^{m\times m}$ be a symmetric matrix. Then $\overline{M} \in \{0,1,*\}^{m\times m}$ denotes the **complement** of M and, for all $i, j \in \mathbb{N}_m^*$, the entries of $\overline{M}$ satisfy the conditions $M_{i,j} = 0 \Leftrightarrow \overline{M}_{i,j} = 1$ and $M_{i,j} = 1 \Leftrightarrow \overline{M}_{i,j} = 0$.*

Definition 17. *Let $M \in \{0,1,*\}^{m\times m}$ be a symmetric matrix and $P, Q \subseteq \mathbb{N}_m^*$. The matrix $M_{P,Q} \in \{0,1,*\}^{|P|\times|Q|}$ is defined as the submatrix of M where only the rows in P and only the columns in Q are taken. $M_P \in \{0,1,*\}^{|P|\times|P|}$ is a short form for the partition submatrix $M_P \coloneqq M_{P,P}$.*

2.1. Generalized Colorings with lists

Feder et al. [FHKM03] and Cameron et al. [CEHS04] applied the list concept [AT92, FS92] to the matrix partition problem as a further generalization. Here, the vertices cannot be placed into any arbitrary part, instead there is a predefined list of parts for every vertex determining to which parts the vertex may belong:

Definition 18. *Let the matrix $M \in \{0,1,*\}^{m\times m}$ with m rows and m columns be symmetric and each of its entries is either 0, 1, or $*$.*

Let there be a graph $G = (V, E)$. The function $L : V \to \mathcal{P}(\mathbb{N}_m^*)$ assigns a set of parts to each vertex in G.

*Each set $L(v) \subset \mathbb{N}_m^*$ is called a **list**, while the function L is referred to as **lists**.*

The graph G with lists L admits M if there is an M-partition $A_1, \ldots, A_m$ such that

$$v \in A_i \Rightarrow i \in L(v) \qquad\qquad (v \in V, 1 \le i \le m)$$

*In this case, $A_1, \ldots, A_m$ is called a **list M-partition** (or just M-partition if it is clear from the context that lists are involved).*

The matrix partition problem without lists is identical to the matrix partition problem with lists if all lists $L(v)$ are set to $\mathbb{N}_m^*$. With these lists, there are no restrictions on where the vertices can be placed. Therefore, all algorithms for list matrix partition problems can also be used for matrix partition problems without lists.

Remark 5. *A graph G with lists L has an M-partition $A_1, \ldots, A_m$ if and only if $A_1, \ldots, A_m$ is an $\overline{M}$-partition of $\overline{G}$ with lists L.*

Definition 19. *Let the matrix $M \in \{0,1,*\}^{m\times m}$ be symmetric and let $G = (V, E)$ be a graph with lists L. Let $G' = (V', E')$ be an induced subgraph of G.*

Then the lists $L_{V'} : V' \to \mathcal{P}(\mathbb{N}_m^*)$ are defined as $L_{V'}(v') \mapsto L(v')$ for all $v \in V'$. The form $L_{G'} := L_{V'}$ may also be used for the same function.

With this definition, list M-partitions for subgraphs can be described by restricting the list to the subgraph's vertex subset:

Definition 20. *Let the matrix $M \in \{0,1,*\}^{m\times m}$ be symmetric and let G be a graph with lists L. G is a **minimal M-obstruction with lists** L if G obstructs M with lists L and every induced subgraph G' of G admits M with lists $L_{G'}$.*

If partition submatrices of M are involved, every list's parts have to be changed such that the numbers in the list still refers to the same rows and columns in the matrix. This is accomplished with the help of a mapping function:

Definition 21. *Let $m \in \mathbb{N}^*$, $P \subseteq \mathbb{N}_m^*$, then $\eta_P : P \to \mathbb{N}_{|P|}^*$ is defined as $\eta_P : x \mapsto |P \cap \mathbb{N}_x^*|$, i.e. x is the $\eta_P(x)$th element of P regarding the natural order of $\mathbb{N}$). Its inverse function $\eta_P^{-1} : \mathbb{N}_{|P|}^* \to P$ therefore is $\eta_P^{-1}(y) = \min\{x \in \mathbb{N} \mid y = |\mathbb{N}_x^* \cap P|\}$, i. e. $\eta_P^{-1}(y)$ is the yth element of P.*

Let $M_P \in \{0,1,*\}^{|P|\times|P|}$ be a partition submatrix of $M \in \{0,1,*\}^{m\times m}$ and let $L : V \to \mathcal{P}(\mathbb{N}_m^*)$ be lists for M. Then the lists ${}_PL : V \to \mathcal{P}\left(\mathbb{N}_{|P|}^*\right)$ are defined as ${}_PL(v) \mapsto \eta_P(P \cap L(v)) = \{\eta_P(x) \mid x \in P \cap L(v)\}$, i.e. the parts in P are enumerated as $1, \ldots, |P|$ in the lists while parts not in P are discarded.

If M' is a partition submatrix of M, the relationship between list M'-partitions and list M-partitions can be described with these restrictions of L:

Definition 22. *Let the matrix $M \in \{0,1,*\}^{m\times m}$ be symmetric and let $\mathcal{M} \subseteq \bigcup_{\mu=1}^{m} \{0,1,*\}^{\mu\times\mu}$ be a collection of partition submatrices of M. Let G be a graph with lists $L : V \to \mathcal{P}(\mathbb{N}_m^*)$.*

*Then G is an $\mathcal{M}$**-obstruction with lists** L if, for every matrix $M' \in \mathcal{M}$ and every $P \subseteq \mathbb{N}_m^*$ with $M_P = M'$, G obstructs M' with lists ${}_PL$.*

*G is a **minimal** $\mathcal{M}$**-obstruction with lists** L if G is an $\mathcal{M}$-obstruction with lists L and no induced subgraph G' of G other than G itself is an $\mathcal{M}$-obstruction with lists $L_{G'}$.*

In this thesis, a set of matrices $\mathcal{M}$ and lists L may be given without formally mentioning which matrix M the matrices in $\mathcal{M}$ are partition submatrices of and for which matrix the lists L are defined. This applies especially to the following Chapter 3, which discusses list M-partitions. In these cases, the matrices in the set are partition submatrices of the matrix with the symbol M, which is defined as a symmetric matrix $M \in \{0,1,*\}^{m\times m}$ throughout the major part of the chapter.

Part II.

Main Part

3. List M-partitions of P_4-sparse Graphs

3.1. Preliminary remarks about list M-partitions

Theorems very similar to the following two lemmas have already been proved by Feder et al. ([FHH06], Lemma 2.1 and Corollary 2.2), the following proofs are slightly adapted for the lemmas found here.

Lemma 1. *Let $G = G_1 \cup G_2$ be a disconnected graph and $M \in \{0, 1, *\}^{m \times m}$.*

G admits M with lists L if and only if there are sets of parts $P, Q \subset \mathbb{N}_m^*$ such that G_1 admits M_P with lists ${}_PL_{G_1}$, G_2 admits M_Q with lists ${}_QL_{G_2}$, and $M_{P,Q}$ contains no 1.

Proof. "$\Leftarrow$" Let ${}_1A_i$ $(i \in P)$ be an M_P-partition of G_1 with lists ${}_PL_{G_1}$ and let ${}_2A_i$ $(i \in Q)$ be an M_Q-partition of G_2 with lists ${}_QL_{G_2}$ and let $A_i := {}_1A_i \cup {}_2A_i$ $(1 \leq i \leq m)$ be the join of these partitions, where ${}_1A_i$ and ${}_2A_i$ are empty sets for $i \notin P$ and $i \notin Q$, respectively. The partition $A_1, \ldots, A_m$ of G is compatible with the lists L, because for every $j \in \{1, 2\}$, $i \in \mathbb{N}_m^*$, and every $v \in {}_jA_i$, we have $\eta_P(i) \in {}_PL_{G_j}(v)$ and thus $i \in L(v)$. Hence, we just have to validate that the M-partition condition is not violated.

Assume the vertices $v, w \in V$ violate the M-partition condition of $A_1, \ldots, A_m$. Then the vertices v, w will not both be within the same graph G_1 or G_2, because then they would also violate the partition condition of ${}_1A_i$ $(i \in P)$ or ${}_2A_i$ $(i \in Q)$. Therefore, one of the vertices is in G_1 and the other is in G_2 and therefore they are not adjacent. As v, w violate the partition condition, the corresponding matrix entry must be $M_{j,k} = 1$ for $v \in A_j$, $w \in A_k$. On the other hand, $M_{P,Q}$ does not contain a 1 and this also means $M_{j,k} \neq 1$, a contradiction.

"$\Rightarrow$" Let $A_1, \ldots, A_m$ be an M-partition of G with lists L. Define ${}_jA_i := A_i \cap V_j$ for $G_j = (V_j, E_j)$, $j \in \{1, 2\}$ and $i \in \{1, \ldots, m\}$. Let P, Q be defined via $i \in P \Leftrightarrow {}_1A_i \neq \varnothing$ and $i \in Q \Leftrightarrow {}_2A_i \neq \varnothing$.

Then, obviously, ${}_1A_i$ $(i \in P)$ is a valid M_P-partition with lists ${}_PL_{G_1}$ and ${}_2A_i$ $(i \in Q)$ is a valid M_Q-partition with lists ${}_QL_{G_2}$, so G_1 admits M_P with lists ${}_PL_{G_1}$ and G_2 admits M_Q with lists ${}_QL_{G_2}$.

Assume that $M_{P,Q}$ contains a 1. Then there are $j \in P$, $k \in Q$ with $M_{j,k} = 1$ and, by the definition of P and Q, there are vertices $v \in A_j \cap V_1$ and $w \in A_k \cap V_2$. These vertices are non-adjacent, because v is in G_1 and w in G_2 and the two subgraphs are disconnected. Therefore $A_1, \ldots, A_m$ is no valid partition of G, which contradicts the assumptions. Therefore $M_{P,Q}$ contains no 1. □

Corollary 1. *Let $G = G_1 \cup G_2$ be a disconnected graph and $M \in \{0, 1, *\}^{m\times m}$.*

G obstructs M with lists L if and only if for all sets of parts $P, Q \subset \mathbb{N}^*_m$ the subgraph G_1 obstructs M_P with lists ${}_PL_{G_1}$, G_2 obstructs M_Q with lists ${}_QL_{G_2}$, or $M_{P,Q}$ contains a 1.

Lemma 2. *Let $\mathcal{M} \subset \bigcup_{\mu=1}^{m} \{0, 1, *\}^{\mu\times\mu}$ be a set of partition submatrices of $M \in \{0, 1, *\}^{m\times m}$ and let $G = G_1 \cup G_2$ be a disconnected graph with lists L. For $i \in \{1, 2\}$, let the matrix set $\mathcal{M}_i \subset \bigcup_{\mu=1}^{m} \{0, 1, *\}^{\mu\times\mu}$ only contain partition submatrices of matrices in $\mathcal{M}$ and let G_i be an $\mathcal{M}_i$-obstruction with lists L_{G_i}. For every matrix $\tilde{M} \in \mathcal{M}$ and all sets of parts $P, Q \subset \mathbb{N}^*_m$, let there be a* 1 *in $\tilde{M}_{P,Q}$, or $\tilde{M}_P \in \mathcal{M}_1$, or $\tilde{M}_Q \in \mathcal{M}_2$.*

If G is a minimal $\mathcal{M}$-obstruction with lists L, then G_1 is a minimal $\mathcal{M}_1$-obstruction with lists L_{G_1} and G_2 is a minimal $\mathcal{M}_2$-obstruction with lists L_{G_2}.

Proof. Assume G_1 is not a minimal $\mathcal{M}_1$-obstruction with lists L_{G_1}, then G_1 has an induced subgraph $G'_1 \neq G_1$ that obstructs $\mathcal{M}_1$ with lists $L_{G'_1}$. By Corollary 1, the induced subgraph $G' = G'_1 \cup G_2 \neq G$ of G is an $\mathcal{M}$-obstruction with lists $L_{G'}$, which contradicts that G is a minimal $\mathcal{M}$-obstruction with lists L. Thus, the initial assumption is wrong and G_1 is a minimal $\mathcal{M}_1$-obstruction with lists L_{G_1}.

Swapping G_1 and G_2 as well as $\mathcal{M}_1$ and $\mathcal{M}_2$ in the proof above shows that G_2 is a minimal $\mathcal{M}_2$-obstruction with lists L_{G_2}. □

If no lists are used, the condition that the matrices in $\mathcal{M}$ are partition submatrices of a matrix $M \in \{0, 1, *\}^{m\times m}$ may be omitted, as it is required only to have a proper definition for the lists of the submatrices; a matrix $M \in \{0, 1, *\}^{m'\times m'}$ with lists $L(v) := \mathbb{N}^*_{m'}$ ($v \in V$) can always be found by putting all matrices in $\mathcal{M}$ in the diagonal of M (m' is the sum of all matrix sizes in $\mathcal{M}$ in this case):

Corollary 2. *Let $\mathcal{M} \subset \bigcup_{\mu=1}^{m} \{0, 1, *\}^{\mu\times\mu}$ be a set of matrices and let $G = G_1 \cup G_2$ be a disconnected graph. For $i \in \{1, 2\}$, let the matrix set $\mathcal{M}_i \subset \bigcup_{\mu=1}^{m} \{0, 1, *\}^{\mu\times\mu}$ only contain partition submatrices of matrices in $\mathcal{M}$ and let G_i be an $\mathcal{M}_i$-obstruction. For every matrix $\tilde{M} \in \mathcal{M}$ and all sets of parts $P, Q \subset \mathbb{N}^*_m$, let there be a* 1 *in $\tilde{M}_{P,Q}$, or $\tilde{M}_P \in \mathcal{M}_1$, or $\tilde{M}_Q \in \mathcal{M}_2$.*

If G is a minimal $\mathcal{M}$-obstruction, then G_1 is a minimal $\mathcal{M}_1$-obstruction and G_2 is a minimal $\mathcal{M}_2$-obstruction.

As a result of Corollary 1, matrix sets $\mathcal{M}_1$, $\mathcal{M}_2$ as required by Lemma 2 can always be chosen: for example, $\mathcal{M}_i$ could contain all partition submatrices $\tilde{M}$ of all matrices in $\mathcal{M}$ such that G_i obstructs $\tilde{M}$ with lists L_{G_i}. Thus, if $G = G_1 \cup G_2$ is a minimal $\mathcal{M}$-obstruction with lists for some set $\mathcal{M}$ of partition submatrices of a matrix M, then G_1 is a minimal $\mathcal{M}_1$-obstruction with lists for some matrix set $\mathcal{M}_1$.

The following lemma limits the number of vertices that certain spider graphs have. This ensures that the time complexity for some algorithms (namely the ones in Section 3.2 and Section 4.2) on P_4-sparse graphs presented in this thesis does not depend on the number of vertices n but instead on the number of partitions m. For a fixed matrix size m, these algorithms can be shown to become polynomial in n.

Definition 23. *Let $G = (V, E)$ be a graph. Two pairs of vertices (a, b), $(a', b') \in V \times V$ are called a **twin couple** if $a \times b \in E \Leftrightarrow a' \times b' \in E$, and $a \times c \in E \Leftrightarrow a' \times c \in E$, and $b \times c \in E \Leftrightarrow b' \times c \in E$ ($c \in V \setminus \{a, b, a', b'\}$).*

Lemma 3. *Let $M \in \{0, 1, *\}^{m \times m}$ denote a partition matrix and $G = (V, E)$ a graph with lists L. Let the pairs of vertices $(a_1, b_1), \dots, (a_{m+2}, b_{m+s}) \in V \times V$ be pairwise disjoint and twin couples such that $L(a_i) = L(a_j)$, $L(b_i) = L(b_j)$ ($1 \le i, j \le m + 2$). Then $G \setminus \{a_1, b_1\}$ with lists $L_{V \setminus \{a_1, b_1\}}$ admits M if and only if G admits M with lists L.*

Proof. "$\Leftarrow$" If G admits an M-partition with lists L, then $G \setminus \{a_1, b_1\}$ also admits an M-partition with lists $L_{V \setminus \{a_1, b_1\}}$.

"$\Rightarrow$" Assume $G \setminus \{a_1, b_1\}$ has an M-partition with lists $L_{V \setminus \{a_1, b_1\}}$ with parts $A_1, \dots, A_m$. The $m+1$ vertices $a_2, \dots, a_{m+2}$ are elements of the m sets $A_1, \dots, A_m$, so by the pigeon hole principle, there must be a part with at least two different vertices from $\{a_2, \dots, a_{m+2}\}$. Without loss of generality, we may assume $a_2, a_3 \in A_1$. Adding a_1 to A_1 and b_1 to the part A_j containing b_2 yields an M-partition of G, which will now be proved. First, we observe $1 \in L(a_2) = L(a_1)$ and $j \in L(b_2) = L(b_1)$, so our choice of classes for a_1 and b_1 is compatible with the lists.

Now assume that there exists a pair of vertices x, y violating the partition condition. Since there is no violation without a_1 and b_1, one of the vertices x, y must be a_1 or b_1. Without loss of generality, the two cases $x = a_1$ and $y = b_1$ may be looked at separately.

In the case $x = a_1$, the twin couple property $a_1 \times c \in E \Leftrightarrow a_2 \times c \in E$ for all $c \in V \setminus \{a_1, b_1, a_2, b_2\}$ implies $y \in \{b_1, a_2, b_2\}$. Because (a_1, b_1) and (a_3, b_3) are also twin couples, we observe $y \in \{b_1, a_3, b_3\}$, which means $y = b_1$.

The case $y = b_1$ similarly leads to $x = a_1$: Because (a_1, b_1), (a_2, b_2), and (a_3, b_3) are pairwise twin couples, we have $b_1 \times c \in E \Leftrightarrow b_2 \times c \in E$ ($c \in V \setminus \{a_1, b_1, a_2, b_2\}$) and $b_1 \times c' \in E \Leftrightarrow b_3 \times c' \in E$ ($c \in V \setminus \{a_1, b_1, a_3, b_3\}$) and therefore $x \in \{a_1, a_2, b_2\} \cup \{a_1, a_3, b_3\}$.

This means that the two cases are in fact only one and the pair of vertices $(x, y) = (a_1, b_1)$ are the only vertices violating the partition condition. The vertex pairs (a_1, b_1) and (a_2, b_2) are a twin couple, hence $a_1 \times b_1 \in E \Leftrightarrow a_2 \times b_2 \in E$. Since $a_1, a_2 \in A_1$ and $b_1, b_2 \in A_j$, (a_2, b_2) also violates the partition condition, which contradicts the choice of $A_1, \dots, A_m$ as classes for an M-partition of $G \setminus \{a_1, b_1\}$.

Hence, the original assumption must be wrong and G admits an M-partition. $\square$

3.2. The list M-partition problem with fixed-size matrices

The list M-partition problem is NP-complete because it is a generalization of the NP-complete coloring-problem (see Section 2.1). If the matrix size m is fixed, the M-partition problem is still NP-complete. For some matrices, the M-partition problem is NP-complete, even if the whole matrix is fixed, because for example Planar 3-colorability reduces to an M-partition with a fixed matrix [Sto73].

If the graph is a cograph and the size $m \in \mathbb{N}^*$ of the matrix $M \in \{0, 1, *\}^{m\times m}$ is fixed, there is an algorithm that decides the list M-partition problem in linear time [FHH06]. This section generalizes this algorithm to P_4-sparse graphs. The algorithm solves the list M-partition problem with a fixed size m of the matrix for P_4-sparse graphs in linear time.

Proposition 3. *The list M-partition problem for headless spider graphs can be solved in time $O\left(nm + m^2 \cdot 16^m\right)$, linear in n.*

Proof. A headless spider graph $G = (\{c_1, \ldots, c_v, s_1, \ldots, s_v\}, E)$ with lists L satisfies the conditions in Lemma 3 if $p \geq m + 2$ vertex pairs $(c_1, s_1), \ldots, (c_p, s_p)$ have the same lists. These pairs of vertices are twin couples, so removing the vertices $c_{m+2}, \ldots, c_p$ and $s_{m+2}, \ldots, s_p$ yields, by Lemma 3, an induced subgraph of G that admits exactly the same matrices of size m with corresponding lists.

Removing these redundant vertices for all different list pairs leads to the induced subgraph $G' = (V', E')$ of G, which contains at most $m+1$ vertex pairs with identical list pairs for each list pair. G' with lists $L' := L_{G'}$ still admits the same matrices with size m as G with lists L.

As there are $(2^m)^2 = 2^{2m}$ different combinations of two lists and G' contains at most $m+1$ vertex pairs for each combination, the number of vertices in G' is $n' := |V'| \leq 2\cdot(m+1)\cdot 2^{2m}$. Obviously, this upper limit for n' does not depend on the number n of vertices in G, although calculating G' from G takes linear time in n. Therefore, checking all partitions whether they constitute a valid M-partition does not increase the time in regards to n:

For every partition to check, every vertex is an element of one of the m classes. This means that there are n'^m possibilities to distribute the vertices to m classes. First, each partition is checked whether it is compatible with the lists L', which takes the time $O(n' \cdot m)$ per partition. Partitions violating the partition condition given by the matrix M can be dropped in time $O\left(n'^2\right)$ by comparing the adjacency of every pair of vertices in parts $i, j \in \{1, \ldots, m\}$ to the value of $M_{i,j}$. All other partitions are valid, so G' with lists L' and thereby G with lists L admits M if and only if at least one valid partition is left.

The steps of this algorithm taking the most time are the calculation of G' from G, which takes the time $O(n \cdot m)$, and the adjacency comparison, which takes the time $O\left(n'^2\right)$. As a whole, the algorithm therefore takes the time $O\left(mn + n'^2\right) \subset O\left(nm + 4m^2(2^{2m})^2\right) = O\left(nm + m^2 \cdot 16^m\right)$. $\square$

The algorithm described in this section exploits the tree representation of spider graphs. The matrices admitted by the graph of a node in the tree will be calculated from the node's children. The next lemma is a prerequisite for such an operation on spider nodes.

Lemma 4. *Let $G = (S \cup R, E)$ be a spider graph with lists L. The vertices R are the head of the spider and the rest of the vertices $S := \{c_1, \dots, c_v, s_1, \dots, s_v\}$ split into the spider's body $\{c_1, \dots, c_v\}$ and legs $\{s_1, \dots, s_v\}$.*

Then G admits a matrix $M \in \{0, 1, *\}^{m \times m}$ if and only if there are submatrices $M_{\mathcal{S}}, M_{\mathcal{R}}$ of M (with $\mathcal{S}, \mathcal{R} \subset \mathbb{N}_m^*$) such that $G \cap R$ with lists L_R admits $M_{\mathcal{R}}$ and $G \cap S$ admits $M_{\mathcal{S}}$ with lists L' defined as follows:

$$L'(c_i) := L(c_i) \setminus \left\{x \in \mathbb{N}_m^* \mid \exists y \in \mathcal{R}. M_{x,y} = 0\right\}$$
$$L'(s_i) := L(s_i) \setminus \left\{x \in \mathbb{N}_m^* \mid \exists y \in \mathcal{R}. M_{x,y} = 1\right\}$$

Proof. "$\Rightarrow$" Let the graph G admit a matrix $M \in \{0, 1, *\}^{m \times m}$. Then G has an M-partition with classes $A_1, \dots, A_m$. Let $\mathcal{S} \subset \mathbb{N}_m^*$ denote the set of indices i of classes A_i containing vertices of S. Similarly, let $\mathcal{R} \subset \mathbb{N}_m^*$ be the set of indices j of parts A_j containing vertices of R.

Since G with lists L has an M-partition with classes $A_1, \dots, A_m$, the same M-partition will be valid if it is restricted to the vertices of the induced subgraph $G \cap R$ with lists L_R. Restricting the M-partition to the induced subgraph $G \cap S$ with lists L_S also yields a valid partition for the induced subgraph. As all vertices of R are in parts A_i with $i \in \mathcal{R}$ and all vertices of S are in parts A_i with $i \in \mathcal{S}$, $G \cap R$ with lists L_R will also admit $M_{\mathcal{R}}$ and $G \cap S$ with lists L_S will also admit $M_{\mathcal{S}}$.

The last thing to prove is that $G \cap S$ still admits $M_{\mathcal{S}}$ if the lists L' instead of L_S are used. By the definition of L', this claim is true if there are no parts $x \in \mathcal{S}$ and $y \in \mathcal{R}$ with

- a vertex $c_i \in A_x$ and $M_{x,y} = 0$ or
- a vertex $s_i \in A_x$ and $M_{x,y} = 1$.

The first case is not possible, because $y \in \mathcal{R}$ implies $A_y \cap R \neq \emptyset$ and therefore c_i is adjacent to a vertex $r \in A_y \cap R$. Then $M_{x,y} = 0$ contradicts that $A_1, \dots, A_m$ constitute an M-partition for G. The latter case is not possible either, because the vertex $r \in A_y \cap R$ is non-adjacent to s_i, which contradicts $M_{x,y} = 1$ and $A_1, \dots, A_m$ being a valid M-partition for G.

Hence, $G \cap S$ with lists L' admits $M_{\mathcal{S}}$.

"$\Leftarrow$" Let there be sets $\mathcal{S}, \mathcal{R} \subset \mathbb{N}_m^*$, such that $G \cap R$ with lists L_R admits $M_{\mathcal{R}}$ and $G \cap S$ with lists L' admits $M_{\mathcal{S}}$.

Then $G \cap R$ has an $M_{\mathcal{R}}$-partition ${}_RA_i$ ($i \in \mathcal{R}$) with lists ${}_{\mathcal{R}}L_R$ and $G \cap S$ has an $M_{\mathcal{S}}$-partition ${}_SA_i$ ($i \in \mathcal{S}$) with lists L'. The rest of the sets are treated as empty sets, ${}_RA_i := \emptyset$ ($i \in \mathbb{N}_m^* \setminus \mathcal{R}$) and ${}_SA_i := \emptyset$ ($i \in \mathbb{N}_m^* \setminus \mathcal{S}$). The join $A_i := {}_RA_i \cup {}_SA_i$ ($1 \leq i \leq m$) of these classes constitutes an M-partition for G with lists L:

The sets $A_1, \ldots, A_m$ are supersets of the sets ${}_RA_1, \ldots, {}_RA_m$ and ${}_SA_1, \ldots, {}_SA_m$ and the lists L' are stricter than L, so any two vertices both within the same induced subgraph $G \cap R$ or $G \cap S$ will not violate the partition condition for $A_1, \ldots, A_m$. Therefore, if the vertex pair r, q violates the partition condition, there will be one vertex $r \in R \cap A_y$ with some $y \in \mathcal{R}$ and one vertex $q \in S \cap A_x$ with some $x \in \mathcal{S}$. If $q = c_i$ for an $i \in \{1, \ldots, \nu\}$, then $x \in L'(q) = L'(c_i)$ implies $M_{x,y} \neq 0$ by the definition of L'. This cannot be a violation of the partition condition, because c_i and r are adjacent, so we may assume $q = s_i$ for an $i \in \{1, \ldots, \nu\}$. Then $x \in L'(s_i) = L'(q)$ implies $M_{x,y} \neq 1$, which is also no violation of the partition condition, because s_i and r are not adjacent. Therefore the partition condition is not violated and the classes $A_1, \ldots, A_m$ constitute a partition of G with lists L. Thus, G with lists L admits M. □

Lemma 4 requires a given partition submatrix $M_{\mathcal{S}}$ for the spider body and legs in order to determine whether G admits the matrix M. If no such submatrix is given, the algorithm from Proposition 3 efficiently calculates all possible partition submatrices. With this prerequisites, the matrices admitted by a given spider graph can be calculated in linear time:

Proposition 4. *Let the graph $G = (S \cup R, E)$ with lists L be a spider graph, where S is the spider's body and legs and R is the spider's head. Let $2\nu := |S|$ be the number of vertices in S and let $M \in \{0, 1, *\}^{m \times m}$ be a matrix. Let the set of matrices $\mathcal{M}_R$ contain all partition submatrices of M that the induced subgraph $G \cap R$ of G with lists L_R admits.*

Given $G = (S \cup R, E)$, L, M, and $\mathcal{M}_R$, the set of partition submatrices of M admitted by G can be calculated in time $O\left(\nu \cdot m \cdot 4^m + m^2 \cdot 64^m\right)$, linear in ν.

Proof. For each set of parts $\mathcal{R} \subset \mathbb{N}_m^*$ such that $G \cap R$ with lists L_R admits the submatrix $M_{\mathcal{R}}$ of M, three steps are processed. Note that $M_{\mathcal{R}} \in \mathcal{M}_R$.

Step 1: Let the lists L' for the graph $G \cap S$ be defined like the lists L' in Lemma 4. Calculating L' as follows takes the time $O\left(\nu \cdot m^2\right)$: There are $2 \cdot \nu$ lists to calculate and each of them contains at most m parts. For each part $i \in \{1, \ldots, m\}$, the $|\mathcal{R}| \leq m$ entries from the row $M_{i,1}, \ldots, M_{i,m}$ of the matrix M have to be checked in order to decide whether i lies within the list or not.

Step 2: The set $\mathcal{M}_S$ of all matrices that $G \cap S$ admits with lists L' can be calculated in time $O\left(2^m \cdot (\nu m + m^2 \cdot 16^m)\right) = O\left(\nu \cdot m \cdot 2^m + m^2 \cdot 32^m\right)$ by using the algorithm in Proposition 3 on every partition submatrix of M, of which 2^m exist.

Step 3: For each set of parts $\mathcal{S}$ such that $M_{\mathcal{S}} \in \mathcal{M}_S$, Lemma 4 implies that G with lists ${}_{\mathcal{R} \cup \mathcal{S}}L$ admits the matrix $M_{\mathcal{R} \cup \mathcal{S}}$. This takes the time $O\left(m \cdot 2^m\right)$, as for each of the at most 2^m partition submatrices in $\mathcal{M}_S$, only the at most m parts have to be joined from the two sets $\mathcal{R}$ and $\mathcal{S}$.

Step 1 to 3 as a whole need $O\left(\nu \cdot m \cdot 2^m + m^2 \cdot 32^m\right)$ time, as step 2 dominates the time requirements. With Step 1 to 3 repeated for each of the 2^m possible sets of parts $\mathcal{R} \subset \mathbb{N}_m^*$, all partition submatrices admitted by G with lists L are calculated in time $O\left(2^m \cdot \left(v \cdot m \cdot 2^m + m^2 \cdot 32^m\right)\right) = O\left(v \cdot m \cdot 4^m + m^2 \cdot 64^m\right)$ □

Theorem 2. *The M-partition problem with lists can be decided in linear time for P_4-sparse graphs if the size of M is constant and the graph is given in its tree representation.*

Proof. Let $G = (V, E)$ be a P_4-sparse graph with lists L. For each vertex set $V' \subset V$, let the matrix set $\mathcal{M}_{V'} \subset \mathcal{M}$ be the set of partition submatrices admitted by $G \cap V'$ with lists $L_{V'}$. Each node t of the tree representation of G is assigned a vertex set $V_t \subset V$, such that the subtree rooted at t represents the induced subgraph $G \cap V_t$. The set $\mathcal{M}_{V_t}$ containing all partition submatrices of M admitted by the induced subgraph $G \cap V_t$ with lists L_{V_t} will be calculated for each node t from the leaves to the root:

A leaf node maps to exactly one vertex $v \in V$. For any partition $P \subset \mathbb{N}_m^*$, the leaf's graph $G \cap \{v\}$ with lists ${}_P L_{\{v\}}$ admits M_P if and only if $L(v) \cap P \neq \varnothing$. Calculating $\mathcal{M}_{\{v\}}$ therefore takes the time $O\left(2^m\right)$ or maybe $O\left(m^2\right)$, depending on how $\mathcal{M}_{\{v\}}$ is stored.

For an inner node t labeled $\cup$ or $+$ with the vertex set V_t, the set of admitted matrices $\mathcal{M}_{V_t}$ can be calculated in time $2^{O(m)}$ if the sets of admitted matrices of the child nodes are known. This was proven by Feder et al. ([FHH06], Lemma 2.1 and its complement).

Assume that the inner node t is a spider node labeled $\circledast$ or $\star$, so the vertex set V_t assigned to t is $V_t = \{c_1, \ldots, c_v, s_1, \ldots, s_v\} \cup R$. The set of admitted matrices $\mathcal{M}_{V_t}$ with lists L_{V_t} can be calculated in time $O\left(v \cdot m \cdot 4^m + m^2 \cdot 64^m\right)$ if the set of admitted matrices $\mathcal{M}_R$ is known. This is the result of Proposition 4. Because t has $2v$ leaves as child nodes and the whole tree contains n leaves, the complexity for calculating all sets of admitted matrices for all spider nodes is $O\left(n \cdot m \cdot 4^m + m^2 \cdot 64^m\right) \in 2^{O(m)} \cdot n$.

The tree contains n leaves and thereby at most $n - 1$ inner nodes, so calculating the sets of admitted matrices for every node in the tree takes the time $2^{O(m)} \cdot n$. With this information available, the list M-partition problem can be decided since G with lists L admits M if and only if $M \in \mathcal{M}_V$ and $\mathcal{M}_V$ was calculated as the set of admitted matrices for the root node. □

Corollary 3. *The M-partition problem with lists can be decided in linear time for P_4-sparse graphs if the size of M is constant.*

3.3. The list M-partition problem with variable-sized M

If both the matrix M and the graph G with lists L are variable, the list M-partition problem is NP-complete, since 3-Satisfiability (3SAT) can be reduced to the problem. This fact will be shown in this section.

The 3SAT problem is about deciding whether an arbitrary boolean formula in conjunctive normal form (CNF) such that every clause contains exactly three literals is satisfiable. This problem is NP-complete [Coo71]. A boolean formula is in CNF if each clause is a logical OR of its literals and the clauses are connected by a logical AND. A boolean formula is satisfiable if the variables can be assigned values of *true* and *false* such that the formula evaluates *true*.

Let E be a boolean formula in CNF with three literals per clause. Let $A \in \mathbb{N}^*$ be the number of variables appearing in E and let $B \in \mathbb{N}^*$ be the number of clauses. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_A$ denote these variables.

$$E = \bigwedge_{\beta=1}^{B} \left({}_1x_\beta \vee {}_2x_\beta \vee {}_3x_\beta \right)$$

Every ${}_ix_\beta$ $(\beta \in \mathbb{N}^*_B, i \in \{1, 2, 3\})$ is a literal, a possibly negated boolean variable of E. Variables can appear more than once in the formula, that means ${}_ix_{\beta_1}$ and ${}_jx_{\beta_2}$ might be the same variable, even if $i \neq j$ or $\beta_1 \neq \beta_2$.

Let $G = (\mathcal{A} \cup \mathcal{B}, \{x \times y \mid x, y \in \mathcal{A} \cup \mathcal{B} \wedge x \neq y\})$ be the complete graph with $A + B$ ($A := |\mathcal{A}|$, $B := |\mathcal{B}|$) vertices. The vertex $a_\alpha \in \mathcal{A}$ $(1 \leq \alpha \leq A)$ represents the variable $\mathfrak{a}_\alpha$ of E. As explained later in detail, every vertex in $\mathcal{A}$ will have two possible parts of an M-partition that determine whether the corresponding variable is *true* or *false* in a specific assignment. Each vertex $b_\beta \in \mathcal{B}$ $(1 \leq \beta \leq B)$ corresponds to one clause ${}_1x_\beta \vee {}_2x_\beta \vee {}_3x_\beta$ of E. These vertices choose between three parts of an M-partition, where every part requires a different literal of the clause to be *true*. This will also be described more formally later.

The symmetric matrix $M \in \{0, 1, *\}^{2\cdot A+3\cdot B \times 2\cdot A+3\cdot B}$ has $2 \cdot A + 3 \cdot B$ rows and the same number of columns. M splits into two diagonal blocks $1_{2A} \in {}_{2A}\mathcal{M}_{2A}$ and $1_{3B} \in {}_{3B}\mathcal{M}_{3B}$ containing only 1 entries and a submatrix $C \in {}_{2A}\mathcal{M}_{3B}$ as well as C's transposed C^T in the upper right and lower left corner. Therefore, it looks like this:

$$\begin{pmatrix} 1_{2A} & C \\ C^T & 1_{3B} \end{pmatrix}$$

The columns of C can be grouped into B submatrices ${}_\beta C \in \{0, 1, *\}^{2A \times 3} (1 \leq \beta \leq B)$. The matrix ${}_\beta C$ encodes the clause ${}_1x_\beta \vee {}_2x_\beta \vee {}_3x_\beta$: For $1 \leq \alpha \leq A$, the i-th column $(1 \leq i \leq 3)$ of the $(2\alpha - 1)$-th and (2α)-th row of ${}_\beta C$ are

- $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ if the literal ${}_ix_\beta$ does not represent the variable $\mathfrak{a}_\alpha$,
- $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ if the literal ${}_ix_\beta$ unnegatedly represents the variable $\mathfrak{a}_\alpha$, and
- $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ if the literal ${}_ix_\beta$ negates the variable $\mathfrak{a}_\alpha$.

The vertex a_α is only allowed to be placed in partitions $2\alpha - 1$ and 2α by its list $L(a_\alpha) = \{2\alpha - 1, 2\alpha\}$. The vertex b_β must be in only one of the partitions $2A + 3\beta - 2$, $2A + 3\beta - 1$, or $2A + 3\beta$ because of its list $L(b_\beta) = \{2A + 3\beta - 2, 2A + 3\beta - 1, 2A + 3\beta\}$.

As the construction of G, L and M is straightforward, it is obviously possibly to construct them in polynomial time, given a boolean formula E. The following theorem reduces the satisfiability problem for E to the list M-partition problem and therefore shows that it is NP-complete.

Proposition 5. *The previously defined graph G with lists L admits an M-partition if and only if the boolean formula E is satisfiable.*

Proof. "⇒" Assume that $G = (\mathcal{A} \cup \mathcal{B}, E)$ admits an M-partition $\mathcal{A}_1, \dots, \mathcal{A}_{2A}, \mathcal{B}_1, \dots, \mathcal{B}_{3B}$ with lists L. $\mathcal{B}_i$ is the $(2A + i)$-th part of the partition. By the definition of the lists L, we obviously have $\mathcal{A} = \bigcup_{i=1}^{2A} A_i$ and $\mathcal{B} = \bigcup_{i=1}^{3B} B_i$.

For every $\alpha \in \{1, \dots, A\}$, let the variable $\mathfrak{a}_\alpha$ have the value *true* if $a_\alpha \in \mathcal{A}_{2\alpha-1}$ and *false* if $a_\alpha \in \mathcal{A}_{2\alpha}$ (by construction of the list $L(a_\alpha) = \{2\alpha - 1, 2\alpha\}$, there are no further possibilities).

For every $\beta \in \{1, \dots, B\}$, the vertex b_β is in a part $\mathcal{B}_{3\beta-3+i}$ with $i \in \{1, 2, 3\}$ by construction of the list $L\left(b_\beta\right) = \{2A + 3\beta - 2, 2A + 3\beta - 1, 2A + 3\beta\}$. Let $\alpha \in \{1, \dots, A\}$ be chosen such that $\mathfrak{a}_\alpha$ is the variable represented by the literal ${}_ix_\beta$, then

$$\begin{pmatrix} M_{2\alpha-1,2A+3\beta-3+i} \\ M_{2\alpha,2A+3\beta-3+i} \end{pmatrix} = \begin{pmatrix} C_{2\alpha-1,3\beta-3+i} \\ C_{2\alpha,3\beta-3+i} \end{pmatrix} = \begin{pmatrix} {}_\beta C_{2\alpha-1,i} \\ {}_\beta C_{2\alpha,i} \end{pmatrix}$$

is either

- $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ if ${}_ix_\beta$ does not negate $\mathfrak{a}_\alpha$, or
- $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ if ${}_ix_\beta$ negates $\mathfrak{a}_\alpha$.

As the vertices b_β and a_α are adjacent, a_α must be in part $2\alpha - 1$ if the literal ${}_ix_\beta$ does not negate $\mathfrak{a}_\alpha$, in this case let $\mathfrak{a}_\alpha$ be assigned *true*, or in partition 2α if the literal ${}_ix_\beta$ negates $\mathfrak{a}_\alpha$, then let $\mathfrak{a}_\alpha$ be assigned *false*. In both cases, ${}_ix_\beta$ evaluates *true*, so the whole clause ${}_1x_\beta \vee {}_2x_\beta \vee {}_3x_\beta$ is *true* with the given assignment.

This applies to every clause and therefore, the assignment satisfies the boolean formula E if the graph G with lists L admits an M-partition.

"⇐" Assuming there is an assignment to the variables $\mathfrak{a}_1, \dots, \mathfrak{a}_A$ satisfying E, an M-partition $\mathcal{A}_1, \dots, \mathcal{A}_{2A}, \mathcal{B}_1, \dots, \mathcal{B}_{3B}$ admitted by G can be constructed:

For every $\alpha \in \{1, \dots, A\}$,

- if $\mathfrak{a}_\alpha$ is *true*, then $\mathcal{A}_{2\alpha-1} := \{a_\alpha\}$, $\mathcal{A}_{2\alpha} := \varnothing$,
- if $\mathfrak{a}_\alpha$ is *false*, then $\mathcal{A}_{2\alpha-1} := \varnothing$, $\mathcal{A}_{2\alpha} := \{a_\alpha\}$.

For every $\beta \in \{1, \dots, B\}$, the clause ${}_1x_\beta \vee {}_2x_\beta \vee {}_3x_\beta$ holds *true* for the observed assignment. Therefore, an $i \in \{1, 2, 3\}$ exists such that ${}_ix_\beta$ is *true*. After defining $\mathcal{B}_{3\beta-3+i} := \left\{b_\beta\right\}$, $\mathcal{B}_{3\beta-3+j} := \varnothing$ $(j \in \{1, 2, 3\} \setminus \{i\})$, all vertices are assigned to parts with respect to their lists.

Since the diagonal blocks of M will obviously not prevent G from admitting the M-partition (they contain only 1 entries and G is a complete graph), only the submatrix C needs to be analyzed more thoroughly. G would obstruct the M-partition with lists L only if there were $\alpha \in \mathbb{N}_A^*$, $\beta \in \mathbb{N}_B^*$ with $a_\alpha \in \mathcal{A}_{2\alpha-i}$ for some $i \in \{0,1\}$ and $b_\beta \in \mathcal{B}_{3\beta-3+j}$ for some $j \in \{1,2,3\}$ such that $C_{2\alpha-i,3\beta-3+j} = 0$. By the construction of C, this can happen only if either $i = 0$ and the literal ${}_jx_\beta$ unnegatedly represents the variable $\mathfrak{a}_\alpha$ or $i = 1$ and the literal ${}_jx_\beta$ negates the variable $\mathfrak{a}_\alpha$. By the definition of the parts $\mathcal{A}_{2\alpha-1}$ and $\mathcal{A}_{2\alpha}$, $\mathfrak{a}_\alpha$ would be *false* if ${}_jx_\beta$ negates $\mathfrak{a}_\alpha$ and *true* if ${}_jx_\beta$ is $\mathfrak{a}_\alpha$. In both cases, the literal ${}_jx_\beta$ evaluates *false*. This contradicts the choice of j, since, by the construction of $\mathcal{B}_{3\beta-3+j}$, $b_\beta \in \mathcal{B}_{3\beta-3+j}$ implies that ${}_ix_\beta$ is *true*.

Therefore G with lists L admits the M-partition. □

Theorem 3. *The list M-partition problem is NP-complete for complete graphs if the graph, its lists and M are variable.*

Proof. The boolean formula E is an arbitrary boolean formula in CNF with exactly three literals per clause. Deciding its satisfiability is the NP-complete 3SAT problem [Coo71]. The preceding proposition therefore reduces the 3SAT problem to a list M-partition problem, so this is also NP-hard.

Whether given sets are a valid partition can be decided in polynomial time by comparing the adjacency of every pair of vertices with the matrix entry determined by their parts. Thus, the list M-partition problem is in NP and so the list M-partition problem is NP-complete for complete graphs. □

The theorem is still true if G and M are complemented:

Corollary 4. *The list M-partition problem is NP-complete for empty graphs if the graph, its lists and M are variable.*

As the problem is NP-complete for complete graphs, of course it is also NP-complete for all larger classes of graphs.

Corollary 5. *If the graph, its lists and M are variable, the list M-partition problem is NP-complete for complete bipartite graphs, for cographs, and for P_4-sparse graphs.*

3.4. Maximum size of a minimal M-obstruction graph with lists

Theorem 4. *Let $f : m \mapsto 4^{m+1} \cdot (m+1)!$. For every $m \in \mathbb{N}^*$, every m-by-m matrix M, and every P_4-sparse graph G with lists that is a minimal M-obstruction, G has at most $f(m)$ vertices.*

Proof. Let M be a matrix an m-by-m matrix and let G be a P_4-sparse graph with lists L that is a minimal M-obstruction. Either G or its complement is disconnected or G is a spider graph since G is P_4-sparse. The theorem will be proved by induction over $m \in \mathbb{N}^*$.

Let $m = 1$. If a graph G obstructs M with lists L, then $G = (V, E)$ has a vertex $u \in V$ with the list $L(u) = \emptyset$ or G contains two vertices $v, w \in V$ that are adjacent or non-adjacent while the matrix is $M = (0)$ or $M = (1)$, respectively. In the first case, $G \cap \{u\}$ obstructs M with lists $L_{\{u\}}$ and since G is a minimal M-obstruction with lists L, we have $V = \{u\}$ and $|G| = 1 < f(1)$. In the second case, $G \cap \{v, w\}$ obstructs M with lists $L_{\{v,w\}}$ and this implies $|G| = 2 < f(1)$.

Now assume that the theorem is correct for all $m' < m \in \mathbb{N}^* \setminus \{1\}$.

Case 1: Assume that G or its complement is disconnected. If G is not a spider graph, we may assume that $G = G_1 \cup G_2$ is disconnected by complementing G and M if necessary. For $j \in \{1, 2\}$, let $\mathcal{M}_j$ contain all partition submatrices of M that $G_j = (V_j, E_j)$ obstructs with lists L_{G_j}, all of which have a size lower than m because $G_j = (V_j, E_j)$ admits M with lists L_{G_j}. Using $\mathcal{M} := \{M\}$ in Corollary 1 and Lemma 2 shows that G_j is a minimal $\mathcal{M}_j$-obstruction with lists L_{G_j}. For every matrix $\tilde{M} \in \mathcal{M}_j$, the largest P_4-sparse graph that minimally obstructs $\tilde{M}$ with lists has a size of at most $f(i)$ by the inductive assumption where i is the size of $\tilde{M}$. For each $\tilde{M} \in \mathcal{M}_j$, G_j obstructs $\tilde{M}$ with lists and therefore contains a minimal $\tilde{M}$-obstruction $G_{j,\tilde{M}} = (V_{j,\tilde{M}}, E_{j,\tilde{M}})$ as an induced subgraph. $G_j \cap (\bigcup_{\tilde{M} \in \mathcal{M}_j} V_{j,\tilde{M}})$ obstructs $\mathcal{M}_j$ and as G_j is a minimal $\mathcal{M}_j$-obstruction, we see $V_j = \bigcup_{\tilde{M} \in \mathcal{M}_j} V_{j,\tilde{M}}$. This implies $\left|G_j\right| \leq \sum_{\tilde{M} \in \mathcal{M}_j} |V_{j,\tilde{M}}| \leq \sum_{0<i<m} \binom{m}{i} f(i)$. Combining the results from $j = 1$ and $j = 2$ yields $|G| = |G_1| + |G_2| \leq 2 \sum_{0<i<m} \binom{m}{i} f(i)$. By the inductive assumption, this implies the proposition:

$$\begin{aligned}
|G| &\leq 2 \sum_{0<i<m} \binom{m}{i} f(i) \\
&= 2 \sum_{0<i<m} \frac{m!}{i! \cdot (m-i)!} 4^{i+1} (i+1)! \\
&= 2 \sum_{0<i<m} (m+1)! 4^{m+1} \frac{4^{i-m}}{(m-i)!} \frac{i+1}{m+1} \\
&< 4^{m+1} (m+1)! \cdot 2 \sum_{0<k<m} \frac{4^{-k}}{k!} \\
&< 4^{m+1} (m+1)! \cdot 2 (e^{\frac{1}{4}} - 1) \\
&< 4^{m+1} (m+1)! = f(m)
\end{aligned}$$

Case 2: Assume that G is a spider graph. $G = (V, E)$ is a spider graph with the vertex set $V = S \cup C \cup R$ partitioned into the spider's body $C = \{c_1, \ldots, c_v\}$, legs $S = \{s_1, \ldots, s_v\}$, and head R. As G is a minimal M-obstruction with lists L, Lemma 3 implies that, for each pair of lists $(L(c_i), L(s_i)) \subset \mathbb{N}_m^* \times \mathbb{N}_m^*$, there are at most $m + 1$ pairs of vertices in $\{(c_1, s_1); \ldots; (c_v, s_v)\}$ that have these lists. There are $(2^m)^2$ possible pairs of lists, so we see $|S \cup C| \leq (m+1) \cdot 2 \cdot (2^m)^2$.

For $R = \emptyset$, this implies the claim $|G| = |S \cup C| \leq 2(m+1)4^m < f(m)$. Thus, assume $R \neq \emptyset$.

For each $r \in R$, the induced subgraph $G \setminus \{r\}$ admits M with lists $L_{G \setminus \{r\}}$ since G is a minimal M-obstruction with lists L. By Lemma 4, there are sets $\mathcal{S}_r, \mathcal{R}_r \subset \mathbb{N}_m^*$ such that $G \cap (R \setminus \{r\})$ with lists $L_{R\setminus\{r\}}$ admits $M_{\mathcal{R}_r}$ and $G \cap (S \cup C)$ admits $M_{\mathcal{S}_r}$ with lists $L'_r : S \cup C \to \mathcal{P}(\mathbb{N}^*_m)$ defined as follows:

$$\begin{aligned}
L'_r(c_i) &:= L(c_i) \setminus \left\{x \in \mathbb{N}^*_m \mid \exists y \in \mathcal{R}_r . M_{x,y} = 0\right\} \\
L'_r(s_i) &:= L(s_i) \setminus \left\{x \in \mathbb{N}^*_m \mid \exists y \in \mathcal{R}_r . M_{x,y} = 1\right\}
\end{aligned}$$

As G obstructs M with lists L, by Lemma 4 it is not possible that $G \cap R$ admits $M_{\mathcal{R}_r}$ with lists L_R while $G \cap (S \cup C)$ admits $M_{\mathcal{S}_r}$ with lists L'_r. Thus, $G \cap R$ obstructs $M_{\mathcal{R}_r}$ with lists L_R. This implies $\mathcal{R}_r \neq \mathbb{N}^*_m$ since $G \cap R$ as an induced subgraph of the minimal M-obstruction G admits $M = M_{\mathbb{N}^*_m}$.

Let $\mathcal{M}_R := \{M_{\mathcal{R}_r} \mid r \in R\}$. Then $G \cap R$ obstructs $\mathcal{M}_R$ with lists L_R and, for any $r \in R$, $G \cap (R \setminus \{r\})$ admits a matrix in $\mathcal{M}_R$. Hence, $G \cap R$ is a minimal $\mathcal{M}_R$-obstruction. Similar to Case 1, we see that the size of $G \cap R$ is at most the sum of the sizes of all minimal $M_{\mathcal{R}_r}$-obstructions. The matrices in $\mathcal{M}_R$ have a size of at most $m - 1$, so we conclude

$$\begin{aligned}
|R| &\leq \sum_{0<i<m} \binom{m}{i} f(i) \\
&< 4^{m+1}(m+1)! \cdot (e^{\frac{1}{4}} - 1) \\
&< 4^{m+1}(m+1)! \cdot \frac{1}{3}
\end{aligned}$$

The size of G therefore is

$$\begin{aligned}
|G| &= |S \cup C| + |R| \\
&= 2(m+1)4^m + 4^{m+1}(m+1)! \cdot \frac{1}{3} \\
&= (m+1)4^{m+1} \cdot \frac{1}{2} + 4^{m+1}(m+1)! \cdot \frac{1}{3} \\
&< \left(4^{m+1}(m+1)!\right) \cdot \left(\frac{1}{2} + \frac{1}{3}\right) \\
&< 4^{m+1}(m+1)! = f(m)
\end{aligned}$$

□

4. Minimal M-obstruction, P_4-sparse Graph without lists

This chapter has its focus on the M-partition problem without lists. The matrix M has no diagonal $*$s, since otherwise any graph would admit M and the problem becomes trivial. By permutating the rows and columns, M may be written in block form with k diagonal 0s in the first rows and then l diagonal 1s.

4.1. Upper Bounds for minimal M-obstruction, P_4-sparse Graphs with bound clique sizes

Proposition 6. *Let $\mathcal{M} \subset \bigcup_{\mu=1}^{m} \{0, 1, *\}^{\mu\times\mu}$ be a non-empty set of matrices with a maximum size of $m \in \mathbb{N}^*$. Let G be a P_4-sparse graph that obstructs $\mathcal{M}$ minimally. Let the maximum clique size in G be $r \in \mathbb{N}^*$.*

Then G has at most $g(m,r) := 2\binom{m+r}{r} - m - 1$ vertices.

Proof. The proposition will be proved by induction over $m + r$.

The lowest possible value for m is 1, in which case there is only one matrix $M \in \mathcal{M}$, and then G has at least one vertex, as it is an $\mathcal{M}$-obstruction. This implies $r \geq 1$. The lowest possible value for $m + r$ therefore is $m + r = 2$ with $\mathcal{M} = \{(1)\}$ and $G = (\{v, w\}, \varnothing\}$. This implies $|G| = 2 = 2 \cdot 2 - 1 - 1 = g(1, 1)$.

Let $m, r \in \mathbb{N}^*$ and assume the proposition to be true for all r', m' with $r' + m' < r + m$. Let $\mathcal{M}$ be a set of matrices with maximum size m and let G be a P_4-sparse graph with maximum clique size r.

The root node of G in its tree representation is either one of the two types of spider nodes $\circledast$ or $\star$, a node $\cup$ for a disjoint union, or a node $+$ for a join. The following proof distinguishes between these cases. The upper limit of vertices of G for roots $\circledast$, $\star$ and $+$ will be shown to be $|G| \leq g(m,t) + g(m, r - t)$ with $t \in \{1, \ldots, r - 1\}$. If the root is labeled $\cup$, the upper limit is only $|G| \leq g(m - 1, r) + g(m, t) + g(m, r - t)$.

Case 1: Assume the root node is labeled $+$. In this case, there are two disjoint, P_4-sparse graphs G_1, G_2 with $G = G_1 + G_2$. Let t be the size of the largest clique in G_1 and let t' be the largest clique in G_2. The union of these cliques is a clique in G of size $t + t'$. $r \geq t + t'$ being the size of the maximum clique in G implies $1 \leq t \leq r - 1$ and $t' \leq r - t$. Using Lemma 2, we conclude that G_1 and G_2 are minimal obstruction graphs of matrix sets with maximum size m, which means $|G| \leq g(m,t) + g(m, r-t)$ by induction.

Case 2: Assume the root node is a spider node. Before the main part of the proof, note that the following formula is correct for all $m \geq 1$ and $\rho \geq 1$:

$$2\rho = 2\rho + 2 - 2 = 2\binom{1+\rho}{\rho} - 1 - 1 \leq 2\binom{m+\rho}{\rho} - m - 1 = g(m,\rho) \tag{4.1.1}$$

Since $G = (V, E)$ is a spider graph, its node set may be written as $V = \{c_1, \dots, c_\nu\} \cup \{s_1, \dots, s_\nu\} \cup R$, where $\{c_1, \dots, c_\nu\}$ is the spider's body, $\{s_1, \dots, s_\nu\}$ is the spider's legs, and R is the spider's head, which is the same terminology as in Definition 11. If R is empty, G has exactly $2\nu \leq 2r$ vertices. Using Inequality 4.1.1, this results in $|G| \leq 2r = 2r - 2t + 2t \leq g(m, r-t) + g(m,t)$ for any $t \in \{1, \dots, r-1\}$.

Now assume $R \neq \emptyset$. Any clique in $G \cap R$ is part of a clique in G with the ν additional vertices $c_1, \dots, c_\nu$. As the maximum clique size in G is r, the maximum clique size in $G \cap R$ is $r - \nu$. By the inductive assumption, the number of vertices in $G \cap R$ is $|R| \leq g(m, r-\nu)$, which limits the number of vertices in G to

$$|G| = |G \cap \{s_1, \dots, s_\nu, c_1, \dots, c_\nu\}| + |G \cap R| = 2\nu + |R| \leq g(m,\nu) + g(m, r-\nu)$$

The inequality in the preceding formula is a result of Inequality 4.1.1. By the definition of spider nodes, we have $1 \leq \nu$, and as the maximum clique size $r - \nu$ of $G \cap R$ is positive, we have $\nu < r$. Defining $t := \nu$ results in the same inequality as in the case $R = \emptyset$: $|G| \leq g(m, r-t) + g(m,t)$ with $1 \leq t \leq r-1$.

Case 3: Assume the root node is labeled $\cup$. For all $1 \leq r' < r$, $g(m,r') = 2\binom{m+r'}{r'} - m - 1 < 2\binom{m+r}{r} - m - 1 = g(m,r)$ is true. Therefore and by the inductive assumption, we conclude $|G| < g(m,r)$ if the largest clique in G contains less than r vertices. Hence, we will assume that the largest clique in G contains exactly r vertices.

G is a P_4-sparse graph and therefore perfect and r-colorable. Let $F_r \in \{0, 1, *\}^{r \times r}$ be the matrix with only 0s on the diagonal and only $*$s everywhere else. Due to the fact that the graphs admitting F_r are exactly the r-colorable graphs and since G obstructs all matrices in $\mathcal{M}$, no matrix in $\mathcal{M}$ contains F_r as a submatrix.

Since the root node is labeled $\cup$, G is a disconnected graph. Let G_1 be the induced subgraph of G containing only the vertices of the connected component with the largest clique, which has size r, while the induced subgraph G_2 contains all other vertices. Thus, G_1 and G_2 are two disjoint,

P_4-sparse induced subgraphs of G with $G = G_1 \cup G_2$. As G obstructs the set $\mathcal{M}$ minimally, $G_1 \subsetneq G$ admits a matrix $M \in \mathcal{M}$. Let $\mathcal{M}_M$ be the set of partition submatrices M_Q of M such that there are sets of parts $P, Q \subset \mathbb{N}_m^*$ with $M_{P,Q}$ not containing a 1 and G_1 having an M_P-partition. G_2 obstructs $\mathcal{M}_M$ as otherwise $G_1 \cup G_2 = G$ would admit M by Lemma 1. Because F_r is not a submatrix of M and G_1 contains a clique of size r, every M_P contains a 1 in its diagonal. Because $M_{P,Q}$ does not contain a 1, we conclude $M_Q \neq M$ and therefore M_Q has a size smaller than m. G_2 obstructs $\mathcal{M}_2 := \bigcup_{M \in \mathcal{M}} \mathcal{M}_M$, as it is a union of matrix sets obstructed by G_2.

Assume there is an induced subgraph G_2' of G_2 that obstructs $\mathcal{M}_2$. Let M be any matrix in $\mathcal{M}$. Then for all sets of parts $P, Q \subset \mathbb{N}_m^*$ for which G_1 admits M_P and $M_{P,Q}$ contains no 1, we have $M_Q \in \mathcal{M}_2$ and therefore G_2' obstructs M_Q. This means $G_2' \cup G_1$ obstructs M by Corollary 1. As M was chosen arbitrarily, $G_2' \cup G_1$, an induced subgraph of G, also obstructs $\mathcal{M}$. This contradicts that G is a minimal $\mathcal{M}$-obstruction.

Hence, G_2 is a minimal $\mathcal{M}_2$-obstruction and the matrices in $\mathcal{M}_2$ have a size of at most $m-1$. By the inductive assumption, G_2 has only $|G_2| \leq g(m-1, r)$ vertices. Since G_1 is P_4-sparse and connected, the root node in the tree representation of G_1 is either labeled + or a spider node. As shown, there is a $t \in \{1, \ldots, r-1\}$ in these cases such that G_1 has $|G_1| \leq g(m,t) + g(m, r-t)$ vertices. Adding the two inequalities yields $|G| = |G_2| + |G_1| \leq g(m-1, r) + g(m,t) + g(m, r-t)$.

All cases: Inductive step. The rest of this proof requires an inequality that will be proved first. Iteratively using the well-known identity $\forall a, x \in \mathbb{N}. \binom{a+x}{x} = \binom{a+x+1}{x+1} - \binom{a+x}{x+1}$, we find $\binom{a+x}{x} - \binom{a+x'}{x'} = -\sum_{i=x}^{x'-1} \binom{a+i}{i+1}$ for all $x' > x$. Thus, for $t \leq r-1$ and $m \in \mathbb{N}^*$ we have

$$
\begin{aligned}
&\binom{m+t}{t} + \binom{m+(r-t)}{r-t} \\
=&\binom{m+1}{1} - \binom{m+1}{1} + \binom{m+t}{t} + \binom{m+(r-1)}{r-1} - \binom{m+(r-1)}{r-1} + \binom{m+(r-t)}{r-t} \\
=&\binom{m+1}{1} + \binom{m+(r-1)}{r-1} + \sum_{i=1}^{t-1} \binom{m+i}{i+1} - \sum_{i=r-t}^{r-2} \binom{m+i}{i+1} \\
=&\binom{m+1}{1} + \binom{m+(r-1)}{r-1} + \sum_{i=1}^{t-1} \left(\binom{m+i}{i+1} - \binom{m+i+r-t-1}{i+r-t} \right) \\
\leq&\binom{m+1}{1} + \binom{m+(r-1)}{r-1}
\end{aligned}
\tag{4.1.2}
$$

Even in the worst case – if the root node is labeled $\cup$ –, the proposition can be proved with the help of Inequality 4.1.2:

$$\begin{aligned}
|G| &\leq g(m-1,r) + g(m,t) + g(m,r-t) \\
&= 2\binom{m+r-1}{r} - m - 2 + 2\binom{m+t}{t} - m - 1 + 2\binom{m+r-t}{r-t} - m - 1 \\
&\leq 2\left(\binom{m+r-1}{r} + \binom{m+r-1}{r-1}\right) + 2\binom{m+1}{1} - 3m - 4 \\
&= 2\binom{m+r}{r} + (2m+2) - 3m - 4 \\
&< 2\binom{m+r}{r} - m - 1 \\
&= g(m,r)
\end{aligned}$$

□

Complementing G in Proposition 6 yields the following corollary:

Corollary 6. *Let $\mathcal{M}$ be a matrix set with maximum matrix size $m \in \mathbb{N}^*$. Let G be a P_4-sparse graph and a minimal $\mathcal{M}$-obstruction. The largest stable set in G contains at most r vertices. Then G has at most $g(m,r) = 2\binom{m+r}{r} - m - 1$ vertices.*

4.2. Upper Bounds for minimal M-obstruction, P_4-sparse Graphs

Theorem 5. *Any minimal M-obstruction, P_4-sparse graph G has at most $O\left(16^m\right)$ vertices.*

Proof. Let G be in its tree representation and assign every node t of the tree a set $\mathcal{M}_t$ of partition submatrices of M such that the graph G_t corresponding to the node t is a minimal $\mathcal{M}_t$-obstruction. The set $\mathcal{M}_{t_0}$ of the root t_0 consists of the matrix M only. Obviously, the number of vertices of G_t is the number of leaves of the subtree rooted at t and therefore, the size of $G = G_{t_0}$ is the number of leaves in the whole tree.

Let $h\left(m, \tilde{m}\right)$ denote the maximum number of vertices that a graph G_t can have if the greatest size of a matrix in $\mathcal{M}_t$ is $\tilde{m}$ and all matrices in $\mathcal{M}_t$ are submatrices of a matrix of size m. In the first step of this proof, the inequality $h\left(m, \tilde{m}\right) \leq 2^m \left(2\binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1\right) + 2h\left(m, \tilde{m} - 1\right)$ will be shown.

If G_t has a maximum clique size of at most $\tilde{m}$, then the size of G_t is $|G_t| \leq 2\binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1$ by Proposition 6. The size of G_t is also $|G_t| \leq 2\binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1$ if the largest independent set in G_t has size at most $\tilde{m}$ by Corollary 6.

If instead the maximum clique size of G_t is greater than the maximum matrix size $\tilde{m}$ of $\mathcal{M}_t$ and the maximum independent set size of G_t is also greater than $\tilde{m}$, there are different cases to consider:

Case 1: Assume the node t is labeled $\cup$. Let k be the maximum number of diagonal 0s of the matrices in $\mathcal{M}_t$. The node t has two child nodes t' and t'' with $G_t = G_{t'} \cup G_{t''}$. There are two subcases, of which only the first will be proved completely here. In the second subcase, G_t will be shown to consist of the induced subgraphs G' with an independent set of size greater than $\tilde{m}$ and an induced subgraph G'' with $|G''| \leq \binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1$. The second subcase will be shown later to also lead to a number of vertices of G_t of at most $2^m \left(2\binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1\right) + 2h\left(m, \tilde{m} - 1\right)$.

In the first subcase, $G_{t'}$ and $G_{t''}$ both contain a clique of size greater than k. Let $\mathcal{M}_{t''}$ be the set of all matrices $\tilde{M}_Q$ such that there is a matrix $\tilde{M} \in \mathcal{M}_t$ with size $\hat{m}$ and and parts $P, Q \subset \mathbb{N}^*_{\hat{m}}$ such that $\tilde{M}_{P,Q}$ contains no 1 and $G_{t'}$ admits $\tilde{M}_P$. There are more vertices in the largest clique of $G_{t'}$ than there are diagonal 0s in $\tilde{M}_P$. Because vertices of the same clique cannot be placed in the same part i if $\tilde{M}_{i,i} = 0$, every $\tilde{M}_P$-partition of $G_{t'}$ uses a part $j \in P$ with $\tilde{M}_{j,j} = 1$. From the facts that $\tilde{M}_{P,Q}$ contains no 1, $\tilde{M}_{j,j} = 1$ and $j \in P$ follows $j \notin Q$, and therefore the matrix $\tilde{M}_Q$ is smaller than $\tilde{M}_{P \cup Q}$ and especially $|Q| < \hat{m} \leq \tilde{m}$. By Corollary 1, $G_{t''}$ obstructs the matrix $\tilde{M}_Q$ and therefore $G_{t''}$ obstructs $\mathcal{M}_{t''}$. By Lemma 2, $G_{t''}$ is also a minimal $\mathcal{M}_{t''}$-obstruction. As the maximum matrix size of $\mathcal{M}_{t''}$ is smaller than $\tilde{m}$, this means $|G_{t''}| \leq h\left(m, \tilde{m} - 1\right)$.

Because $G_{t''}$ also contains a clique of size greater than k, the graphs $G_{t'}$ and $G_{t''}$ in the proof above can be exchanged, which proves the inequality $|G_{t'}| \leq h\left(m, \tilde{m} - 1\right)$. Therefore, the size of G_t is $|G_t| = |G_{t'}| + |G_{t''}| \leq 2h\left(m, \tilde{m} - 1\right)$.

In the other subcase, let $G'' := G_{t''}$ have a maximum clique size of at most k, without loss of generality, and let the largest clique in $G' := G_{t'}$ have a size greater than $\tilde{m}$. Therefore the number of vertices in $G_{t''}$ is $|G''| \leq \binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1$. Obviously no matrix can contain more diagonal 0s than it has rows, so k is smaller than $\tilde{m}$. This subcase will be analyzed together with similar subcases of different labels for the node t later in this proof.

Case 2: Assume the node t is labeled $+$. Let l be the maximum number of diagonal 1s of the matrices in $\mathcal{M}_t$. Let t' and t'' be the child nodes of t, which means that $G_t = G_{t'} + G_{t''}$. Complementing G and M in the case where t is labeled $\cup$ leads to the conclusion that G_t has at most $2h\left(m, \tilde{m} - 1\right)$ vertices if both $G_{t'}$ and $G_{t''}$ contain an independent set of size greater than l.

Otherwise and without loss of generality, let the graph $G'' := G_{t''}$ have a maximum independent set with size of at most l and therefore its number of vertices is $|G''| \leq \binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1$. In this subcase, $G' := G_{t'}$ has an independent set of size greater than $\tilde{m}$. This subcase will be analyzed later in the proof, together with similar subcases of different labels for the node t.

Case 3: Assume the node t is a spider node. Using the terminology from Definition 11, let $G_t = \left(V_t, E_t\right)$ contain the vertices in the set $V_t = \{c_1, \ldots, c_v\} \cup \{s_1, \ldots, s_v\} \cup R$.

Because G_t is a minimal $\mathcal{M}_t$-obstruction graph, Lemma 3 can be applied if lists are defined as $L(v) := V_t \setminus \{v\}$ for every vertex $v \in V_t$, which effectively means that there are no restrictions where vertices can be placed. Lemma 3 implies that the number $2v$ of vertices in the spider's body and legs $G_t \setminus R$ is at most $2\tilde{m} + 2$.

If the spider's head $G_t \cap R$ has a maximum clique size or maximum independent set size of at most $\tilde{m}$, then the number of vertices of $G_t \cap R$ is at most $2\binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1$. In this subcase, the number of vertices in G_t is $|G_t| = 2\nu + |R| \leq 2\tilde{m} + 2 + 2\binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1 = 2\binom{2\tilde{m}}{\tilde{m}} + \tilde{m} + 1$.

Otherwise, $G' := G \cap R$ has a clique and an independent set of size greater than $\tilde{m}$ and $G'' := G \setminus R$ has at most $2\tilde{m}+2$ vertices. This subcase will now be analyzed together with the similar subcases of nodes labeled $\cup$ or $+$.

At this point, G_t has been shown to either have less than $\max\left\{2\binom{2\tilde{m}}{\tilde{m}} + \tilde{m} + 1, 2h(m, \tilde{m} - 1)\right\}$ vertices or that all of the vertices of G_t are in the disjoint induced subgraphs G' and G'', where G'' has at most $\max\left\{2\tilde{m} + 2, \binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1\right\}$ vertices and G' has an independent set or clique of size greater than $\tilde{m}$.

$G'_1 := G'$ may have less than $\max\left\{2\binom{2\tilde{m}}{\tilde{m}} + \tilde{m} + 1, 2h(m, \tilde{m} - 1)\right\}$ vertices or, again, G'_1 splits up into two induced subgraphs G'_2 and G''_2 with G'_2 being a graph with an independent set or clique of size greater than $\tilde{m}$ and G''_2 having at most $\max\{2\tilde{m}+2, \binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1\}$ vertices. Further repetition leads to $2s$ graphs $G'_1, \ldots, G'_s$ and $G''_1, \ldots, G''_s$, such that for every $1 \leq i \leq s$, G'_i contains a clique and an independent set with more than $\tilde{m}$ vertices and G''_i has at most $\max\{2\tilde{m} + 2, \binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1\}$ vertices. Let G'_s have at most $\max\left\{2\binom{2\tilde{m}}{\tilde{m}} + \tilde{m} + 1, 2h(m, \tilde{m} - 1)\right\}$ vertices. An upper limit for s will now be proved, namely $s \leq 2^m$, which will eventually lead to an upper limit for the number of vertices of G_t.

For $1 \leq i \leq s$, let N_i be the set $N_i := \{P \subset \mathbb{N}^*_m \mid G'_i \text{ obstructs } M_P\}$. For two indices i, j ($1 \leq i < j \leq s$), G'_j is an induced subgraph of G'_i and G'_i is a minimal obstruction graph for some set $\mathcal{M}_t$ of submatrices of M. Therefore G'_j does not obstruct $\mathcal{M}_t$ and hence $N_i \subsetneq N_j$. As there are only 2^m different subsets of $\mathbb{N}^*_m$, this implies $s \leq 2^m$,

Hence, the s graphs $G''_1, \ldots, G''_s$ together have at most $2^m \cdot \max\{2\tilde{m} + 2, \binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1\}$ vertices while G'_s has at most $\max\left\{2\binom{2\tilde{m}}{\tilde{m}} + \tilde{m} + 1, 2h(m, \tilde{m} - 1)\right\}$ vertices. Because every vertex of G_t lies in exactly one of the induced subgraphs $G''_1, \ldots, G''_s, G'_s$, G_t has at most $2^m \cdot \max\{2\tilde{m} + 2, \binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1\} + \max\left\{2\binom{2\tilde{m}}{\tilde{m}} + \tilde{m} + 1, 2h(m, \tilde{m} - 1)\right\}$ vertices, which together with the other cases completes the first step of the proof: For every $m \geq \tilde{m} > 1$, the maximum number of vertices of a minimal $\mathcal{M}$-obstruction, P_4-sparse graph where all matrices in $\mathcal{M}$ are submatrices of a matrix $M \in \{0, 1, *\}^{m \times m}$ is

$$\begin{aligned} h(m, \tilde{m}) \leq& 2^m \cdot \max\{2\tilde{m} + 2, \binom{2\tilde{m}}{\tilde{m}} - \tilde{m} - 1\} + \max\left\{2\binom{2\tilde{m}}{\tilde{m}} + \tilde{m} + 1, 2h(m, \tilde{m} - 1)\right\} \\ <& (2^m + 1) \cdot \left(2\binom{2\tilde{m}}{\tilde{m}} + \tilde{m} + 1\right) + 2h(m, \tilde{m} - 1) \end{aligned}$$

In the case $\tilde{m} = 1$, we have $h(m, 1) \leq 3 < (2^m + 1) \cdot \left(2\binom{2}{1} + 1 + 1\right)$.

In the second step of the proof, the recursion will be resolved to a sum. The formula will be simplified even further with the help of the inequality $\sum_{\tilde{m}=1}^{m} \binom{2\tilde{m}}{\tilde{m}} \leq \sum_{\tilde{m}=1}^{2m} \binom{2m}{\tilde{m}} = 2^{2m}$:

$$\begin{aligned} h\left(m, m\right) &< \sum_{\tilde{m}=1}^{m} 2^{m-\tilde{m}} \cdot \left(2^m + 1\right) \cdot \left(2\binom{2\tilde{m}}{\tilde{m}} + \tilde{m} + 1\right) \\ &\leq \left(2^{2m+1}\right) \cdot \sum_{\tilde{m}=1}^{m} \left(2\binom{2\tilde{m}}{\tilde{m}} + \tilde{m} + 1\right) \\ &\leq \left(2^{2m+1}\right) \cdot \left(2^{2m+1} + \frac{m(m+1)}{2} + m\right) \end{aligned}$$

Therefore, the number of vertices of any minimal M-obstruction, P_4-sparse graph is at most $h\left(m, m\right) \in O\left(\left(2^{2m+1}\right) \cdot \left(2^{2m+1} + \frac{m(m+1)}{2} + m\right)\right) = O(16^m)$. $\square$

5. Minimal M-obstruction, P_4-sparse Graphs with constant matrices M

In this chapter, let the matrix M be an (a,b,c)-block matrix. This means that its first k diagonal entries are all 0 and all other l diagonal entries are 1. The diagonal of M does not contain any $*$, so the matrix size is $m = k + l$. The entries of M can be divided into four blocks: the upper left block with all diagonal 0s, the lower right block with all diagonal 1s, and the two remaining blocks on the lower left and the upper right. In each of these blocks, all entries have the same value, except those in the diagonal of M. The off-diagonal entries in the upper left block all have the value a, while the off-diagonal entries in the lower right block all have the value b. All other entries, which are those in the upper right and lower left block, have the value c. a, b, and c can be 0, 1, and $*$, but we may assume $a \neq 0$ and $b \neq 1$ because such blocks are equal to a single 0 and 1 entry. A matrix meeting those criteria is also called a *constant* matrix.

An (a,b,c)-block matrix is completely determined by the parameters a,b,c, the number k of 0s in its diagonal and the number l of 1s in its diagonal. If a,b,c are clear from the context, then $M[k',l']$ stands for the (a,b,c)-block matrix with k' diagonal 0s and l' diagonal 1s. Note that $M[k',l']$ is a partition submatrix of M if and only if $k' \leq k$ and $l' \leq l$. If $k' < 0$ or $l' < 0$, then $M[k',l']$ shall be treated like a matrix that every graph obstructs.

Lemma 5. *Let M be an (a,b,c)-block matrix with $a = *, b \in \{0,*\}, c \in \{0,*\}$, let $G = G_1 \cup G_2$ be a disconnected graph and let $k,x,y \in \mathbb{N}$ be natural numbers. If G_1 admits $M[k,x]$ and G_2 admits $M[k,y]$ then G admits $M[k,x+y]$.*

Proof. Let $A_1,\ldots,A_{k+x}$ be an $M[k,x]$-partition of G_1 and let $B_1,\ldots,B_{k+y}$ be an $M[k,y]$-partition of G_2. Then obviously $A_1 \cup B_1,\ldots,A_k \cup B_k, A_{k+1},\ldots,A_{k+x}, B_{k+1},\ldots,B_{k+y}$ is an $M[k,x+y]$-partition of G. □

5.1. $(*,*,*)$-block matrices

Theorem 6. *Let M be a $(*,*,*)$-block matrix and let G be a minimal M-obstruction, P_4-sparse graph. Then G is a cograph.*

Proof. This proof distinguishes between two cases: Either one of the values k or l are zero or both k and l are greater than 0.

Starting with the first case, either k or l is 0. We may assume $l = 0$, the other subcase follows by complementation of G and M. A graph admits M if and only if it is k-colorable. Since G as a P_4-sparse graph is perfect, it is k-colorable if and only if its largest clique consists of at most k vertices. Therefore, a P_4-sparse graph obstructs M if and only if it has the complete graph K_{m+1} with $m + 1 = k + 1$ vertices as an induced subgraph. K_{m+1} is P_4-sparse already, so it is the only minimal M-obstructing P_4-sparse graph. K_{m+1} is also a cograph, so all minimal M-obstruction P_4-sparse graphs are cographs if $k = 0$ or $l = 0$.

Looking at the second case, let both k and l be at least 1. First assume the root node in the tree representation of G is a spider node. Using the terminology as in Definition 11, the node set V of $G = (V, E)$ may be written as $V = \{c_1, \dots, c_v, s_1, \dots, s_v\} \cup R$ where $\{c_1, \dots, c_v\}$ is the spider's body, $\{s_1, \dots, s_v\}$ is the spider's legs, and R is the spider's head. Since G is a minimal M-obstruction, every induced subgraph of G admits M, so there is an M-partition $A_1, \dots, A_m$ of $G \cap R$, where $A_1, \dots, A_k$ are independent sets and $A_{k-1}, \dots, A_m$ are l cliques. The vertices $s_1, \dots, s_v$ are non-adjacent to each other and they are non-adjacent to every vertex in R, so $A_1 \cup \{s_1, \dots, s_v\}$ is also an independent set. Similarly, the vertices $c_1, \dots, c_v$ are adjacent to each other and adjacent to every vertex in R, so $A_m \cup \{c_1, \dots, c_v\}$ constitutes a clique. Hence, $A_1 \cup \{s_1, \dots, s_v\}, A_2, \dots, A_{m-1}, A_m \cup \{c_1, \dots, c_v\}$ is an M-partition of the M-obstructing graph G, which is a contradiction. Therefore the root node of G is not a spider node.

The rest of the proof uses induction on the matrix size m. The induction base $m = 1$ has already been proved, as this case implies either $k = 0$ or $l = 0$. The induction step was already proved for all matrices smaller than the given matrix M. As shown above, the root node in the tree representation of G is not a spider node, so by complementing G and M if necessary, we may assume that $G = G_1 \cup G_2$ is a disconnected graph.

G is a minimal M-obstruction, so the induced subgraphs G_1 and G_2 both admit $M = M[k, l]$. This implies that G_1 obstructs $M[k, 0]$, as otherwise G would admit $M[k, 0 + l] = M[k, l] = M$ by Lemma 5. Now let l_1 be the smallest number such that G_1 admits $M[k, l_1]$. As G_1 obstructs $M[k, 0]$ but admits $M[k, l]$, we have $0 < l_1 \leq l$. G obstructs $M = M[k, l_1 + (l - l_1)]$ and hence G_2 obstructs $M[k, l - l_1]$. Let $\mathcal{M}_1$, $\mathcal{M}_2$ be matrix sets defined as $\mathcal{M}_1 := \{M[k, l'] \mid 0 \leq l' < l_1\}$ and $\mathcal{M}_2 := \{M[k, l'] \mid 0 \leq l' \leq l - l_1\}$. This satisfies the conditions of Lemma 2: G_1 obstructs $\mathcal{M}_1$ and G_2 obstructs $\mathcal{M}_2$. For any two sets of parts $P, Q \subseteq \mathbb{N}_m^*$ with $M_P \notin \mathcal{M}_1$ and $M_Q \notin \mathcal{M}_2$, the matrix M_P has the partition submatrix $M[0, l_1]$ and M_Q has the partition submatrix $M[0, l - l_1 + 1]$. Since M_P and M_Q together contain at least $l_1 + l - l_1 + 1 > l$ rows with a 1 in the diagonal entry, the submatrix M_{PQ} of M contains at least one 1 in the diagonal. By Lemma 2, G_1 is a minimal $\mathcal{M}_1$-obstruction and G_2 is a minimal $\mathcal{M}_2$-obstruction. All matrices obstructing $M[k, l_1 - 1]$ also obstruct $\mathcal{M}_1$ and all matrices obstructing $M[k, l - l_1]$ also obstruct $\mathcal{M}_2$. Thus, G_1 is a minimal $M[k, l_1 - 1]$-obstruction and G_2 is a minimal $M[k, l - l_1]$-obstruction, so by the inductive assumption, both G_1 and G_2 are cographs. This makes $G = G_1 \cup G_2$ a cograph, too. □

Because Feder et al. have determined the number of vertices of cograph minimal M-obstructions for $(*, *, *)$-block matrices ([FHKM99], Corollary 4.2), the number of vertices of G can be exactly specified using the preceding theorem:

Corollary 7. *Let M be a $(*,*,*)$-block matrix. Then every P_4-sparse, minimal M-obstruction graph has exactly $(k+1)(l+1)$ vertices.*

5.2. Staircase-like (a,b,c)-block matrices

By excluding the redundant cases $a \neq 0$ and $b \neq 1$, there are 12 combinations of a, b, and c to consider. In the first of these combinations, $(a,b,c) = (*,*,*)$, minimal M-obstructing P_4-sparse graphs were already shown to have the same upper bound of vertices as M-obstruction cographs in Theorem 6. In this section, upper bounds for the other combinations will be proved. In order to do that, spider graphs and disconnected graphs will be shown to have a couple of properties that are needed for the proof of the upper bound.

Lemma 6. *Let M be an $(1,*,*)$-block matrix and let G be an M-obstruction spider graph with the vertex set $V = \{c_1,\dots,c_v,s_1,\dots,s_v\} \cup R$ where $\{c_1,\dots,c_v\}$ is the spider's body, $\{s_1,\dots,s_v\}$ is the spider's legs, and R is the spider's head.*

Then all of the following holds:

- $G \cap R$ *obstructs* $M[k,l-v]$
- $k=0$ *or* $l=0$ *or* $G \cap R$ *obstructs* $M[1,l]$.

Proof. We assume that $G \cap R$ does not obstruct $M[k,l-v]$, so there is an $M[k,l-v]$-partition $A_1,\dots,A_k,B_1,\dots,B_{l-v}$ of $G \cap R$. The vertices $c_1,\dots,c_v,s_1,\dots,s_v$ can be partitioned into the v cliques $C_i := \{c_i,s_i\}$ $(1 \leq i \leq v)$ if G is a slim spider or $C_1 := \{c_1,s_v\}$, $C_i := \{c_i,s_{i-1}\}$ $(2 \leq i \leq v)$ if G is a fat spider. Then $A_1,\dots,A_k,B_1,\dots,B_{l-v},C_1,\dots,C_v$ are an $M[k,l-v+v]$-partition, which contradicts the fact that G obstructs $M = M[k,l]$. Figure 5.1 illustrates this $M[k,l]$-partition of G in the case of a slim spider graph.

Assume next that $k > 0$ and $l > 0$ although $G \cap R$ does not obstruct $M[1,l]$. Then there is an $M[1,l]$-partition $A,B_1,\dots,B_l$ of $G \cap R$ such that A is an independent set and $B_1,\dots,B_l$ are cliques. Since the vertices $s_1,\dots,s_v$ are non-adjacent to all vertices in R, especially A, and since the vertices $c_1,\dots,c_v$ are adjacent to all vertices in R, the set $A \cup \{s_1,\dots,s_v\}$ is an independent set in G and the set $B_1 \cup \{c_1,\dots,c_v\}$ is a clique in G. Therefore, $A \cup \{s_1,\dots,s_v\}, B_1 \cup \{c_1,\dots,c_v\}, B_2,\dots,B_l$ is an $M[1,l]$-partition of G. Since $k > 0$, the matrix $M[1,l]$ is a partition submatrix of $M[k,l]$ and therefore G does not only admit $M[1,l]$ but also $M[k,l] = M$, which is a contradiction. Figure 5.2 illustrates this $M[1,l]$-partition of G in the case of a fat spider graph. □

Proposition 7. *Let M be an $(1,*,*)$-block matrix with $l > 0$ and let G be a spider graph with the vertex set $V = \{c_1,\dots,c_v,s_1,\dots,s_v\} \cup R$ where $\{c_1,\dots,c_v\}$ is the spider's body, $\{s_1,\dots,s_v\}$ is the spider's legs, and R is the spider's head. G obstructs M if and only if $G \setminus \{c_1,\dots,c_v\}$ obstructs M.*

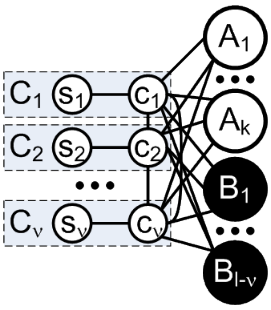

Figure 5.1.: Possible partition of G if $G \cap R$ does not obstruct $M[k, l - v]$

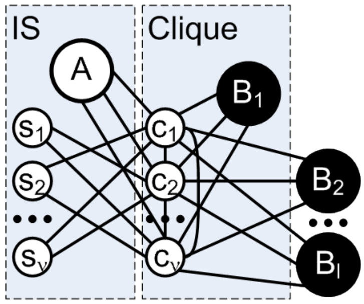

Figure 5.2.: Possible partition of G if $G \cap R$ does not obstruct $M[1, l]$ (IS is an independent set)

Proof. "⇒" Assume that G obstructs M. By Lemma 6, $G \cap R$ obstructs $M[k, l - v]$ and, if k is greater than 0, $G \cap R$ also obstructs $M[1, l]$.

We assume contrary to the proposition that $G' := G \setminus \{c_1, \dots, c_v\}$ admits M. Then there is an M-partition $A_1, \dots, A_k, B_1, \dots, B_l$ of G'.

Assume all vertices $s_1, \dots, s_v$ in the spider's legs are elements of clique parts. All vertices in the spider's legs are pairwise non-adjacent, therefore we may assume $s_1 \in B_1, \dots, s_v \in B_v$, without loss of generality. As the vertices in R are non-adjacent to the vertices in the spider's legs, none of the vertices in R can be an element of a clique together with a vertex s_j ($1 \leq j \leq v$). This implies $B_j \cap R = \emptyset$ ($1 \leq j \leq v$) and therefore the parts $A_1 \cap R, \dots, A_k \cap R, B_{v+1} \cap R, \dots, B_l \cap R$ constitute an $M[k, l - v]$-partition of $G \cap R$. This contradicts Lemma 6.

Assume that at least one of the vertices $s_1, \dots, s_v$ in the spider's legs is an element of an independent set part. This implies $k > 0$ and then, by Lemma 6, $G \cap R$ obstructs $M[1, l]$. Assuming $s_1 \in A_1$ without loss of generality, as M is a $(1, *, *)$-block matrix, all vertices in $A_2, \dots, A_k$ must be adjacent to s_1. Because s_1 is non-adjacent to the vertices in R, we get $A_j \cap R = \emptyset$ ($2 \leq j \leq k$). This makes $A_1 \cap R, B_1 \cap R, \dots, B_l \cap R$ an $M[1, l]$-partition of $G \cap R$, a contradiction to the initial assumption that $G \cap R$ obstructs $M[1, l]$.

The initial assumption must be wrong and G' obstructs M.

"⇐" Trivial. □

Complementing M yields the following corollary:

Corollary 8. *Let M be a $(*, 0, *)$-block matrix with $k > 0$ and let G be a spider graph with the vertex set $V = \{c_1, \dots, c_v, s_1, \dots, s_v\} \cup R$ where $\{c_1, \dots, c_v\}$ is the spider's body, $\{s_1, \dots, s_v\}$ is the spider's legs, and R is the spider's head. G obstructs M if and only if $G \setminus \{s_1, \dots, s_v\}$ obstructs M.*

Lemma 7. *Let M be an (a, b, c)-block matrix with $a = 1$ and let the graph G have an induced subgraph $\overline{P_3} := (\{x, y, z\}, \{x \sim z\})$. Then G obstructs $M[k, 0]$.*

Proof. Assume G had an $M[k,0]$-partition $A_1,\dots,A_k$. Because $a = 1$ and because there are no clique parts, two vertices are adjacent to each other if and only if they are in different independent set parts. The non-adjacent vertices x and y must be elements of the same independent set part, let this part be $x, y \in A_1$, without loss of generality. z is adjacent to x, which implies $z \notin A_1$, while it is non-adjacent to y, which implies $z \in A_1$. This is a contradiction and so G obstructs $M[k,0]$. □

Corollary 9. *Let M be an (a,b,c)-block matrix with $a = 1$ and let G be a spider graph with the vertex set $V = \{c_1,\dots,c_v,s_1,\dots,s_v\} \cup R$ where $\{c_1,\dots,c_v\}$ is the spider's body, $\{s_1,\dots,s_v\}$ is the spider's legs, and R is the spider's head.*

Then $G \setminus \{c_1\}$ obstructs $M[k,0]$ and for $G \neq P_4$, the following statements are true:

- *if G is a fat spider graph, the induced subgraph $G \setminus \{s_1,c_1\}$ obstructs $M[k,0]$.*
- *if G is a slim spider graph, the induced subgraph $G \setminus \{s_1,c_2\}$ obstructs $M[k,0]$.*

Proof. By Lemma 7, we have to show that these graphs contain $\overline{P_3}$ as an induced subgraph.

$G \setminus \{c_1\}$ contains the induced subgraph $G \cap \{c_2,s_1,s_2\} = \overline{P_3}$.

If G is a fat spider graph, $G' := G \setminus \{s_1,c_1\}$ contains the non-adjacent vertices $s := s_2$ and $c := c_2$. If G is a slim spider graph, let the vertices s and c in $G' := G \setminus \{s_1,c_2\}$ be defined as $s := s_2$ and $c := c_1$, so they are non-adjacent again. If there is a vertex $r \in R$, G' contains the induced subgraph $G \cap \{r,s,c\} = \overline{P_3}$, as shown in Figure 5.3. For $R = \varnothing$, we must have $v > 2$ as otherwise G would be P_4. In this case, G' contains $G \cap \{s_2,s_3,c_3\} = \overline{P_3}$, which is shown in Figure 5.4. □

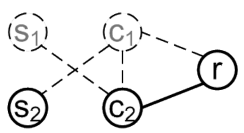

Figure 5.3.: Example of the induced subgraph $G \setminus \{s_1,c_1\}$ of a fat spider graph G

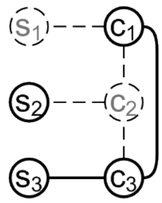

Figure 5.4.: Example of a slim spider graph $G \neq P_4$ and its induced subgraph $G \setminus \{s_1,c_2\}$

Combining the results of Proposition 7, Corollary 8, and Corollary 9 as well as its complement yields the following proposition since every M-obstruction spider graph G has an M-obstruction induced subgraph $G' \subsetneq G$:

Proposition 8. *Let M be a (a,b,c)-block matrix with $(a,b,c) \in \{(1,*,*),(*,0,*)\}$ and let G be a spider graph. Then G is not a minimal M-obstruction.*

Lemma 8. *Let M be an $(1, 0, *)$-block matrix and let G be a spider graph with the vertex set $V = \{c_1, \ldots, c_\nu, s_1, \ldots, s_\nu\} \cup R$ where $\{c_1, \ldots, c_\nu\}$ is the spider's body, $\{s_1, \ldots, s_\nu\}$ is the spider's legs, and R is the spider's head. G obstructs M if and only if $k = 0$ or $l = 0$ or all of the following conditions hold:*

- *$G \cap R$ obstructs $M[1, 1]$.*
- *$G \cap R$ obstructs $M[k - \nu, l - \nu]$.*
- *If G is a fat spider graph and $k \geq \nu$, then $G \cap R$ obstructs $M[0, l - \nu + 1]$.*
- *If G is a slim spider graph and $l \geq \nu$, then $G \cap R$ obstructs $M[k - \nu + 1, 0]$.*

Proof. "⇒" Let G obstruct M, $k > 0$, and $l > 0$.

Assume $G \cap R$ does not obstruct $M[1, 1]$. Then $G \cap R$ is partitioned into an independent set A and a clique B. Obviously, $A \cup \{s_1, \ldots, s_\nu\}, B \cup \{c_1, \ldots, c_\nu\}$ is an $M[1, 1]$-partition of G, so G would admit the submatrix $M[1, 1]$ of $M[k, l] = M$, which is not possible since G is an M-obstruction.

Assume that $G \cap R$ does not obstruct $M[k - \nu, l - \nu]$. Let the sets $A_1, \ldots, A_{k-\nu}, B_1, \ldots, B_{l-\nu}$ be an $M[k - \nu, l - \nu]$-partition of $G \cap R$. $A_1, \ldots, A_{k-\nu}, \{c_1\}, \ldots, \{c_\nu\}, B_1, \ldots, B_{l-\nu}, \{s_1\}, \ldots \{s_\nu\}$ comprise an $M[k, l]$-partition of G. No partition condition is violated, as all independent sets $A_1, \ldots, A_{k-\nu}, \{c_1\}, \ldots, \{c_\nu\}$ are adjacent to each other and all cliques $B_1, \ldots, B_{l-\nu}, \{s_1\}, \ldots \{s_\nu\}$ are non-adjacent to each other. As this is a contradiction to G obstructing M, this proves that $G \cap R$ obstructs $M[k - \nu, l - \nu]$.

By complementing G and M if necessary, we may assume that G is a fat spider graph. Assume $k \geq \nu$ and that $G \cap R$ does not obstruct $M[0, l - \nu + 1]$, so there is an $M[0, l - \nu + 1]$-partition $B_1, \ldots, B_{l-\nu+1}$ of R. We claim that $\{c_1, s_1\}, \{c_2\}, \ldots, \{c_\nu\}, B_1, \ldots, B_{l-\nu+1}, \{s_2\}, \ldots, \{s_\nu\}$ is an $M[\nu, l]$-partition of G, where $\{c_1, s_1\}, \{c_2\}, \ldots, \{c_\nu\}$ are ν independent sets and the other parts $B_1, \ldots, B_{l-\nu+1}, \{s_2\}, \ldots, \{s_\nu\}$ are $l - \nu + 1 + (\nu - 1)$ cliques. As G is a fat spider graph, c_1 and s_1 are non-adjacent and therefore $\{c_1, s_1\}$ really is an independent set. In a fat spider graph, s_1 is adjacent to the vertices $c_2, \ldots, c_\nu$, so the partition condition induced by $a = 1$ (all independent sets must be adjacent to each other) is not violated. As the vertices $s_2, \ldots, s_\nu$ are non-adjacent to each other as well as non-adjacent to the vertices in R, especially to the vertices in the other cliques $B_1, \ldots, B_{l-\nu+1}$, the partition condition following from $b = 0$ (all cliques must be non-adjacent to each other) is also not violated. Therefore G admits $M[\nu, l]$ and since $k \geq \nu$, G would also admit $M[k, l]$.

"⇐" By complementing G and M if necessary, we may assume that G is a fat spider graph. Let G admit M, so there is an M-partition $A_1, \ldots, A_k, B_1, \ldots, B_l$ of G. As a spider graph and because $a = 0$ and $b = 1$, Corollary 9 implies that G obstructs $M[k, 0]$ and $M[0, l]$, so we may assume $k > 0$ and $l > 0$. We may also assume that R is not empty, as otherwise $G \cap R$ admits $M[1, 1]$.

Let $n \in \{0, \ldots, \nu\}$ be the number of vertices in the spider's legs that are in one of the independent set parts, $n := |\{s_1, \ldots, s_\nu\} \cap \bigcup_{i=1}^{k} A_i|$. Let these vertices without loss of generality be $s_1, \ldots, s_n \in \bigcup_{i=1}^{k} A_i$ and let the other $\nu - n$ vertices $s_{n+1}, \ldots, s_\nu \in \bigcup_{j=1}^{l} B_j$ be in clique parts. Because of $a = 1$, a vertex in an independent set part must be adjacent to all vertices in different independent set parts. As $s_1, \dots, s_n$ are non-adjacent to each other, these vertices must be elements of the same independent set part, $s_1, \dots, s_n \in A_1$ without loss of generality.

Case $n > 1$**.** The vertices $c_1, \dots, c_n$ cannot be in one of the parts $A_1, \dots, A_k$: Each of these vertices is adjacent to one of the vertices $s_1, \dots, s_n \in A_1$ and so cannot be in the independent set A_1. Also, every vertex $c_1, \dots, c_n$ is non-adjacent to one of the vertices $s_1, \dots, s_n$ and therefore cannot be placed in one of the other independent set parts $A_2, \dots, A_k$, as $a = 1$ requires vertices in different independent set parts to be adjacent. As the vertices $c_1, \dots, c_n$ are adjacent to each other and as $b = 0$ implies that vertices in different clique parts are non-adjacent, all of the vertices $c_1, \dots, c_n$ are in the same clique part, $c_1, \dots, c_n \in B_1$ without loss of generality. All vertices in $R \cap A_j$ with $2 \leq j \leq k$ must be adjacent to all vertices in A_1, especially to s_1. Because R does not contain any vertices adjacent to s_1, $R \cap A_j$ is empty. Similarly, $R \cap B_j$ is empty for $2 \leq j \leq l$, because R does not contain vertices non-adjacent to $c_1 \in B_1$. Because of $R \subset A_1 \cup B_1$, the graph $G \cap R$ admits $M[1, 1]$.

Case $n = 1$**.** We have $s_1 \in A_1$ and the other vertices in the spider's leg $s_2, \dots, s_\nu$ are elements of clique parts. As $s_2, \dots, s_\nu$ are non-adjacent to each other, these vertices are in $\nu - 1$ different parts, $s_j \in B_{j-1}$ $(2 \leq j \leq \nu)$ without loss of generality. Any vertex in $A_2, \dots, A_k$ must be adjacent to $s_1 \in A_1$, which is not the case for the vertices in R. Because the clique parts $B_1, \dots, B_{\nu-1}$ all contain one of the vertices $s_2, \dots, s_\nu$ and each of those vertices are non-adjacent to the vertices in R, we have $R \subset A_1 \cup \left(\bigcup_{j=\nu}^{l} B_j\right)$.

$c_1 \in \bigcup_{i=2}^{k} A_i$ is not possible, as c_1 is non-adjacent to $s_1 \in A_1$ and so $a = 1$ prevents c_1 being an element of any independent set part other than A_1. If we have $c_1 \in A_1$, the vertices in R are adjacent to $c_1 \in A_1$ and thus are not elements of the independent set A_1. This implies that $G \cap R$ admits $M[0, l - \nu + 1]$, so we may assume $c_1 \in \bigcup_{j=1}^{l} B_j$. The vertex c_1 is not an element of a clique part different to B_1, because c_1 is adjacent to $s_2 \in B_1$ in spite of $b = 0$. Thus, $c_1 \in B_1$, which implies $R \cap B_j = \varnothing$ $(2 \leq j \leq l)$ because c_1 and the vertices in R are adjacent and vertices in different clique parts must be non-adjacent since we have $b = 0$. This results in $R \subset B_1 \cup A_1$ and so $G \cap R$ admits $M[1, 1]$.

Case $n = 0$**.** Every vertex in the spider's legs has its own clique part, $s_j \in B_j$ $(1 \leq j \leq \nu)$ without loss of generality. If in addition every vertex in the spider's body has its own independent set part, $c_j \in A_j$ $(1 \leq j \leq \nu)$ without loss of generality, then no vertex in R can be an element of one of the parts $A_1, \dots, A_\nu, B_1, \dots, B_\nu$ as R is adjacent to a vertex in each A_j but non-adjacent to a vertex in each B_j $(1 \leq j \leq \nu)$. $G \cap R$ therefore admits $M[k - \nu, l - \nu]$. Thus, we may assume $c_1 \in B_j$ for some $j \in \{1, \dots, k\}$. $j \neq 2$ is impossible since $s_2 \in B_2$ is adjacent to c_1 and vertices in different clique parts must be non-adjacent because $b = 0$. For $\nu > 2$, $s_3 \in B_3$ is also adjacent to c_1 and $j \neq 3$ is also impossible, so ν must be 2. c_2 cannot be an element of a clique part B_j with $j > 1$ since $s_1 \in B_1$ and c_2 are adjacent. On the other hand, $c_2 \in B_1$ is also impossible because $c_1 \in B_2$ and c_2 are adjacent. Therefore, c_2 must be an element of an independent set part, $c_2 \in A_1$ without loss of generality. The vertices in R cannot be elements of A_1 because $c_2 \in A_1$ and they cannot be elements of any B_j ($1 \leq j \leq l$) because $s_2, c_1 \in B_2$. This implies $R \subset \bigcup_{j=2}^{k} A_j$ and eventually that $G \cap R$ admits $M[k-1, 0] = M[k-\nu+1, 0]$. □

Lemma 9. *Let M be an (a,b,c)-block matrix with $c = 0$ and let G be a connected graph.*

G admits M if and only if G admits $M[k,0]$ or G admits $M[0,l]$.

Proof. "⇒" Let G admit M. Because $c = 0$, if two vertices are adjacent, they must both be elements of independent set parts or both must be elements of clique parts. Repeated usage of this fact on all vertex pairs in a path between two connected vertices implies that connected vertices must be both elements of independent set parts or both elements of clique parts. As G has only one connected component, either all independent set parts are empty or all clique parts of an M-partition are empty.

"⇐" Trivial. □

All spider graphs are connected and their complemented graphs are also connected, so by possibly complementing M and G, the preceding lemma can be applied to an even greater number of matrices if G is a spider graph:

Corollary 10. *Let M be an (a,b,c)-block matrix with $c \neq *$ and let G be a spider graph.*

G admits M if and only if G admits $M[k,0]$ or G admits $M[0,l]$.

Lemma 10. *Let M be an (a,b,c)-block matrix with $c \neq *$ and let G be a spider graph with the vertex set $V = \{c_1, \ldots, c_\nu, s_1, \ldots, s_\nu\} \cup R$ where $\{c_1, \ldots, c_\nu\}$ is the spider's body, $\{s_1, \ldots, s_\nu\}$ is the spider's legs, and R is the spider's head.*

If $G \setminus \{s_1\}$ admits $M[k,0]$ then $G \cap R$ admits $M[k-\nu, 0]$.

Proof. Let $G' := G \setminus \{s_1\}$ admit $M[k,0]$, then there is an $M[k,0]$-partition $A_1, \ldots, A_k$ of G'.

The vertices $c_1, \ldots, c_\nu$ are adjacent to each other, so they must be in ν different parts, $c_1 \in A_1, \ldots, c_\nu \in A_\nu$ without loss of generality. Each vertex in R is adjacent to each of the vertices $c_1, \ldots, c_\nu$, so the vertices in R cannot be elements of the independent sets $A_1, \ldots, A_\nu$. $A_1 \cap R, \ldots, A_k \cap R$ is an $M[k,0]$-partition of $G \cap R$ and $A_j \cap R = \varnothing$ ($1 \leq j \leq \nu$), so $A_{\nu+1}, \ldots, A_k$ is an $M[k-\nu, 0]$-partition. □

Lemma 11. *Let M be an (a,b,c)-block matrix with $c \neq *$ and let G be a spider graph with the vertex set $V = \{c_1, \ldots, c_\nu, s_1, \ldots, s_\nu\} \cup R$ where $\{c_1, \ldots, c_\nu\}$ is the spider's body, $\{s_1, \ldots, s_\nu\}$ is the spider's legs, and R is the spider's head.*

G admits $M[k,0]$ if and only if $G \cap R$ admits $M[k-\nu, 0]$ and $a = *$.

Proof. "⇒" This is an immediate result of Lemma 10 and Corollary 9.

"⇐" Let $G \cap R$ admit $M[k-\nu, 0]$ and $a = *$. There is an $M[k-\nu,0]$-partition $A_{\nu+1}, \dots, A_k$ of $G \cap R$. Let the parts $A_1, \dots, A_\nu$ be defined as

$$A_j := \begin{cases} \{s_j, c_j\} & \text{if } G \text{ is a fat spider,} \\ \{s_j, c_{j+1}\} & \text{if } G \text{ is a slim spider and } j < \nu, \\ \{s_j, c_1\} & \text{if } G \text{ is a slim spider and } j = \nu. \end{cases}$$

Obviously, the parts $A_1, \dots, A_k$ are independent sets. $a = *$ does not impose any restrictions on the adjacency of vertices in different independent sets and there are no further partition conditions as there are no clique parts. Thus, $A_1, \dots, A_k$ are a $M[k,0]$-partition of G. □

Complementing G and M in Lemmas 10, Lemma 11, and Corollary 9 yields the following corollary:

Corollary 11. *Let M be an (a,b,c)-block matrix with $c \neq *$ and let G be a spider graph with the vertex set $V = \{c_1, \dots, c_\nu, s_1, \dots, s_\nu\} \cup R$ where $\{c_1, \dots, c_\nu\}$ is the spider's body, $\{s_1, \dots, s_\nu\}$ is the spider's legs, and R is the spider's head.*

- *G admits $M[0,l]$ if and only if $G \cap R$ admits $M[0, l-\nu]$ and $b = *$.*
- *If $G \setminus \{c_1\}$ admits $M[0,l]$ and $b = *$ then G admits $M[0,l]$.*
- *If $b = 0$ then $G \setminus \{s_1\}$ obstructs $M[0,l]$.*
- *If $b = 0$ and $G \neq P_4$, then $G \setminus \{s_1, c_1\}$ or $G \setminus \{s_2, c_1\}$ obstructs $M[0,l]$.*

Table 5.1 summarizes the results found in Lemmas 10 and 11 as well as in Corollaries 9 and 11. In the table, assume $c \neq *$ and that G are a spider graph. The first column contains the possible combinations of a, b, and c while the second column contains statements that are equivalent to "G obstructs M" and the third column contains statements that must be true if G obstructs M. The abbreviation "obs." stands for "obstructs".

(a,b,c)	The spider graph G obstructs M...	
	equivalent to	precondition
$(*,*,0)$	$G \cap R$ obs. $M[k-\nu,0]$ and $M[0,l-\nu]$	-
$(*,*,1)$	$G \cap R$ obs. $M[k-\nu,0]$ and $M[0,l-\nu]$	-
$(1,*,0)$	$G \cap R$ obs. $M[0,l-\nu]$	$G \setminus \{c_1\}$ obs. $M[k,0]$ and $M[0,l]$
$(1,*,1)$	$G \cap R$ obs. $M[0,l-\nu]$	$G \setminus \{c_1\}$ obs. M
$(*,0,0)$	$G \cap R$ obs. $M[k-\nu,0]$	$G \setminus \{s_1\}$ obs. M
$(*,0,1)$	$G \cap R$ obs. $M[k-\nu,0]$	$G \setminus \{s_1\}$ obs. $M[k,0]$ and $M[0,l]$
$(1,0,0)$	always	-
$(1,0,1)$	always	-

Table 5.1.: Summarized results for spider graphs and $c \neq *$

Lemma 12. *Let M be an (a,b,c)-block matrix with $a = *, b \in \{0,*\}, c \in \{0,*\}$ and let $G = G_1 \cup G_2$ be a disconnected graph. Then G obstructs M if and only if*

$$\exists j \in \{0,\ldots,l+1\} \text{ such that } G_1 \text{ obstructs } M[k,j-1] \text{ and } G_2 \text{ obstructs } M[k,l-j]$$

Proof. "⇒" Assume that G obstructs M. Let $j-1$ be the greatest number smaller than $l+1$ such that G_1 obstructs $M[k,j-1]$. Such a number exists as obviously G_1 obstructs $M[k,-1]$. If G_1 admits $M[k,j]$, then G_2 obstructs $M[k,l-j]$ by Lemma 5. If G_1 does not admit $M[k,j]$, then j must be $l+1$ and therefore G_2 also obstructs $M[k,l-j] = M[k,l-(l+1)] = M[k,-1]$.

"⇐" Let $j \in \{0,\ldots,l+1\}$ be such that G_1 obstructs $M[k,j-1]$ and G_2 obstructs $M[k,l-j]$. For every $j' \le j$, G_1 obstructs $M[k,j'-1]$ because it is a submatrix of $M[k,j-1]$. For every $j' \ge j$, G_2 obstructs $M[k,l-j']$ because it is a submatrix of $M[k,j-1]$.

Assume contrary to the original claim that G admits M. By Lemma 1, there exist sets of parts $P,Q \subset \mathbb{N}^*_{k+l}$ such that G_1 admits M_P, G_2 admits M_Q, and $M_{P,Q}$ contains no 1. M contains 1s only in the diagonal, so the last condition implies that P and Q do not share parts with 1 in the diagonal. Therefore M_P is a submatrix of $M[k,j'-1]$ and M_Q is a submatrix of $M[k,l-j']$ with $j' \in \{1,\ldots,l\}$. As G_1 admits the submatrix M_P, it also admits $M[k,j'-1]$ and, similarly, M_Q also admits $M[k,l-j']$. This contradicts the fact that no $j' \in \mathbb{N}$ with such a property exists. Thus, G obstructs M. □

Lemma 13. *Let M be an (a,b,c)-block matrix with $a = 1, b \in \{0,*\}, c \in \{0,*\}$ and let $G = G_1 \cup G_2$ be a disconnected graph. Then G obstructs M if and only if there exist $i,j,\lambda \in \{0,\ldots,l+1\}$ such that the following holds*

1. *G_1 obstructs $M[k,i-1]$ and G_2 obstructs $M[0,l-i]$,*
2. *G_2 obstructs $M[k,j-1]$ and G_1 obstructs $M[0,l-j]$,*
3. *$i+j < l+2$, and*
4. *if $k > 0$ then G_1 obstructs $M[1,\lambda-1]$ and G_2 obstructs $M[1,l-\lambda]$.*

Proof. "⇒" Assume that $G = G_1 \cup G_2$ obstructs M.

Let i be the smallest number in $\{0,\ldots,l\}$ such that G_1 admits $M[k,i]$ or $i := l+1$ if no such number exists. By definition, G_1 obstructs $M[k,i-1]$. For $i = l+1$, G_2 obviously obstructs $M[0,l-i]$, so assume $i \le l$. If G_2 admits $M[0,l-i]$, then there are an $M[0,l-i]$-partition $B_1,\ldots,B_{l-i}$ of G_2 and an $M[k,i]$-partition $A_1,\ldots,A_{k+i}$ of G_1. $A_1,\ldots,A_{k+i},B_1,\ldots,B_{l-i}$ is an $M[k,l]$-partition of G, a contradiction, so G_2 obstructs $M[0,l-i]$.

By exchanging G_1 and G_2 as well as replacing i with j in the proof above, the claim that G_2 obstructs $M[k,j-1]$ and G_1 obstructs $M[0,l-j]$ is proved.

Hence, there exist $i,j \in \{0,\ldots,l+1\}$ such that the first two conditions of the claim are satisfied. Assume now that $i,j \in \{0,\ldots,l+1\}$ are the lowest numbers such that the first two conditions are satisfied. Then G_1 obstructs $M[k,i-1]$ and thus G_1 also obstructs $M[k,i-2]$. So if G_2 obstructs

$M[0,l-(i-1)]$, then not only i but also $i-1$ satisfies the first condition, in which case we have $i=0$. For $i>0$, G_2 admits $M[0,l-(i-1)]$ and so i is the smallest number such that G_2 obstructs $M[0,l-i]$. G_2 obstructs $M[k,j-1]$ and therefore G_2 obstructs $M[0,j-1]$ and thus $l-i \geq j-1$, for $i=0$ as well as $i>0$. Hence, we have $l-i \geq j-1 \Leftrightarrow j+i<l+2$.

For the proof of the last of the four conditions, assume $k>0$. Let λ be the smallest number in $\{0,\ldots,l\}$ such that G_1 admits $M[1,\lambda]$ or $\lambda := l+1$ if G_1 obstructs $M[1,x]$ for all $x \in \{0,\ldots,l\}$. For $\lambda = l+1$, G_2 obviously obstructs $M[1,-1] = M[1,l-\lambda]$. Assume $\lambda \leq l$ and assume G_2 had an $M[1,l-\lambda]$-partition $C, D_1,\ldots,D_{l-\lambda}$. Together with the $M[1,\lambda]$-partition $A, B_1,\ldots,B_\lambda$ of G_1, there is an $M[1,l]$-partition $A \cup C, B_1,\ldots,B_\lambda, D_1,\ldots,D_{l-\lambda}$ of G. Contradicting to the first assumption of this proof, G admits $M[1,l]$ and therefore also admits $M[k,l]$. Hence, the assumption that G_2 admits $M[1,l-\lambda]$ must be wrong. By the definition of λ, G_1 obstructs $M[1,\lambda-1]$ which proves the fourth condition.

"$\Leftarrow$" Assume that $G = G_1 \cup G_2$ admits M. This proof will show that one of the four conditions is violated. Let $A_1,\ldots,A_k,B_1,\ldots,B_l$ be an $M[k,l]$-partition of G. As $B_1,\ldots,B_l$ are cliques, each of them is a subset of V_1 or a subset of V_2. By reordering the parts, we may assume $(B_1 \cup \ldots \cup B_\mu) \subset V_1$ and $(B_{\mu+1} \cup \ldots \cup B_l) \subset V_2$ with $\mu \in \{0,\ldots,l\}$. Depending on which of the independent set parts are non-empty, two cases can be distinguished:

Case 1: There are at least two independent set parts that are non-empty. Without loss of generality, there is a vertex $v \in A_1$ and a vertex $w \in A_2$. Because $a=1$, vertices in different independent set parts must be adjacent to each other and must therefore be in the same induced subgraph, either G_1 or G_2. All vertices in $A_2,\ldots,A_k$, especially w, must therefore be in the same induced subgraph as v, and all vertices in A_1 must be in the same induced subgraph as w, which is the same induced subgraph as v. We may assume, without loss of generality, $\bigcup_{x=1}^{k} A_x \subset V_1$ with $G_1 = (V_1,E_1)$. Thus, G_1 has an $M[k,\mu]$-partition $A_1,\ldots,A_k,B_1,\ldots,B_\mu$ and G_2 has an $M[0,l-\mu]$-partition $B_{\mu+1},\ldots,B_l$. For any $i \in \{0,\ldots,\mu\}$, G_2 admits $M[0,l-i]$, and for $i \in \{\mu+1,\ldots,l+1\}$, G_1 admits $M[k,i-1]$, which violates the first of the four conditions.

Case 2: At most one of the independent set parts is non-empty. Without loss of generality, $A_x = \varnothing$ $(2 \leq x \leq k)$. $A_1,B_1,\ldots,B_\mu$ is an $M[1,\mu]$-partition of G_1 and $A_1,B_{\mu+1},\ldots,B_l$ is an $M[1,l-\mu]$-partition of G_2. For any $\lambda \in \{0,\ldots,\mu\}$, G_2 admits $M[1,l-\lambda]$ and for any $\lambda \in \{\mu+1,\ldots,l+1\}$, G_1 admits $M[1,\lambda-1]$. In any case, the fourth condition is violated. $\square$

With the results already found in this chapter, we can obtain an upper limit for the size of a minimal M-obstruction, P_4-sparse graph. In order to prove this upper limit, the limit will be shown to be correct for so called staircase-like collections of matrices and then applied to the special case of a single matrix M. Therefore, the definition of staircase-like matrix collections precedes the proof that these collections adhere to the upper limit mentioned above. The following definition is taken from [FHH06], Theorem 4.3.

Definition 24. *Let $\mathcal{M} = \{M_0, \dots, M_r\}$ be a collection of (a,b,c)-block matrices with $r \in \mathbb{N}$ and let $k_i, l_i \in \mathbb{N}$ be chosen such that $M_i = M[k_i, l_i]$ $(0 \le i \le r)$. $\mathcal{M}$ is* ***staircase-like*** *if $k_i \le k_j$ and $l_i \ge l_j$ for all $i < j$.*

For any staircase-like collection of matrices $\mathcal{M}$ as defined above, we define $k_{-1} := -1$ and $l_{r+1} := -1$.

Lemma 14. *Let $a \in \{1, *\}$, $b \in \{0, *\}$, $c \in \{0, 1, *\}$ be fixed and let $\mathcal{M}_{(a,b,c)}$ be defined as the set of all (a,b,c)-block matrices. Let the function $f :\subset \mathcal{P}(\mathcal{M}_{(a,b,c)}) \longrightarrow \mathbb{N}$ be defined for each finite, non-empty, staircase-like collection of (a,b,c)-block matrices $\mathcal{M} = \{M[k_0, l_0]; \dots; M[k_r, l_r]\}$ as*

$$f : \mathcal{M} \longmapsto \sum_{i=0}^{r} (k_i - k_{i-1})(l_i + 1) = \sum_{i=0}^{r} (l_i - l_{i+1})(k_i + 1)$$

Then f is well defined and, for $r > 0$ and each $j \in \{0, \dots, r\}$, $f(\mathcal{M}) \ge f(\mathcal{M} \setminus \{M[k_j, l_j]\})$.

Proof. We will first show that f is well-defined and start by showing that the two sums are equal.

$$\begin{aligned}
\sum_{i=0}^{r} (k_i - k_{i-1})(l_i + 1) =& k_r(l_{r+1} + 1) - k_{-1}(l_0 + 1) + \sum_{i=0}^{r} (k_i(l_i + 1) - k_i(l_{i+1} + 1)) \\
=& l_0 + 1 + \sum_{i=0}^{r} k_i(l_i - l_{i+1}) + \sum_{i=0}^{r} (l_i - l_{i+1}) - \sum_{i=0}^{r} (l_i - l_{i+1}) \\
=& l_0 + 1 + \sum_{i=0}^{r} (k_i + 1)(l_i - l_{i+1}) - l_0 + l_{r+1} \\
=& \sum_{i=0}^{r} (k_i + 1)(l_i - l_{i+1})
\end{aligned}$$

For $r > 0$ and $j = r$, the inequality $f(\mathcal{M}) = \sum_{i=0}^{r} (k_i - k_{i-1})(l_i + 1) \ge \sum_{i=0}^{r-1} (k_i - k_{i-1})(l_i + 1) = f(\mathcal{M} \setminus \{M[k_r, l_r]\})$ follows immediately from the definition of f and $k_r - k_{r-1} \ge 0$, $l_r \ge 0$. Now the inequality $f(\mathcal{M}) \ge f(\mathcal{M} \setminus \{M[k_j, l_j]\})$ for $j \in \{0, \dots, r-1\}$ is shown. As $\mathcal{M}$ is staircase-like, we have $l_j \ge l_{j+1}$ and $k_j - k_{j-1} \ge 0$, so we see

$$\begin{aligned}
f(\mathcal{M}) =& (k_j - k_{j-1})(l_j + 1) + (k_{j+1} - k_j)(l_{j+1} + 1) + \sum_{\substack{0 \le i \le r \\ i \notin \{j, j+1\}}} (k_i - k_{i-1})(l_i + 1) \\
\ge& (k_j - k_{j-1} + k_{j+1} - k_j)(l_{j+1} + 1) + \sum_{\substack{0 \le i \le r \\ i \notin \{j, j+1\}}} (k_i - k_{i-1})(l_i + 1) \\
=& f(\mathcal{M} \setminus \{M[k_j, l_j]\})
\end{aligned}$$

□

Theorem 7. *Let $a \in \{1, *\}$, $b \in \{0, *\}$, $c \in \{0, 1, *\}$ be fixed. Let $\mathcal{M} = \{M[k_0, l_0]; \dots; M[k_r, l_r]\}$ be a staircase-like collection of (a,b,c)-block matrices. Let G be a P_4-sparse, minimal $\mathcal{M}$-obstruction graph. Then G has at most $f(\mathcal{M})$ vertices.*

Proof. The proof will use induction over the size of the graph G. For the induction basis, let G_0 be a minimal $\mathcal{M}$-obstruction, P_4-sparse graph consisting of only one vertex. Because $(k_0 - k_{-1})(l_0 + 1) \geq 1$ and $(k_i - k_{i-1})(l_i + 1) \geq 0$ $(1 \leq i \leq r)$, we have $f(\mathcal{M}) \geq 1 = |G|$ and so the proposition is correct for $|G| = 1$.

Assume all graphs G' with a lower number of vertices than G have at most $f(\mathcal{M}')$ vertices if there is an (a, b, c)-block matrix set $\mathcal{M}'$ such that G' is a minimal $\mathcal{M}'$-obstruction. As a P_4-sparse graph, G has a tree representation and its root node is either a spider node, in which case G is a spider graph, or G has a node labeled $\cup$ or $\cap$, in which case G or its complement is disconnected. Each of these two cases has subcases depending on the values of a, b, and c.

Case 1: G is a spider graph. In this case, G is a spider graph with the vertex set $V = \{c_1, \ldots, c_v, s_1, \ldots, s_v\} \cup R$ where $\{c_1, \ldots, c_v\}$ is the spider's body, $\{s_1, \ldots, s_v\}$ is the spider's legs, and R is the spider's head.

Subcase 1a: $(a, b, c) = (*, *, *)$ For $0 = l_0 \geq l_i$ $(0 \leq i \leq r)$, G is obviously a minimal $\mathcal{M}$-obstruction if and only if it is a minimal $M[k_r, 0]$-obstruction. By Corollary 7, G is a cograph in this case and therefore cannot be a spider graph. Similarly, for $0 = k_r \geq k_i$ $(0 \leq i \leq r)$, G is a minimal $M[0, l_0]$-obstruction and therefore a cograph and not a spider graph. Hence, assume $l_0 > 0$ and $k_r > 0$.

$G \cap R$ admits a matrix $M[k_j, l_j] \in \mathcal{M}$ because $G \cap R \neq G$ is an induced subgraph of the minimal $\mathcal{M}$-obstruction G. Thus, there is an $M[k_j, l_j]$-partition $A_1, \ldots, A_{k_j}, B_1, \ldots, B_{l_j}$, where $A_1, \ldots, A_{k_j}$ are independent sets and $B_1, \ldots, B_{l_j}$ are cliques. The vertices $s_1, \ldots, s_v$ are non-adjacent to the vertices in R and therefore $A_1 \cup \{s_1, \ldots, s_v\}$ is an independent set. The vertices $c_1, \ldots, c_v$ are adjacent to the vertices in R and so $B_1 \cup \{c_1, \ldots, c_v\}$ is a clique. Thus, $A_1 \cup \{s_1, \ldots, s_v\}, A_2, \ldots, A_{k_j}, B_1 \cup \{c_1, \ldots, c_v\}, B_2, \ldots, B_{l_j}$ is an $M[k_j, l_j]$-partition of G, which contradicts that G is an $\mathcal{M}$-obstruction. This implies that in the subcase $(a, b, c) = (*, *, *)$, G must not be a spider graph.

Subcase 1b: $(a, b, c) \in \{(1, *, *), (*, 0, *)\}$. By Proposition 8, G cannot be a minimal $\mathcal{M}$-obstruction, so this subcase is impossible.

Subcase 1c: $(a, b, c) = (1, 0, *)$. Assume $G = P_4$, the smallest possible spider graph. We have $f(\mathcal{M}) < 4 = |P_4|$ only if

$$\mathcal{M} \subset \{M[0, 1]; M[0, 2]; M[1, 0]; M[2, 0]\} \text{ and } \{M[0, 2]; M[2, 0]\} \not\subset \mathcal{M}.$$

The graph P_3 is an induced subgraph of P_4 and P_3 obstructs $\{M[0, 1]; M[0, 2]; M[1, 0]\}$. The graph $\overline{P_3}$, the complement of P_3, is also an induced subgraph of P_4 and obstructs the matrix set $\{M[0, 1]; M[1, 0]; M[2; 0]\}$. In all cases with $f(\mathcal{M}) < |P_4|$, P_4 is not a minimal $\mathcal{M}$-obstruction, because P_4 has an $\mathcal{M}$-obstruction as an induced subgraph. We may assume $G \neq P_4$ in the following. As a result, we may further assume $M[k,l] \in \mathcal{M}$ with $k > 0 \wedge l > 0$, as otherwise the induced subgraph P_4 of G would obstruct $\mathcal{M}$, which is a result of Lemma 8: Every spider graph and especially P_4 obstructs $M[0,l]$ and $M[k,0]$ for every $k,l \in \mathbb{N}$. Because G obstructs a matrix $M[k,l]$ with $k > 0$ and $l > 0$, $G \cap R$ obstructs $M[1,1]$ by Lemma 8.

By possibly complementing G and M, we may assume that G is a fat spider graph with 2ν vertices in the spider's body and legs.

The induced subgraph $G' := G \setminus \{s_1, c_1\}$ of G is a spider graph for $\nu > 2$. As a result of Lemma 8, G' obstructs $M[\nu-2, x]$ and $M[x, \nu-2]$ for all $x \in \mathbb{N}$, as well as $M[\nu-1,\nu-1]$, $M[\nu,\nu-1]$. G is a minimal $\mathcal{M}$-obstruction, so G' does not obstruct $\mathcal{M}$ and therefore $\mathcal{M}$ contains a matrix $M[k_j, l_j]$ with $j \in \{0,\dots,r\}$ that G' admits. This obviously implies $\big(k_j > \nu-2\big) \wedge \big(l_j > \nu-2\big) \wedge \big(l_j = \nu-1 \Rightarrow k_j > \nu\big)$. Therefore, we may distinguish the following cases, of which at least one occurs:

- $\exists M[k_j,l_j] \in \mathcal{M}$ such that $k_j > \nu \wedge l_j > \nu$
- $\exists M[k_j,l_j] \in \mathcal{M}$ such that $k_j > \nu \wedge l_j = \nu$
- $\exists M[k_j,l_j] \in \mathcal{M}$ such that $k_j = \nu \wedge l_j \geq \nu$
- $\nu > 2 \wedge \exists M[k_j,l_j] \in \mathcal{M}$ such that $k_j = \nu-1 \wedge l_j \geq \nu$
- $\exists M[k_j,l_j] \in \mathcal{M}$ such that $k_j > \nu \wedge l_j = \nu-1$
- $\nu = 2$ and $\mathcal{M} \subset (\{M[0,y] \mid y \in \mathbb{N}\} \cup \{M[1,z] \mid z \in \mathbb{N}\} \cup \{M[2,1]\} \cup \{M[x,0] \mid x \in \mathbb{N}\})$

Assume that $\mathcal{M}$ contains a matrix $M[k_j,l_j]$ with $k_j > \nu \wedge l_j \geq \nu$. Let $s := \min\{s' \mid k_{s'} \geq \nu\}$, $t := \min\{t' \mid k_{t'} > \nu\}$, and $u := \max\{u' \mid l_{u'} \geq \nu\}$. Note that $l_s \geq \nu$, $l_t \geq \nu$, and $k_u > \nu$ because $\mathcal{M}$ is staircase-like. By Lemma 8, $G \cap R$ is a minimal $\tilde{\mathcal{M}}$-obstruction with

$$\tilde{\mathcal{M}} := \begin{cases} \{M[0, l_s-\nu+1], M[1,1], M[k_t-\nu, l_t-\nu], \dots, M[k_u-\nu, l_u-\nu]\} & \text{if } l_t = \nu, \\ \{M[0, l_s-\nu+1], M[k_t-\nu, l_t-\nu], \dots, M[k_u-\nu, l_u-\nu]\} & \text{if } l_t > \nu. \end{cases}$$

In both cases, $\tilde{\mathcal{M}}$ is staircase-like so the inductive assumption can be applied to $G \cap R$ and $\tilde{\mathcal{M}}$. For $l_t > \nu$ ($\Leftrightarrow \exists M[k_j,l_j] \in \mathcal{M}.k_j > \nu \wedge l_j > \nu$), this implies

$$\begin{aligned}
|G \cap R| \leq& f(\tilde{\mathcal{M}}) \\
=& l_s - \nu + 1 - (l_t - \nu) + \sum_{i=t}^{u-1} (l_i - \nu - (l_{i+1} - \nu))(k_i - \nu + 1) + (l_u - \nu + 1)(k_u - \nu + 1) \\
=& l_s - l_t + 1 + \sum_{i=t}^{u-1} (l_i - l_{i+1})(k_i - \nu + 1) - \nu(l_u + k_u - \nu + 2) + (l_u + 1)(k_u + 1) \\
=& l_s - l_t + \sum_{i=t}^{u-1} (l_i - l_{i+1})(k_i + 1) + (l_u + 1)(k_u + 1) - \nu(l_t - l_u) - \nu(l_u + k_u - \nu + 2) + 1 \\
\leq& \sum_{i=0}^{r} (l_i - l_{i+1})(k_i + 1) - \nu(l_t + k_u - \nu + 2) + 1 \\
=& f(\mathcal{M}) + 1 - \nu(l_t + k_u - \nu + 2)
\end{aligned}$$

Remembering $l_t > \nu \geq 2$ and $k_u > \nu \geq 2$, we see $|G \cap R| \leq f(\mathcal{M}) - 2\nu$ and therefore $|G| = |G \cap R| + 2\nu \leq f(\mathcal{M})$.

For $l_t = \nu$, we see

$$\begin{aligned}
|G \cap R| \leq & f(\tilde{\mathcal{M}}) \\
&= l_s - \nu + 1 - 1 + (1+1) \cdot (1 - (l_t - \nu)) \\
&\quad + \sum_{i=t}^{u-1} (l_i - \nu - (l_{i+1} - \nu))(k_i - \nu + 1) + (k_u - \nu + 1)(l_u - \nu - (-1)) \\
&= l_s - l_t + \sum_{i=t}^{u-1} (l_i - l_{i+1})(k_i + 1) - \nu(l_t - l_u) + (k_u + 1)(l_u + 1) + 2 \\
&\quad - \nu(k_u + l_u - \nu + 2) \\
&\leq \sum_{i=0}^{r} (k_i + 1)(l_i - l_{i+1}) + 2 - \nu(k_u + l_t - \nu + 2) \\
&= f(\mathcal{M}) + 2 - \nu(k_u + 1) < f(\mathcal{M}) - 2\nu
\end{aligned}$$

Thus, G has at most $|G| = 2\nu + |G \cap R| \leq f(\mathcal{M})$ vertices.

Now assume that the first two cases do not apply but that there is a matrix $M[k_j, l_j] \in \mathcal{M}$ with $k_j = \nu \wedge l_j \geq \nu$. Let s be the lowest number satisfying this condition, $s := \min\{s' \mid k_{s'} = \nu\}$ (this implies $l_s \geq \nu$, as $\mathcal{M}$ is staircase-like). By Lemma 8, $G \cap R$ is a minimal $\tilde{\mathcal{M}}$-obstruction with

$$\tilde{\mathcal{M}} := \{M[0, l_s - \nu + 1], M[1,1]\}$$

$\tilde{\mathcal{M}}$ obviously is staircase-like, so the inductive assumption applies:

$$\begin{aligned}
|G \cap R| \leq f(\tilde{\mathcal{M}}) &= l_s - \nu + 1 - 1 + (1+1)(1 - (-1)) \\
&= l_s - \nu + 2 \cdot 2 \\
&= 2 \cdot 2 + l_s + k_s - k_s - \nu \\
&\leq l_s \cdot k_s + l_s + k_s - 2\nu \\
&< (l_s + 1)(k_s + 1) - 2\nu \\
&\leq \sum_{i=0}^{r} (l_i - l_{i+1})(k_i + 1) - 2\nu = f(\mathcal{M}) - 2\nu
\end{aligned}$$

Again we conclude $|G| = |G \cap R| + 2\nu \leq f(\mathcal{M})$.

If the first three cases do not apply, which implies $k_i < \nu \vee l_i < \nu$ ($0 \leq i \leq r$), assume that we have $\nu > 2$ and that there is a matrix $M[k_j, l_j] \in \mathcal{M}$ with $k_j = \nu - 1 \wedge l_j \geq \nu$. By Lemma 8 and for any $F \subset R$, the induced subgraph $G \setminus F$ of G would obstruct $\mathcal{M}$ if $G \cap (R \setminus F)$ obstructed $M[1,1]$. G is a minimal $\mathcal{M}$-obstruction, so this is only true for $F = \emptyset$. Hence, $G \cap R$ is a minimal $M[1,1]$-obstruction and so we have $|R| \leq f(\{M[1,1]\}) = 4$ by the inductive assumption. Considering $\nu > 2$, the number of vertices of G is

$$|G| \leq 2\nu + 4 \leq (\nu - 1)\nu + \nu - \nu + 4 < \nu^2 + \nu = \nu \cdot (\nu + 1) \leq (k_j + 1)(l_j + 1) \leq f(\mathcal{M})$$

Now assume $l_i < \nu$ $(0 \le i \le r)$ and that there is a matrix $M[k_j, l_j] \in \mathcal{M}$ with $k_j > \nu \wedge l_j = \nu - 1$. For the same reasons as in the previous case, $G \cap R$ is a minimal $M[1,1]$-obstruction, $|R| \le 4$, and so we have

$$|G| \le 2\nu + 4 \le \nu^2 + 2\nu = \nu \cdot (\nu + 2) \le (l_j + 1)(k_j + 1) \le f(\mathcal{M})$$

As the last possibility in subcase $(a, b, c) = (1, 0, *)$, we consider $\nu = 2$ and

$$\mathcal{M} \subset (\{M[0,y] \mid y \in \mathbb{N}\} \cup \{M[1,z] \mid z \in \mathbb{N}\} \cup \{M[2,1]\} \cup \{M[x,0] \mid x \in \mathbb{N}\})$$

As in the previous two cases, R is a minimal $M[1,1]$-obstruction because if $G \cap (R \setminus F)$ with $F \subset R$ and $F \neq \emptyset$ obstructs $M[1,1]$, then $G \setminus F$ obstructs $\mathcal{M}$ because of Lemma 8. Thus, we have $|R| \le f(\{M[1,1]\}) = 4$ and therefore $|G| = 2\nu + |R| \le 8$. Hence, we may assume $\{M[1,z] \mid z \in \mathbb{N} \wedge z > 2\} \cap \mathcal{M} = \emptyset$ as otherwise we would have $f(\mathcal{M}) \ge f(\{M[1,3]\}) = 8 \ge |G|$ by Lemma 14.

Assume the induced subgraph $G_S := G \setminus \{c_1, c_2\}$ of G had an $M[x,1]$-partition $A_1, \ldots, A_x, B_1$ with $x \in \mathbb{N}$. Because $G \cap R \subset G_S$ obstructs $M[1,1]$, at least two of the independent set parts contain vertices of R, let $r_1 \in R \cap A_1$, $r_2 \in R \cap A_2$ without loss of generality. Vertices in two different independent set parts must be adjacent because $a = 1$. Since s_1, s_2 and $r_2 \in A_2$ are non-adjacent, s_1 as well as s_2 can neither be in A_1 nor in A_i $(2 \le i \le x)$ as s_1, s_2 and $r_1 \in A_1$ are also non-adjacent. This implies $s_1, s_2 \in B_1$, which is also impossible since B_1 is a clique although s_1 and s_2 are non-adjacent. This disproves the assumption that G_S admits $M[x,1]$. Assume instead that G_S has an $M[0,2]$-partition B_1, B_2. As $G \cap R$ obstructs $M[1,1]$ and therefore also $M[0,1]$, there are two vertices $r_1 \in R \cap B_1$ and $r_2 \in R \cap B_2$. The vertex s_1 is non-adjacent to r_1 and r_2 and therefore cannot be in any of the cliques B_1 and B_2, which shows that G_S has no $M[0,2]$-partition. Taking the two results together, G_S obstructs $\{M[0,2]; M[x,1]\}$ for all $x \in \mathbb{N}$. Similarly, $G_C := G \setminus \{s_1, s_2\}$ obstructs $\{M[1,y]; M[2,0]\}$ for all $y \in \mathbb{N}$.

Because G is a minimal $\mathcal{M}$-obstruction, every induced subgraph of G must admit some matrix $M \in \mathcal{M}$. If $\mathcal{M}$ contains neither $M[2,1]$ nor $M[1,2]$, then G_S or G_C obstructs $\mathcal{M}$, so this is not possible. If $\mathcal{M}$ contains only one of these two matrices, without loss of generality $M[2,1] \in \mathcal{M}$, there must be a matrix $M[0,y] \in \mathcal{M}$ with $y > 2$ as otherwise G_S obstructs $\mathcal{M}$. Let y be the greatest number such that $M[0,y] \in \mathcal{M}$. Then we have

$$f(\mathcal{M}) \ge f(\{M[0,y]; M[2,1]\}) = (0 - (-1))(y+1) + (2 - 0)(1 + 1) = y + 5 \ge 8 \ge |G|$$

Similarly, $M[x,0], M[1,2] \in \mathcal{M}$ with $x > 2$ also implies $f(\mathcal{M}) \ge |G|$. For $\{M[1,2]; M[2,1]\} \subset \mathcal{M}$, we see

$$f(\mathcal{M}) \ge f(\{M[1,2]; M[2,1]\}) = (1 - (-1))(2+1) + (2-1)(1+1) = 8 \ge |G|$$

Subcase 1d: $(a,b,c) \in \{(*,*,0);(*,*,1)\}$. Assume first $l_0 > \nu \wedge k_r > \nu$. G is a minimal $\mathcal{M}$-obstruction, so we see in Table 5.1 that $G \cap R$ is a minimal $\tilde{\mathcal{M}}$-obstruction with

$$\tilde{\mathcal{M}} := \{M[0,l_0-\nu];M[k_r-\nu,0]\}$$

$\tilde{\mathcal{M}}$ is staircase-like, so the inductive assumption limits the size of R to

$$\begin{aligned}|G\cap R| \leq & f(\tilde{\mathcal{M}})\\ =&(0-(-1))(l_0-\nu+1)+(k_r-\nu-0)(0+1)\\ =&l_0+k_r+1-2\nu\\ \leq & f(\mathcal{M})-2\nu\end{aligned}$$

This proves the proposition: $|G| = |R| + 2\nu \leq f(\mathcal{M}) - 2\nu + 2\nu = f(\mathcal{M})$.

For $l_0 \leq \nu \vee k_r \leq \nu$, note that $G \setminus \{s_1\}$ is a $\{M[k_0,0];\ldots;M[k_r,0]\}$-obstruction by Lemma 11 and Lemma 10. Corollary 11 implies that $G \setminus \{c_1\}$ is a $\{M[0,l_0];\ldots;M[0,l_r]\}$-obstruction. Neither $G \setminus \{s_1\}$ nor $G \setminus \{c_1\}$ are complete or empty, so both obstruct $M[1,0]$ and $M[0,1]$. By Lemma 9, possibly with complemented G and M, if $c = 0$, $G \setminus \{s_1\}$ obstructs $M[1,1]$ and, if $c = 1$, $G \setminus \{c_1\}$ obstructs $M[1,1]$. This implies $l_0 > 0$, $k_r > 0$, and $l_0 > 1 \vee k_r > 1$, so $f(\mathcal{M}) \geq 4$ and we may assume $G \neq P_4$.

Now assume $\nu > 2$. As shown in Table 5.1, G obstructs a matrix $M[k_i,l_i] \in \mathcal{M}$ $(0 \leq i \leq r)$ if and only if $G \cap R$ obstructs $M[k_i-\nu,0]$ as well as $M[0,l_i-\nu]$. Similarly, the spider graph $G \setminus \{s_1,c_1\}$ obstructs $M[k,l]$ if and only if $(G \setminus \{s_1,c_1\}) \cap R$ obstructs $M[k-(\nu-1),0]$ as well as $M[0,l-(\nu-1)]$ $(k,l \in \mathbb{N})$. Thus, $G \setminus \{s_1,c_1\}$ obstructs $\{M[k_i,l_i] \mid k_i < \nu \wedge l_i < \nu \wedge 0 \leq i \leq r\}$. $G \setminus \{s_1,c_1\}$ is an induced subgraph of both $G \setminus \{s_1\}$ and $G \setminus \{c_1\}$, so $G \setminus \{s_1\}$ obstructs $\{M[k_0,0];\ldots;M[k_r,0]\} \cup \{M[k_i,l_i] \mid k_i < \nu \wedge l_i < \nu \wedge 0 \leq i \leq r\}$ and $G \setminus \{c_1\}$ obstructs $\{M[0,l_0];\ldots;M[0,l_r]\} \cup \{M[k_i,l_i] \mid k_i < \nu \wedge l_i < \nu \wedge 0 \leq i \leq r\}$. Since $G \setminus \{c_1\}$ and $G \setminus \{s_1\}$ do not obstruct $\mathcal{M}$, there must be matrices $M[k_s,l_s], M[k_t,l_t] \in \mathcal{M}$ with $k_s > 0 \wedge (l_s \geq \nu \vee k_s \geq \nu)$ and $k_t > 0 \wedge (l_t \geq \nu \vee k_t \geq \nu)$. As we have $l_0 \leq \nu \vee k_r \leq \nu$, we may assume $l_0 \geq l_s \geq \nu \wedge 0 < k_s \leq k_r \leq \nu$ by possibly complementing G and $\mathcal{M}$. As another result of Table 5.1, we see that $G \cap R$ is a minimal $\{M[0,l_0-\nu]\}$-obstruction and, by the inductive assumption, $G \cap R$ therefore has a size of at most $f(\{M[0,l_0-\nu]\}) = l_0 - \nu + 1$, so the size of G is $|G| = 2\nu + |R| \leq \nu + l_0 + 1$.

$k_0 = 0$ implies $s \neq 0$ and then $f(\mathcal{M})$ is

$$f(\mathcal{M}) = \sum_{i=0}^{r}(l_i+1)(k_i-k_{i-1}) \geq l_0 + 1 + (l_s+1)\cdot k_s > l_0 + 1 + \nu \geq |G|$$

$k_0 > 0$ on the other hand implies

$$f(\mathcal{M}) = \sum_{i=0}^{r}(l_i+1)(k_i-k_{i-1}) \geq (l_0+1)(k_0+1) \geq 2l_0 + 2 > l_0 + \nu + 1 \geq |G|$$

All cases except $\nu = 2 \wedge l_0 > 0 \wedge k_r > 0 \wedge (l_0 \leq \nu \vee k_r \leq \nu)$ have now been dealt with. If $l_0 \leq \nu \wedge k_r \leq \nu$ then $G \cap R$ obviously obstructs $M[k_i-\nu,0]$ and $M[0,l_i-\nu]$, so $G \setminus R$ obstructs $\mathcal{M}$ as shown in Table 5.1. This is only possible for $G \setminus R = G$ since G is a minimal $\mathcal{M}$-obstruction, which shows $R = \varnothing$ although this is impossible since we have assumed $G \neq P_4$. Not both l_0 and k_r can be lower or equal to ν and, by possibly complementing G and $\mathcal{M}$, we may assume $l_0 > \nu$ and $0 < k_r \leq \nu \Leftrightarrow k_r \in \{1,2\}$. In this case, $G \cap R$ is a minimal $\{M[0, l_0 - \nu]\}$-obstruction, as can be seen in Table 5.1, and by the inductive assumption, $G \cap R$ has $|G \cap R| \leq f(\{M[0, l_0 - \nu]\}) = l_0 - 1$ vertices. If there is an index $u \in \{0, \dots, r\}$ with $k_u = 1 \wedge l_u > 0$, the number of vertices in G is

$$|G| = 2\nu + |R| = l_0 + 3 \leq l_0 + 1 + l_u + 1 \leq \sum_{i=0}^{r} (l_i + 1)(k_i - k_{i-1}) = f(\mathcal{M})$$

If there is no such index u but $k_r = 1$, then $G \setminus \{c_1\}$ is an $\mathcal{M}$-obstruction because $G \setminus \{c_1\}$ obstructs $\{M[0, l_0]; \dots; M[0, l_r]\}$ and also obstructs $M[1,0]$ as it is not an empty graph. We may assume $k_r = 2$ now, so the size of G is

$$|G| = 2\nu + |R| = l_0 + 3 = l_0 + k_r + 1 \leq f(\mathcal{M})$$

Subcase 1e: $(a,b,c) \in \{(1,*,1); (*,0,0)\}$. As shown in Table 5.1, one of the induced subgraphs $G \setminus \{c_1\}$ or $G \setminus \{s_1\}$ of G obstructs $\mathcal{M}$ and therefore G is not a minimal $\mathcal{M}$-obstruction. Thus, this subcase is not possible.

Subcase 1f: $(a,b,c) \in \{(1,*,0); (*,0,1)\}$. By complementing M and G if necessary, we may assume $(a,b,c) = (1,*,0)$. Assume $G = P_4$, the smallest possible spider graph. We have $f(\mathcal{M}) < 4 = |P_4|$ only if $\mathcal{M} \subset \{M[0,1]; M[0,2]; M[1,0]; M[2,0]\}$. P_4 admits $M[0,2]$, which implies $M[0,2] \notin \mathcal{M}$, and then the induced subgraph $G \setminus \{c_1\}$ of G obstructs $\mathcal{M} \subset \{M[0,1]; M[1,0]; M[2,0]\}$. This contradicts the assumption that G is a minimal $\mathcal{M}$-obstruction and so we may assume $G \neq P_4$.

Case 1f.1: Assume $\nu = 2$. By Corollary 9, $G \setminus R$ as a spider graph obstructs all matrices $M[k,0]$. Also, $G \setminus R$ obstructs $M[0,1]$ as it is not a complete graph. $G \setminus R$ is connected and so by Lemma 9, $G \setminus R$ obstructs $M[k,1]$ for all $k \in \mathbb{N}$. This implies $l_0 \geq 2 = \nu$, as otherwise $G \setminus R$ obstructs $\mathcal{M}$ and, as G is a minimal $\mathcal{M}$-obstruction, this implies $R = \varnothing$ and $G = P_4$, which was assumed not to be the case at the beginning of this proof.

As shown in Table 5.1, $G \cap R$ is a minimal $\{M[0, l_0 - \nu], \dots, M[0, l_r - \nu]\}$-obstruction and by the inductive assumption, we see $|G \cap R| \leq l_0 - \nu + 1$.

$G \setminus \{c_1\}$ obstructs $M[k,0]$ ($k \in \mathbb{N}$) because of Corollary 9. Also, $G \setminus \{c_1\}$ obstructs $M[0, l_0]$ by Corollary 11. As an induced subgraph of the minimal $\mathcal{M}$-obstruction G, $G \setminus \{c_1\}$ does not obstruct $\mathcal{M}$ and therefore there must be a matrix $M[k_s, l_s]$ with $k_s > 0 \wedge l_s > 0$. Now two cases can be distinguished: We may have $s = 0$, in which case the size of G is

$$|G| = 2\nu + |R| \leq l_0 + \nu + 1 \leq 2l_0 + 1 \leq (k_0 + 1)l_0 + 1 \leq f(\mathcal{M})$$

For $s \neq 0$, the size of G can be determined as

$$|G| \leq l_0 + \nu + 1 \leq l_0 + 1 + (l_s + 1)k_s \leq \sum_{i=0}^{r} (l_i + 1)(k_i - k_{i-1}) = f(\mathcal{M})$$

Case 1f-2: Assume $\nu > 2$. By Corollary 11 and because $b = *$, for all $l \in \mathbb{N}$, the spider graph $G \setminus \{s_1, c_1\}$ (we have $\nu > 2$) admits $M[0,l]$ if and only if $(G \setminus \{s_1, c_1\}) \cap R$ admits $M[0, l-(\nu-1)]$. By the same corollary, G obstructs $M[0,l]$ if and only if $G \cap R$ obstructs $M[0, l-\nu]$. Because G obstructs $M[0, l_0]$, $G \cap R$ obstructs $M[0, l_0 - \nu]$ and so $G \setminus \{s_1, c_1\}$ obstructs $M[0, l_0 - 1]$. By Corollary 9 and because $\nu > 2$, $G \setminus \{s_1, c_1\}$ is a spider graph and therefore obstructs $M[k, 0]$ for all $k \in \mathbb{N}$. By Corollary 10, $G \setminus \{s_1, c_1\}$ also obstructs $M[k, l_0 - 1]$ for all $k \in \mathbb{N}$. For $l_0 < \nu$, $G \cap R$ obstructs $M[0, l_0 - (\nu - 1)]$ and then $G \setminus \{s_1, c_1\}$ obstructs even $M[0, l_0]$ and $M[k, l_0]$. Then the induced subgraph $G \setminus \{s_1, c_1\}$ of the minimal $\mathcal{M}$-obstruction G obstructs $\mathcal{M}$, which is obviously not possible, so we may assume $l_0 \geq \nu$. $G \setminus \{s_1, c_1\}$ is an induced subgraph of $G \setminus \{c_1\}$ and obstructs $M[k, l_0 - 1]$. Hence, $G \setminus \{c_1\}$ also obstructs $M[k, l_0 - 1]$ for all $k \in \mathbb{N}$. Additionally, $G \setminus \{c_1\}$ obstructs $M[0, l_0]$ as can be seen in Table 5.1. $G \setminus \{c_1\}$ does not obstruct $\mathcal{M}$ and therefore there must be a matrix $M[k_s, l_s] \in \mathcal{M}$ with $k_s > 0$ and $l_s \geq l_0 \geq \nu$.

Again, $G \cap R$ is a minimal $\{M[0, l_0 - \nu], \ldots, M[0, l_r - \nu]\}$-obstruction and, by the inductive assumption, we see $|G \cap R| \leq l_0 - \nu + 1$. The size of G therefore is

$$|G| = l_0 + \nu + 1 \leq l_s + l_s + 1 \leq l_s(k_s + 1) + 1 \leq f(\mathcal{M})$$

Subcase 1g: $(a,b,c) \in \{(1,0,0); (1,0,1)\}$**.** As shown in Table 5.1, all spider graphs obstruct $\mathcal{M}$. Therefore, G must be the smallest possible spider graph, P_4. As shown in Corollary 11, $G \setminus \{s_1\}$ obstructs $M[0,l]$ and, as shown in Corollary 9, $G \setminus \{c_1\}$ obstructs $M[k,0]$ for all $k, l \in \mathbb{N}$. Since both graphs are neither empty nor complete, both obstruct $M[1,0]$ and $M[0,1]$ as well. For $c = 0$, Lemma 9 implies that $G \setminus \{s_1\}$ obstructs $M[1,1]$ and, for $c = 1$, Lemma 9 with complemented M and G implies that $G \setminus \{c_1\}$ obstructs $M[1,1]$. Neither $G \setminus \{s_1\}$ nor $G \setminus \{c_1\}$ obstruct $\mathcal{M}$ as they are induced subgraphs of the minimal $\mathcal{M}$-obstruction G. Thus, $\mathcal{M}$ contains a matrix $M[k_s, l_s]$ with $k_s > 0 \wedge l_s > 0 \wedge (k_s > 1 \vee l_s > 1)$ or two matrices $M[k_t, l_t], M[k_u, l_u]$ with $k_t = 0 \wedge l_t > 1$ and $k_u > 1 \wedge l_u = 0$. In the first case, we have $f(\mathcal{M}) \geq (k_s + 1)(l_s + 1) \geq 4 = |P_4|$. In the second case, we have $f(\mathcal{M}) \geq (k_t + 1)(l_t + 1) + (k_u - k_t)(l_u + 1) \geq 4 = |P_4|$.

Case 2: G or its complement is disconnected. By possibly complementing G and M, we may assume that $G = G_1 \cup G_2$ is disconnected.

Subcase 2a: $(a,b,c) \in \{(*,0,0); (*,*,0); (*,0,*); (*,*,*)\}$ By Lemma 12, there is a number $j_i \in \{0, \ldots, l_i + 1\}$ for every index $i \in \{0, \ldots, r\}$ such that G_1 is an $M[k_i, j_i - 1]$-obstruction and G_2 is an $M[k_i, l_i - j_i]$-obstruction. The values j_i may be chosen such that $j_i \geq j_{i+1}$ and $l_i - j_i \geq l_{i+1} - j_{i+1}$:

For $j_i < j_{i+1}$, $M[k_i, j_{i+1} - 1]$ is a submatrix of $M[k_{i+1}, j_{i+1} - 1]$ and so G_1 also obstructs $M[k_i, j_{i+1} - 1]$. Additionally, $M[k_i, l_i - j_{i+1}]$ is a submatrix of $M[k_i, l_i - j_i]$ and so G_2 also obstructs $M[k_i, l_i - j_{i+1}]$. Thus, we may define $j_i := j_{i+1}$ instead of a j_i lower than j_{i+1}. For $j_i = j_{i+1}$, the second inequality $l_i - j_i \geq l_{i+1} - j_{i+1}$ is true because of $l_i \geq l_{i+1}$. For $l_i - j_i < l_{i+1} - j_{i+1}$ ($\Leftrightarrow j_{i+1} + l_i - l_{i+1} < j_i$), G_1 obstructs $M[k_i, (j_{i+1} + l_i - l_{i+1}) - 1]$ as a submatrix of $M[k_i, j_i]$ and G_2 obstructs $M[k_i, l_i - (j_{i+1} + l_i - l_{i+1})] = M[k_i, l_{i+1} - j_{i+1}]$ as a submatrix of $M[k_{i+1}, l_{i+1} - j_{i+1}]$. Thus, we may define $j_i := j_{i+1} + l_i - l_{i+1} \geq j_{i+1}$ instead of a value with $l_i - j_i < l_{i+1} - j_{i+1}$.

Obviously, G_1 obstructs $\mathcal{M}_1 := \{M[k_0, j_0 - 1]; \ldots; M[k_r, j_r - 1]\}$ and G_2 obstructs $\mathcal{M}_2 := \{M[k_0, l_0 - j_0]; \ldots; M[k_r, l_r - j_r]\}$. If G_1 has an induced subgraph $G_1' \subsetneq G_1$ that obstructs $\mathcal{M}_1$, then $(G_1' \cup G_2) \subsetneq G$ obstructs $\mathcal{M}$ by Lemma 12. Hence, G_1 is a minimal $\mathcal{M}_1$-obstruction. Similarly, G_2 is a minimal $\mathcal{M}_2$-obstruction. By the inductive assumption, the number of vertices in G is

$$\begin{aligned}
|G| =& |G_1| + |G_2| \leq f(\mathcal{M}_1) + f(\mathcal{M}_2) \\
=& \sum_{i=0}^{r-1} (j_i - 1 - (j_{i+1} - 1))(k_i + 1) + (j_r - 1 + 1)(k_r + 1) \\
&+ \sum_{i=0}^{r-1} (l_i - j_i - (l_{i+1} - j_{i+1}))(k_i + 1) + (l_r - j_r + 1)(k_r + 1) \\
=& \sum_{i=0}^{r-1} (k_i + 1)(j_i - j_{i+1} + l_i - j_i - l_{i+1} + j_{i+1}) + (l_r - j_r + j_r + 1)(k_r + 1) \\
=& \sum_{i=0}^{r} (k_i + 1)(l_i - l_{i+1}) = f(\mathcal{M})
\end{aligned}$$

Subcase 2b: $(a, b, c) \in \{(1, 0, 0); (1, *, 0); (1, 0, *); (1, *, *)\}$ By Lemma 13, there are numbers $u_i, v_i, w_i \in \{0, \ldots, l + 1\}$ for every $i \in \{0, \ldots, r\}$ such that

- G_1 obstructs $M[k_i, u_i - 1]$ and G_2 obstructs $M[0, l_i - u_i]$,
- G_2 obstructs $M[k_i, v_i - 1]$ and G_1 obstructs $M[0, l_i - v_i]$,
- $u_i + v_i < l_i + 2$, and
- if $k_i > 0$ then G_1 obstructs $M[1, w_i - 1]$ and G_2 obstructs $M[1, l_i - w_i]$.

We may further assume $u_i \geq u_{i+1}$ and $v_i \geq v_{i+1}$ ($0 \leq i < r$): Let $x_1, x_2 \in \mathbb{N}_0$ be the greatest values such that G_1 obstructs $M[0, x_1]$ and G_2 obstructs $M[0, x_2]$. Then we may assume $u_i := \max\{0, l_i - x_2\}$ and $v_i := \max\{0, l_i - x_1\}$; these are the lowest values such that G_2 obstructs $M[0, l_i - u_i]$ and G_1 obstructs $M[0, l_i - v_i]$ and so the conditions that G_2 obstructs $M[0, l_i - u_i]$, G_1 obstructs $M[0, l_i - v_i]$, and especially that G_1 obstructs $M[k_i, u_i - 1]$, G_2 obstructs $M[k_i, v_i - 1]$, and $u_i + v_i < l_i + 2$ are satisfied by u_i, v_i as long as there are any values u_i, v_i at all that satisfy them, which is ensured by Lemma 13. Let $t \in \{0, \ldots, r\}$ be the smallest number with $k_t > 0$. For $i > t$, we may assume $w_i := \min\{w_t, l_i + 1\}$ because G_1 obstructs the submatrix $M[1, w_i - 1]$ of

$M[1, w_t - 1]$ and G_2 obstructs the submatrix $M[1, l_i - w_i]$ of $M[1, l_t - w_t]$. This obviously implies $w_i \geq w_{i+1}$ and $l_i - w_i \geq l_{i+1} - w_{i+1}$ $(0 \leq i < r)$.

Thus, G_1 is a minimal $\mathcal{M}_1$-obstruction and G_2 is a minimal $\mathcal{M}_2$-obstruction with

$$\mathcal{M}_1 := \left\{ \begin{array}{l} M[0, l_0 - v_0], \\ M[k_0, u_0 - 1], \ldots, M[k_{t-1}, u_{t-1} - 1], \\ M[1, w_t - 1], \\ M[k_t, u_t - 1], \ldots, M[k_r, u_r - 1] \end{array} \right\}$$

$$\mathcal{M}_2 := \left\{ \begin{array}{l} M[0, l_0 - u_0], \\ M[k_0, v_0 - 1], \ldots, M[k_{t-1}, v_{t-1} - 1], \\ M[1, l_t - w_t], \\ M[k_t, v_t - 1], \ldots, M[k_r, v_r - 1] \end{array} \right\}$$

A graph that obstructs $M[1, d]$ for $d \in \mathbb{N}$ also obstructs $M[0, d]$, so for $0 \leq i < t$, we see from the definition of u_i, w_t, v_i that $l_i - v_i \geq l_t - v_t \geq w_t - 1$ and $l_i - u_i \geq l_t - u_t \geq l_t - w_t$, which immediately implies $w_t - 1 \geq u_t - 1$ and $l_t - w_t \geq v_t - 1$. $\mathcal{M}_1$ and $\mathcal{M}_2$ are staircase-like and so the inductive assumption applies to G_1 and G_2.

For $k_0 \geq 1$, we have $t = 0$. By the inductive assumption, the size of G is

$$\begin{aligned}
|G| =& |G_1| + |G_2| \leq f(\mathcal{M}_1) + f(\mathcal{M}_2) \\
=& (0 - (-1))(l_0 - v_0 + 1) + (1 - 0)(w_0 - 1 + 1) + (k_0 - 1)(u_0 - 1 + 1) \\
&+ \sum_{i=1}^{r} (k_i - k_{i-1})(u_i - 1 + 1) \\
&+ (0 - (-1))(l_0 - u_0 + 1) + (1 - 0)(l_0 - w_0 + 1) + (k_0 - 1)(v_0 - 1 + 1) \\
&+ \sum_{i=1}^{r} (k_i - k_{i-1})(v_i - 1 + 1) \\
=& 3(l_0 + 1) - v_0 - u_0 + (k_0 - 1)(u_0 + v_0) + \sum_{i=1}^{r} (k_i - k_{i-1})(v_i + u_i) \\
=& 3(l_0 + 1) + (k_0 - 2)(u_0 + v_0) + \sum_{i=1}^{r} (k_i - k_{i-1})(v_i + u_i)
\end{aligned}$$

For $k_0 > 1$, we have $k_0 - 2 \geq 0$ and $u_0 + v_0 \leq l_0 + 1$, so we may conclude further

$$\begin{aligned}
|G| \leq& 3(l_0 + 1) + (k_0 - 2)(l_0 + 1) + \sum_{i=1}^{r} (k_i - k_{i-1})(v_i + u_i) \\
=& (k_0 + 1)(l_0 + 1) + \sum_{i=1}^{r} (k_i - k_{i-1})(v_i + u_i) = f(\mathcal{M})
\end{aligned}$$

For $k_0 = 1$, G_1 obstructs $M[1, w_0 - 1]$ and therefore also obstructs $M[0, w_0 - 1]$. G_2 obstructs $M[1, l_0 - w_0]$ and thus G_2 also obstructs $M[0, l_0 - w_0]$. Also, we have $w_0 \geq u_0 \geq u_i$ and $l_0 - w_0 \geq$

$v_0 - 1 \geq v_i - 1$ ($0 \leq i \leq r$), so we may assume $u_0 := w_0$ and $v_0 := l_0 + 1 - w_0$ and all previous assumptions are still true (without, of course, the original definitions of u_0 and v_0). This also implies $u_0 + v_0 = l_0 + 1$ and so we see

$$\begin{aligned}|G| \leq& 3(l_0+1) + (k_0-2)(u_0+v_0) + \sum_{i=1}^{r}(k_i - k_{i-1})(v_i + u_i)\\ =&(k_0+1)(l_0+1) + \sum_{i=1}^{r}(k_i-k_{i-1})(v_i+u_i) = f(\mathcal{M})\end{aligned}$$

For $t > 0$ (and therefore $k_0 = 0$), the first three conditions for $i = 0$ state that G_1 obstructs $M[0, u_i - 1]$, G_2 obstructs $M[0, l_i - u_i]$, G_2 obstructs $M[0, v_i - 1]$, G_1 obstructs $M[0, l_i - v_i]$, and $u_i + v_i \leq l_i + 1$. Obviously, we may increase u_i to $u_i := l_i + 1 - v_i$ while still preserving these conditions and we still have $u_i \geq u_{i+1}$. In this case, the inductive assumption yields

$$\begin{aligned}|G| =&|G_1| + |G_2| \leq f(\mathcal{M}_1) + f(\mathcal{M}_2)\\ =&(k_0 - (-1))(u_0 - 1 + 1) + \sum_{i=1}^{t-1}(k_i - k_{i-1})(u_i - 1 + 1) + (1-0)(w_t - 1 + 1)\\ &+ (k_t - 1)(u_t - 1 + 1) + \sum_{i=t+1}^{r}(k_i - k_{i-1})(u_i - 1 + 1)\\ &+ (k_0 - (-1))(v_0 - 1 + 1) + \sum_{i=1}^{t-1}(k_i - k_{i-1})(v_i - 1 + 1) + (1-0)(l_t - w_t + 1)\\ &+ (k_t - 1)(v_t - 1 + 1) + \sum_{i=t+1}^{r}(k_i - k_{i-1})(v_i - 1 + 1)\\ =&u_0 + v_0 + l_t + 1 + (k_t - 1)(u_t + v_t) + \sum_{i=t+1}^{r}(k_i - k_{i-1})(u_i + v_i)\\ \leq&u_0 + v_0 + l_t + 1 + (k_t - 1)(l_t + 1) + \sum_{i=t+1}^{r}(k_i - k_{i-1})(l_i + 1)\\ =&(l_0 + 1) + k_t(l_t + 1) + \sum_{i=t+1}^{r}(k_i - k_{i-1})(l_i + 1) \leq f(\mathcal{M})\end{aligned}$$

Subcase 2c: $c = 1$ Before the main proof in this subcase, minimal $M[k, 0]$- and $M[0, l]$-obstructions ($k, l \in \mathbb{N}$) are looked at more generally: For $a = *$, a graph G admits $M[k, 0]$ if and only if it is k-colorable. A P_4-sparse and therefore perfect graph is k-colorable if and only if the size of its largest clique is at most k. Obviously, the only minimal $M[k, 0]$-obstruction perfect graph is the complete graph with $k + 1$ vertices K_{k+1}. Inversely, the only minimal $M[0, l]$-obstruction perfect graph for $b = *$ is the empty graph with $l + 1$ vertices $\overline{K_{l+1}}$. For $a = 1$, every $M[k, 0]$-partition $A_1, \ldots, A_k$ of a graph $G = (V, E)$ splits the graphs vertices into k parts such that two vertices are in different parts if and only if they are adjacent to each other. A graph $G \neq K_{k+1}$ that is an $M[k, 0]$-obstruction therefore contains two non-adjacent vertices and another vertex that distinguishes between the former two, which is an induced subgraph $\overline{P_3}$. For $k \geq 2$, $\overline{P_3}$ does not contain K_{k+1} as an induced subgraph and thus is the only other minimal $M[k,0]$-obstruction. Similarly, for $l \geq 2$ and $b = 0$ the only minimal $M[0,l]$-obstruction other than $\overline{K_{l+1}}$ is P_3.

For $l_0 = 0$, the graphs K_{k_r+1} and possibly $\overline{P_3}$ are the only minimal $M[k_r,0]$-obstructions and therefore also the only minimal $\mathcal{M}$-obstructions, so we either have $G = K_{k_r+1}$ or $G = \overline{P_3}$. Both cases result in $|G| \leq k_r+1 = (k_r+1)(l_r+1) \leq f(\mathcal{M})$, so we may assume $l_0 > 0$. Similarly, we may assume $k_r > 0$ as otherwise we had $G = \overline{K_{l_0+1}}$ or $G = P_3$ and $|G| \leq l_0+1 = (k_0+1)(l_0+1) \leq f(\mathcal{M})$.

By complementing M and G in Lemma 9, we see that a disconnected graph obstructs a matrix $M[k,l]$ if and only if G obstructs $M[k,0]$ as well as $M[0,l]$. $\mathcal{M} = \{M[k_0,l_0]; \ldots; M[k_r,l_r]\}$ is staircase-like, so a disconnected graph obstructs $\mathcal{M}$ if and only if it obstructs $M[k_r,0]$ and $M[0,l_0]$. So if G as a disconnected graph and minimal $\mathcal{M}$-obstruction has a disconnected induced subgraph H that obstructs $M[k_r,0]$ and $M[0,l_0]$ then G and H must be identical. The rest of this proof shows that G contains such an induced subgraph and that H has at most $k_r + l_0 + 1 \leq f(\mathcal{M})$ vertices. Note that G as an $M[k_r,0]$-obstruction and $M[0,l_0]$-obstruction contains induced subgraphs G_c and G_s such that G_c is a minimal $M[k_r,0]$-obstruction and G_s is a minimal $M[0,l_0]$-obstruction.

First we assume $G_c = K_{k_r+1} = (V_c,E_c)$ and $G_s = \overline{K_{l_0+1}} = (V_s,E_s)$. If $G \cap (V_c \cup V_s)$ is connected, there is a vertex $x \in V \setminus (V_c \cup V_s)$ that is non-adjacent to all vertices in $V_c \cup V_s$, because G is disconnected. In this case, we may exchange any vertex $y \in V_s$ with x and $(G_s \cup \{x\}) \setminus \{y\}$ is still an empty induced subgraph of G with l_0+1 vertices. Thus, we may assume that $H := G \cap (V_c \cup V_s)$ is disconnected. As a disconnected graph that obstructs $M[k_r,0]$ and $M[0,l_0]$, we see that H obstructs $\mathcal{M}$ and must therefore be identical to G. If there is a vertex $v \in V_c \setminus V_s$ that is non-adjacent to all vertices in V_s, then, for any $w \in V_s$, the set $V_s \cup \{v\} \setminus \{w\}$ is an independent set in G with $l_0 + 1$ vertices. $G \setminus \{w\}$ therefore obstructs $M[0,l_0]$ and of course $M[k_r,0]$, which makes the disconnected graph $G \setminus \{w\}$ an $\mathcal{M}$-obstruction, contradicting G being a minimal $\mathcal{M}$-obstruction. We may therefore assume that $V_c \setminus V_s$ does not contain vertices that are non-adjacent to all vertices in V_s and, similarly, $V_s \setminus V_s$ does not contain vertices that are adjacent to all vertices in V_c. Thus, V_c and V_s are the maximum clique and the maximum independent set in G, respectively, and, as there are no other vertices in G, they are the only maximum clique and the only maximum independent set. By Theorem 1, there is a maximum clique and a maximum independent set in G that meet in exactly one vertex. As there is only one maximum clique and only one maximum independent set in G, we see $|V_c \cap V_s| = 1$ and therefore

$$|G| = |G \cap (V_c \cup V_s)| = |V_c| + |V_s| - |V_c \cap V_s| = k_r + 1 + l_0 + 1 - 1 \leq f(\mathcal{M})$$

In the case $G_c = K_{k_r+1} = (V_c,E_c)$ and $G_s = P_3 = (\{p_1,p_2,p_3\},\{p_1 \times p_2, p_2 \times p_3\})$, we may assume $l_0 \geq 2$ and $a = 1$. Now assume that $G \cap (V_c \cup \{p_1,p_2,p_3\})$ is connected and that p_i for some $i \in \{1,2,3\}$ distinguishes between two vertices $y,z \in V_c$. Without loss of generality let i be 1. Then $H := G \setminus \{p_2,p_3\} \neq G$ is disconnected and contains G_c and $P_3 = G \cap \{p_1,y,z\}$ as induced subgraphs and thus H obstructs $\mathcal{M}$, which contradicts that G is a minimal $\mathcal{M}$-obstruction. Thus, every p_i ($1 \leq i \leq 3$) is either adjacent to all vertices in V_c or non-adjacent to all vertices in V_c, as otherwise $G \cap (V_c \cup \{p_1,p_2,p_3\})$ would be connected and some p_i would distinguish between vertices in V_c. If there is a p_i that is adjacent to all vertices in V_c then, for any $y \in V_c$,

$H := G \setminus \{y\} \neq G$ is disconnected, contains P_3 as an induced subgraph, and contains $K_{k_r+1} = G \cap ((V_c \setminus \{y\}) \cup \{p_i\})$ as an induced subgraph. H obstructs $\mathcal{M}$, which is not possible, so we may assume that all p_i are non-adjacent to the vertices in V_c, which makes $H := G \cap (V_c \cup \{p_1, p_2, p_3\})$ a disconnected graph. H obstructs $M[k_r, 0]$ and $M[0, l_0]$, so we have $H = G$. For $l_0 > 2$, the claim $|G| = k_r + 1 + 3 \leq k_r + 1 + l_0 \leq f(\mathcal{M})$ follows immediately, so we may assume $l_0 = 2$. $G \setminus \{p_2\}$ contains the empty graph with three vertices $\overline{K_3} = (\{p_1, p_3, x\}, \varnothing)$ (with $x \in V_c$) as an induced subgraph as well as the complete graph $G_c = K_{k_r+1}$ with $k_r + 1$ vertices. $G \setminus \{p_2\}$ is disconnected and obstructs $M[k_r, 0]$ and $M[0, 2] = M[0, l_0]$, so it obstructs $\mathcal{M}$, which is impossible for an induced subgraph of G. Hence, the case $l_0 = 2$ is not possible.

For $G_c = \overline{P_3} = (\{n_1, n_2, n_3\}, \{n_1 \times n_3\})$ and $G_s = \overline{K_{l_0+1}} = (V_s, \varnothing)$, we may assume $k_r \geq 2$ and $b = 0$. If there is a vertex $z_1 \in V \setminus V_s$ that distinguishes between two vertices $x_1, y_1 \in V_s$, without loss of generality $z_1 \times x_1 \in E$ and $z_1 \times y_1 \notin E$, then $H := G \cap (V_s \cup \{z_1\})$ is disconnected, contains the induced subgraphs G_s and $\overline{P_3} = (\{x_1, y_1, z_1\}, \{z_1 \times x_1\})$, and therefore obstructs $\mathcal{M}$ as an induced subgraph of G, which implies $G = H$. The size of G is then $|H| = l_0 + 1 + 1 \leq l_0 + k_r + 1 \leq f(\mathcal{M})$. If there is a vertex $x_2 \in V \setminus V_s$ that is adjacent to all vertices in V_s, then the vertices in V_s and x_2 are obviously in the same connected component of G. As G is disconnected, there is another connected component that contains a vertex y_2. y_2 is non-adjacent to x_2 and non-adjacent to the vertices in V_s, so $H := G \cap (V_s \cup \{x_2, y_2\})$ is disconnected, contains the induced subgraph G_s, and contains an induced subgraph $\overline{P_3} = (\{x_2, y_2, z_2\}, \{x_2 \times z_2\})$ ($z_2 \in V_s$). Therefore H obstructs $\mathcal{M}$ and is identical to G. This implies $|G| = |H| = |V_s| + 2 = l_0 + 3 \leq l_0 + k_r + 1 \leq f(\mathcal{M})$. As the last case to look at, we assume that no vertex in $V \setminus V_s$ is adjacent to a vertex in V_s, which implies that $\{n_1, n_3\} \cap V_s = \varnothing$. Then $H := G \setminus \{x_3\}$ ($x_3 \in V_s$) is disconnected and contains the induced subgraphs $\overline{K_{l_0+1}} = ((V_S \cup \{n_1\}) \setminus \{x_3\}, \varnothing)$ and G_c. Thus, $H \neq G$ obstructs $\mathcal{M}$, which contradicts that G is a minimal $\mathcal{M}$-obstruction.

For $G_c = \overline{P_3}$ and $G_s = P_3 = (\{p_1, p_2, p_3\}, \{p_1 \times p_2, p_2 \times p_3\})$, we assume $l_0 \geq 2$ and $k_r \geq 2$. G_s is connected while G is not, so there is a vertex $x \in V$ with $x \notin \{p_1, p_2, p_3\}$ that is non-adjacent to the vertices p_1,p_2, and p_3. $H := G \cap \{x, p_1, p_2, p_3\}$ is disconnected and has the induced subgraphs $\overline{P_3} = (\{x, p_1, p_2\}, \{p_1 \times p_2\})$ and obviously P_3. Thus, H obstructs $\mathcal{M}$, and so we have $H = G$ and $|G| = |H| = 4 \leq l_0 + k_r < f(\mathcal{M})$. □

The preceding theorem has the following important special case.

Corollary 12. *If M is a constant matrix and G is a P_4-sparse, minimal M-obstruction graph, then G has at most $(k+1)(l+1)$ vertices.*

6. Summary

Most of the properties that Feder et al. have proved for cographs [FHH06] have been proved for P_4-sparse graphs as well in this work.

When the matrix size is fixed, Feder et al. have shown how to determine whether a cograph with lists obstructs a matrix in linear time ([FHH06], Corollary 2.4). This thesis shows the same for P_4-sparse graphs in Corollary 3. Theorem 3 shows that the condition of a fixed matrix size cannot be omitted since otherwise the problem becomes NP-complete.

Feder et al. have shown that a minimal M-obstruction cograph with lists has at most $a^m m!$ vertices, where $a = \ln^{-1}(\frac{3}{2})$ ([FHH06], Theorem 2.3) and m is the size of the matrix. This is the only property of cographs proved in the article of Feder et al. [FHH06] that this thesis does not show to also apply to P_4-sparse graphs. Instead, Theorem 4 only shows that P_4-sparse minimal M-obstructions with lists have less than $4^{m+1} \cdot (m+1)!$ vertices.

Let $f_{\text{cograph}} : \mathbb{N} \longrightarrow \mathbb{N}$ be the function that maps $m \in \mathbb{N}$ to the size of the largest cograph minimal M-obstruction of a matrix of size m. Feder et al. have proved that $f_{\text{cograph}} \in O(8^m/\sqrt{m})$ ([FHH06], Theorem 3.2). This thesis shows in Theorem 5 that the upper bound $f_{P_4\text{-sparse}} \in O(16^m)$ applies if $f_{P_4\text{-sparse}}$ maps to the size of the largest, minimal M-obstruction, P_4-sparse graph instead of only a cograph.

For constant matrices, Feder et al. have shown that a minimal M-obstruction cograph has exactly $(k+1)(l+1)$ vertices if M is a $(*,*,*)$-block matrix with k diagonal 0s and l diagonal 1s ([FHH06], Corollary 4.2). For other constant matrices, minimal M-obstruction cographs have at most $(k+1)(l+1)$ vertices ([FHH06], Corollary 4.4). In Chapter 5 of this thesis, these results could be extendend to P_4-sparse graphs. For $(*,*,*)$-block matrices, Corollary 7 shows that a minimal M-obstruction, P_4-sparse graph has exactly $(k+1)(l+1)$ vertices. For constant matrices in general, Corollary 12 shows that a minimal M-obstruction, P_4-sparse graph has at most $(k+1)(l+1)$ vertices.

These results show that P_4-sparse graphs share many properties with cographs. This indicates to have a closer look on P_4-sparse graphs if a problem is easy to solve on the class of cographs but difficult for graphs in general. As another research direction, maybe the results presented in this thesis can be extended to an even larger class of graphs, for example the graphs with few P_4s that were introduced by Babel and Olariu [BO95]. Also, either an upper bound for the size of P_4-sparse minimal M-obstructions with lists exists that equals the upper bound already found for cographs or, if this is not possible, a counterexample for such an upper bound must exist, a P_4-sparse minimal M-obstruction with lists whose size is greater than the upper bound $a^m m!$, which Feder et al. have proved for cographs.

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Eidesstattliche Erklärung

Ich versichere, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe. Zitate und dem Sinne nach anderen Quellen entnommene Textpassagen sind entsprechend gekennzeichnet. Die Arbeit hat in gleicher oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegen.

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